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Reachability analysis of switched linear discrete singular systems

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Abstract: This paper studies the reachability problem of the switched linear discrete singular (SLDS) systems. Under the condition that all subsystems are regular, the reachability of the SLDS systems is characterized based on a peculiar repeatedly introduced switching sequence. The necessary and sufficient conditions are obtained for the reachability of the SLDS systems.

Keywords: Switched system; Linear discrete singular system; Reachability; Switching sequence

1 Introduction

During the past few years, the study of switched systems has been revivified (see e.g. [1∼4] and the references therein). Various conditions and subtle results on controllability, reachability and observability etc. are presented in [5∼10] for continuous-time periodic, general (non-periodic) switched control systems and discrete-time switched control systems, respectively.

Switched linear *singular* systems are important class of switched systems, which arise in, for example, electrical networks and economic systems [11∼17]. Due to the existence of switching actions, state-inconsistence phenomena often occurs in the electrical networks. This may result in discontinuity of network variables and presence of impulse voltage and currents at the switching instants. Physically, some problems like sparks and short circuits etc. may occur [13]. For dynamic economic systems, as pointed out by Cantó et al. [12], when the interrelationships among different industrial sectors are described, and the capital and the demand are variable depending on seasons, the system can be modelled as a periodically switched singular systems. Therefore, the study of state consistence, stability, impulse, and controllability and observability of switched linear singular systems are important research topics in the area of switching control.

Due to the complexity of the structure and behavior, the analysis and synthesis of the switched singular systems are much more complicated. Especially, state discontinuity, impulse (noncausality) phenomenon and regularity should be considered at the same time. Recently, some preliminary results on switched linear singular systems have been given (e.g. [12, 15, 18, 19]). In [18], the reachability of switched linear continuous singular systems is studied, and some necessary conditions and sufficient conditions are presented. In [19], asymptotic properties, including complexity reduction and limit behavior, of large-scale hybrid singular systems are analyzed. The controllability and solvability of the switched linear discrete singular (SLDS) systems with periodical switching law are considered in [12, 15] respectively. Up to now, there is no results available for the SLDS systems with designable switching laws.

The reachability problem of the SLDS systems with designable switching laws is studied under the condition that all subsystems are regular in this paper. Due to the noncausality of the discrete singular systems, it is difficult to give the geometric characterization for the SLDS systems with a general switching sequence. Thus, we construct a peculiar repeatedly switching sequence and give the geometric char-

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acterization of reachability of SLDS systems. Based on the geometric characterization a necessary condition and a sufficient condition are presented for the reachability of the SLDS systems.

2 Notations and preliminary results

Consider a switched linear discrete singular (SLDS) control system described by

$$E_\sigma x(k+1) = A_\sigma x(k) + B_\sigma u(k), \tag{1}$$

where σ: $\{0, 1, \cdots\} \to \Lambda$, $\Lambda = \{1, 2, \cdots, m\}$, is a designable switching law; $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, are the state and input, respectively; $E_i \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $\mathrm{rank}(E_i) < n$, $i \in \Lambda$.

First, we introduce the definitions of the reachability of the SLDS systems and the switching sequence.

Definition 1 State $x \in \mathbb{R}^n$ is reachable, if there exist a switching law σ and a time instant $k > 0$ such that the state trajectory $x(k)$ starting from the initial point $x(0) = 0$ is equal to x, i.e. $x(k) = x$.

The reachable set of system (1) is the set consisting of all reachable states.

System (1) is said to be completely reachable, if its reachable set is $\mathbb{R}^n$.

In the sequel, we will use π_k to denote a switching sequence with length k of the form

$$\pi_k = \{i_0, \cdots, i_{k-1}\},$$

with $i_l = \sigma(l) \in \Lambda$, $l = 0, \cdots, k-1$.

By Definition 1, the reachable set of system (1) can be obtained as follows:

$$\mathscr{R} = \bigcup_{k=1}^{\infty} \bigcup_{\pi_k} \mathscr{R}(\pi_k), \tag{2}$$

where $\mathscr{R}(\pi_k)$ denotes the reachable set under the switching sequence π_k.

From [20], it is known that a necessary and sufficient condition for the existence and uniqueness of the solution to (1) is that for all $i \in \Lambda$, (E_i, A_i) are regular. In this paper, we make the following assumption.

Assumption 1 For all $i \in \Lambda$, (E_i, A_i) are regular.

By [20], under this assumption there exist nonsingular matrices $P_i \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}$, $i \in \Lambda$, such that

$$P_i E_i Q_i = \begin{bmatrix} I_{i_1} & 0 \\ 0 & N_i \end{bmatrix}, \quad P_i A_i Q_i = \begin{bmatrix} G_i & 0 \\ 0 & I_{i_2} \end{bmatrix}, \tag{3}$$

where $N_i \in \mathbb{R}^{(n-n_i)\times(n-n_i)}$ is nilpotent with nilpotent index h_i, $G_i \in \mathbb{R}^{n_i\times n_i}$, I_{i1} and I_{i2} are unit matrices with proper dimensions. Let $Q_i = [\bar{Q}_{i1},\ \bar{Q}_{i2}]$, $Q_i^{-1} = [Q_{i1}^{\mathrm{T}},\ Q_{i2}^{\mathrm{T}}]^{\mathrm{T}}$, $P_i B_i = [B_{i1}^{\mathrm{T}},\ B_{i2}^{\mathrm{T}}]^{\mathrm{T}}$, $Q_i^{-1}x = [x_{i1}^{\mathrm{T}},\ x_{i2}^{\mathrm{T}}]^{\mathrm{T}}$, with $\bar{Q}_{i1} \in \mathbb{R}^{n\times n_i}$, $Q_{i1} \in \mathbb{R}^{n_i\times n}$, $B_{i1} \in \mathbb{R}^{n_i\times m}$, $x_{i1} \in \mathbb{R}^{n_i}$.

It is obvious that if one of the subsystems is reachable, then the SLDS system (1) is reachable. In this paper, we focus on the nontrivial case that none of the subsystem is reachable.

For any given matrices $A \in \mathbb{R}^{k\times k}$, $B \in \mathbb{R}^{k\times p}$, and subspace $\mathscr{W} \subseteq \mathbb{R}^k$, set $\langle A \mid \mathscr{W}\rangle = \sum_{i=0}^{n-1} A^i \mathscr{W}$, and denote $\langle A \mid \mathscr{C}(B)\rangle$ by $\langle A \mid B\rangle$, where $\mathscr{C}(B)$ denotes the column subspace of the matrix B. It can be shown that $\langle A \mid \mathscr{W}\rangle$ is invariant with respect to A. For convenience of citation, we introduce the following subspaces:

$$\mathscr{V}_1 = \sum_{i=1}^{m} \mathscr{C}_i, \quad \mathscr{V}_k = \sum_{i=1}^{m} \langle F_i \mid \mathscr{V}_{k-1}\rangle, \tag{4}$$

$$\mathscr{W}_1 = \sum_{i=1}^{m} \mathscr{D}_i, \quad \mathscr{W}_k = \sum_{i=1}^{m} \langle F_i^z \mid \mathscr{W}_{k-1}\rangle, \tag{5}$$

$k = 2, 3, \cdots$, where $\mathscr{C}_i = Q_i(\mathscr{C}(B_{i1}) \oplus \langle N_i \mid B_{i2}\rangle)$, $\mathscr{D}_i = Q_i(\langle G_i \mid B_{i1}\rangle \oplus \langle N_i \mid B_{i2}\rangle)$, $F_i = \bar{Q}_{i1} G_i Q_{i1}$, $i \in \Lambda$, $z = \max(n_i + h_j)$, $i, j = 1, 2, \cdots, m$, and $\oplus$ is the direct sum. From $Q_{i1}\bar{Q}_{i1} = I_{n_i}$, it is easy to know the following results that $F_i^k = \bar{Q}_{i1} G_i^k Q_{i1}$, $k \geqslant 2$. Imitating the proofs of Proposition 1 and 2 of [18], we have the following results.

Proposition 1 Under Assumption 1, the subspaces defined in (4) and (5) are independent of the choices of nonsingular matrices P_i and Q_i.

Proposition 2 Under Assumption 1, $\mathscr{W}_n \subseteq \mathscr{V}_n$, $\mathscr{V}_i \subseteq \mathscr{V}_n$, and $\mathscr{W}_i \subseteq \mathscr{W}_n$, $\forall i \in \mathbb{Z}^+$.

Lemma 1 For any given matrices $A \in \mathbb{R}^{n_1\times n_2}$, $B \in \mathbb{R}^{n_1\times n_2}$, and $C \in \mathbb{R}^{n_1\times n_3}$, if there exists a matrix $Z \in \mathbb{R}^{n_3\times n_2}$, such that $B = CZ$, then

$$\begin{aligned} &\{x:\ x = (A+B)y + Cz,\ \forall y \in \mathbb{R}^{n_2}, z \in \mathbb{R}^{n_3}\} \\ &= \{x:\ x = Ay + Cz,\ \forall y \in \mathbb{R}^{n_2}, z \in \mathbb{R}^{n_3}\}. \end{aligned}$$

3 Main results

3.1 Geometric characterization

In this subsection, we give the geometric characterization of the reachable set $\mathscr{R}(\pi_k)$. For a general switching sequence, we have the following result.

Lemma 2 Under Assumption 1, for a given switching sequence π_k, we have

$$\mathscr{R}(\pi_k) \subseteq \mathscr{V}_n. \tag{6}$$

Proof By [20], it is easy to know that the states of systems (1) at $k = 1$ with the initial value $x(0) = 0$ under the switching law π_1 and input $u(k)(k = 0, 1, \cdots,)$ are

$$x(1) = Q_{i_0}\begin{bmatrix} B_{i_0 1}u_{i_0}(0) \\ -\sum\limits_{j=0}^{h_{i_0}-1} N_{i_0}^j B_{i_0 2}u(j+1) \end{bmatrix}$$

By the definition of $\mathscr{R}(\pi_1)$, $\mathscr{C}_{i_0}$ and $\mathscr{V}_1$, and Proposition 2, we have

$$\mathscr{R}(\pi_1) = \mathscr{C}_{i_0} \subseteq \mathscr{V}_1 \subseteq \mathscr{V}_n,$$

that is, when $k = 1$, (6) holds. We now investigate (6) for $k = 2$. From [20], we have

$$x(2) = Q_{i_1}\begin{bmatrix} G_{i_1}Q_{i_1 1}x(1) + B_{i_1 1}u_{i_1}(1) \\ -\sum\limits_{j=0}^{h_{i_1}-1} N_{i_1}^j B_{i_1 2}u(j+2) \end{bmatrix}$$

This, together with $x(1) \in \mathscr{V}_1$ and the definitions of $\mathscr{R}(\pi_2)$, F_{i_1}, and $\mathscr{C}_{i_1}$, gives that

$$\begin{aligned}\mathscr{R}(\pi_2) &\subseteq Q_{i_1}((G_{i_1}Q_{i_1 1}\mathscr{R}(\pi_1) \\ &\quad +\mathscr{C}(B_{i_1 1})) \oplus \langle N_{i_1} \mid B_{i_1 2}\rangle) \\ &= F_{i_1}\mathscr{C}_{i_0} + \mathscr{C}_{i_1} \subseteq \mathscr{V}_2 \subseteq \mathscr{V}_n.\end{aligned}$$

Iteratively, for $j = 1, \cdots, k$ we have

$$\begin{aligned}\mathscr{R}(\pi_j) &\subseteq F_{i_{j-1}} \cdots F_{i_1}\mathscr{C}_{i_0} \\ &\quad +F_{i_{j-1}} \cdots F_{i_2}\mathscr{C}_{i_1} + \cdots + \mathscr{C}_{i_{j-1}} \\ &\subseteq \mathscr{V}_j \subseteq \mathscr{V}_n.\end{aligned} \tag{7}$$

Thus, (6) is true.

Due to the noncausality of the discrete singular systems, there is no inclusion between the reachable sets $\mathscr{R}(\pi_{s+1})$ and $\mathscr{R}(\pi_s)$. Hence, it is difficult to give the geometric characterization of a general switching sequence. Here, we construct a peculiar repeatedly switching sequence (RSS).

Definition 2 The switching sequence $\pi_{z\times j}$ is called a z-RSS if $\pi_{z\times j}$ has the following form:

$$(\underbrace{i_0, \cdots, i_0}_{z}, \cdots, \underbrace{i_s, \cdots, i_s}_{z}, \cdots, \underbrace{i_{j-1}, \cdots, i_{j-1}}_{z}),$$

where z is given in (5). In this case, we will denote such a z-RSS $\pi_{z\times j}$ simply as π_{zj}.

Now, we give the geometric characteristic of the reachable set $\mathscr{R}(\pi_{zj})$.

Lemma 3 Under Assumption 1, for a given z-RSS π_{zj}, we have

$$\begin{aligned}\mathscr{R}(\pi_{zj}) &= F_{i_{j-1}}^z \cdots F_{i_1}^z \mathscr{D}_{i_0} + F_{i_{j-1}}^z \cdots F_{i_2}^z \mathscr{D}_{i_1} \\ &\quad + \cdots + \mathscr{D}_{i_{j-1}}, \quad j = 1, 2, \cdots.\end{aligned} \tag{8}$$

Proof From [20], we can obtain the state $x(z)$ starting from $x(0) = 0$:

$$x(z) = Q_{i_0}\begin{bmatrix} \sum\limits_{k=0}^{z-1} G_{i_0}^k B_{i_0 1}u(z-1-k) \\ -\sum\limits_{k=0}^{h_{i_0}-1} N_{i_0}^k B_{i_0 2}u(k+z) \end{bmatrix}. \tag{9}$$

By the definitions of $\mathscr{D}_{i_0}$ and z, we have that $z \geqslant n_{i_0}$ and

$$\mathscr{R}(\pi_{z1}) = \{x(z) :\ x_0 = 0\} = \mathscr{D}_{i_0}.$$

That is, when $j = 1$, (8) holds. We now investigate (8) for $j = 2$. From [20], we have

$$\begin{aligned}&x(2z) \\ &= Q_{i_1}\begin{bmatrix} G_{i_1}^z Q_{i_1 1}x(z) + \sum\limits_{k=z}^{2z-1} G_{i_1}^{2z-k-1}B_{i_1 1}u(k) \\ -\sum\limits_{k=0}^{h_{i_1}-1} N_{i_1}^k B_{i_1 2}u(k+2z) \end{bmatrix},\end{aligned}$$

which together with (9) gives

$$\begin{aligned}&x(2z) \\ &= Q_{i_1}\begin{bmatrix} G_{i_1}^z Q_{i_1 1}\bar{Q}_{i_0 1}\sum\limits_{k=0}^{z-1} G_{i_0}^k B_{i_0 1}u(k) - G_{i_1}^z Q_{i_1 1}\bar{Q}_{i_0 2}\sum\limits_{k=0}^{h_{i_0}-1} N_{i_0}^k B_{i_0 2}u(k+z) + \sum\limits_{k=z}^{2z-1} G_{i_1}^{2z-k-1}B_{i_1 1}u(k) \\ -\sum\limits_{k=0}^{h_{i_1}-1} N_{i_1}^k B_{i_1 2}u(k+2z) \end{bmatrix} \\ &= Q_{i_1}\begin{bmatrix} G_{i_1}^z Q_{i_1 1}\bar{Q}_{i_0 1}\sum\limits_{k=0}^{z-1} G_{i_0}^k B_{i_0 1}u(k) + \sum\limits_{k=z+h_{i_0}}^{2z-1} G_{i_1}^{2z-k-1}B_{i_1 1}u(k) - \sum\limits_{k=z}^{z+h_{i_0}-1}(G_{i_1}^z Q_{i_1 1}\bar{Q}_{i_0 2}N_{i_0}^{k-z}B_{i_0 2} - G_{i_1}^{2z-k-1}B_{i_1 1})u(k) \\ -\sum\limits_{k=0}^{h_{i_1}-1} N_{i_1}^k B_{i_1 2}u(k+2z) \end{bmatrix}.\end{aligned}$$

Noticing that $z - h_{i_0} - 1 \geqslant n_{i_1} - 1$, we have

$$\{\sum_{k=z+h_{i_0}}^{2z-1} G_{i_1}^{2z-k-1} B_{i_1 1} u(k)\,, \forall u(k),$$
$$k = z + h_{i_0}, \cdots, 2z-1\}$$
$$= \langle G_{i_1} \mid B_{i_1 1} \rangle.$$

By the Cayley-Hamilton theorem [21], it is easy to know that

$$\mathscr{C}(G_{i_1}^{2z-k-1} B_{i_1 1}) \subseteq \langle G_{i_1} \mid B_{i_1 1} \rangle, k = z, \cdots, z + h_{i_0} - 1.$$

Therefore, by Lemma 1, we have

$$\begin{aligned} &\mathscr{R}(\pi_{z2}) \\ &= \{x(2z) : \ x_0 = 0\} \\ &= F_{i_1}^z \mathscr{D}(i_0) + Q_{i_1}(\langle G_{i_1} \mid B_{i_1 1} \rangle \oplus \langle N_{i_1} \mid B_{i_1 2} \rangle) \\ &= F_{i_1}^z \mathscr{D}(i_0) + \mathscr{D}(i_1). \end{aligned} \tag{10}$$

With the similar argument, we can prove that (8) holds for $j = 1, 2, \cdots$.

From (8) and the definition of $\mathscr{W}_n$, it is easy to know that

$$\mathscr{R}(\pi_{zj}) \subseteq \mathscr{W}_n, \ j = 1, 2, \cdots. \tag{11}$$

Remark 1 The main difference between $\mathscr{R}(\pi_{zj})$ given in (8) and the reachable set of the normal switched linear systems given in [7] is that $F_{i_k}^z$, $k = 0, \cdots, j-1$, are singular here (since $\bar{Q}_{i_k 1}$ are singular), while A_{i_k}, $k = 0, \cdots, j-1$, are assumed nonsingular, which is the key to the reachability proof in [7].

3.2 Reachability

In this subsection a necessary condition and a sufficient condition are presented for the complete reachability of the SLDS systems.

Imitating to proof of the corollary of the theorem 2 in [8], we have the following lemma.

Lemma 4 The SLDS system (1) is reachable if and only if there exist an integer $k < \infty$ and a switching sequence π_k such that

$$\mathscr{R}(\pi_k) = \mathbb{R}^n.$$

By this lemma we give the following necessary condition on reachability of the SLDS systems.

Theorem 1 Under Assumption 1, if the SLDS system (1) is completely reachable, then $\mathscr{V}_n = \mathbb{R}^n$ and there exists an $i \in \Lambda$ such that $\langle N_i \mid B_{i2} \rangle = \mathbb{R}^{n-n_i}$.

Proof By Definition 1, the complete reachability of the SLDS system (1) implies $\mathscr{R} = \mathbb{R}^n$, which together with $\mathscr{R} \subseteq \mathscr{V}_n$ gives $\mathscr{V}_n = \mathbb{R}^n$. Now we show the second part of the theorem.

It follows from Lemma 4 that if SLDS system (1) is completely reachable, then there exists a switching sequence π_k such that

$$\mathscr{R}(\pi_k) = \mathbb{R}^n. \tag{12}$$

From (7), we have

$$\begin{aligned} &\mathscr{R}(\pi_k) \\ &\subseteq F_{i_{k-1}} \cdots F_{i_1} \mathscr{C}_{i_0} + F_{i_{k-1}} \cdots F_{i_2} \mathscr{C}_{i_1} + \cdots + \mathscr{C}_{i_{k-1}}, \end{aligned}$$

which together with (12) gives

$$F_{i_{k-1}} \cdots F_{i_1} \mathscr{C}_{i_0} + F_{i_{k-1}} \cdots F_{i_2} \mathscr{C}_{i_1} + \cdots + \mathscr{C}_{i_{k-1}} = \mathbb{R}^n,$$

or equivalently,

$$\begin{aligned} &\bar{Q}_{i_{k-1}1}(G_{i_{k-1}} Q_{i_{k-1}1}(F_{i_{k-2}} \cdots F_{i_1} \mathscr{C}_{i_0} + \cdots \\ &+ \mathscr{C}_{i_{k-2}}) + \mathscr{C}(B_{i_k 1})) + \bar{Q}_{i_{k-1}2} \langle N_{i_{k-1}} \mid B_{i_{k-1}2} \rangle \\ &= \mathbb{R}^n. \end{aligned}$$

This, together with the definitions of $\bar{Q}_{i_{k-1}1}$ and $\bar{Q}_{i_{k-1}2}$, gives

$$\bar{Q}_{i_{k-1}2} \langle N_{i_{k-1}} \mid B_{i_{k-1}2} \rangle = \mathbb{R}^{n-n_{i_{k-1}}}.$$

Since $\bar{Q}_{i_{k-1}2}$ has full column rank, we have

$$\langle N_{i_{k-1}} \mid B_{i_{k-1}2} \rangle = \mathbb{R}^{n-n_{i_{k-1}}}.$$

This completes the proof.

System (1) is said to be reversible, if for each $i \in \Lambda$, A_i is nonsingular. It is easy to know that a sampled-data singular system is reversible [22].

Denote the z-RSS that realizes the maximal set as π_{zk}^r, that is,

$$\begin{aligned} &\dim(\mathscr{R}(\pi_{zk}^r)) \\ &= \max\{\dim(\mathscr{R}(\pi_{zj})), \ j = 0, 1, \cdots\}. \end{aligned} \tag{13}$$

We now give the sufficient condition of the reachability of the SLDS systems. First, we introduce the following lemma.

Lemma 5 If system (1) is reversible and $\langle N_i \mid B_{i2} \rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \Lambda$, then

$$\begin{aligned} &\dim(F_{i_{k-1}}^z \cdots F_{i_0}^z \mathscr{W}_n + F_{i_{k-1}}^z \cdots F_{i_1}^z \bar{Q}_{i_0 2} \langle N_{i_0} \mid \\ &B_{i_0 2} \rangle + \cdots + \bar{Q}_{i_{k-1}2} \langle N_{i_{k-1}} \mid B_{i_{k-1}2} \rangle) \\ &\geqslant \dim(\mathscr{W}_n). \end{aligned}$$

Proof From (3) and the nonsingularity of A_i, $i = 1, 2, \cdots, m$, one can see that G_i, $i = 1, 2, \cdots, m$, are nonsingular. By the definitions of $F_{i_{k-1}}$, $\bar{Q}_{i_{k-1}1}$ and $\bar{Q}_{i_{k-1}2}$, we have

$$\begin{aligned}&\dim(F^z_{i_{k-1}} \cdots F^z_{i_0}\mathscr{W}_n + F^z_{i_{k-1}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid \\ &\quad B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ &= \dim(Q_{i_{k-1}}(G^z_{i_{k-1}} Q_{i_{k-1}1}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots \\ &\quad + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)) \\ &\quad \oplus \langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle).\end{aligned}$$

From the nonsingularity of matrices $Q_{i_{k-1}}$ and $G_{i_{k-1}}$ and Dimension Theorem [21] it follows that

$$\begin{aligned}&\dim(Q_{i_{k-1}}(G^z_{i_{k-1}} Q_{i_{k-1}1}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots \\ &\quad + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)) \\ &\quad \oplus \langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ &= \dim(Q_{i_{k-1}1}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots \\ &\quad + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)) \\ &\quad + \dim(\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle).\end{aligned}$$

On the other hand, by $\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle = \mathbb{R}^{n-n_{i_{k-1}}}$ we can obtain

$$\begin{aligned}&\dim(\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ &\geqslant \dim(Q_{i_{k-1}2}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle \\ &\quad + \cdots + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)),\end{aligned}$$

which implies that

$$\begin{aligned}&\dim(Q_{i_{k-1}1}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots \\ &\quad + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)) \\ &\quad + \dim(\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ &\geqslant \dim(Q^{-1}_{i_{k-1}}(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n \\ &\quad + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots \\ &\quad + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle)) \\ &= \dim(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid \\ &\quad B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle).\end{aligned}$$

Hence, by similar deduction we have

$$\begin{aligned}&\dim(F^z_{i_{k-1}} \cdots F^z_{i_0}\mathscr{W}_n + F^z_{i_{k-1}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid \\ &\quad B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ &\geqslant \dim(F^z_{i_{k-2}} \cdots F^z_{i_0}\mathscr{W}_n + F^z_{i_{k-2}} \cdots F^z_{i_1}\bar{Q}_{i_0 2}\langle N_{i_0} \mid \\ &\quad B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-2}2}\langle N_{i_{k-2}} \mid B_{i_{k-2}2}\rangle) \\ &\geqslant \cdots \\ &\geqslant \dim(\mathscr{W}_n).\end{aligned}$$

We complete the proof.

Theorem 2 If system (1) is reversible and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \varLambda$, then

$$\mathscr{R}(\pi^r_{zk}) = \mathscr{W}_n,$$

where π^r_{zk} are given in (13).

Proof Let

$$\begin{aligned}&\pi_{z(j+1+k)} \\ &= (\underbrace{l_0, \cdots, l_0}_{z}, \cdots, \underbrace{l_s, \cdots, l_s}_{z}, \cdots, \underbrace{l_j, \cdots, l_j}_{z}, \pi^r_{zk}),\end{aligned}$$

where π^r_{zk} are given in (13) and $l_0, \cdots, l_j \in \varLambda$ are any given arbitrary integers. Then, from Lemma 2, we have

$$\begin{aligned}&\mathscr{R}(\pi_{z(j+1+k)}) \\ &= F^z_{i_{k-1}} \cdots F^z_{i_0} F^z_{l_j} \cdots F^z_{l_1}\mathscr{D}_{l_0} + F^z_{i_{k-1}} \cdots F^z_{i_0} F^z_{l_j} \cdots \\ &\quad \times F^z_{l_2}\mathscr{D}_{l_1} + \cdots + F^z_{i_{k-1}} \cdots F^z_{i_1}\mathscr{D}_{i_0} + \cdots + \mathscr{D}_{i_{k-1}}.\end{aligned}$$

It is easy to see that

$$F^z_{i_{k-1}} \cdots F^z_{i_1}\mathscr{D}_{i_0} + \cdots + \mathscr{D}_{i_{k-1}} \subseteq \mathscr{R}(\pi_{z(j+1+k)}),$$

or equivalently,

$$\mathscr{R}(\pi^r_{zk}) \subseteq \mathscr{R}(\pi_{z(j+1+k)}). \tag{14}$$

By (13) we have

$$\dim(\mathscr{R}(\pi^r_{zk})) \geqslant \dim(\mathscr{R}(\pi_{z(j+1+k)})),$$

which, together with (14), implies

$$\mathscr{R}(\pi^r_{zk}) = \mathscr{R}(\pi_{z(j+1+k)}).$$

This, together with Lemma 5 and the definitions of $\mathscr{D}_{i_q}$, $q = 0, \cdots, k-1$, gives

$$\begin{aligned}&F^z_{i_{k-1}} \cdots F^z_{i_0} F^z_{l_j} \cdots F^z_{l_1}\mathscr{D}_{l_0} + F^z_{i_{k-1}} \cdots F^z_{i_1}\bar{Q}_{i_0 2} \\ &\times\langle N_{i_0} \mid B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle \\ &\subseteq \mathscr{R}(\pi_{z(j+1+k)}) \\ &= \mathscr{R}(\pi^r_{zk}).\end{aligned} \tag{15}$$

Noticing that

$$\mathscr{W}_n = \sum_{p_0,p_1,p_2,\cdots,p_n=1,2,\cdots,m}^{q_1,q_2,\cdots,q_n=0,z,\cdots,(n-1)z} F_{p_n}^{q_n} F_{p_{n-1}}^{q_{n-1}} \cdots F_{p_1}^{q_1} \mathscr{D}_{p_0},$$

and the arbitrariness of $l_0, l_1, \cdots, l_j$ and j, we have by (15)

$$\begin{aligned} & F_{i_{k-1}}^z \cdots F_{i_0}^z \mathscr{W}_n + F_{i_{k-1}}^z \cdots F_{i_1}^z \bar{Q}_{i_0 2}\langle N_{i_0} \mid B_{i_0 2}\rangle \\ & + \cdots + \bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle \\ & \subseteq \mathscr{R}(\pi_{zk}^r). \end{aligned} \tag{16}$$

From Lemma 5 we have

$$\begin{aligned} & \dim(F_{i_{k-1}}^z \cdots F_{i_0}^z \mathscr{W}_n + F_{i_{k-1}}^z \cdots F_{i_1}^z \bar{Q}_{i_0 2}\langle N_{i_0} \mid \\ & \quad B_{i_0 2}\rangle + \cdots + \bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) \\ & \geqslant \dim(\mathscr{W}_n), \end{aligned}$$

which, together with (16), gives

$$\dim(\mathscr{W}_n) \leqslant \dim(\mathscr{R}(\pi_{zk}^r)).$$

Noticing that (11), we have

$$\mathscr{W}_n = \mathscr{R}(\pi_{zk}^r). \tag{17}$$

Similar to the above analysis procedure, if we define subspaces and switching sequences as

$$\begin{cases} \overline{\mathscr{W}}_1 = \sum_{i=1}^{m} \mathscr{D}_i, \\ \overline{\mathscr{W}}_k = \sum_{i=1}^{m} F_i^z \langle F_i \mid \overline{\mathscr{W}}_{k-1}\rangle, \ k = 2, 3, \cdots, \end{cases} \tag{18}$$

and

$$\pi_{p_0,\cdots,p_{j-1}} = (\underbrace{i_0, \cdots, i_0}_{p_0}, \cdots, \underbrace{i_s, \cdots, i_s}_{p_s}, \cdots, \underbrace{i_{j-1}, \cdots, i_{j-1}}_{p_{j-1}}),$$

where $p_k \geqslant z$, $k = 0,\cdots,j-1$, z and $\mathscr{D}_i$, $i = 1, \cdots, m$, are given in (5), then we can obtain the following theorem.

Theorem 3 If system (1) is reversible and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \Lambda$, then there exists a switching sequence $\pi_{p_0,\cdots,p_{j-1}}$ such that

$$\mathscr{R}(\pi_{p_0,\cdots,p_{j-1}}) = \overline{\mathscr{W}}_n.$$

Remark 2 From Theorem 3 it is obvious that when $\overline{\mathscr{W}}_n = \mathbb{R}^n$, system (1) is completely reachable. From (18) we can verify that $\mathscr{W}_1 = \overline{\mathscr{W}}_1$ and $\mathscr{W}_k \subset \overline{\mathscr{W}}_k$ for $k = 2, 3, \cdots$. This, together with Theorems 2-3, implies when A_i is nonsingular and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \Lambda$, one can reach more by switching sequences of the form $\pi_{p_0,\cdots,p_{j-1}}$ than by switching sequences of the form π_{zk}. In other words, the more general the switching sequence is, the larger the reachable state set is. We guess that when A_i is nonsingular and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \Lambda$, the largest reachable set is $\widehat{\mathscr{W}_n}$, which can be iteratively defined as follows:

$$\widehat{\mathscr{W}_1} = \sum_{i=1}^{m} \mathscr{D}_i, \quad \widehat{\mathscr{W}_k} = \sum_{i=1}^{m} \langle F_i \mid \widehat{\mathscr{W}_{k-1}}\rangle, \ k = 2, 3, \cdots.$$

4 Conclusions

The reachability problem of the switched linear discrete singular systems is studied under the condition that all subsystems are regular. Due to the noncausality of the discrete singular systems, generally speaking, it is difficult to give the geometric characterization of reachable set with a general switching sequence. Thus, we construct a peculiar repeatedly switching sequence and obtain the geometric characterization of the reachability of the SLDS systems. Based on the geometric characterization a necessary condition and a sufficient condition are presented for the reachability of the SLDS systems.

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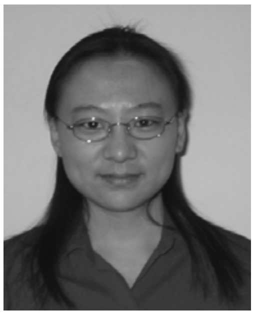

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