# Reachability Conditions for Switched Linear Singular Systems

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***Abstract*—The reachability problem of switched linear singular systems is investigated in this note. By using the geometric approach and the admissible control set introduced, a necessary condition and a sufficient condition on complete reachability are presented. The conditions are the same as those of the conventional (nonsingular) switched linear systems and normal (nonswitching) singular systems given in previous papers when the systems degenerate to normal singular systems and conventional switched systems, respectively.**

***Index Terms*—Reachability, singular system, switched linear system.**

## I. Introduction

Switched systems are frequently encountered in practice, for example, in robot manipulators [11], power systems [25], and even in human behaviors. It is for this reason that the study of such systems has attracted considerable attention, and a lot of new results have emerged recently (e.g., [1], [4], [7], [12], [17], [18], [23], [26], and [27]). In [7] and [23], conditions were given for verifying the observability and detectability of continuous-time switched linear systems. In [18], necessary and sufficient conditions on controllability and observability were obtained for continuous-time switched linear systems by taking advantage of the geometric characterization of the controllable sets of switched linear systems and the dual principle of controllability and observability of linear time-varying systems. In [27], a switching law was constructed so that the controllable set with this single switching law is equal to that with all switching laws. This means that the controllability can be realized by using only one switching sequence. As for the study on controllability and reachability of discrete-time switched systems, some basic results have been given in [9], [15], and [28], etc.

Switched linear *singular* (SLS) systems constitute an important class of switched systems, which arises, for example, in electrical networks and economic systems [2], [3], [8], [14], [16], [19], [21], [22], [24]. Due to the existence of switching actions, state-inconsistence phenomena often occurs in the electrical networks. This may result in the discontinuity of network variables and in the presence of impulse voltage and currents at the switching instants. Physically, some problems like sparks and short circuits etc. may occur [8]. For dynamic economic systems, as pointed out by Cantó *et al.* [3], when the interrelationships among different industrial sectors are described, and the capital and the demand are variable depending on seasons, the system can be modeled as periodically switched singular systems. Therefore, states consistence, stability, impulse, and controllability and observability of SLS systems are important research topics in the area of switching control.

Both analysis and synthesis of switched singular systems are more difficult, since stability, regularity, impulse elimination and state consistence of such systems should be considered at the same time. Recently, some preliminary results on such systems have been given [3], [13], [19], [21], [22], [24], [26]. In [21], [22], and [24], the problems such as how to obtain consistent initial conditions and switching transformation matrices (to express the discontinuity of the state variables at the switching instants) were investigated by using Laplace transformation and differential-algebraic equation. In [3] and [19], the solvability and controllability of periodically switched singular systems were studied. In [13], for the case where the switching laws were designable, both state feedback gain matrices and switching laws were designed such that the closed-loop SLS system was admissible and the states were continuous. In [26], asymptotic properties, including complexity reduction and limit behavior, of large-scale hybrid singular systems were analyzed.

This note is devoted to the reachability problem of SLS systems. For any given switching sequence, an admissible control set is defined to characterize the impact of the input choice to the continuity of the system states. Under the regularity condition, a necessary condition and a sufficient condition on complete reachability of SLS systems are obtained, which are the same as those of the conventional (nonsingular) switched linear systems and normal (nonswitching) singular systems given in [6] and [20] when the systems degenerate to normal singular systems and conventional switched systems, respectively.

The remainder of the note is organized as follows. In Section II, some notations and preliminary results are given. In Section III, a necessary condition and a sufficient condition for reachability are obtained. In Section IV, some concluding remarks are presented.

## II. Notations and Preliminary Results

Consider an SLS control system described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u_{\sigma(t)}(t) \tag{1}$$

where $\sigma(t) : [0, +\infty) \to \Lambda = \{1, 2, \ldots, m\}$ is a right-continuous piecewise constant mapping; $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$, $i \in \Lambda$ are the state and input, respectively; $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $E_i \in \mathbb{R}^{n\times n}$, $\mathrm{rank}E_i = r_i \le n$, $i \in \Lambda$.

Throughout this note, $\mathbb{C}$ denotes the set of all complex numbers; $\mathbb{R}^n$ denotes the real $n$-dimensional space; $\mathbb{R}^{n\times n}$ denotes the real $n \times n$-dimensional space; for a given vector or matrix $X$, $X^T$ denotes its transpose; and $\Lambda$ denotes the integer set $\{1, 2, \ldots, m\}$.

From [6], it is known that a necessary and sufficient condition for the existence and uniqueness of the solution of (1) is that for all $i \in \Lambda$, $(E_i, A_i)$ are regular. So, in this note, we assume the following.

*Assumption 1:* For all $i \in \Lambda$, $(E_i, A_i)$ are regular, i.e., for every $i \in \Lambda$, there exists $s_i \in \mathbb{C}$ such that $\det(s_iE_i - A_i) \neq 0$.

By Assumption 1, there exist nonsingular matrices $P_i \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}, i \in \Lambda$, such that [6]

$$P_iE_iQ_i = \begin{bmatrix} I_{i_1} & 0 \\ 0 & N_i \end{bmatrix} \quad P_iA_iQ_i = \begin{bmatrix} G_i & 0 \\ 0 & I_{i_2} \end{bmatrix} \tag{2}$$

where $N_i \in \mathbb{R}^{(n-n_i)\times(n-n_i)}$ is a nilpotent matrix with nilpotent index $h_i$; $G_i \in \mathbb{R}^{n_i\times n_i}$; $I_{i_1}$ and $I_{i_2}$ are identity matrices with proper dimensions. Let $Q_i^{-1} = [Q_{i1}^T\ Q_{i2}^T]^T$ and $P_iB_i = [B_{i1}^T\ B_{i2}^T]^T$, where $Q_{i1} \in \mathbb{R}^{n_i\times n}$ and $B_{i1} \in \mathbb{R}^{n_i\times m_i}$. Then, it is easy to see that $Q_{i1}Q_i = [I_{n_i}\ 0]$.

From [6], we know that for any fixed regular singular subsystem

$$E_i\dot{x} = A_ix + B_iu_i \quad x(t_0) = x_0 \tag{3}$$

the solution of (3) with initial value $x(t_0) = x_0$ and input $u_i$ can be expressed as shown in (4) at the bottom of the next page, where $f^{(k)}(t)$ and $f^{(r)}(t^+)$ denote the $k$-derivative and right $r$-derivative at $t$ of the generalized function $f(t)$ respectively.

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For clarity, let $x(t;t_0,x_0,u,\sigma)$ denote the state trajectory at time $t$ of system (1) starting from $t_0$ with initial value $x_0$, input $u$ and switching law $\sigma$. For any given time interval $[t_1,t_2]$, suppose that $\sigma(t)$ has $k$ switching (discontinuous) points $t_{11}, t_{12}, \ldots, t_{1k}$ ($t_1 < t_{11} < t_{12} < \cdots < t_{1k} < t_2$), i.e., for any $t \in [t_{1j}, t_{1(j+1)})$, $\sigma(t) = \sigma(t_{1j}) \in \Lambda$, $j = 0, 1, \ldots, k-1$, $t_{10} = t_1$, and for any $t \in [t_{1k}, t_2]$, $\sigma(t) = \sigma(t_{1k}) \in \Lambda$. We denote this switching sequence as $\{\sigma(t_{1j}), t_{1j}\}_{j=0}^{k}$; and for any given initial value $x(t_1) = x_0$, time interval $[t_1,t_2]$, and switching sequence $\sigma = \{\sigma(t_{1j}), t_{1j}\}_{j=0}^{k}$, we define an admissible control set $\mathcal{U}_\sigma[t_1,t_2]$ as follows:

$$\mathcal{U}_\sigma[t_1,t_2] = \Bigg\{ u : u = \left[u_1^T, u_2^T, \ldots, u_m^T\right]^T \\ u_i \in C^{h-1}[t_1,t_2] \\ \sum_{r=0}^{h_{\sigma(t_{1j})}-1} N_{\sigma(t_{1j})}^r B_{\sigma(t_{1j})2} u_{\sigma(t_{1j})}^{(r)}(t_{1j}^+) \\ = -Q_{\sigma(t_{1j})2} x\left(t_{1j}^-; t_1, x_0, u, \sigma\right) \\ j = 0, 1, \ldots, k, t_{10}^- = t_1 \Bigg\} \tag{5}$$

where $h = \max\{h_1, h_2, \ldots, h_m\}$; $C^h[t_1,t_2]$ is the set of all $h$-differentiable functions in the interval $[t_1,t_2]$; $u_{\sigma(t_{1j})}^{(r)}(t_{1j}^+)$ and $x(t_{1j}^-; t_1, x_0, u, \sigma)$ are the right $r$-derivative of $u_{\sigma(t_{1j})}(t)$ at $t = t_{1j}$ and left limit of $x(t;t_1,x_0,u,\sigma)$ at $t = t_{1j}$, respectively.

*Remark 1:* At first glance, the admissible set $\mathcal{U}_\sigma[t_1,t_2]$ is very complex and hard to know if it is empty or not. But, in some special cases, for instance, when $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_{i1}}$, $i = 1, \ldots, m$, one can easily see that $\mathcal{U}_\sigma[t_1,t_2]$ is nonempty.

By the definition of the admissible control set $\mathcal{U}_\sigma[t_1,t_2]$ we see that for any given switching law $\sigma$ and time interval $[t_1,t_2]$, the states of system (1) are continuous in $[t_1,t_2]$ under the action of all admissible input $u \in \mathcal{U}_\sigma[t_1,t_2]$. In fact, by (4), we have

$$x(t;t_1,x_0,u,\sigma) = Q_{\sigma(t_{1j})} \begin{bmatrix} e^{G_{\sigma(t_{1j})}(t-t_{1j})} Q_{\sigma(t_{1j})1} x(t_{1j}^-; t_1, x_0, u, \sigma) \\ + \int_{t_{1j}}^t e^{G_{\sigma(t_{1j})}(t-\tau)} B_{\sigma(t_{1j})} u_{\sigma(t_{1j})}(\tau) d\tau \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -\sum_{r=0}^{h_{\sigma(t_{1j})}-1} N_{\sigma(t_{1j})}^r B_{\sigma(t_{1j})2} u_{\sigma(t_{1j})}^{(r)}(t) \end{bmatrix} \quad \forall t \in [t_{1j}, t_{1j+1}).$$

Letting $t \to t_{1j}^+$, by (5) we have

$$\begin{aligned} x(t_{1j}^+; t_1, x_0, u, \sigma) &= Q_{\sigma(t_{1j})} \begin{bmatrix} Q_{\sigma(t_{1j})1} x(t_{1j}^-; t_1, x_0, u, \sigma) \\ -\sum_{r=0}^{h_{\sigma(t_{1j})}-1} N_{\sigma(t_{1j})}^r B_{\sigma(t_{1j})2} u_{\sigma(t_{1j})}^{(r)}(t_{1j}^+) \end{bmatrix} \\ &= Q_{\sigma(t_{1j})} \begin{bmatrix} Q_{\sigma(t_{1j})1} x(t_{1j}^-; t_1, x_0, u, \sigma) \\ Q_{\sigma(t_{1j})2} x(t_{1j}^-; t_1, x_0, u, \sigma) \end{bmatrix} \\ &= x(t_{1j}^-; t_1, x_0, u, \sigma). \end{aligned}$$

This implies that $x(t;t_1,x_0,u,\sigma)$ is continuous at switching points and, hence, in $[t_1,t_2]$.

*Definition 1:* SLS system (1) is said completely reachable, if for any given initial time $t_0 \in \mathbb{R}$ and state $x_f \in \mathbb{R}^n$, there exist a real number $t_f > t_0$, a switching law $\sigma(t)$ and an admissible input $u \in \mathcal{U}_\sigma[t_0, t_f]$, such that $x_f = x(t_f; t_0, 0, u, \sigma)$.

Obviously, if there exists an $i \in \Lambda$ such that $(E_i, A_i, B_i)$ is reachable, then by setting $\sigma(t) = i$, SLS system (1) is completely reachable. Thus, without loss of generality, in this note we will consider the nontrivial case where all the subsystems are not reachable, that is, $\forall i \in \Lambda$, $(E_i, A_i, B_i)$ is not reachable.

For any given matrices $A \in \mathbb{R}^{k\times k}$, $B \in \mathbb{R}^{k\times p}$, and subspace $\mathcal{W} \subseteq \mathbb{R}^k$, denote $\mathcal{R}(B)$ the subspace spanned by the columns of $B$, $\langle A \mid \mathcal{W}\rangle = \sum_{i=1}^k A^{i-1}\mathcal{W}$, and $\langle A \mid \mathcal{R}(B)\rangle$ as $\langle A \mid B\rangle$. It can be shown that $\langle A \mid \mathcal{W}\rangle$ is invariant with respect to $A$. For convenience of citation, we introduce the following subspaces:

$$\mathcal{V}_1 = \sum_{i=1}^m Q_i \mathcal{C}_i, \mathcal{V}_k = \sum_{i=1}^m Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \mid B_{i2}\rangle), \quad k = 2, \ldots \tag{6}$$

where $\mathcal{C}_i = \langle G_i \mid B_{i1}\rangle \oplus \langle N_i \mid B_{i2}\rangle$, $i \in \Lambda$, and $\oplus$ is the direct sum in vector space.

*Remark 2:* When $E_i = I$, $i \in \Lambda$, the subspaces $\mathcal{V}_i$ defined above for SLS systems are the same as those defined for conventional switching systems in [20].

We now show that the subspaces defined in (6) are independent of the choices of nonsingular matrices $P_i$ and $Q_i$. For regular matrices pencils $(E_i, A_i)$, $i \in \Lambda$, suppose that there exist nonsingular matrices $\bar{P}_i \in \mathbb{R}^{n\times n}$ and $\bar{Q}_i \in \mathbb{R}^{n\times n}$, different from $P_i$ and $Q_i$, such that

$$\bar{P}_i E_i \bar{Q}_i = \begin{bmatrix} \bar{I}_{i_1} & 0 \\ 0 & \bar{N}_i \end{bmatrix} \quad \bar{P}_i A_i \bar{Q}_i = \begin{bmatrix} \bar{G}_i & 0 \\ 0 & \bar{I}_{i_2} \end{bmatrix} \tag{7}$$

where $\bar{N}_i \in \mathbb{R}^{(n-\bar{n}_i)\times(n-\bar{n}_i)}$ is a nilpotent matrix with nilpotent index $\bar{h}_i$; $\bar{G}_i \in \mathbb{R}^{\bar{n}_i\times\bar{n}_i}$; $\bar{I}_{i1}$ and $\bar{I}_{i_2}$ are identity matrices with proper dimensions. Let $\bar{Q}_i^{-1} = [\bar{Q}_{i1}^T \ \bar{Q}_{i2}^T]^T$, $\bar{P}_i B_i = [\bar{B}_{i1}^T \ \bar{B}_{i2}^T]^T$, and $\bar{\mathcal{C}}_i = \langle \bar{G}_i \mid \bar{B}_{i1}\rangle \oplus \langle \bar{N}_i \mid \bar{B}_{i2}\rangle$, where $\bar{Q}_{i1} \in \mathbb{R}^{\bar{n}_i\times n}$ and $\bar{B}_{i1} \in \mathbb{R}^{\bar{n}_i\times m_i}$.

*Proposition 1:* Under Assumption 1, if we define subspaces $\overline{\mathcal{V}}_j$, $j = 1, 2, \ldots, n$, according to (6) with matrices $\bar{Q}_i, \bar{Q}_{i1}, \bar{G}_i, \bar{N}_i, \bar{B}_{i1}, \bar{B}_{i2}$ and $\bar{\mathcal{C}}_i$, $i = 1, 2, \ldots, m$, then $\mathcal{V}_j = \bar{\mathcal{V}}_j$, $j = 1, 2, \ldots, n$.

*Proposition 2:* Under Assumption 1, we have

1.) $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n$;
2.) if there exists $1 < i \le n$ such that $\mathcal{V}_i = \mathcal{V}_{i-1}$, then for all $l > i$, $\mathcal{V}_l = \mathcal{V}_i$.

The proofs of these two propositions are given in the Appendix.

*Remark 3:* From Proposition 2, we see that $\mathcal{V}_i \subseteq \mathcal{V}_n$, $\forall i < n$; and $\mathcal{V}_j = \mathcal{V}_n$, $\forall j > n$.

In order to prove the main results in the next section, we first give several basic lemmas. Among them, Lemma 1 is from [6]; Lemmas 2–5 are proved in the Appendix; Lemmas 6 and 7 are obvious, and hence, with the proofs omitted.

*Lemma 1:* [6] For any given matrix $A \in \mathbb{R}^{n\times n}$, there exist continuous functions $f_0(t), f_1(t), \ldots, f_{n-1}(t)$ such that

$$e^{At} = f_0(t)I + f_1(t)A + \cdots + f_{n-1}(t)A^{n-1}.$$

*Lemma 2:* For any given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$, and any nonzero polynomial $f(t)$, let

$$W(f,t) = \int_0^t f(s) e^{A(t-s)} BB^T e^{A^T(t-s)} f(s)\, ds.$$

Then, $\mathcal{R}(W(f,t)) = \langle A \mid B\rangle$, $\forall t > 0$.

$$\begin{cases} Q_{i1}x(t) = e^{G_i(t-t_0)} Q_{i1} x_0 + \int_{t_0}^t e^{G_i(t-\tau)} B_{i1} u_i(\tau) d\tau \\ Q_{i2}x(t) = -\sum_{k=0}^{h_i-1} N_i^k B_{i2} u_i^{(k)}(t) - \sum_{k=1}^{h_i-1} N_i^k \delta^{(k-1)}(t-t_0) \left(Q_{i2}x_0 - \sum_{r=0}^{h_i-1} N_i^r B_{i2} u_i^{(r)}(t_0^+)\right) \end{cases} \tag{4}$$

*Lemma 3:* Under Assumption 1, for all $i \in \Lambda$, $\langle G_i \,|\, B_{i1}\rangle \subseteq \langle G_i \,|\, Q_{i1}\mathcal{V}_k\rangle$, $k = 1, 2, \ldots, n$.

*Lemma 4:* For any given $t_2 > t_1$, matrices $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times s}$, $D \in \mathbb{R}^{(n-r)\times s}$, nilpotent matrix $N \in \mathbb{R}^{(n-r)\times(n-r)}$ with nilpotent index $h$, and $y \in \langle N \,|\, D\rangle$, denote

$$\begin{aligned}
&\mathcal{U}_1(y,[t_1,t_2]) \\
&\quad = \{u(t):\ u(t) \in C^{h-1}[t_1,t_2],\ \sum_{j=0}^{h-1} N^j D u^{(j)}(t_1) = -y\}\\
\mathcal{S}_1 &= \{x_1:\ \text{there exists } u(t) \in \mathcal{U}_1(y,[t_1,t_2]) \text{ such that } x_1 \\
&\qquad = \int_{t_1}^{t_2} e^{A(t_2-\tau)} B u(\tau) d\tau\}\\
\mathcal{S}_2 &= \{x_2:\ \text{there exists } u(t) \in \mathcal{U}_1(y,[t_1,t_2]) \text{ such that } x_2 \\
&\qquad = -\sum_{j=0}^{h-1} N^j D u^{(j)}(t_2)\}.
\end{aligned}$$

Then, $\mathcal{S}_1 = \langle A \,|\, B\rangle, \mathcal{S}_2 = \langle N \,|\, D\rangle$.

*Lemma 5:* Under Assumption 1, if for all $i \in \Lambda$, $\langle N_i \,|\, B_{i2}\rangle = \mathbb{R}^{n-n_i}$, then for any given subspace $\mathcal{W} \subseteq \mathbb{R}^n$, we have

$$Q_i((Q_{i1}\mathcal{W}) \oplus \langle N_i \,|\, B_{i2}\rangle) \supseteq \mathcal{W} \qquad \forall i \in \Lambda.$$

*Lemma 6:* For any given matrices $A_1 \in \mathbb{R}^{n\times r}$, $A_2 \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times p_1}$ and $B_2 \in \mathbb{R}^{n\times p_2}$, we have

$$\operatorname{rank}[A_1 e^{A_2 t} B_1 \quad B_2] \geq \operatorname{rank}[A_1 B_1 \quad B_2] \text{ for almost all } t \in \mathbb{R}.$$

*Remark 4:* When $n = r$, the proof of Lemma 6 is given in [18]. When $n \neq r$, it can be proved similarly.

*Lemma 7:* For any given subspaces $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, if $\mathcal{A} \subseteq \mathcal{B}$, then $\dim(\mathcal{A} + \mathcal{C}) \leq \dim(\mathcal{B} + \mathcal{C})$, where $\dim(\cdot)$ is the dimension of subspace.

## III. Reachability Conditions of SLS Systems

In this section, we will give a necessary condition and a sufficient condition for the complete reachability of SLS systems.

A necessary condition for complete reachability is summarized in Theorem 1.

*Theorem 1:* Under Assumption 1, if SLS system (1) is completely reachable, then $\mathcal{V}_n = \mathbb{R}^n$.

*Proof:* Suppose that (1) is completely reachable. Then by Definition 1, for any given $x \in \mathbb{R}^n$, there exist a switching sequence $\{i_j, t_j\}_{j=0}^{s}$, a time $t_f > t_s$, and an admissible input $u \in \mathcal{U}_\sigma[t_0, t_f]$, such that $x(t_f; t_0, 0, u, \sigma) = x$.

We now show $x(t_1; t_0, 0, u, \sigma) \in \mathcal{V}_1$. In fact, by Remark 1, the states of system (1) with admissible control is continuous. Thus, $x(t_1; t_0, 0, u, \sigma)$ is the state of subsystem $(E_{i_0}, A_{i_0}, B_{i_0})$ at $t_1$, and can be expressed as

$$x(t_1;t_0,0,u,\sigma) = Q_{i_0}\begin{bmatrix} \int_{t_0}^{t_1} e^{G_{i_0}(t_1-\tau)} B_{i_0 1} u_{i_0}(\tau) d\tau \\ -\sum_{j=0}^{h_{i_0}-1} N_{i_0}^j B_{i_0 2} u_{i_0}^{(j)}(t_1)\end{bmatrix}$$

which together with Lemma 1 gives

$$\begin{aligned}
&x(t_1;t_0,0,u,\sigma)\\
&\quad = Q_{i_0}\begin{bmatrix} \sum_{j=0}^{n_{i_0}-1} G_{i_0}^j B_{i_0 1}\int_{t_0}^{t_1} f_j(t_1-\tau) u_{i_0}(\tau)\, d\tau \\ -\sum_{j=0}^{h_{i_0}-1} N_{i_0}^j B_{i_0 2} u_{i_0}^{(j)}(t_1)\end{bmatrix}.
\end{aligned}\tag{8}$$

So, by the definition of $\mathcal{V}_1$ we know that $x(t_1; t_0, 0, u, \sigma) \in Q_{i_0}(\langle G_{i_0} \,|\, B_{i_0 1}\rangle \oplus \langle N_{i_0} \,|\, B_{i_0 2}\rangle) \subseteq \mathcal{V}_1$.

We now investigate the states of system (1) at $t_2$. Similar to the previous discussion, we have the equation shown at the bottom of the page. By $x(t_1; t_0, 0, u, \sigma) \in \mathcal{V}_1$ and Lemma 1 we have

$$\begin{aligned}
e^{G_{i_1}(t_2-t_1)} Q_{i_1 1} x(t_1;t_0,0,u,\sigma) &\in e^{G_{i_1}(t_2-t_1)} Q_{i_1 1}\mathcal{V}_1\\
&\subseteq \langle G_{i_1} \,|\, Q_{i_1 1}\mathcal{V}_1\rangle.
\end{aligned}\tag{9}$$

Similar to (8), from Lemmas 1 and 3 it follows that

$$\int_{t_1}^{t_2} e^{G_{i_1}(t_2-\tau)} B_{i_1 1} u_{i_1}(\tau)\, d\tau \in \langle G_{i_1} \,|\, B_{i_1 1}\rangle \subseteq \langle G_{i_1} \,|\, Q_{i_1 1}\mathcal{V}_1\rangle. \tag{10}$$

Noticing that

$$-\sum_{j=0}^{h_{i_1}-1} N_{i_1}^j B_{i_1 2} u_{i_1}^{(j)}(t_2) \in \langle N_{i_1} \,|\, B_{i_1 2}\rangle \tag{11}$$

by (9) and (10) and the definition of $\mathcal{V}_2$ we have

$$x(t_2;t_0,0,u,\sigma) \in Q_{i_1}(\langle G_{i_1} \,|\, Q_{i_1 1}\mathcal{V}_1\rangle \oplus \langle N_{i_1} \,|\, B_{i_1 2}\rangle) \subseteq \mathcal{V}_2.$$

Similarly, we can show $x(t_f; t_0, 0, u, \sigma) \in \mathcal{V}_{s+1}$. From Remark 3 we know that $\mathcal{V}_{s+1} \subseteq \mathcal{V}_n$, which in turn implies $x = x(t_f; t_0, 0, u, \sigma) \in \mathcal{V}_n$. Thus, $\mathcal{V}_n = \mathbb{R}^n$.

*Remark 5:* It is worth pointing out that the condition of Theorem 1 is not sufficient for the reachability of the SLS system (1). For instance, in the case where $n_1 = n_2 = \cdots = n_m = 0$, $P_i = Q_i = I$, $\sum_{i=1}^m \langle N_i \,|\, B_{i2}\rangle = \mathbb{R}^n$, $\langle N_i \,|\, B_{i2}\rangle \cap \langle N_j \,|\, B_{j2}\rangle = \{0\}$, $\forall i \neq j$, and $\langle N_i \,|\, B_{i2}\rangle \neq \mathbb{R}^n$, from $\mathcal{V}_n = \sum_{i=1}^m \langle N_i \,|\, B_{i2}\rangle$ and $\sum_{i=1}^m \langle N_i \,|\, B_{i2}\rangle = \mathbb{R}^n$ it follows that the necessary condition $\mathcal{V}_n = \mathbb{R}^n$ is satisfied, but the system is not reachable because of the following facts.

1) From [6], the reachable set of the subsystem $(N_i, B_{i2})$ is $\langle N_i \,|\, B_{i2}\rangle$, $i = 1, \ldots, m$.
2) For the case where the control has at least one switching, let $\{\sigma(t_{1j}), t_{1j}\}_{j=0}^{k}$, $k \geq 1$ be the switching sequence. For clarity of discussion, here we consider only the case of $k = 1$. For any given admissible input, from (4) and the definition of the admissible input we have

$$x(t_{11}) = -\sum_{r=0}^{h_{\sigma(t_{10})}-1} N_{\sigma(t_{10})}^r B_{\sigma(t_{10})2} u_{\sigma(t_{10})}^{(r)}(t_{11})$$

and

$$x(t_{11}) = -\sum_{l=0}^{h_{\sigma(t_{11})}-1} N_{\sigma(t_{11})}^l B_{\sigma(t_{11})2} u_{\sigma(t_{11})}^{(l)}(t_{11}^+).$$

This means that $x(t_{11}) \in \langle N_{\sigma(t_{10})} \,|\, B_{\sigma(t_{10})2}\rangle \cap \langle N_{\sigma(t_{11})} \,|\, B_{\sigma(t_{11})2}\rangle$. Hence, from condition $\langle N_i \,|\, B_{i2}\rangle \cap$

$$x(t_2;t_0,0,u,\sigma) = Q_{i_1}\begin{bmatrix} e^{G_{i_1}(t_2-t_1)} Q_{i_1 1} x(t_1;t_0,0,u,\sigma) + \int_{t_1}^{t_2} e^{G_{i_1}(t_2-\tau)} B_{i_1 1} u_{i_1}(\tau)\, d\tau \\ -\sum_{j=0}^{h_{i_1}-1} N_{i_1}^j B_{i_1 2} u_{i_1}^{(j)}(t_2)\end{bmatrix}$$

$\langle N_j \mid B_{j2}\rangle = \{0\}$, $\forall i \neq j$, it follows that $x(t_{11}) = 0$, which together with 1) implies that under the switching sequence $\{\sigma(t_{1j}), t_{1j}\}_{j=0}^{1}$, the reachable set of the system is $\langle N_{\sigma(t_{11})} \mid B_{\sigma(t_{11})2}\rangle$. Thus, the reachable set of the system under all possible switching sequences is

$$\mathcal{R} = \bigcup_{i=1}^{m} \langle N_i \mid B_{i2}\rangle$$

which is not equal to $\mathbb{R}^n$, since $\langle N_i \mid B_{i2}\rangle \neq \mathbb{R}^n$, $i = 1,\ldots,m$, and $\mathbb{R}^n$ cannot be expressed as a union of finite lower dimensional subspaces.

Now, by using the concept of reachable set and the geometric approach we would like to give a sufficient condition for the complete reachability of system (1). For a given switching sequence $\sigma = \{i+1, t_i\}_{i=0}^{m-1}$, $t_0 < t_1 < \cdots < t_{m-1} < t_m$, define reachable sets of (1) as:

$$\mathcal{R}_i = \{x : x = x(t_i; t_0, 0, u, \sigma),\ u \in \mathcal{U}_\sigma[t_0, t_m]\}, \quad i = 1,2,\ldots,m \tag{12}$$

where $\mathcal{U}_\sigma[t_0, t_m]$ is given by (5). Reachable set $\mathcal{R}_i$ consists of all the states reached at $t_i$ by system (1) under all possible admissible control in $\mathcal{U}_\sigma[t_0, t_m]$ in accordance with switching law $\sigma$. Let $l_i = t_i - t_{i-1}$, $i = 1,2,\ldots,m$. Then, we have the following lemma.

*Lemma 8:* For (1) and a switching sequence $\sigma = \{i+1, t_i\}_{i=0}^{m-1}$, $t_0 < t_1 < \cdots < t_{m-1} < t_m$, if Assumption 1 holds, and for any $i \in \Lambda$, $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, then

$$\begin{aligned}\mathcal{R}_1 &= Q_1(\langle G_1 \mid B_{11}\rangle \oplus \langle N_1 \mid B_{12}\rangle)\\ \mathcal{R}_i &= Q_i((e^{G_i l_i} Q_{i1}\mathcal{R}_{i-1} + \langle G_i \mid B_{i1}\rangle) \oplus \langle N_i \mid B_{i2}\rangle), \quad i = 2,3,\ldots,m.\end{aligned}$$

*Proof:* From the definition of

$$\mathcal{U}_\sigma[t_0, t_m]$$

and

$$u = [u_1^T(t), u_2^T(t), \ldots, u_m^T(t)]^T \in \mathcal{U}_\sigma[t_0, t_m]$$

it follows that

$$\sum_{r=0}^{h_i-1} N_i^r B_{i2} u_i^{(r)}(t_{i-1}^+) = -Q_{i2}x(t_{i-1}^-; t_0, 0, u, \sigma), \quad i = 1,2,\ldots,m \quad t_0^- = t_0$$

and $u_i(t) \in \mathcal{U}_1(Q_{i2}x(t_{i-1}^-; t_0, 0, u, \sigma), [t_{i-1}, t_i])$, $i = 1,2,\ldots,m$, where $\mathcal{U}_1(\cdot,\cdot)$ is given in Lemma 4. By Remark 1, we know that the states of system (1) are continuous in $[t_0, t_m]$. Noticing that $0 \in \langle N_1 \mid B_{12}\rangle$, by the definition of $\mathcal{R}_1$ and Lemma 4 we have

$$\begin{aligned}\mathcal{R}_1 &= \{x : x = x(t_1; t_0, 0, u, \sigma), u \in \mathcal{U}_\sigma[t_0, t_m]\}\\ &= \left\{x : x = Q_1 \begin{bmatrix} \int_{t_0}^{t_1} e^{G_1(t_1-\tau)} B_{11} u_1(\tau)\, d\tau \\ -\sum_{j=0}^{h_1-1} N_1^j B_{12} u_1^{(j)}(t_1)\end{bmatrix},\right.\\ &\qquad \left. u_1(t) \in \mathcal{U}_1(0, [t_0, t_1])\right\}\\ &= Q_1(\langle G_1 \mid B_{11}\rangle \oplus \langle N_1 \mid B_{12}\rangle).\end{aligned}$$

By Lemma 4, $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, and the definitions of $\mathcal{R}_i$ $(i = 2,3,\ldots,m)$, we have the equation shown at the bottom of the page, or, equivalently, $\mathcal{R}_i = Q_i((e^{G_i l_i} Q_{i1}\mathcal{R}_{i-1} + \langle G_i \mid B_{i1}\rangle) \oplus \langle N_i \mid B_{i2}\rangle)$, $i = 2,3,\ldots,m$. ■

In the following theorem, a sufficient condition for complete reachability of SLS systems is obtained.

*Theorem 2:* Under Assumption 1, if $\mathcal{V}_1 = \mathbb{R}^n$ and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $\forall i \in \Lambda$, then SLS system (1) is completely reachable.

*Proof:* By Lemma 8 and $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i \in \Lambda$, we have

$$\begin{aligned}\mathcal{R}_m &= Q_m((e^{G_m l_m} Q_{m1}\mathcal{R}_{m-1} + \langle G_m \mid B_{m1}\rangle) \oplus \langle N_m \mid B_{m2}\rangle)\\ &= \bar{Q}_{m1} e^{G_m l_m} Q_{m1}\mathcal{R}_{m-1} + \bar{Q}_{m1}\langle G_m \mid B_{m1}\rangle\\ &\quad + \bar{Q}_{m2}\langle N_m \mid B_{m2}\rangle\end{aligned}$$

where $Q_m = [\bar{Q}_{m1}\ \bar{Q}_{m2}]$ and $\bar{Q}_{m1} \in \mathbb{R}^{n \times n_m}$. By Lemma 16, we know that for almost all real number $l_m > 0$

$$\begin{aligned}\dim \mathcal{R}_m &= \dim(\bar{Q}_{m1} e^{G_m l_m} Q_{m1}\mathcal{R}_{m-1}\\ &\quad + \bar{Q}_{m1}\langle G_m \mid B_{m1}\rangle + \bar{Q}_{m2}\langle N_m \mid B_{m2}\rangle)\\ &\geq \dim(\bar{Q}_{m1} Q_{m1}\mathcal{R}_{m-1}\\ &\quad + \bar{Q}_{m1}\langle G_m \mid B_{m1}\rangle + \bar{Q}_{m2}\langle N_m \mid B_{m2}\rangle)\\ &= \dim(Q_m((Q_{m1}\mathcal{R}_{m-1}\\ &\quad + \langle G_m \mid B_{m1}\rangle) \oplus \langle N_m \mid B_{m2}\rangle))\\ &= \dim(Q_m(((Q_{m1}\mathcal{R}_{m-1}) \oplus \langle N_m \mid B_{m2}\rangle)\\ &\quad + (\langle G_m \mid B_{m1}\rangle \oplus \langle N_m \mid B_{m2}\rangle)))\\ &= \dim(Q_m((Q_{m1}\mathcal{R}_{m-1}) \oplus \langle N_m \mid B_{m2}\rangle) + Q_m\mathcal{C}_m).\end{aligned}$$

Notice by Lemma 5 that $\mathcal{R}_{m-1} \subseteq Q_m((Q_{m1}\mathcal{R}_{m-1}) \oplus \langle N_m \mid B_{m2}\rangle)$. Then, it follows from Lemma 7 that

$$\begin{aligned}\dim \mathcal{R}_m &\geq \dim(Q_m((Q_{m1}\mathcal{R}_{m-1}) \oplus \langle N_m \mid B_{m2}\rangle) + Q_m\mathcal{C}_m)\\ &\geq \dim(\mathcal{R}_{m-1} + Q_m\mathcal{C}_m).\end{aligned}$$

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$$\begin{aligned}\mathcal{R}_i &= \{x : x = x(t_i; t_0, 0, u, \sigma), u \in \mathcal{U}_\sigma[t_0, t_m]\}\\ &= \left\{x : x = Q_i \begin{bmatrix} e^{G_i l_i} Q_{i1} x(t_{i-1}^-; t_0, 0, u, \sigma) + \int_{t_{i-1}}^{t_i} e^{G_i(t_i-\tau)} B_{i1} u_i(\tau) d\tau \\ -\sum_{j=0}^{h_i-1} N_i^j B_{i2} u_i^{(j)}(t_i)\end{bmatrix},\right.\\ &\qquad \left. u \in \mathcal{U}_\sigma[t_0, t_m], \quad u_i(t) \in \mathcal{U}_1(Q_{i2}x(t_{i-1}^-; t_0, 0, u, \sigma), [t_{i-1}, t_i])\right\}\\ &= Q_i \begin{bmatrix} e^{G_i l_i} Q_{i1}\mathcal{R}_{i-1} + \langle G_i \mid B_{i1}\rangle \\ \langle N_i \mid B_{i2}\rangle \end{bmatrix}\end{aligned}$$

Repeating the aforementioned analysis procedures, for almost all real numbers $l_i > 0$, $i = 2, \ldots, m-1$, we have

$$\dim(\mathcal{R}_i + \sum_{k=i+1}^{m} Q_k \mathcal{C}_k) \geq \dim(\mathcal{R}_{i-1} + \sum_{k=i}^{m} Q_k \mathcal{C}_k).$$

Thus, there exist real numbers $l_i > 0$, $i = 1, \ldots, m$, (correspondingly, switching law $\sigma$) such that

$$\begin{aligned}\dim(\mathcal{R}_m) &\geq \dim(\mathcal{R}_{m-1} + Q_m \mathcal{C}_m)\\ &\geq \dim(\mathcal{R}_{m-2} + Q_{m-1}\mathcal{C}_{m-1} + Q_m \mathcal{C}_m)\\ &\vdots\\ &\geq \dim(Q_1\mathcal{C}_1 + \cdots + Q_{m-1}\mathcal{C}_{m-1} + Q_m\mathcal{C}_m)\\ &= \dim(\mathcal{V}_1) = \dim(\mathbb{R}^n) = n\end{aligned}$$

which implies that for any $x \in \mathbb{R}^n$, there exist switching sequence $\{i+1, t_i\}_{i=0}^{m-1}$ and admissible input such that $x = x(t_m; t_0, 0, u, \sigma)$. Therefore, SLS system (1) is completely reachable. ■

*Remark 6:* When $E_i = I$, $i \in \Lambda$, the reachable conditions given in Theorems 1 and 2 degenerate to those given in [20] for conventional switching systems.

*Remark 7:* When there is only one subsystem and there exists no switching phenomenon (i.e., $m = 1$), by $\mathcal{V}_n = \mathcal{V}_1 = Q_1(\langle A_1 \mid B_{11}\rangle \oplus \langle N_1 \mid B_{12}\rangle)$ we see that the reachable conditions given in Theorems 1 and 2 degenerate to those given in [6] for regular (no switching) singular systems.

*Remark 8:* It is worth noticing that the condition $\mathcal{V}_1 = \mathbb{R}^n$ of Theorem 2 is not necessary for the reachability of the system (1). For example, when $m = 2$, $n_1 = n_2 = n = 2$, $G_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have

$$\mathcal{V}_1 = \mathcal{R}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq \mathbb{R}^2 \quad \mathcal{V}_2 = \mathbb{R}^2.$$

From $\mathcal{V}_2 = \mathbb{R}^2$ and [18] we know that the system (1) is reachable. So, $\mathcal{V}_1 = \mathbb{R}^n$ is not necessary.

*Remark 9:* The condition $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$ of Theorem 2 is not necessary for the reachability of the system (1). To see this, let us consider the case where $n_1 = n_2 = \cdots = n_m < n$; $P_i = Q_i = I$, $i = 1, \ldots, m$; $B_{i2} = 0$, $i = 1, 2, \ldots, m-1$; $\langle N_m \mid B_{m2}\rangle = \mathbb{R}^{n-n_m}$, and $\sum_{i=1}^{m}\langle G_i \mid B_{i1}\rangle = \mathbb{R}^{n_1}$. It is easy to see that $\mathcal{V}_1 = \sum_{i=1}^{m}\langle G_i \mid B_{i1}\rangle \oplus \langle N_m \mid B_{m2}\rangle = \mathbb{R}^n$; and for switching sequence $\{i+1, t_i\}_{i=0}^{m-1}$, similar to Lemma 8 and Theorem 2,

$$\begin{aligned}\mathcal{R}_m = (&e^{G_m l_m} \cdots e^{G_2 l_2}\langle G_1 \mid B_{11}\rangle + \cdots + \langle G_m \mid B_{m1}\rangle)\\ &\oplus \langle N_m \mid B_{m2}\rangle\end{aligned}$$

and

$$\dim(\mathcal{R}_m) \geq \dim(\mathcal{V}_1) = n$$

for almost all $l_1, \ldots, l_m \in \mathbb{R}$. Thus, the system is completely reachable, although $\langle N_i \mid B_{i2}\rangle = \{0\} \neq \mathbb{R}^{n-n_i}$, $i = 1, 2, \ldots, m-1$.

Remarks 5, 8, and 9 imply that there is a gap between the necessary condition and the sufficient condition given in Theorems 1 and 2 for the reachability of SLS system (1). The existence of admissible inputs and the switching laws are coupled, which makes the reachability analysis of SLS systems complicated. So, it needs more efforts to find out a necessary and sufficient condition.

## IV. Conclusion

The reachability problem of SLS system has been investigated in this note under the regularity assumption of all subsystems. To ensure the continuity of the states of SLS systems, for a given switching sequence we have introduced an admissible control set. By using the structure characteristic of the solution of SLS system state equation and the geometric approach a necessary condition and a sufficient condition have been given for complete reachability, which are exactly the same as those of the conventional (nonsingular) switched system and normal (nonswitching) singular system cases given in [20] and [6] when the systems degenerate to conventional systems and normal singular systems, respectively.

## Appendix

In order to prove Proposition 1, we need the following lemma.

*Lemma 9:* [6] Under Assumption 1, if matrices $P_i$, $Q_i$, $G_i$, $N_i$, $B_{i1}$, $B_{i2}$, $\bar{P}_i$, $\bar{Q}_i$, $\bar{G}_i$, $\bar{N}_i$, $\bar{B}_{i1}$, $\bar{B}_{i2}$, and real numbers $n_i$ and $\bar{n}_i$, $i = 1, 2, \ldots, m$, are defined according to (2) and (7), then $n_i = \bar{n}_i$; and there exist nonsingular matrices $T_{i1} \in \mathbb{R}^{n_i \times n_i}$ and $T_{i2} \in \mathbb{R}^{(n-n_i)\times(n-n_i)}$ such that

$$\begin{aligned}P_i = \mathrm{diag}(T_{i1}, T_{i2})\bar{P}_i \quad Q_i = \bar{Q}_i \mathrm{diag}(T_{i1}^{-1}, T_{i2}^{-1})\\ G_i = T_{i1}\bar{G}_i T_{i1}^{-1} \quad N_i = T_{i2}\bar{N}_i T_{i2}^{-1}\\ B_{i1} = T_{i1}\bar{B}_{i1} \quad B_{i2} = T_{i2}\bar{B}_{i2}.\end{aligned}$$

*Proof of Proposition 1:* The proof is given by induction. By Lemma 9 and straightforward calculations we have $h_i = \bar{h}_i$, $i = 1, 2, \ldots, m$

$$\begin{aligned}\mathcal{C}_i &= \langle G_i \mid B_{i1}\rangle \oplus \langle N_i \mid B_{i2}\rangle\\ &= (\mathcal{R}(B_{i1}) + G_i\mathcal{R}(B_{i1}) + \cdots + G_i^{n_i-1}\mathcal{R}(B_{i1}))\\ &\quad \oplus (\mathcal{R}(B_{i2}) + N_i\mathcal{R}(B_{i2}) + \cdots + N_i^{h_i-1}\mathcal{R}(B_{i2}))\\ &= (T_{i1}\mathcal{R}(\bar{B}_{i1}) + T_{i1}\bar{G}_i\mathcal{R}(\bar{B}_{i1}) + \cdots + T_{i1}\bar{G}_i^{n_i-1}\mathcal{R}(\bar{B}_{i1}))\\ &\quad \oplus (T_{i2}\mathcal{R}(\bar{B}_{i2}) + T_{i2}\bar{N}_i\mathcal{R}(\bar{B}_{i2}) + \cdots + T_{i2}\bar{N}_i^{h_i-1}\mathcal{R}(\bar{B}_{i2}))\\ &= \begin{bmatrix} T_{i1} & \\ & T_{i2}\end{bmatrix}\bar{\mathcal{C}}_i, \qquad i = 1, 2, \ldots, m\end{aligned}$$

and $Q_i\mathcal{C}_i = \bar{Q}_i\bar{\mathcal{C}}_i$, , $i = 1, 2, \ldots, m$. So, from the definitions of $\mathcal{V}_1$ and $\bar{\mathcal{V}}_1$ we have $\mathcal{V}_1 = \sum_{i=1}^{m} Q_i\mathcal{C}_i = \sum_{i=1}^{m}\bar{Q}_i\bar{\mathcal{C}}_i = \bar{\mathcal{V}}_1$, i.e., the proposition holds for $j = 1$.

Suppose that $\mathcal{V}_j = \bar{\mathcal{V}}_j$ for all $j < k$. We now consider the case of $j = k$. From Lemma 9 and the definitions of $Q_{i1}$ and $\bar{Q}_{i1}$, it follows that $Q_{i1} = T_{i1}\bar{Q}_{i1}$, which leads to

$$\begin{aligned}\mathcal{V}_k &= \sum_{i=1}^{m} Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \mid B_{i2}\rangle)\\ &= \sum_{i=1}^{m} Q_i((\mathcal{R}(Q_{i1}\mathcal{V}_{k-1}) + G_i\mathcal{R}(Q_{i1}\mathcal{V}_{k-1}) + \cdots\\ &\quad + G_i^{n_i-1}\mathcal{R}(Q_{i1}\mathcal{V}_{k-1}))\\ &\quad \oplus (\mathcal{R}(B_{i2}) + N_i\mathcal{R}(B_{i2}) + \cdots + N_i^{h_i-1}\mathcal{R}(B_{i2})))\\ &= \sum_{i=1}^{m} Q_i((T_{i1}\mathcal{R}(\bar{Q}_{i1}\bar{\mathcal{V}}_{k-1}) + T_{i1}\bar{G}_i\mathcal{R}(\bar{Q}_{i1}\bar{\mathcal{V}}_{k-1}) + \cdots\\ &\quad + T_{i1}\bar{G}_i^{n_i-1}\mathcal{R}(\bar{Q}_{i1}\bar{\mathcal{V}}_{k-1}))\\ &\quad \oplus (T_{i2}\mathcal{R}(\bar{B}_{i2}) + T_{i2}\bar{N}_i\mathcal{R}(\bar{B}_{i2}) + \cdots\\ &\quad + T_{i2}\bar{N}_i^{h_i-1}\mathcal{R}(\bar{B}_{i2})))\\ &= \sum_{i=1}^{m}\bar{Q}_i(\langle\bar{G}_i \mid \bar{Q}_{i1}\bar{\mathcal{V}}_{k-1}\rangle \oplus \langle\bar{N}_i \mid \bar{B}_{i2}\rangle)\\ &= \bar{\mathcal{V}}_k.\end{aligned}$$

Thus, by the induction principle we have $\mathcal{V}_j = \bar{\mathcal{V}}_j$, $j = 1, 2, \ldots, n$. ■

*Proof of Proposition 2:*

1) For any given $k > 1$, by the definition of $\mathcal{V}_{k+1}$ we have

$$\begin{aligned}\mathcal{V}_{k+1} &= \sum_{i=1}^{m} Q_i(\langle G_i \,|\, Q_{i1}\mathcal{V}_k\rangle \oplus \langle N_i \,|\, B_{i2}\rangle) \\ &\supseteq \sum_{i=1}^{m} Q_i(Q_{i1}\mathcal{V}_k \oplus \langle N_i \,|\, B_{i2}\rangle) \\ &\supseteq \sum_{i=1}^{m} Q_i(Q_{i1}(Q_i(\langle G_i \,|\, Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \,|\, B_{i2}\rangle)) \\ &\quad \oplus \langle N_i \,|\, B_{i2}\rangle) \\ &\supseteq \sum_{i=1}^{m} Q_i(\langle G_i \,|\, Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \,|\, B_{i2}\rangle) = \mathcal{V}_k\end{aligned}$$

where $Q_{i1}Q_i = [I_{n_i} 0]$ has been used for the last $\supseteq$.

2) If there exists $i \le n$ such that $\mathcal{V}_i = \mathcal{V}_{i-1}$, then by the definition of $\mathcal{V}_{i+1}$ we have

$$\begin{aligned}\mathcal{V}_{i+1} &= \sum_{k=1}^{m} Q_k(\langle G_k \,|\, Q_{k1}\mathcal{V}_i\rangle \oplus \langle N_k \,|\, B_{k2}\rangle) \\ &= \sum_{k=1}^{m} Q_k(\langle G_k \,|\, Q_{k1}\mathcal{V}_{i-1}\rangle \oplus \langle N_k \,|\, B_{k2}\rangle) \\ &= \mathcal{V}_i.\end{aligned}$$

This implies that $\mathcal{V}_l = \mathcal{V}_i, \forall l > i$. ■

*Proof of Lemma 2:* Since $W^T(f,t) = W(f,t)$, we need only to show that

$$\ker(W(f,t)) = \bigcap_{i=0}^{n-1} \ker(B^T(A^T)^i)$$

where $\ker(B) = \{x : \ Bx = 0, \ x \in \mathbb{R}^r\}$. Since $W(f,t)$ is semi-positive definite, $x \in \ker(W(f,t))$ is equivalent to $x^T W(f,t)x = 0$. Notice that

$$\begin{aligned}0 = x^T W(f,t)x &= \int_0^t x^T f(s) e^{A(t-s)} B B^T e^{A^T(t-s)} f(s)\, x ds \\ &= \int_0^t \|B^T e^{A^T(t-s)} f(s)x\|^2\, ds.\end{aligned} \tag{13}$$

Then, by the continuity of $\|B^T e^{A^T(t-s)} f(s)x\|^2$, we have

$$B^T e^{A^T(t-s)} f(s)x = 0 \qquad \forall\, 0 \le s \le t. \tag{14}$$

Since $f(s)$ has at most a finite number of zeros in $[0, \ t]$, by (14) we get

$$B^T e^{A^T s} x = 0 \qquad \forall\, 0 \le s \le t. \tag{15}$$

Differentiating $B^T e^{A^T s} x$ with respect to $s$ up to $n-1$ times and setting $s = 0$, we have

$$B^T x = 0 \quad B^T A^T x = 0, \ldots, B^T (A^T)^{n-1} x = 0 \tag{16}$$

which implies that

$$x \in \ker(B^T) \bigcap \ker(B^T A^T) \bigcap \cdots \bigcap \ker(B^T (A^T)^{n-1}).$$

Thus

$$\ker(W(f,t)) \subseteq \bigcap_{i=0}^{n-1} \ker(B^T(A^T)^i).$$

So, the remaining is to prove

$$\ker(W(f,t)) \supseteq \bigcap_{i=0}^{n-1} \ker(B^T(A^T)^i). \tag{17}$$

In fact, for any given $x \in \bigcap_{i=0}^{n-1} \ker(B^T(A^T)^i)$, by Lemma 1 we can get (15) and, hence, get (13), i.e., $x \in \ker(W(f,t))$. Thus, (17) is true. ■

*Proof of Lemma 3:* From the definitions of $\langle \cdot \,|\, \cdot \rangle$ and $\mathcal{V}_1$, and $Q_{i1}Q_i = [I_{n_i} \ 0]$, we have

$$\langle G_i \,|\, Q_{i1}\mathcal{V}_1\rangle \supseteq Q_{i1}\mathcal{V}_1 \supseteq Q_{i1}Q_i\mathcal{C}_i = \langle G_i \,|\, B_{i1}\rangle$$

which implies that $\langle G_i \,|\, B_{i1}\rangle \subseteq \langle G_i \,|\, Q_{i1}\mathcal{V}_1\rangle, \forall i \in \Lambda$. This together Proposition 2 gives the conclusion immediately. ■

In order to prove Lemma 4, we need the following lemma.

*Lemma 10:* For any given $2h$ vectors $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, h-1$, and $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, $t_2 > t_1$, there exists a polynomial vector $f(t) \in \mathbb{R}^n$ of order $2h-1$ such that $f^{(i)}(t_1) = x_i$, $f^{(i)}(t_2) = y_i$, $i = 0, 1, \ldots, h-1$.

*Proof of Lemma 4:* For any given $x_1 \in \mathcal{S}_1$ and $x_2 \in \mathcal{S}_2$, by the definitions of $\mathcal{S}_1$ and $\mathcal{S}_2$ we know that there exists $u(t) \in \mathcal{U}_1(y, [t_1, t_2])$ such that

$$\begin{aligned}x_1 &= \int_{t_1}^{t_2} e^{A(t_2-\tau)} Bu(\tau)\, d\tau \\ &= \sum_{j=0}^{r-1} A^j B \int_{t_1}^{t_2} f_j(t_2-\tau)u(\tau)\, d\tau \in \langle A \,|\, B\rangle \\ x_2 &= -\sum_{j=0}^{h-1} N^j D u^{(j)}(t_2) \in \langle N \,|\, D\rangle\end{aligned}$$

where $f_j(t)$, $j = 0, 1, \ldots, r-1$, are the continuous functions defined by Lemma 1. This implies

$$\mathcal{S}_1 \subseteq \langle A \,|\, B\rangle, \mathcal{S}_2 \subseteq \langle N \,|\, D\rangle. \tag{18}$$

So, it suffices to show $\langle A \,|\, B\rangle \subseteq \mathcal{S}_1$ and $\langle N \,|\, D\rangle \subseteq \mathcal{S}_2$. To this end, for any given $x_1 \in \langle A \,|\, B\rangle$ and $x_2 \in \langle N \,|\, D\rangle$ we need only to construct an admissible input $u(t) \in \mathcal{U}_1(y, [t_1, t_2])$ such that

$$x_1 = \int_{t_1}^{t_2} e^{A(t_2-\tau)} Bu(\tau) d\tau \quad x_2 = -\sum_{j=0}^{h-1} N^j D u^{(j)}(t_2).$$

From $x_2 \in \langle N \,|\, D\rangle$, there exist $x_{2j} \in \mathbb{R}^{(n-r)}$, $j = 0, 1, \ldots, h$, such that

$$x_2 = -\sum_{j=0}^{h-1} N^j D x_{2j}. \tag{19}$$

From $y \in \langle N \,|\, D\rangle$, there exist $\alpha_{2j} \in \mathbb{R}^{(n-r)}$, $j = 0, 1, \ldots, h-1$ such that

$$y = -\sum_{j=0}^{h-1} N^j D \alpha_{2j}. \tag{20}$$

By Lemma 10, there exists a polynomial vector $f(s)$ of order $2h-1$ such that

$$f^{(j)}(t_1) = \alpha_{2j} \quad f^{(j)}(t_2) = x_{2j}, \qquad j = 0, 1, \ldots, h-1. \tag{21}$$

Set

$$\tilde{x}_1 = x_1 - \int_{t_1}^{t_2} e^{A(t_2-\tau)} B f(\tau)\, d\tau. \tag{22}$$

Then, from $x_1 \in \langle A \mid B \rangle$ and Lemma 1, we know that $\tilde{x}_1 \in \langle A \mid B \rangle$. Let $f_1(s) = (s-t_1)^h (s-t_2)^h$ and $f_2(s) = f_1(s+t_1)$. Then $f_2(s)$ is not identically equal to zero. By Lemma 2, there is a vector $z \in \mathbb{R}^r$ such that $\tilde{x}_1 = W(f_2, t_2 - t_1)z$. Denote

$$u_1(s) = f_1^2(s) B^T e^{A^T(t_2-s)} z, \qquad s \in [t_1, t_2]$$

and

$$u(s) = u_1(s) + f(s), s \in [t_1, t_2].$$

Then, by (22), we have

$$\begin{aligned} x_1(t_2) &= \int_{t_1}^{t_2} e^{A(t_2-\tau)} B f_1^2(\tau) B^T e^{A^T(t_2-\tau)} z d\tau \\ &\quad + \int_{t_1}^{t_2} e^{A(t_2-\tau)} B f(\tau) d\tau \\ &= \int_0^{t_2-t_1} e^{A(t_2-t_1-\tau)} B f_2^2(\tau) B^T e^{A^T(t_2-t_1-\tau)} z d\tau \\ &\quad + \int_{t_1}^{t_2} e^{A(t_2-\tau)} B f(\tau)\, d\tau \\ &= W(f_2, t_2-t_1) z + x_1 - \tilde{x}_1 \\ &= x_1. \end{aligned}$$

Noticing that $u_1^{(j)}(t_1) = 0$, $u_1^{(j)}(t_2) = 0$, $j = 0, 1, \ldots, h-1$, by (19)–(21) we obtain

$$x_2(t_1) = -\sum_{j=0}^{h-1} N^j D(u_1^{(j)}(t_1) + f^{(j)}(t_1)) = y$$

$$x_2(t_2) = -\sum_{j=0}^{h-1} N^j D(u_1^{(j)}(t_2) + f^{(j)}(t_2)) = x_2.$$

Since $u(t)$ is infinitely differentiable, $u(t) \in \mathcal{U}_1(y, [t_1, t_2])$. Thus, $S_1 \subseteq \langle A \mid B \rangle$ and $S_2 \subseteq \langle N \mid B_2 \rangle$. This together with (18) implies $S_1 = \langle A \mid B \rangle$ and $S_2 = \langle N \mid D \rangle$. ■

*Proof of Lemma 5:* For any given $x \in \mathcal{W}$, denote $y_1 = Q_{i1}x$, $y_2 = Q_{i2}x$, and $y = [y_1^T\, y_2^T]^T$. Then $y_1 \in Q_{i1}\mathcal{W}$, and by

$$\langle N_i \mid B_{i2} \rangle = \mathbb{R}^{n-n_i}, y_2 \in \langle N_i \mid B_{i2} \rangle.$$

Thus

$$x = Q_i y \in Q_i((Q_{i1}\mathcal{W}) \oplus \langle N_i \mid B_{i2} \rangle).$$

This together with the arbitrariness of $x \in \mathcal{W}$ implies that $\mathcal{W} \subseteq Q_i((Q_{i1}\mathcal{W}) \oplus \langle N_i \mid B_{i2} \rangle)$. ■

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