International Journal of Control
Vol. 78, No. 7, May 2005, 474–490

Output-feedback adaptive stabilization control design for non-holonomic systems with strong non-linear drifts

YUN-GANG LIU† and JI-FENG ZHANG‡*

†School of Control Science and Engineering, Shandong University, Jinan, 250061, China
‡LSC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100080, China

(*Received 22 December 2003; in final form 24 February 2005*)

This paper investigates the problem of output-feedback adaptive stabilization control design for non-holonomic chained systems with strong non-linear drifts, including modelled non-linear dynamics, unmodelled dynamics, and those modelled but with unknown parameters. An observer and an estimator are introduced for state and parameter estimates, respectively. By using the integrator backstepping approach and based on the observer and parameter estimator, a constructive design procedure for output-feedback adaptive stabilization control is given. It is shown that, under some conditions, the control design ensures the closed-loop system is globally asymptotically stable when there is no non-linear drift in the first subsystem, and semiglobally asymptotically stable, otherwise. An example is given to show the effectiveness of the theory.

1. Introduction

Non-holonomic systems, i.e., systems with non-holonomic or non-integrable constraints are quite often encountered in practice (Astolfi 1996, Bloch *et al*. 1992, Do and Pan 2002, Ge *et al*. 2001, Jiang and Nijmeijer 1999 and Kolmanovsky and McClamroch 1995). The representative examples of such systems are unicycle, four-wheel car, *n*-level trailer systems, etc. Therefore, it is important from the view point of applications to study non-holonomic control systems. However, to date, it has not yet been clear what control approaches are more suitable for studying such systems (Huang 2002), since there exists no smooth (or even continuous) state-feedback control for such systems (Bloch *et al*. 1992 and Jiang and Nijmeijer 1999), and the well-developed smooth non-linear control theory and methodology can not be directly used to such systems. During the past decade, the study of control problems for such systems has received considerable attention, and greatly initiated and accelerated by the rapid development of other branches of control theory, such as hybrid or switching technique, adaptive scheme, nonsmooth (or discontinuous) control and time-varying control, etc. (Kolmanovsky and McClamroch 1995).

Astolfi (1996), Bloch *et al*. (1992), Ge *et al*. (2001) and Jiang and Nijmeijer (1999) studied the control problems of non-holonomic systems with standard structure (i.e., drift-free non-holonomic systems). Do and Pan (2002) investigated the adaptive stabilization control design problem of non-holonomic systems with non-linear drifts, and presented a constructive control design procedure. The work of Do and Pan (2002) is characterized by full-state feedback and the overparametrized scheme which will increase the dynamic order of the resulting adaptive controller and the closed-loop systems, and so, is undesirable (Kanellakopoulos 1995). More comments on the work of Do and Pan (2002) are referred to Ge (2003), who points out that there is a technical problem inherent in the input based switching, which may cause unexpected failure when the scheme is used in practice.

In this paper, the output-feedback adaptive stabilization control design of non-holonomic systems with strong non-linear drifts is considered. To our knowledge, this problem is still open, since in general, output-feedback-based control design of non-holonomic systems is more challenging than that based on

*Corresponding author. Email: jif@issn.ac.cn

International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170500080280

state-feedback, and the conventional observer design methods in being cannot be directly used for state reconstruction of the non-holonomic systems. To solve this problem, in this paper, a new observer design method is proposed, and based on the observer, the unmeasurable states of the system involved are reconstructed. Unlike Do and Pan (2002), only one estimator is used to estimate the unknown parameters. This avoids the undesirable overparametrization estimate. By using the integrator backstepping approach and based on the observer and parameter estimator given, a constructive design procedure of output-feedback adaptive stabilization control is presented. It is shown that, under some conditions, the control designed ensures the closed-loop system is globally asymptotically stable when there is no non-linear drift in the first subsystem, and semiglobally asymptotically stable, otherwise.

This paper is organized as follows. In § 2, some notations and preliminary results are introduced. In § 3, the model and structure of the systems involved are described, and the problem to be studied is formulated. In § 4, an observer and an estimator are proposed for the reconstruction of the unmeasurable states and the estimate of unknown parameters, respectively, and then, a constructive design procedure of output-feedback control is presented. Under some conditions, the stability of closed-loop system is proved. Section 5 gives a simulation example to illustrate the theoretical findings of this paper. Section 6 summarizes the paper.

2. Notations and preliminary results

In the sequel, we will use the following notations. For a given vector or matrix X, X^{T} denotes its transpose; $\|X\|$ denotes the Euclidean norm for vectors or the corresponding induced norm for matrices. For a given vector $x=[x_1,\dots,x_n]^T$, $x_{[i]}$ denotes $[x_1,\dots,x_i]^T$; $\hat{x}$ denotes its estimate associated with an observer, and $\widetilde{x}$ denotes the estimation error, i.e., $\widetilde{x}=x-\widehat{x}$. For a given scalar number x, $|x|$ denotes its absolute value. I_i denotes the identity matrix with i-dimension. C^∞ denotes the set of all infinitely differentiable functions. For simplicity of expression, we sometimes drop the arguments of functions when no confusion is caused.

Definition 1 (Battilotti 2001)**:** Consider system $\dot{x}=f(t,x)$ with $f(t,0)=0$. $\Omega\subset\mathbb{R}^n$ denotes some compact set including the origin $x=0$. If for any given initial value $x(0)\in\Omega$, the corresponding solution $x(t)$ of the system satisfies $\sup_{t\geq 0}\|x(t)\|<\infty$, then the system is said semiglobally stable; if for any given initial value $x(0)\in\Omega$, the corresponding solution $x(t)$ of the system satisfies $\lim_{t\to\infty}\|x(t)\|=0$, then the system is said semiglobally asymptotically stable. Particularly, when $\Omega=\mathbb{R}^n$, the system is said globally stable and globally asymptotically stable, respectively.

3. Problem formulation

3.1. *System model*

Consider the following non-holonomic system with strong non-linear drifts

$$\left.\begin{aligned}
&\dot{x}_0=u_0+f_0(y)+\varphi_0(t,x_0,x,u_0)+\phi_0^T(y)\theta,\\
&\dot{x}_1=x_2u_0+f_1(y,u_0)+\varphi_1(t,x_0,x,u_0)+\phi_1^T(y,u_0)\theta,\\
&\quad\vdots\\
&\dot{x}_{n-1}=x_nu_0+f_{n-1}(y,u_0)+\varphi_i(t,x_0,x,u_0)+\phi_{n-1}^T(y,u_0)\theta,\\
&\dot{x}_n=u_1+f_n(y,u_0)+\varphi_n(t,x_0,x,u_0)+\phi_n(y,u_0)^T\theta,\\
&y=[x_0,x_1]^T,
\end{aligned}\right\}\tag{1}$$

where $[x_0,x^T]^T=[x_0,x_1,\dots,x_n]^T\in\mathbb{R}^{n+1}$, $u=[u_0,\ u_1]^T\in\mathbb{R}^2$ and $y\in\mathbb{R}^2$ are the system state, control input and system measurable output, respectively; $f_i(\cdot)\in\mathbb{R}$, $i=0,1,\dots,n$ are the modelled (known) dynamics, depending on y and u_0; $\varphi_i(\cdot)\in\mathbb{R}$, $i=0,1,\dots,n$ are the unmodelled (unknown) dynamics; $\theta\in\mathbb{R}^r$ is unknown time-invariant parameter; $\phi_i(\cdot)\in\mathbb{R}^r$, $i=0,1,\dots,n$ are known and depend on y and u_0 only.

The functions f_i, φ_i and $\phi_i^T\theta$, $i=0,1,\dots,n$ are called as the non-linear drifts of the system (1). If the non-linear drifts do not exist, i.e., $f_i\equiv 0$, $\varphi_i\equiv 0$ and $\phi_i^T\theta\equiv 0$, $i=0,1,\dots,n$, then the system (1) degenerates to the standard (or normal) form of non-holonomic systems widely studied in the literature.

Suppose that the system (1) satisfies the following assumptions, which will be the base of the coming control design and performance analysis.

A1 There is a known constant $M\geq 0$ such that the unknown parameter θ satisfies: $\|\theta\|\leq M$.

A2 The non-linear functions $f_i(\cdot)$ and $\phi_i(\cdot)$, $i=0,1,\dots,n$ are smooth, and for some smooth known functions $\overline{f}_0(\cdot)$, $\overline{\phi}_0(\cdot)$, $\overline{\varphi}_0(\cdot)$, $\overline{f}_{i0}(\cdot)$, $\overline{f}_{i1}(\cdot)$, $\overline{\phi}_{i0}(\cdot)$, $\overline{\phi}_{i1}(\cdot)$, $\overline{\varphi}_{i0}(\cdot)$ and $\overline{\varphi}_{i1}(\cdot)$, $i=1,\dots,n$, the non-linear functions $f_i(\cdot)$ and $\phi_i(\cdot)$, $i=0,1,\dots,n$ can be expressed as:

$$f_0(y)=x_0\overline{f}_0(y),\quad f_i(y,u_0)=x_0^{n-i+1}\overline{f}_{i0}(y)+x_1\overline{f}_{i1}(y,u_0),\quad i=1,\dots,n,\tag{2}$$

$$\phi_0(y)=x_0\overline{\phi}_0(y),\quad \phi_i(y,u_0)=x_0^{n-i+1}\overline{\phi}_{i0}(y)+x_1\overline{\phi}_{i1}(y,u_0),\quad i=1,\dots,n,\tag{3}$$

and the non-linear functions φ_i, $i = 0, 1, \ldots, n$ satisfy:

$$|\varphi_0(t, x_0, x, u_0)| \le |x_0|\overline{\varphi}_0(y),$$
$$|\varphi_i(t, x_0, x, u_0)| \le |x_0|^{n-i+1}\overline{\varphi}_{i0}(y) + |x_1|\overline{\varphi}_{i1}(y, u_0),$$
$$i = 1, \ldots, n.$$

Remark 1: In the full-state feedback case, Assumption A2 together with (2), (3) and (4) can be generalized to:

$$f_i(x, u_0) = x_0^{n-i+1}\overline{f}_{i0}(x_0, x_1) + \sum_{i=1}^{n} x_i\overline{f}_{i1}(x_0, x_{[i]}, u_0),$$
$$i = 1, \ldots, n, \tag{5}$$
$$\phi_i(x, u_0) = x_0^{n-i+1}\overline{\phi}_{i0}(x_0, x_1) + \sum_{i=1}^{n} x_i\overline{\phi}_{i1}(x_0, x_{[i]}, u_0),$$
$$i = 1, \ldots, n, \tag{6}$$
$$|\varphi_i(t, x_0, x, u_0)| \le |x_0|^{n-i+1}\overline{\varphi}_{i0}(x_0, x_1) + \|x_{[i]}\|\overline{\varphi}_{i1}(x_0, x_{[i]}, u_0),$$
$$i = 1, \ldots, n. \tag{7}$$

This, compared with the existing work, includes more non-linear drifts. For instance, in Do and Pan (2002), it is implicitly assumed that $\overline{\phi}_{i0} = 0$ in the equality (6), and that the modelled dynamics f_i and unmodelled dynamics φ_i, $i = 0, 1, \ldots, n$, are not existent, i.e., $f_i = 0$ and $\varphi_i = 0$, $i = 1, \ldots, n$. In Xi *et al.* (2003), it is assumed not only that $\overline{\varphi}_{i0} = 0$ in the inequality (7), but also that the modelled dynamics f_i and the dynamics with unknown parameters $\phi_i^T(y)\theta$ do not exist, i.e., $f_i = 0$ and $\phi_i^T(y)\theta = 0$, although the gains of $u_0, x_2u_0, \ldots, x_{n-1}u_0$ and u on the right hand sides of the system (1) are not assumed to be ones and allowed to be in some known finite intervals.

Remark 2: Assumption A1 implies that the unknown parameter θ belongs to a known hyperball in R^r with center 0 and radius M. Assumption A2 ensures the feasibility of the state coordinates transformation below, and implies that $f_0(0, x_1) \equiv 0$, $\phi_0(0, x_1) \equiv 0$, $\varphi_0(t, 0, 0, u_0) \equiv 0$ and $f_i(0, u_0) \equiv 0$, $\varphi_i(t, 0, 0, u_0) \equiv 0, \phi_i(0, u_0) \equiv 0, i = 1, \ldots, n$, that is, the origin ($x_0 = x_1 = \cdots = x_n = 0$) is the equilibrium point of the open-loop system.

3.2. *Control objective*

The objective of this paper is to design an output-feedback adaptive stabilization control in the form:

$$\dot{\widehat{x}} = \vartheta(\widehat{x}, y), \quad \dot{\widehat{\theta}} = \kappa(\widehat{\theta}, y),$$
$$u_0 = \mu_0(y, \widehat{\theta}), \quad u_1 = \mu_1(\widehat{x}, \widehat{\theta}, y), \tag{8}$$

so that the resulting closed-loop system is globally asymptotically stable when there is no non-linear drift (i.e., $f_0 \equiv 0$, $\varphi_0 \equiv 0$ and $\phi_0^T\theta \equiv 0$) in the first subsystem, and semiglobally asymptotically stable, otherwise.

4. Output-feedback adaptive stabilization control design

According to the special structure of the system (1), the design procedures of u_0 and u_1 are usually carried out separately (Astolfi 1996, Bloch *et al.* 1992, Do and Pan 2002, Ge *et al.* 2001, Jiang and Nijmeijer 1999, and Kolmanovsky and McClamroch 1995): design u_0 to asymptotically stabilize the subsystem x_0 first, and then, design u_1 to stabilize the other subsystems $x_1 \sim x_n$. However, as x_0 asymptotically converges to 0, $x_1 \sim x_n$ will become uncontrollable. This will cause difficulty when designing control u_1. An effective method to deal with this difficulty is to introduce a suitable coordinate transformation (Do and Pan 2002 and Jiang and Nijmeijer 1999) transforming the original system into a new one (denoted by $\chi_0, \chi_1, \ldots, \chi_n$). In this new framework, the subsystems $\chi_1 \sim \chi_n$ can be stabilized to zero faster than χ_0 can. Then, we can design u_0 and u_1 separately to realize the asymptotical stabilization of χ_0 and $\chi_1 \sim \chi_n$, respectively.

4.1. *State coordinate transformation*

The state coordinate transformation is designed in form (Do and Pan 2002 and Jiang and Nijmeijer 1999):

$$\chi_0 = x_0, \quad \chi_1 = \frac{x_1}{x_0^{n-1}}, \ldots, \chi_{n-1} = \frac{x_{n-1}}{x_0}, \quad \chi_n = x_n. \tag{9}$$

Under this transformation, the system (1) becomes

$$\left.\begin{aligned}
\dot{\chi}_0 &= u_0 + f_0 + \varphi_0 + \phi_0^T\theta,\\
\dot{\chi}_1 &= \frac{u_0}{\chi_0}\chi_2 - \frac{n-1}{\chi_0}\chi_1\big(u_0 + f_0 + \varphi_0 + \phi_0^T\theta\big)\\
&\quad + \frac{1}{\chi_0^{n-1}}\big(f_1 + \varphi_1 + \phi_1^T\theta\big),\\
\dot{\chi}_2 &= \frac{u_0}{\chi_0}\chi_3 - \frac{n-2}{\chi_0}\chi_2\big(u_0 + f_0 + \varphi_0 + \phi_0^T\theta\big)\\
&\quad + \frac{1}{\chi_0^{n-2}}\big(f_2 + \varphi_2 + \phi_2^T\theta\big),\\
&\vdots\\
\dot{\chi}_{n-1} &= \frac{u_0}{\chi_0}\chi_n - \frac{1}{\chi_0}\chi_{n-1}(u_0 + f_0 + \varphi_0 + \phi_0^T\theta)\\
&\quad + \frac{1}{\chi_0}\big(f_{n-1} + \varphi_{n-1} + \phi_{n-1}^T\theta\big),\\
\dot{\chi}_n &= u_1 + f_n + \varphi_n + \phi_n^T\theta.
\end{aligned}\right\} \tag{10}$$

The transformation (9) defines a diffeomorphism excepting the origin, and if the state $[\chi_0, \chi_1, \ldots, \chi_n]^T$ of the system (10) converges to zero, so does the state $[x_0, x_1, \ldots, x_n]^T$ of the system (1). Thus, we need only to consider the control problem of the system (10).

Remark 3: It is worth pointing out that the transformation (9) has no definition at $x_0 = 0$. In order to guarantee the feasibility of the coming control design and performance analysis, without loss of generality, we assume that $x_0(0) \neq 0$.

If $x_0(0) = 0$, one can first set controls u_0, u_1 in the following form in an initial period of time, for instance, $[0, t_0]$ with some small $t_0 > 0$,

$$u_0 = -f_0(y) + 1 + |x_0||\overline{\varphi}_0(y)| + M|\phi_0(y)|, \quad u_1 \equiv 0. \tag{11}$$

Then, substituting (11) into the first subsystem of (1), and by Assumptions A1–A2, we get, in the time interval $[0, t_0]$,

$$\begin{aligned}\dot{x}_0 &= -f_0(y) + 1 + |x_0||\overline{\varphi}_0(y)| + M|\phi_0(y)| \\ &\quad + f_0(y) + \varphi_0(t, x_0, x, u_0) + \phi_0^T(y)\theta \geq 1.\end{aligned} \tag{12}$$

This together with $x_0(0) = 0$ implies $x_0(t_0) \geq t_0 > 0$. Thus, time t_0 can be regarded as the new initial time of the control system with initial condition $x_0(t_0) \neq 0$.

Note that the control u_0 designed in the sequel (of this paper) is of the form $u_0 = -\beta x_0$ for all $t \geq t_0$, where β is a smooth feedback gain function. Then, one can conclude that $x_0(t)$ is not equal to zero for all $t \geq t_0$. This is because the closed-loop system is well-defined in $[0, \infty)$ and the solution of the first equation of the closed-loop system can be expressed as

$$x_0(t) = x_0(t_0) \exp\left(\int_{t_0}^{t} \left(\frac{u_0 + \varphi_0}{x_0} + \overline{f}_0 + \overline{\phi}_0\theta\right) ds\right),$$

which implies

$$\begin{aligned}|x_0(t)| &\geq |x_0(t_0)| \exp\left(-\int_{t_0}^{t} \left|\frac{u_0 + \varphi_0}{x_0} + \overline{f}_0 + \overline{\phi}_0\theta\right| ds\right) \\ &\geq |x_0(t_0)| \exp\left(-\int_{t_0}^{t} \left(|\beta| + \overline{\varphi}_0 + |\overline{f}_0| + M|\overline{\phi}_0|\right) ds\right).\end{aligned} \tag{13}$$

By the smoothness of $\beta, \overline{f}_0, \varphi_0$ and $\overline{\phi}_0$ guaranteed by Assumption A2, it is easy to see that, for any given finite time $t \geq t_0$, the integral $\int_{t_0}^{t} \left(|\beta| + \overline{\varphi}_0 + |\overline{f}_0| + M|\overline{\phi}_0|\right) ds$ is finite. This together with (13) implies that $|x_0(t)| > 0$ for all $t \geq t_0$, although $x_0(t)$ may converge to zero.

4.2. *Observer design*

We design the following observer associated with the system (10)

$$\left.\begin{aligned}\dot{\widehat{\chi}}_1 &= \frac{u_0}{\chi_0}\widehat{\chi}_2 - \frac{(n-1)u_0}{\chi_0}\widehat{\chi}_1 + \frac{k_1 u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1) + \frac{f_1}{\chi_0^{n-1}}, \\ \dot{\widehat{\chi}}_2 &= \frac{u_0}{\chi_0}\widehat{\chi}_3 - \frac{(n-2)u_0}{\chi_0}\widehat{\chi}_2 + \frac{k_2 u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1) + \frac{f_2}{\chi_0^{n-2}}, \\ &\vdots \\ \dot{\widehat{\chi}}_i &= \frac{u_0}{\chi_0}\widehat{\chi}_{i+1} - \frac{(n-i)u_0}{\chi_0}\widehat{\chi}_i + \frac{k_i u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1) + \frac{f_i}{\chi_0^{n-i}}, \\ &\vdots \\ \dot{\widehat{\chi}}_n &= u_1 + \frac{k_n u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1) + f_n,\end{aligned}\right\} \tag{14}$$

where $k_1, \ldots, k_n$ are design parameters to be determined later.

The estimation error $\widetilde{\chi} = \chi - \widehat{\chi}$ satisfies the dynamical equations

$$\left.\begin{aligned}\dot{\widetilde{\chi}}_1 &= \frac{u_0}{\chi_0}\widetilde{\chi}_2 - \frac{(n-1)u_0}{\chi_0}\widetilde{\chi}_1 - \frac{k_1 u_0}{\chi_0}\widetilde{\chi}_1 + \frac{\varphi_1 + \phi_1^T\theta}{\chi_0^{n-1}} \\ &\quad - \frac{(n-1)\chi_1}{\chi_0}(f_0 + \varphi_0 + \phi_0^T\theta), \\ \dot{\widetilde{\chi}}_2 &= \frac{u_0}{\chi_0}\widetilde{\chi}_3 - \frac{(n-2)u_0}{\chi_0}\widetilde{\chi}_2 - \frac{k_2 u_0}{\chi_0}\widetilde{\chi}_1 + \frac{\varphi_2 + \phi_2^T\theta}{\chi_0^{n-2}} \\ &\quad - \frac{(n-2)(\widehat{\chi}_2 + \widetilde{\chi}_2)}{\chi_0}(f_0 + \varphi_0 + \phi_0^T\theta), \\ &\vdots \\ \dot{\widetilde{\chi}}_i &= \frac{u_0}{\chi_0}\widetilde{\chi}_i + 1 - \frac{(n-i)u_0}{\chi_0}\widetilde{\chi}_i - \frac{k_i u_0}{\chi_0}\widetilde{\chi}_1 + \frac{\varphi_i + \phi_i^T\theta}{\chi_0^{n-i}} \\ &\quad - \frac{(n-i)(\widehat{\chi}_i + \widetilde{\chi}_i)}{\chi_0}(f_0 + \varphi_0 + \phi_0^T\theta), \\ &\vdots \\ \dot{\widetilde{\chi}}_n &= -\frac{k_n u_0}{\chi_0}\widetilde{\chi}_1 + \varphi_n + \phi_n^T\theta.\end{aligned}\right\} \tag{15}$$

The differential equations (15) can be rewritten into the compact form

$$\dot{\widetilde{\chi}} = \frac{u_0}{\chi_0}A\widetilde{\chi} + \Upsilon + \Phi\theta - (n-1)B_1F_0\chi_1 - B_2F_0(\widehat{\chi} + \widetilde{\chi}), \tag{16}$$

where $F_0 = \overline{f}_0 + (\varphi_0/\chi_0) + \overline{\phi}_0^T\theta$, and

$$A = \begin{bmatrix} -k_1 - n + 1 & 1 & 0 & 0 & \cdots & 0 \\ -k_2 & -n+2 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & -n+3 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -k_{n-1} & \vdots & \vdots & \ddots & -1 & 1 \\ -k_n & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{17}$$

$$\Upsilon = \begin{bmatrix} \frac{\varphi_1}{\chi_0^{n-1}} \\ \frac{\varphi_2}{\chi_0^{n-2}} \\ \vdots \\ \frac{\varphi_{n-1}}{\chi_0} \\ \varphi_n \end{bmatrix}, \quad \Phi = \begin{bmatrix} \frac{\phi_1^T}{\chi_0^{n-1}} \\ \frac{\phi_2^T}{\chi_0^{n-2}} \\ \vdots \\ \frac{\phi_{n-1}^T}{\chi_0} \\ \phi_n^T \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}. \tag{18}$$

About matrix A defined by (17), there exists the following lemma.

Lemma 1: *The eigenvalues of the matrix A defined by* (17) *can be arbitrarily assigned by a proper selection of the design parameters $k_1, \ldots, k_n$.*

Proof: For the sake of simplicity, we first introduce some notations $p_i(m), i = 0, 1, 2, \ldots, m$ as follows:

$$p_0(m) = 1,$$

$$p_1(m) = \sum_{i=1}^{m-1} i,$$

$$\vdots$$

$$p_i(m) = \sum_{j_1,\ldots,j_i \in \{1,\ldots,m-1\}, j_1 > j_2 > \cdots > j_i} (m-j_1)(m-j_2)\cdots(m-j_i),$$

$$\vdots$$

$$p_{m-1}(m) = (m-1)!,$$

$$p_m(m) = 0.$$

Then, the eigenpolynomial of the matrix A is

$$\begin{aligned} \det(sI - A) &= (s + k_1 + n - 1)(s + n - 2)\cdots(s+1)s \\ &\quad + k_2(s + n - 3)\cdots(s+1)s \\ &\quad + \cdots + k_{n-2}(s+1)s + k_{n-1}s + k_n \\ &= s^n + (p_0(n-1)k_1 + p_1(n))s^{n-1} \\ &\quad + \cdots + (p_{n-2}(n-1)k_1 + p_{n-1}(n))s \\ &\quad + k_2 p_0(n-2)s^{n-2} + k_2 p_1(n-2)s^{n-3} \\ &\quad + \cdots + k_2 p_{n-3}(n-2)s \\ &\quad + \cdots + k_{n-2}p_0(2)s^2 + k_{n-2}p_1(2)s \\ &\quad + k_{n-1}p_0(1)s + k_n \\ &= s^n + P_1(n, k_{[1]})s^{n-1} + P_2(n, k_{[2]})s^{n-2} \\ &\quad + P_3(n, k_{[3]})s^{n-3} \\ &\quad + \cdots + P_i(n, k_{[i]})s^{n-i} \\ &\quad + \cdots + P_{n-1}(n, k_{[n-1]})s + k_n, \end{aligned} \tag{19}$$

where $P_i(n, k_{[i]}) = p_i(n) + \sum_{j=1}^{i} k_j p_{i-j}(n-j), i = 1, \ldots, n$. From the expressions of $P_i(\cdot)$, $i = 1, \ldots, n$, it can easily be seen that each coefficient of the eigenpolynomial (19) can be arbitrarily assigned by a proper selection of the design parameters $k_1, \ldots, k_n$, and so can the eigenvalues of A. □

By using Lemma 1, we can choose the design parameters $k_1, \ldots, k_n$ such that all the eigenvalues of A have positive real parts, i.e., the matrix $-A$ is Hurwitz, and then there exists a positive definite matrix P satisfying

$$A^T P + PA = I_n. \tag{20}$$

Remark 4: When the subsystem x_0 is without nonlinear drifts, that is, its dynamical equation degenerates to $\dot{x}_0 = u_0$, we can take control u_0 in the form of $u_0 = -kx_0$, where k is a positive design parameter to be specified later, and take a state observer in the form of

$$\begin{aligned} \dot{\widehat{\chi}}_1 &= -k\widehat{\chi}_2 + \frac{f_1(y, u_0)}{\chi_0^{n-1}} + k(n-1)\widehat{\chi}_1 + k_1(\chi_1 - \widehat{\chi}_1) \\ &\vdots \\ \dot{\widehat{\chi}}_{n-1} &= -k\widehat{\chi}_n + \frac{f_{n-1}(y, u_0)}{\chi_0} + k\widehat{\chi}_{n-1} + k_{n-1}(\chi_1 - \widehat{\chi}_1), \\ \dot{\widehat{\chi}}_n &= u_1 + f_n(y, u_0) + k_n(\chi_1 - \widehat{\chi}_1). \end{aligned} \tag{21}$$

In this case, the estimation error $\widehat{\chi}$ satisfies:

$$\dot{\widehat{\chi}} = A_k\widehat{\chi} + \Upsilon + \Phi\theta, \tag{22}$$

where

$$A_k = \begin{bmatrix} -k_1+k(n-1) & -k & 0 & 0 & \cdots & 0 \\ -k_2 & k(n-2) & -k & 0 & \cdots & 0 \\ \vdots & 0 & k(n-3) & -k & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -k_{n-1} & \vdots & \vdots & \ddots & k & -k \\ -k_n & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}. \tag{23}$$

Comparing (16) with (22) shows that the latter has no the non-linear terms $-(n-1)B_1F_0\chi_1$ and $-B_2F_0(\widehat{\chi}+\widetilde{\chi})$. This reduces the difficulty to design an output-feedback adaptive stabilization control. And so, it is easy to design a globally asymptotical stabilization control.

Similar to matrix A, we have the following lemma for matrix A_k.

Lemma 2: *The eigenvalues of matrix A_k defined by* (23) *can be arbitrarily assigned by a proper selection of the design parameters k and $k_1, \ldots, k_n$.*

Proof: The eigenpolynomial of the matrix A_k is

$$\begin{aligned} \det(sI - A_k) &= (s+k_1-k(n-1))(s-k(n-2))\cdots(s-k)s \\ &\quad + (-1)^3kk_2(s-k(n-3))\cdots(s-k)s \\ &\quad + \cdots + (-1)^{n-1}k^{n-3}k_{n-2}(s-k)s \\ &\quad + (-1)^nk^{n-2}k_{n-1}s + (-1)^{n+1}k^{n-1}k_n \\ &= s^n + (p_0(n-1)k_1 - kp_1(n))s^{n-1} + \cdots \\ &\quad + (p_{n-2}(n-1)(-k)^{n-2}k_1 + (-k)^{n-1}p_{n-1}(n))s \\ &\quad + (-1)^3kk_2p_0(n-2)s^{n-2} \\ &\quad + (-1)^3k(-k)k_2p_1(n-2)s^{n-3} + \cdots \\ &\quad + (-1)^3k(-k)^{n-3}k_2p_{n-3}(n-2)s \\ &\quad + \cdots + (-1)^{n-1}k^{n-3}k_{n-2}p_0(2)s^2 \\ &\quad - (-1)^{n-1}k^{n-2}k_{n-2}p_1(2)s \\ &\quad + (-1)^nk^{n-2}k_{n-1}p_0(1)s + (-1)^{n+1}k^{n-1}k_n \\ &= s^n + Q_1(n,k,k_{[1]})s^{n-1} + Q_2(n,k,k_{[2]})s^{n-2} \\ &\quad + Q_3(n,k,k_{[3]})s^{n-3} \\ &\quad + \cdots + Q_i(n,k,k_{[i]})s^{n-i} \\ &\quad + \cdots + Q_{n-1}(n,k,k_{[n-1]})s + (-1)^{n+1}k^{n-1}k_n, \end{aligned} \tag{24}$$

where $Q_i(n,k,k_{[i]}) = (-k)^ip_i(n) + (-k)^{i-1}\sum_{j=1}^{i} p_{i-j} \times (n-j)k_j, i = 1, \ldots, n$. Notice that $Q_i(n, -1, k_{[i]}) = P_i(n, k_{[i]}), i = 1, \ldots, n$. Then from the expressions of $Q_i(\cdot), i = 1, \ldots, n$, it is easy to see that for any fixed k, each coefficient of the eigenpolynomial (24) can be arbitrarily assigned by a proper selection of the design parameters $k_1, \ldots, k_n$, and so can the eigenvalues of A_k. □

4.3. *Control design*

Due to the existence of the non-linear drifts in the dynamical equation of x_0, a linear feedback control with constant feedback in the form of $u_0 = -kx_0$ (where k is a constant) is not adequate. The reason is that the constant k may not be large enough to dominate the uncontrolled non-linear terms, as will be clear later. Therefore, in this paper, we will design the control u_0 in the form

$$u_0 = -\beta_0(y, \widehat{\theta})x_0, \tag{25}$$

where β_0 is a positive smooth nonlinear function to be specified later.

It follows from (10), (14), (16) and (25) that

$$\left.\begin{aligned} \dot{\widetilde{\chi}} &= -\beta_0A\widetilde{\chi} + \Upsilon + \Phi\theta - (n-1) \\ &\quad \times F_0B_1\chi_1 - F_0B_2(\widehat{\chi}+\widetilde{\chi}), \\ \dot{\chi}_1 &= -\beta_0\widehat{\chi}_2 - \beta_0\widetilde{\chi}_2 + \left(\frac{\varphi_1}{\chi_0^{n-1}} - \frac{(n-1)\varphi_0\chi_1}{\chi_0}\right) \\ &\quad + \left(\frac{\phi_1}{\chi_0^{n-1}} - \frac{(n-1)\chi_1\phi_0}{\chi_0}\right)^T\theta + F_1, \\ \dot{\widehat{\chi}}_2 &= -\beta_0\widehat{\chi}_3 + (n-2)\beta_0\widehat{\chi}_2 - k_2\beta_0\widetilde{\chi}_1 + F_2, \\ &\;\;\vdots \\ \dot{\widehat{\chi}}_{n-1} &= -\beta_0\widehat{\chi}_n + \beta_0\widehat{\chi}_{n-1} - k_{n-1}\beta_0\widetilde{\chi}_1 + F_{n-1}, \\ \dot{\widehat{\chi}}_n &= u_1 - k_n\beta_0\widetilde{\chi}_1 + F_n, \end{aligned}\right\} \tag{26}$$

where

$$F_1 = \frac{f_1}{\chi_0^{n-1}} - \frac{n-1}{\chi_0}\chi_1(u_0+f_0), \quad F_i = \frac{f_i}{\chi_0^{n-i}}, \quad 2 \le i \le n.$$

It is easy to see that when $\widetilde{\chi} = 0$, the system (26) has the lower-triangular structure, and so, the integrator backstepping approach can be used to design an output-feedback adaptive stabilization control.

We start the design procedure with the notations $\alpha_0 \equiv 0$, $z_{n+1} \equiv 0$, and the following coordinate transformations

$$\begin{cases} z_0 = \chi_0, \;\; z_1 = \chi_1, \;\; z_2 = \widehat{\chi}_2 - \alpha_1(\chi_0, \chi_1, \widehat{\theta}), \\ z_i = \widehat{\chi}_i - \alpha_{i-1}(\chi_0, \chi_1, \widehat{\chi}_2, \ldots, \widehat{\chi}_{i-1}, \widehat{\theta}), \;\; i = 3, \ldots, n, \end{cases} \tag{27}$$

where $\alpha_1,\ldots,\alpha_{n-1}$, called as virtual controls, are smooth functions to be specified later; $u_1=-\beta_0\alpha_n(\chi_0,\chi_1,\widehat{\chi}_2,\ldots,\widehat{\chi}_n,\widehat{\theta})$ is the actual control to be specified later.

Under the new variable vector z, the system (26) is transformed into

$$\left.\begin{aligned}
\dot{\widetilde{\chi}}&=-\beta_0A\widetilde{\chi}+\Upsilon+\Phi\theta-(n-1)F_0B_1z_1-F_0B_2(\widehat{\chi}+\widetilde{\chi}),\\
\dot z_1&=-\beta_0(z_2+\alpha_1)+\beta_0N_1\widetilde{\chi}_{[2]}+\Theta_1+\overline{F}_1+\Psi_1^T\theta,\\
\dot z_2&=-\beta_0(z_3+\alpha_2)+\beta_0N_2\widetilde{\chi}_{[2]}+\Theta_2+\overline{F}_2-\frac{\partial\alpha_1}{\partial\widehat{\theta}}\dot{\widehat{\theta}}+\Psi_2^T\theta,\\
&\vdots\\
\dot z_i&=-\beta_0(z_{i+1}+\alpha_i)+\beta_0N_i\widetilde{\chi}_{[2]}+\Theta_i+\overline{F}_i-\frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}}+\Psi_i^T\theta,\\
&\vdots\\
\dot z_n&=u_1+\beta_0N_n\widetilde{\chi}_{[2]}+\Theta_n+\overline{F}_n-\frac{\partial\alpha_{n-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}}+\Psi_n^T\theta,
\end{aligned}\right\}\tag{28}$$

where

$$N_1=[0,\ -1],\quad N_i=\left[-k_i+\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\widehat{\chi}_j}k_j,\frac{\partial\alpha_{i-1}}{\partial\chi_1}\right],\quad i=2,\ldots,n;$$

$$\Theta_1=\frac{\varphi_1}{\chi_0^{n-1}}-\frac{(n-1)\varphi_0\chi_1}{\chi_0},\quad \Theta_i=-\frac{\partial\alpha_{i-1}}{\partial\chi_0}\varphi_0-\frac{\partial\alpha_{i-1}}{\partial\chi_1}\Theta_1,\quad i=2,\ldots,n;$$

$$\Psi_1=\frac{\phi_1}{\chi_0^{n-1}}-\frac{(n-1)\chi_1\phi_0}{\chi_0},\quad \Psi_i=-\frac{\partial\alpha_{i-1}}{\partial\chi_0}\phi_0-\frac{\partial\alpha_{i-1}}{\partial\chi_1}\Psi_1,\quad i=2,\ldots,n;$$

$$\begin{aligned}\overline{F}_1=F_1,\quad \overline{F}_i&=F_i+(n-i)\beta_0\widehat{\chi}_i-\frac{\partial\alpha_{i-1}}{\partial\chi_0}(u_0+f_0)\\&+\frac{\partial\alpha_{i-1}}{\partial\chi_1}(\beta_0\widehat{\chi}_2-F_1)\\&+\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\widehat{\chi}_j}\big(\beta_0\widehat{\chi}_{j+1}-(n-j)\beta_0\widehat{\chi}_j-F_j\big),\\ i&=2,\ldots,n.\end{aligned}$$

For the simplicity of notations, let $\Psi_0=\phi_0,\Theta_0=\varphi_0$.

Lemma 3: *There are non-negative smooth functions $\overline{\Theta}_i(\cdot)$ $i=0,1,\ldots,n$ such that*

$$|\Theta_0|\le|\chi_0|\overline{\Theta}_0(\chi_0,\chi_1),\quad |\Theta_i|\le\big\|[\chi_0,\ \chi_1]\big\|\overline{\Theta}_i(\chi_0,\chi_1,u_0),\quad i=1,2,\ldots,n.$$

Proof: By Assumption A2, we have

$$|\Theta_0|=|\varphi_0|\le|\chi_0|\bar{\Theta}_0(\chi_0,\chi_1),$$

where

$$\overline{\Theta}_0=\{\overline{\varphi}_0(y)\}\Big|_{\substack{x_0=\chi_0\\ x_1=\chi_0^{n-1}\chi_1}}$$

is a non-negative smooth function. For $i=1$, by Assumption A2, we have

$$\begin{aligned}|\Theta_1|&=\left|\frac{\varphi_1}{\chi_0^{n-1}}-\frac{(n-1)\varphi_0\chi_1}{\chi_0}\right|\\&\le\frac{|x_0|^n\overline{\varphi}_{10}+|x_1|\overline{\varphi}_{11}}{|\chi_0|^{n-1}}\\&\quad+\left|\frac{(n-1)|x_0|\overline{\varphi}_0\chi_1}{\chi_0}\right|\\&\le\|[\chi_0,\ \chi_1]\|\overline{\Theta}_1(\chi_0,\chi_1,u_0),\end{aligned}$$

where

$$\bar{\Theta}_1=\{\overline{\varphi}_{10}+\overline{\varphi}_{11}+(n-1)\overline{\varphi}_0\}\Big|_{\substack{x_0=\chi_0\\ x_1=\chi_0^{n-1}\chi_1}}$$

is a non-negative smooth function. For $i\ge2$, by Assumption A2 and the results for cases $i=0,1$, we have

$$\begin{aligned}|\Theta_i|&=\left|-\frac{\partial\alpha_{i-1}}{\partial\chi_0}\varphi_0-\frac{\partial\alpha_{i-1}}{\partial\chi_1}\Theta_1\right|\\&\le|\varphi_0|\sqrt{1+\left(\frac{\partial\alpha_{i-1}}{\partial\chi_0}\right)^2}+\|\Theta_1\|\sqrt{1+\left(\frac{\partial\alpha_{i-1}}{\partial\chi_1}\right)^2}\\&\le|\chi_0|\bar{\Theta}_0\sqrt{1+\left(\frac{\partial\alpha_{i-1}}{\partial\chi_0}\right)^2}\\&\quad+\|[\chi_0,\ \chi_1]\|\overline{\Theta}_1\sqrt{1+\left(\frac{\partial\alpha_{i-1}}{\partial\chi_1}\right)^2}\\&\le\|[\chi_0,\chi_1]\|\bar{\Theta}_i(\chi_0,\chi_1,u_0),\end{aligned}$$

where $\bar{\Theta}_i=\bar{\Theta}_0\sqrt{1+(\partial\alpha_{i-1}/\partial\chi_0)^2}+\bar{\Theta}_1\sqrt{1+(\partial\alpha_{i-1}/\partial\chi_1)^2}$ is a non-negative smooth function. □

We now turn to the constructive design procedure of the controls.

Step 0: This step can be regarded as the initial assignation of the entire design procedure. At this step, we introduce a Lyapunov function for the estimation error $\widetilde{\chi}$

$$V_0=\delta_0\widetilde{\chi}^TP\widetilde{\chi},$$

where δ_0 is a positive design parameter to be specified later, P is the positive definite solution of the

Riccati equation (20). Then, taking time-derivation of V_0 along the solution of (16) and using (25), we have

$$\begin{aligned}\dot{V}_0 = & -\delta_0\beta_0\|\tilde{\chi}\|^2 + 2\delta_0\tilde{\chi}^T P\Upsilon \\ & + 2\delta_0\tilde{\chi}^T P\Phi\theta - 2(n-1)\delta_0\tilde{\chi}^T PF_0B_1z_1 \\ & - 2\delta_0\tilde{\chi}^T PF_0B_2\tilde{\chi} - 2\delta_0\tilde{\chi}^T PF_0B_2\hat{\chi}. \end{aligned} \tag{29}$$

By Assumption A2 and the expression of Υ in (18), we have

$$\|\Upsilon\|^2 \le 2\chi_0^2\overline{\Upsilon}_1(\chi_0, \chi_1) + 2\chi_1^2\overline{\Upsilon}_2(\chi_0, \chi_1, u_0),$$

where

$$\overline{\Upsilon}_1 = \sum_{i=1}^{n}\overline{\varphi}_{i0}^2\Bigg|_{\substack{x_0=\chi_0 \\ x_1=\chi_0^{n-1}\chi_1}} \quad \text{and} \quad \overline{\Upsilon}_2 = \sum_{i=1}^{n}\overline{\varphi}_{i1}^2\Bigg|_{\substack{x_0=\chi_0 \\ x_1=\chi_0^{n-1}\chi_1}}$$

are non-negative smooth functions. Then, for the second term on the right-hand side of (29), by completing the square we have

$$\begin{aligned}2\delta_0\tilde{\chi}^T P\Upsilon = & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + \delta_0\varepsilon_0^{-2}\Upsilon^T PP\Upsilon - \delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Upsilon\|^2 \\ \le & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 - \Delta_{01} + 2\delta_0\varepsilon_0^{-2}\|P\|^2 \\ & \times \left(\chi_0^2\overline{\Upsilon}_1(\chi_0, \chi_1) + \chi_1^2\overline{\Upsilon}_2(\chi_0, \chi_1, u_0)\right), \end{aligned} \tag{30}$$

where $\Delta_{01} = \delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Upsilon\|^2$, and ε_0 is a positive design parameter to be specified later.

From Assumption A2 and the expression of Φ of (18) it follows that

$$\Phi(y, u_0) = \overline{\Phi}_1(z_0, z_1)z_0 + \overline{\Phi}_2(z_0, z_1, u_0)z_1, \tag{31}$$

where $\overline{\Phi}_1(\cdot)$ and $\overline{\Phi}_2(\cdot)$ are known smooth functions and available for feedback design. Then, for the 3rd term on the right-hand side of (29), by using Assumption A1 we have

$$\begin{aligned}2\delta_0\tilde{\chi}^T P\Phi\theta = & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + \delta_0\varepsilon_0^{-2}\theta^T\Phi^T PP\Phi\theta \\ & - \delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta\|^2 \\ \le & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 - \delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta\|^2 \\ & + 2\delta_0\varepsilon_0^{-2}M^2\left(\|\overline{\Phi}_1^T P\|^2z_0^2 + \|\overline{\Phi}_2^T P\|^2z_1^2\right). \end{aligned} \tag{32}$$

The non-positive term $-\delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta\|^2$ is preserved to tackle the 4th term on the right-hand side of (29) through the estimation

$$\begin{aligned} & - \delta_0\varepsilon_0^2\left\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta\right\|^2 - 2\delta_0(n-1)\tilde{\chi}^T PB_1F_0z_1 \\ = & -\delta_0\varepsilon_0^2\left\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta\right\|^2 - 2\delta_0(n-1)\tilde{\chi}^T PB_1(\overline{f}_0 + \overline{\phi}_0^T\theta)z_1 \\ & - 2\delta_0(n-1)\tilde{\chi}^T PB_1\frac{\varphi_0}{\chi_0}z_1 \\ = & -\delta_0\varepsilon_0^2\left\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta + \varepsilon_0^{-2}(n-1)PB_1(\overline{f}_0 + \overline{\phi}_0^T\theta)z_1\right\|^2 \\ & + \delta_0\varepsilon_0^{-2}(n-1)^2B_1^T PPB_1(\overline{f}_0 + \overline{\phi}_0^T\theta)^2z_1^2 \\ & - 2\delta_0\varepsilon_0^{-2}(n-1)B_1^T PP\Phi\theta(\overline{f}_0 + \overline{\phi}_0^T\theta)z_1 \\ & - 2\delta_0(n-1)\tilde{\chi}^T PB_1\frac{\varphi_0}{\chi_0}z_1 \\ \le & -\Delta_{02} + \delta_0\varepsilon_0^{-2}(n-1)B_1^T PP \\ & \quad \times \left((n-1)B_1\overline{f}_0(\overline{f}_0 + 2\overline{\phi}_0^T\hat{\theta})z_1 - 2\overline{f}_0\Phi\hat{\theta}\right)z_1 \\ & + 2\delta_0\varepsilon_0^{-2}(n-1)B_1^T PP\overline{f}_0\left((n-1)B_1\overline{\phi}_0^T z_1 - \Phi\right)\tilde{\theta}z_1 \\ & + \delta_0\varepsilon_0^{-2}n(n-1)M^2B_1^T PPB_1\|\overline{\phi}_0\|^2z_1^2 \\ & + 2\delta_0\varepsilon_0^{-2}(n-1)M^2(\|\overline{\Phi}_1^T P\|^2z_0^2 + \|\overline{\Phi}_2^T P\|^2z_1^2) \\ & + \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + (n-1)^2\delta_0\|PB_1\|^2\varepsilon_0^{-2}\overline{\varphi}_0^2z_1^2, \end{aligned} \tag{33}$$

where
$\Delta_{02} = \delta_0\varepsilon_0^2\|\tilde{\chi} - \varepsilon_0^{-2}P\Phi\theta + \varepsilon_0^{-2}(n-1)PB_1(\overline{f}_0 + \overline{\phi}_0^T\theta)z_1\|^2$, which is non-negative.

For the last term on the right-hand side of (29), by completing the square we obtain the estimation

$$\begin{aligned}-2\delta_0\tilde{\chi}^T PF_0B_2\hat{\chi} \le & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + \delta_0\varepsilon_0^{-2}\|P\|^2\|F_0\|^2\|B_2\hat{\chi}\|^2 \\ \le & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + 2\delta_0\varepsilon_0^{-2}\|P\|^2 \\ & \times \left(\left(\overline{f}_0 + \overline{\phi}_0^T\theta\right)^2 + \overline{\varphi}_0^2\right)\sum_{i=2}^{n-1}(n-i)^2\hat{\chi}_i^2 \\ = & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + 2\delta_0\varepsilon_0^{-2}\|P\|^2 \\ & \times \left(\left(\overline{f}_0 + \overline{\phi}_0^T\theta\right)^2 + \overline{\varphi}_0^2\right) \\ & \times \sum_{i=2}^{n-1}(n-i)^2(z_i + \alpha_{i-1})^2 \\ \le & \, \delta_0\varepsilon_0^2\|\tilde{\chi}\|^2 + 2\delta_0\varepsilon_0^{-2}\|P\|^2 \\ & \times \left(2\overline{f}_0^2 + 2M^2\|\overline{\phi}_0\|^2 + \overline{\varphi}_0^2\right) \\ & \times \sum_{i=2}^{n-1}(n-i)^2z_i(z_i + 2\alpha_{i-1}) \\ & + 2\delta_0\varepsilon_0^{-2}\|P\|^2\left(2\overline{f}_0^2 + 2M^2\|\overline{\phi}_0\|^2 + \overline{\varphi}_0^2\right) \\ & \times \sum_{i=1}^{n-2}(n-i)^2\alpha_i^2. \end{aligned} \tag{34}$$

Substituting (32)–(34) into (29) leads to

$$\dot{V}_0 \le -\delta_0 \widetilde{\chi}^T \Lambda_0 \widetilde{\chi} - \Delta_0 + \delta_0 W + \sum_{i=0}^{n} z_i Q_i + 2\delta_0 \varepsilon_0^{-2}(n-1)B_1^T PP\overline{f}_0 \times \Big((n-1)B_1 \overline{\phi}_0^T z_1 - \phi^T\Big)\widetilde{\theta} z_1, \quad (35)$$

where

$$\Lambda_0 = \beta_0 I_n - 4\varepsilon_0^2 I_n - 2PF_0 B_2; \quad (36)$$

$$\Delta_0 = \Delta_{01} + \Delta_{02}; \quad (37)$$

$$W = 2\|P\|^2 \varepsilon_0^{-2}\Big(2\overline{f}_0^2 + 2M^2\|\overline{\phi}_0\|^2 + \overline{\varphi}_0^2\Big) \times \sum_{i=1}^{n-2}(n-i)^2 \alpha_i^2; \quad (38)$$

$$Q_0 = 2\delta_0 \varepsilon_0^{-2}\|P\|^2 \overline{\Upsilon}_1 z_0 + 2n\delta_0 \varepsilon_0^{-2} M^2 \|\overline{\Phi}_1^T P\|^2 z_0, \quad (39)$$

$$Q_1 = 2\delta_0 \varepsilon_0^{-2}\|P\|^2 \overline{\Upsilon}_2 z_1 + (n-1)^2 \delta_0 \varepsilon_0^{-2}\|PB_1\|^2 \overline{\varphi}_0^2 z_1 + \delta_0 \varepsilon_0^{-2}\Big((n-1)B_1^T PP \times \big((n-1)B_1 \overline{f}_0(\overline{f}_0 + 2\overline{\phi}_0^T \widehat{\theta})z_1 - 2\overline{f}_0 \Phi \widehat{\theta}\big) + 2nM^2\|\overline{\Phi}_2^T P\|^2 + n(n-1)M^2 B_1^T PPB_1 \|\overline{\phi}_0\|^2\Big) z_1, \quad (40)$$

$$Q_i = 2\delta_0 \varepsilon_0^{-2}\|P\|^2 (2\overline{f}_0^2 + 2M^2\|\overline{\phi}_0\|^2 + \overline{\varphi}_0^2)(z_i + 2\alpha_{i-1}), \quad i = 2, \ldots, n. \quad (41)$$

This completes Step 0.

Step 1: Take $V_1 = V_0 + \delta_1 \widetilde{\theta}^T \widetilde{\theta} + z_1^2$ as the Lyapunov function of this step, where δ_1 is a positive design parameter to be determined later, and $(1/\delta_1)$ is called as the gain constant of the adaptive law. Then, by (28) we have

$$\begin{aligned}\dot{V}_1 &= \dot{V}_0 + 2z_1\Big(-\beta_0(z_2 + \alpha_1) + \beta_0 N_1 \widetilde{\chi}_{[2]} + \Theta_1 + \overline{F}_1 + \Psi_1^T \theta\Big) - 2\delta_1 \widetilde{\theta}^T \dot{\widehat{\theta}} \\ &\le -\delta_0 \widetilde{\chi}^T \Lambda_0 \widetilde{\chi} - \Delta_0 + \delta_0 W + \sum_{i=0}^{n} z_i Q_i \\ &\quad + 2\delta_0 \varepsilon_0^{-2}(n-1)B_1^T PP\overline{f}_0\Big((n-1)B_1 \overline{\phi}_0^T z_1 - \Phi\Big)\widetilde{\theta} z_1 \\ &\quad + 2z_1\Big(-\beta_0(z_2 + \alpha_1) + \overline{F}_1 + \Psi_1^T \widehat{\theta}\Big) \\ &\quad + 2z_1 \beta_0 N_1 \widetilde{\chi}_{[2]} + 2z_1 \Theta_1 + 2z_1 \Psi_1^T \widetilde{\theta} - 2\delta_1 \widetilde{\theta}^T \dot{\widehat{\theta}}. \end{aligned} \quad (42)$$

Choose the smooth virtual control as follows:

$$\alpha_1 = \left\{\frac{1}{\beta_0}\left(\overline{F}_1 + \Psi_1^T \widehat{\theta} + \frac{z_1}{2} + \frac{n\bar{\Theta}_1^2 z_1}{2\varepsilon_2^2} + \frac{\varepsilon_2^2 z_1}{2} + \frac{Q_1}{2} + \frac{\beta_0}{2\delta_0 \varepsilon_1^2} z_1 N_1 N_1^T + \frac{(n-1)z_1}{2\delta_1}\right)\right\}_{(27)}, \quad (43)$$

where $\varepsilon_1 \in (0, 1)$ and $\varepsilon_2 \in (0, \infty)$ are design parameters to be specified later. Notice that

$$2z_1 \beta_0 N_1 \widetilde{\chi}_{[2]} = \delta_0 \beta_0 \varepsilon_1^2 \|\widetilde{\chi}_{[2]}\|^2 + \beta_0 \delta_0^{-1} \varepsilon_1^{-2} z_1^2 N_1 N_1^T - \beta_0 \delta_0 \varepsilon_1^2 \big\|\widetilde{\chi}_{[2]} - \delta_0^{-1}\varepsilon_1^{-2} N_1^T z_1\big\|^2,$$

and by Lemma 3, that

$$2z_1 \Theta_1 \le 2|z_1|\,\|[z_0, z_1]\|\bar{\Theta}_1 = \varepsilon_2^2(z_0^2 + z_1^2) + \varepsilon_2^{-2}\bar{\Theta}_1^2 z_1^2 - \varepsilon_2^2\big(\|[z_0, z_1]\| - \varepsilon_2^{-2}\bar{\Theta}_1|z_1|\big)^2.$$

Then from (42) it follows that

$$\dot{V}_1 \le -\delta_0 \widetilde{\chi}^T \Lambda_1 \widetilde{\chi} - \Delta_1 + \delta_0 W - z_1^2 + z_0(Q_0 + \varepsilon_2^2 z_0) + \sum_{i=2}^{n} z_i Q_i - \frac{1}{\delta_1}(n-1)z_1^2 - (n-1)\varepsilon_2^{-2}\bar{\Theta}_1^2 z_1^2 - 2\delta_1 \widetilde{\theta}^T\left(\dot{\widehat{\theta}} - \frac{z_1}{\delta_1} S_1^T\right) - 2\beta_0 z_1 z_2, \quad (44)$$

where $\Lambda_1 = \Lambda_0 - \beta_0 \varepsilon_1^2 I$, and

$$\Delta_1 = \Delta_0 + \beta_0 \delta_0 \varepsilon_1^2 \big\|\widetilde{\chi}_{[2]} - \delta_0^{-1}\varepsilon_1^{-2} N_1^T z_1\big\|^2 + \varepsilon_2^2\big(\|[z_0, z_1]\| - \varepsilon_2^{-2}\bar{\Theta}_1|z_1|\big)^2,$$

$$S_1 = \Psi_1^T + \delta_0 \varepsilon_0^{-2}(n-1)B_1^T PP\Big((n-1)B_1 \overline{\phi}_0^T z_1 - \phi^T\Big)\overline{f}_0.$$

This completes Step 1.

Step i ($i = 2, \ldots, n-1$)**:** Assume at Step $i-1$. Then, we have

$$V_{i-1} = V_0 + \delta_1 \widetilde{\theta}^T \widetilde{\theta} + \sum_{j=1}^{i-1} z_j^2,$$

$$\begin{aligned}\dot{V}_{i-1} &\le -\delta_0 \widetilde{\chi}^T \Lambda_{i-1} \widetilde{\chi} - \Delta_{i-1} + \delta_0 W - \sum_{j=1}^{i-1} z_j^2 + z_0\big(Q_0 + \varepsilon_2^2 z_0\big) \\ &\quad + \sum_{j=i}^{n} z_j Q_j - \frac{1}{\delta_1}(n-1)z_1^2 - \frac{2}{\delta_1}\sum_{j=2}^{i-1}(n-j)z_j^2 - (n-1)\varepsilon_2^{-2} \\ &\quad \times \sum_{j=1}^{i-1}\bar{\Theta}_j^2 z_j^2 - 2\delta_1 \widetilde{\theta}^T\left(\dot{\widehat{\theta}} - \frac{1}{\delta_1}\sum_{j=1}^{i-1} z_j S_j^T\right) - 2\beta_0 z_{i-1} z_i \\ &\quad + 2\varepsilon_2^{-2}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}\bar{\Theta}_j|z_j|\bar{\Theta}_k|z_k| - \sum_{j=2}^{i-1} z_j M_j \dot{\widehat{\theta}} \\ &\quad - \frac{1}{\delta_1}\sum_{j=2}^{i-1} z_j^2 \sqrt{1 + M_j M_j^T S_j S_j^T} \\ &\quad - \frac{1}{2\delta_1}\sum_{j=3}^{i-1} z_j^2 \sum_{k=2}^{j-1}\Big(M_j M_j^T S_k S_k^T + M_k M_k^T S_j S_j^T\Big),\end{aligned}$$

where $\Lambda_{i-1} = \Lambda_1$, $S_j = \Psi_j^T$, $j = 2, \dots, i-1$, $M_j = 2(\partial\alpha_{j-1}/\partial\widehat{\theta})$, $j = 2, \dots, i-1$, and

$$\Delta_{i-1} = \Delta_0 + \beta_0\delta_0\varepsilon_1^2 \left\| \widetilde{\chi}_{[2]} - \delta_0^{-1}\varepsilon_1^{-2}\sum_{j=1}^{i-1} N_j^T z_j \right\|^2 + \varepsilon_2^2\left(\|[z_0, z_1]\| - \sum_{i=1}^{i-1} \varepsilon_2^{-2}\bar{\Theta}_i|z_i| \right)^2.$$

Choose $V_i = V_{i-1} + z_i^2$ as the Lyapunov function for this step. Then, by (28) we have

$$\begin{aligned}
\dot{V}_i &= \dot{V}_{i-1} + 2z_i\bigg(-\beta_0(z_{i+1} + \alpha_i) + \beta_0 N_i\widetilde{\chi}_{[2]} + \overline{F}_i + \Theta_i - \frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}} + \Psi_i^T\theta \bigg) \\
&\le -\delta_0\widetilde{\chi}^T\Lambda_{i-1}\widetilde{\chi} - \Delta_{i-1} + \delta_0 W \\
&\quad - \sum_{j=1}^{i-1} z_j^2 + z_0\big(Q_0 + \varepsilon_2^2 z_0\big) + \sum_{j=i}^{n} z_j Q_j - \frac{1}{\delta_1}(n-1)z_1^2 \\
&\quad - \frac{2}{\delta_1}\sum_{j=2}^{i-1}(n-j)z_j^2 - (n-1)\varepsilon_2^{-2}\sum_{j=1}^{i-1}\bar{\Theta}_j^2 z_j^2 \\
&\quad - 2\delta_1\widetilde{\theta}^T\bigg(\dot{\widehat{\theta}} - \frac{1}{\delta_1}\sum_{j=1}^{i-1} z_j S_j^T\bigg) - 2\beta_0 z_{i-1}z_i \\
&\quad + 2\varepsilon_2^{-2}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}\bar{\Theta}_j|z_j|\bar{\Theta}_k|z_k| - \sum_{j=2}^{i-1} z_j M_j\dot{\widehat{\theta}} \\
&\quad - \frac{1}{\delta_1}\sum_{j=2}^{i-1} z_j^2\sqrt{1 + M_j M_j^T S_j S_j^T} \\
&\quad - \frac{1}{2\delta_1}\sum_{j=3}^{i-1} z_j^2 \sum_{k=2}^{j-1}\Big(M_j M_j^T S_k S_k^T + M_k M_k^T S_j S_j^T \Big) \\
&\quad + 2\beta_0 N_i\widetilde{\chi}_{[2]}z_i \\
&\quad + 2z_i\Big(-\beta_0(z_{i+1} + \alpha_i) + \overline{F}_i + \Theta_i + \Psi_i^T\widehat{\theta} \Big) \\
&\quad - z_i\frac{\partial 2\alpha_{i-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}} + 2z_i\Psi_i^T\widetilde{\theta}.
\end{aligned} \tag{45}$$

Set $S_i = \Psi_i^T$ and $M_i = 2(\partial\alpha_{i-1}/\partial\widehat{\theta})$, and choose the ith smooth virtual control as

$$\alpha_i = \Bigg\{ -z_{i-1} + \frac{N_i}{2\varepsilon_1^2\delta_0}\bigg(N_i z_i + 2\sum_{j=1}^{i-1} N_j z_j \bigg)^T + \frac{1}{\beta_0}\bigg(\overline{F}_i + \Psi_i^T\widehat{\theta} + \frac{z_i}{2} + \frac{Q_i}{2} + \frac{n\bar{\Theta}_i^2 z_i}{2\varepsilon_2^2} \bigg) + \frac{z_i}{\beta_0\delta_1}\bigg((n-i) + \frac{1}{2}\sqrt{1 + M_i M_i^T S_i S_i^T} + \frac{z_i}{4}\sum_{k=2}^{i-1}\Big(M_i M_i^T S_k S_k^T + M_k M_k^T S_i S_i^T \Big) \bigg) \Bigg\}_{(27)}. \tag{46}$$

Notice that

$$\begin{aligned}
&-2\delta_1\widetilde{\theta}^T\bigg(\dot{\widehat{\theta}} - \frac{1}{\delta_1}\sum_{j=1}^{i-1} z_j S_j^T\bigg) + 2z_i S_i\widetilde{\theta} \\
&\quad = -2\delta_1\widetilde{\theta}^T\bigg(\dot{\widehat{\theta}} - \frac{1}{\delta_1}\sum_{j=1}^{i} z_j S_j^T\bigg),
\end{aligned}$$

$$\begin{aligned}
&-\beta_0\delta_0\varepsilon_1^2\left\| \widetilde{\chi}_{[2]} - \delta_0^{-1}\varepsilon_1^{-2}\sum_{j=1}^{i-1} N_j^T z_j \right\|^2 - 2\beta_0 N_i\widetilde{\chi}_{[2]}z_i \\
&\quad = -\beta_0\delta_0\varepsilon_1^2\left\| \widetilde{\chi}_{[2]} - \delta_0^{-1}\varepsilon_1^{-2}\sum_{j=1}^{i} N_j^T z_j \right\|^2 \\
&\qquad + \beta_0\delta_0^{-1}\varepsilon_1^{-2}N_i\bigg(N_i z_i + 2\sum_{j=1}^{i-1} N_j z_j \bigg)^T z_i,
\end{aligned}$$

and

$$\begin{aligned}
&-\varepsilon_2^2\bigg(\|[z_0, z_1]\| - \sum_{j=1}^{i-1}\varepsilon_2^{-2}\bar{\Theta}_j|z_j| \bigg)^2 + 2z_i\Theta_i \\
&\quad \le -\varepsilon_2^2\bigg(\|[z_0, z_1]\| - \sum_{j=1}^{i-1}\varepsilon_2^{-2}\bar{\Theta}_j|z_j| \bigg)^2 + 2|z_i|\|(z_0, z_1)\|\bar{\Theta}_i \\
&\quad = -\varepsilon_2^2\bigg(\|[z_0, z_1]\| - \sum_{j=1}^{i}\varepsilon_2^{-2}\bar{\Theta}_j|z_j| \bigg)^2 \\
&\qquad + 2\varepsilon_2^{-2}\sum_{j=1}^{i-1}\bar{\Theta}_i|z_i|\bar{\Theta}_j|z_j| + \varepsilon_2^{-2}\bar{\Theta}_i^2 z_i^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\dot{V}_i &\le -\delta_0\widetilde{\chi}^T\Lambda_i\widetilde{\chi} - \Delta_i + \delta_0 W - \sum_{j=1}^{i} z_j^2 + z_0\big(Q_0 + \varepsilon_2^2 z_0\big) \\
&\quad + \sum_{j=i+1}^{n} z_j Q_j + \frac{1}{\delta_1}\,(n-1)z_1^2 \\
&\quad - \frac{2}{\delta_1}\sum_{j=2}^{i}(n-j)z_j^2 - (n-1)\varepsilon_2^{-2}\sum_{j=1}^{i}\bar{\Theta}_j^2 z_j^2 \\
&\quad - 2\delta_1\widetilde{\theta}^T\bigg(\dot{\widehat{\theta}} - \frac{1}{\delta_1}\sum_{j=1}^{i} z_j S_j^T\bigg) - 2\beta_0 z_i z_{i+1} \\
&\quad + 2\varepsilon_2^{-2}\sum_{j=2}^{i}\sum_{k=1}^{j-1}\bar{\Theta}_j|z_j|\bar{\Theta}_k|z_k| - \sum_{j=2}^{i} z_j M_j\dot{\widehat{\theta}} \\
&\quad - \frac{1}{\delta_1}\sum_{j=2}^{i} z_j^2\sqrt{1 + M_j M_j^T S_j S_j^T} \\
&\quad - \frac{1}{2\delta_1}\sum_{j=3}^{i} z_j^2\sum_{k=2}^{j-1}\Big(M_j M_j^T S_k S_k^T + M_k M_k^T S_j S_j^T \Big),
\end{aligned} \tag{47}$$

where $\Lambda_i = \Lambda_1$ and $\Delta_i = \Delta_0 + \beta_0\delta_0\varepsilon_1^2\|\widetilde{\chi}_{[2]} - (1/\delta_0\varepsilon_1^2)\sum_{j=1}^{i} N_j^T z_j\|^2 + \varepsilon_2^2(\|[z_0, z_1]\| - \sum_{j=1}^{i}(\bar{\Theta}_j|z_j|)/(\varepsilon_2^2))^2$.

This completes Step i.

Step n: It is easy to see that the results of Step i hold true also for $i=n$ with $u_1=-\beta_0\alpha_n$. So, by (46) we obtain the expression of a_n, and then, design the actual control as follows:

$$u_1(\chi_0,\chi_1,\widehat{\chi}_2,\ldots,\widehat{\chi}_n,\widehat{\theta})=-\beta_0\alpha_n(\chi_0,\chi_1,\widehat{\chi}_2,\ldots,\widehat{\chi}_n,\widehat{\theta}). \tag{48}$$

Substituting (48) into (47) and noticing $i=n$ in this case,

$$\begin{aligned}\dot V_n \le & -\delta_0\widetilde{\chi}^T\Lambda_n\widetilde{\chi}-\Delta_n+\delta_0 W-\sum_{j=1}^n z_j^2+z_0\left(Q_0+\varepsilon_2^2 z_0\right)\\ & -\frac{1}{\delta_0}(n-1)z_1^2\\ & -\frac{2}{\delta_1}\sum_{j=1}^n(n-j)z_j^2-2\delta_1\widetilde{\theta}^T\left(\dot{\widehat{\theta}}-\frac{1}{\delta_1}\sum_{j=1}^n z_jS_j^T\right)-\sum_{j=2}^n z_jM_j\dot{\widehat{\theta}}\\ & -\frac{1}{\delta_1}\sum_{j=2}^n z_j^2\sqrt{1+M_jM_j^TS_jS_j^T}\\ & -\frac{1}{2\delta_1}\sum_{j=3}^n z_j^2\sum_{k=2}^{j-1}\left(M_jM_j^TS_kS_k^T+M_kM_k^TS_jS_j^T\right),\end{aligned} \tag{49}$$

where $2\sum_{j=2}^n\sum_{k=1}^{j-1}\bar\Theta_j|z_j|\bar\Theta_k|z_k|\le(n-1)\sum_{j=1}^n\bar\Theta_j^2 z_j^2$ has been used.

This completes Step n.

The steps above are focused on the constructive design of the control u_1 and with the hypothesis that the control u_0 and adaptive law of $\widehat{\theta}$ are known. The following step will be denoted to the design of the control u_0 and adaptive law of $\widehat{\theta}$.

Step n+1: Take $V=V_n+z_0^2$ as the Lyapunov function of the whole design procedure. Then, we have

$$\begin{aligned}\dot V =& \dot V_n+2(u_0+f_0+\Theta_0+\Psi_0^T\theta)z_0\\ \le & -\delta_0\widetilde{\chi}^T\Lambda_n\widetilde{\chi}-\Delta_n+\delta_0 W-\sum_{j=1}^n z_j^2-\frac{1}{\delta_1}(n-1)z_1^2\\ & -\frac{2}{\delta_1}\sum_{j=2}^n(n-j)z_j^2\\ & -2\delta_1\widetilde{\theta}^T\left(\dot{\widehat{\theta}}-\frac{1}{\delta_1}\sum_{j=0}^n z_jS_j^T\right)-\sum_{j=2}^n z_jM_j\dot{\widehat{\theta}}\\ & -\frac{1}{\delta_1}\sum_{j=2}^n z_j^2\sqrt{1+M_jM_j^TS_jS_j^T}\\ & -\frac{1}{2\delta_1}\sum_{j=3}^n z_j^2\sum_{k=2}^{j-1}\left(M_jM_j^TS_kS_k^T+M_kM_k^TS_jS_j^T\right)\\ & +2(u_0+\overline{F}_0)z_0-\frac{1}{\delta_1}(n-1)z_0^2,\end{aligned} \tag{50}$$

where $S_0=\Psi_0^T$, and $\overline{F}_0=f_0+z_0\bar\Theta_0+\Psi_0^T\widehat{\theta}+\frac{1}{2}\left(Q_0+\varepsilon_2^2 z_0\right)+\frac{1}{2\delta_1}(n-1)z_0$.

Choose the control u_0 and the adaptive law of $\widehat{\theta}$ as follows, respectively:

$$u_0=-\beta_0(y,\widehat{\theta})z_0, \tag{51}$$

$$\dot{\widehat{\theta}}=\frac{1}{\delta_1}\sum_{j=0}^n z_jS_j^T, \tag{52}$$

where $\beta_0=\overline{F}_0+\sqrt{2k^2+2\overline{F}_0^2}+(2\|PB_2\|)/(1-\varepsilon_1^2)(\overline{\varphi}_0+M\sqrt{1+\|\overline{\varphi}_0\|^2})$. Clearly, β_0 is smooth and satisfies

$$\left.\begin{aligned}&\beta_0(y,\widehat{\theta})>k+\left(|\overline{F}_0|+\overline{F}_0\right)\ge k>0,\\ &\beta_0(y,\widehat{\theta})-\overline{F}_0>\sqrt{2}k+1>0,\\ &(1-\varepsilon_1^2)\beta_0(y,\widehat{\theta})-2\|F_0PB_2\|>(1-\varepsilon_1^2)k>0.\end{aligned}\right\} \tag{53}$$

Notice that

$$\begin{aligned}-\sum_{i=2}^n z_iM_i\frac{1}{\delta_1}\sum_{j=0}^n z_jS_j^T \le & \frac{1}{\delta_1}\sum_{i=2}^n\sum_{j=0}^n|z_iz_j|\,\|M_i\|\,\|S_j^T\|\\ =& \frac{1}{\delta_1}\sum_{i=2}^n\sum_{j=0}^{i-1}|z_iz_j|\,\|M_i\|\,\|S_j^T\|+\frac{1}{\delta_1}\sum_{i=2}^n z_i^2\,\|M_i\|\,\|S_i^T\|\\ & +\frac{1}{\delta_1}\sum_{i=2}^n\sum_{j=i+1}^n|z_iz_j|\,\|M_i\|\,\|S_j^T\|\\ =& \frac{1}{\delta_1}\sum_{i=2}^n\sum_{j=0}^{i-1}|z_iz_j|\,\|M_i\|\,\|S_j^T\|+\frac{1}{\delta_1}\sum_{i=2}^n z_i^2\,\|M_i\|\,\|S_i^T\|\\ & +\frac{1}{\delta_1}\sum_{i=3}^n\sum_{j=2}^{i-1}|z_iz_j|\,\|M_j\|\,\|S_i^T\|\\ \le & \frac{1}{\delta_1}\sum_{i=2}^n\left(\sum_{j=0}^{i-1}\left(\frac{1}{4}z_i^2\left(\|M_i\|\,\|S_j^T\|\right)^2+z_j^2\right)\right.\\ & +z_i^2\sqrt{1+\left(\|M_i\|\,\|S_i^T\|\right)^2}\\ & \left.+\sum_{j=2}^{i-1}\left(\frac{1}{4}z_i^2\left(\|M_j\|\,\|S_i^T\|\right)^2+z_j^2\right)\right)\\ =& \frac{1}{\delta_1}(n-1)z_0^2+\frac{1}{\delta_1}(n-1)z_1^2+\frac{2}{\delta_1}\sum_{i=2}^{n-1}(n-i)z_i^2\\ & +\frac{1}{\delta_1}\sum_{i=2}^n z_i^2\sqrt{1+M_iM_i^TS_iS_i^T}\\ & +\frac{1}{2\delta_1}\sum_{i=3}^n z_i^2\sum_{j=2}^{i-1}\left(M_iM_i^TS_jS_j^T+M_jM_j^TS_iS_i^T\right).\end{aligned} \tag{54}$$

Then, substituting (51)–(54) into (50) leads to

$$\dot{V} \le -\delta_0 \widetilde{\chi}^T \Lambda_n \widetilde{\chi} - \Delta_n + \delta_0 W - \sum_{j=1}^{n} z_j^2 - 2\left(\sqrt{2k^2 + 2\overline{F}_0^2} + \frac{2\|PB_2\|}{1-\varepsilon_1^2}\left(\overline{\varphi}_0 + M\sqrt{1+\|\overline{\phi}_0\|^2}\right)\right) z_0^2. \quad (55)$$

4.4. *Design parameters choice*

Before giving the selection of the design parameters, we first present the following lemma.

Lemma 4: *For the function W defined by* (38), *there are positive smooth functions* $W_i\ (z_0,\ z_{[n-2]},\ \widehat{\theta}, \xi_1, \xi_2, \xi_3, \xi_4)$, $i = 0, 1, 2, \dots, n-2$, which is decreasing with respect to ξ_1 and ξ_2, and increasing with respect to ξ_3 and ξ_4, such that

$$W \le \sum_{i=0}^{n-2} W_i(z_0, z_{[n-2]}, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) z_i^2. \quad (56)$$

Proof: This lemma is proven by induction. By Assumption A2 and the expressions of $\overline{F}_1$, Ψ_1 and Q_1, the α_1 given by (43) satisfies:

$$|\alpha_1|^2 \le \overline{\alpha}_{10}(z_0, z_1, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) z_0^2 + \overline{\alpha}_{11}(z_0, z_1, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) z_1^2,$$

where $\overline{\alpha}_{10}(z_0, z_1, \widehat{\theta}, \xi_1, \xi_2, \xi_3, \xi_4)$ and $\overline{\alpha}_{11}(z_0, z_1, \widehat{\theta}, \xi_1, \xi_2, \xi_3, \xi_4)$ are positive smooth functions decreasing with respect to ξ_1 and ξ_2, and increasing with respect to ξ_3 and ξ_4.

Suppose that for all $k = 1, \dots, i-1$, there exist positive smooth functions $\overline{\alpha}_{kj}(z_0, z_{[k]}, \widehat{\theta},\ \xi_1, \xi_2, \xi_3, \xi_4)$, $j = 0, 1, \dots, k$, decreasing with respect to ξ_1 and ξ_2, and increasing with respect to ξ_3 and ξ_4, such that

$$|\alpha_k|^2 \le \sum_{j=0}^{k} \overline{\alpha}_{kj}(z_0, z_{[k]}, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) z_j^2.$$

Then, for $\alpha_i, i = 2, \dots, n-2$, by (46), Assumption A2 and the expressions of $\overline{F}_i$, Q_i and Ψ_i, it is easy to find $i+1$ positive smooth functions $\overline{\alpha}_{ij}(z_0, z_{[i]}, \widehat{\theta}, \xi_1, \xi_2, \xi_3, \xi_4)$, $j = 0, \dots, i$, decreasing with respect to ξ_1 and ξ_2, and increasing with respect to ξ_3 and ξ_4, such that

$$|\alpha_i|^2 \le \sum_{j=0}^{i} \overline{\alpha}_{ij}(z_0, z_{[i]}, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) z_j^2.$$

Thus, from the definition (38) of W, (56) follows for some positive smooth functions $W_i(z_0,\ z_{[n-2]}, \widehat{\theta}, \xi_1, \xi_2, \xi_3, \xi_4)$, $i = 0,\ 1, \dots, n-2$, decreasing with respect to ξ_1 and ξ_2, and increasing with respect to ξ_3 and ξ_4. □

Let

$$\overline{\beta}_0 = 2\sqrt{2}k + 2 - \delta_0 W_0, \quad \overline{\beta}_1 = 1 - \delta_0 W_1, \dots, \quad \overline{\beta}_{n-2} = 1 - \delta_0 W_{n-2}, \quad \overline{\beta}_{n-1} = \overline{\beta}_n = 1.$$

Then, by (55) we have

$$\dot{V} \le -\delta_0 \widetilde{\chi}^T \Lambda_n \widetilde{\chi} - \Delta_n - \sum_{i=0}^{n} \overline{\beta}_i z_i^2. \quad (57)$$

From the design procedure above, we can see that for given initial values, the key is how to choose the design parameters ε_0, ε_1, ε_2, k, δ_0 and δ_1 such that

$$\Lambda_n \ge b I_n > 0, \quad \overline{\beta}_0 \ge b_0 > 0, \quad \overline{\beta}_1 \ge b_1 > 0, \dots, \overline{\beta}_{n-2} \ge b_{n-2} > 0, \quad (58)$$

where $b, b_0, b_1, \dots, b_{n-2}$ are some positive constants.

The following lemma gives the range and the selecting methods of the design parameters.

Lemma 5: *For given initial values, there exist always the design parameters* ε_0, ε_1, ε_2, k, δ_0 *and* δ_1 *such that the inequalities* (58) *hold for some positive constants* b, b_0, $b_1, \dots, b_{n-2}$.

Proof: Define $V_a = \delta_0 \widetilde{\chi}^T P \widetilde{\chi} + \delta_1 \widetilde{\theta}^T \widetilde{\theta} + \sum_{i=0}^{n} z_i^2$. For given initial values, $V_a(t_0)$ is finite. Then, for any given $\varepsilon_1 \in (0, 1)$, $\varepsilon_2 > 0$ and $\delta_1 > 0$, let

$$\Omega_1 = \left\{ \widehat{\theta}:\ \|\widehat{\theta} - \theta\| \le 1 + \sqrt{\frac{V_a(t_0)}{\delta_1}} \right\},$$

$$\Omega_2 = \left\{ [z_0, z_{[n]}^T]^T : \sum_{i=0}^{n} z_i^2 \le 1 + V_a(t_0) \right\}$$

and for $i = 0, 1, \dots, n-2$,

$$\widehat{W}_i(k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}) = \sup_{\substack{\widehat{\theta} \in \Omega_1 \\ [z_0, z_{[n]}^T]^T \in \Omega_2}} W_i(z_0, z_{[n-2]}, \widehat{\theta}, k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}). \quad (59)$$

Then, $\widehat{W}_0(k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1})$, $\widehat{W}_1(k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1}), \dots,$ $\widehat{W}_{n-2}(k, \varepsilon_0, \delta_0, (\varepsilon_0\delta_0)^{-1})$ are finite and positive, and by Lemma 4, are decreasing with respect to k and ε_0, and increasing with respect to δ_0 and $(\varepsilon_0\delta_0)^{-1}$.

Let

$$0 < \delta_0 = \varepsilon_0^{-1} < \frac{1}{2}\min\left\{1,\quad \frac{1}{\widehat{W}_1(1,1,0.5,1)},\ \ldots,\ \frac{1}{\widehat{W}_{n-2}(1,1,0.5,1)}\right\},$$

and

$$k > \max\left\{1,\quad \frac{4\varepsilon_0^2}{1-\varepsilon_1^2},\quad \frac{\widehat{W}_0(1,1,0.5,1)-3}{4\sqrt{2}}\right\}.$$

Then we have

$$\widehat{W}_i(k,\varepsilon_0,\delta_0,(\varepsilon_0\delta_0)^{-1}) \le \widehat{W}_i(1,1,0.5,1), \quad i = 0,1,\ldots,n-2.$$

The design parameters specified above ensure the inequalities (58) hold at initial time t_0 with $b = b_0 = b_1 = \cdots = b_{n-2} = \frac{1}{2}$.

We can prove that (58) holds for any $t \ge t_0$, since otherwise, by the continuity of $\dot{V}$ there would be two time instances t_1 and t_2 satisfying $t_1 > t_2 > t_0$ such that

$$\dot{V} \le 0 \quad \forall t \in [t_0, t_1] \quad \text{and} \quad \dot{V} > 0, \quad \forall t \in (t_1, t_2]. \tag{60}$$

The first inequality of (60) implies that for all $t \in [t_0, t_1]$, $V(t) \le V(t_0)$, which gives

$$\sum_{i=0}^{n} z_i^2(t) \le V_a(t_0) \quad \text{and} \quad \|\widehat{\theta}(t)\| \le \sqrt{\frac{V_a(t_0)}{\delta_1}}, \quad \forall t \in [t_0,\ t_1]. \tag{61}$$

While the second inequality of (60), together with the parameter design procedure stated above, leads to

$$\text{either} \quad \sum_{i=0}^{n} z_i^2(t) > 1 + V_a(t_0) \quad \text{or}$$
$$\|\widehat{\theta}(t)\| > 1 + \sqrt{\frac{V_a(t_0)}{\delta_1}}, \quad \forall t \in (t_1,\ t_2].$$

This together with (61) implies that either $[z_0, z_{[n]}^T]^T$ or $\widehat{\theta}$ is discontinuous at t_1. This contradicts the continuity of $[z_0, z_{[n]}^T]^T$ and $\widehat{\theta}$. Thus, (58) holds for all $t \ge t_0$. □

4.5. *Main results*

The following theorem summarizes the main results of the paper.

Theorem 1: *Consider system* (1). *Suppose that Assumptions A1 and A2 hold. If the design parameters* $\varepsilon_0, \varepsilon_1, \varepsilon_2, k, \delta_0$ *and* δ_1 *are chosen such that* (58) *is satisfied, then,* (i) *the estimation* $\widehat{\theta}$ *given by* (52) *is uniformly bounded;* (ii) *the output-feedback adaptive stabilization controls* (u_0, u_1) *given by* (51), (48), (52) *and* (14) *render the closed-loop system semiglobally asymptotically stable.*

Proof: Under the controls (u_0, u_1) given by (51), (48), (52) and (14), we have

$$\dot{V} \le -\delta_{01}\widetilde{\chi}^T\Lambda_n\widetilde{\chi} - \sum_{i=0}^{n}\overline{\beta}_i z_i^2, \quad \forall t \ge t_0. \tag{62}$$

This together with $\overline{\beta}_{n-1} = \overline{\beta}_n = 1$ and (58) gives $\delta_1\widetilde{\theta}^T\widetilde{\theta} \le V(t_0) < \infty, \forall t \ge t_0$, and together with Assumption A1, gives

$$\sup_{t\ge 0}\|\widehat{\theta}\| < \infty. \tag{63}$$

By (62) we have

$$\sup_{t\ge 0}\|\widetilde{\chi}\| < \infty, \quad \sup_{t\ge 0}\sum_{i=0}^{n} z_i^2 < \infty, \tag{64}$$

and

$$\int_0^\infty \|\widetilde{\chi}\|^2\, dt < \infty, \quad \sum_{i=0}^{n}\int_0^\infty z_i^2\, dt < \infty. \tag{65}$$

From (28), (63) and (64) it follows that the first derivatives of $\widetilde{\chi}$ and z_i $(i = 0, 1, \ldots, n)$ are uniformly bounded. This together with Barbălat's Lemma (Khalil, 2002) implies that $\widetilde{\chi}$ and z_i $(i = 0, 1, \ldots, n)$ are uniformly continuous, and further, together with (65), implies that

$$\lim_{t\to\infty}\widetilde{\chi} = 0, \qquad \lim_{t\to\infty} z_i = 0, \quad \forall i = 0, 1, \ldots, n.$$

So, the closed-loop system is semiglobally asymptotically stable. □

If subsystem x_0 of (1) is without any nonlinear drifts, i.e., $f_0 \equiv 0, \varphi_0 \equiv 0, \phi_0^T\theta \equiv 0$, then, $W \equiv 0$ (so are $W_i \equiv 0$, $i = 0, 1, \ldots, n$), and the choice of design parameters is independent of the initial values. Thus, the closed-loop system is globally asymptotically stable. This is summarized by the following theorem.

Theorem 2: *Consider the system* (1), *whose subsystem* x_0 *is without any nonlinear drifts. Suppose that Assumptions A1 and A2 are satisfied. Then, the output-feedback adaptive stabilization controls* (u_0, u_1) *given by* (51), (48),

(52) *and* (14) *render the closed-loop system globally asymptotically stable*.

5. Example

Consider the third-order non-holonomic system

$$\left.\begin{aligned}\dot{x}_0 &= u_0,\\ \dot{x}_1 &= u_0 x_2 + \frac{1}{2}x_1^2 + \theta x_1,\\ \dot{x}_2 &= u_1,\end{aligned}\right\} \tag{66}$$

where θ is the unknown constant parameter, and assumed to be bounded by 1, i.e. $|\theta| \le 1$. The purpose is to design u_0 and u_1 based on only $y = [x_0, x_1]^T$ such that $[x_0(t), x_1(t), x_2(t)]^T \to \infty$ as $t \to \infty$.

If $x_0(0) = 0$, controls u_0 and u_1 are set as in Remark 3 in interval $[0, t_0)$ such that $x_0(t_0) \neq 0$, then we can adopt the controls developed below. Therefore, without loss of generality, we assume that $x_0(0) \neq 0$. Let $\chi_0 = x_0, \chi_1 = (x_1/x_0), \chi_2 = x_2$. Then by (66), we have

$$\left.\begin{aligned}\dot{\chi}_0 &= u_0,\\ \dot{\chi}_1 &= \frac{u_0}{\chi_0}(\chi_2 - \chi_1) + \frac{1}{2}\chi_0\chi_1^2 + \theta\chi_1,\\ \dot{\chi}_2 &= u_1,\end{aligned}\right\} \tag{67}$$

Design observer to reconstruct χ_1 and χ_2 of system (66) as follows:

$$\begin{aligned}\dot{\widehat{\chi}}_1 &= \frac{u_0}{\chi_0}(\widehat{\chi}_2 - \widehat{\chi}_1) + \frac{1}{2}\chi_0\chi_1^2 + \frac{k_1 u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1), \quad & k_1 = -4,\\ \dot{\widehat{\chi}}_2 &= u_1 + \frac{k_2 u_0}{\chi_0}(\chi_1 - \widehat{\chi}_1), & k_2 = 2.\end{aligned}$$

Then, the estimation error $\widetilde{\chi} = [\widetilde{\chi}_1 - \widehat{\chi}_1, \widetilde{\chi}_2 - \widehat{\chi}_2]^T$ satisfies the equation

$$\dot{\widetilde{\chi}} = \frac{u_0}{\chi_0}\begin{bmatrix} -k_1 - 2 + 1 & 1\\ -k_2 & 0\end{bmatrix}\widetilde{\chi} + \theta\begin{bmatrix}\chi_1\\ 0\end{bmatrix}.$$

In this case, we have

$$A = \begin{bmatrix} 3 & 1\\ -2 & 0\end{bmatrix}.$$

Solving the matrix equation $A^T P + PA = I_2$, we obtain

$$P = \begin{bmatrix} 0.5000 & 0.5000\\ 0.5000 & 0.1000\end{bmatrix}.$$

The eigenvalues of P are 0.1910, 1.3090, and thus $\|P\| = 1.3090$.

Define $z_0 = \chi_0, z_1 = \chi_1, z_2 = \hat{\chi}_2 - \alpha_1(\chi_0, \chi_1, \widehat{\theta})$ and let $u_0 = \alpha_0(y, \widehat{\theta}),\ u_1 = -\beta_0(y, \widehat{\theta})\alpha_2(\chi_0, \chi_1, \widehat{\chi}_2, \widehat{\theta})$, where the expressions of $\alpha_0, \alpha_1, \beta_0, \alpha_2$ and the adaptive law of $\widehat{\theta}$ are as follows, respectively:

$$\alpha_0 = -\beta_0(y, \widehat{\theta})z_0, \quad \text{with} \quad \beta_0 = \overline{F}_0 + \sqrt{2k^2 + 2\overline{F}_0^2},$$

$$\overline{F}_0 = \left(\frac{\varepsilon_2^2}{2} + \frac{1}{2\delta_1}\right)z_0,$$

$$\alpha_1 = -z_1 + \frac{z_1}{2\delta_0\varepsilon_1^2} + \frac{z_1}{\beta_0}\left(\widehat{\theta} + 0.5\chi_0\chi_1 + 1.604\delta_0\varepsilon_0^{-2}M^2 + \frac{1}{2} + \frac{\varepsilon_2^2}{2} + \frac{1}{2\delta_1}\right),$$

$$\alpha_2 = -z_1 + \frac{N_2}{2\varepsilon_1^2\delta_0}(N_2 z_2 + 2N_1 z_1)^T + \frac{1}{\beta_0}\left(\overline{F}_2 + \Psi_2\widehat{\theta} + \frac{z_2}{2}\right) + \frac{z_2}{2\beta_0\delta_1}\sqrt{1 + M_2^2 S_2^2}$$

with

$$N_1 = [0,\ -1], N_2 = \left[-k_2,\ \frac{\partial\alpha_1}{\partial\chi_1}\right],$$

$$\overline{F}_2 = \beta_0\widehat{\chi}_2 - \frac{\partial\alpha_1}{\partial\chi_0}u_0 + \frac{\partial\alpha_1}{\partial\chi_1}\left(\beta_0\widehat{\chi}_2 + \frac{u_0\chi_1}{\chi_0} - \frac{1}{2}\chi_0\chi_1^2\right),$$

$$M_2 = 2\frac{\partial\alpha_1}{\partial\widehat{\theta}} = \frac{2z_1}{\beta_0}, \quad S_2 = \Psi_2 = -\frac{\partial\alpha_1}{\partial\chi_1}\chi_1,$$

$$\dot{\widehat{\theta}} = \frac{1}{\delta_1}\left(z_1 - \frac{\partial\alpha_1}{\partial\chi_1}z_2\right)\chi_1.$$

The design constants are chosen as $k = 4, \delta_0 = 1, \delta_1 = 1, \varepsilon_0 = 0.4, \varepsilon_1 = 0.4, \ \varepsilon_2 = 0.4, M = 1$ and the initial conditions are simply set to $x_0(0) = 2, \ x_1(0) = 0.8, x_2(0) = 0, \widehat{\chi}_1(0) = 0.3, \ \widehat{\chi}_2(0) = 1.5286, \widehat{\theta}(0) = 0.5$.

The simulation results are shown in figures 1–4 given below. In particular, figure 1 is about system states; figure 2 is about observer states; figure 3 is about parameter estimator state; figure 4 is about control inputs u_0 (solid) and u_1 (dashdotted). From figure 1 we can see all the closed-loop system states are regulated to zero.

6. Conclusion

This paper investigates the problem of output-feedback adaptive stabilization control design for non-holonomic chained systems with strong non-linear drifts, including modelled non-linear dynamics, unmodelled dynamics, and those modelled but with unknown parameters. A new observer design method is proposed, and based on the observer, the unmeasurable states of the system involved are reconstructed. Unlike Do and

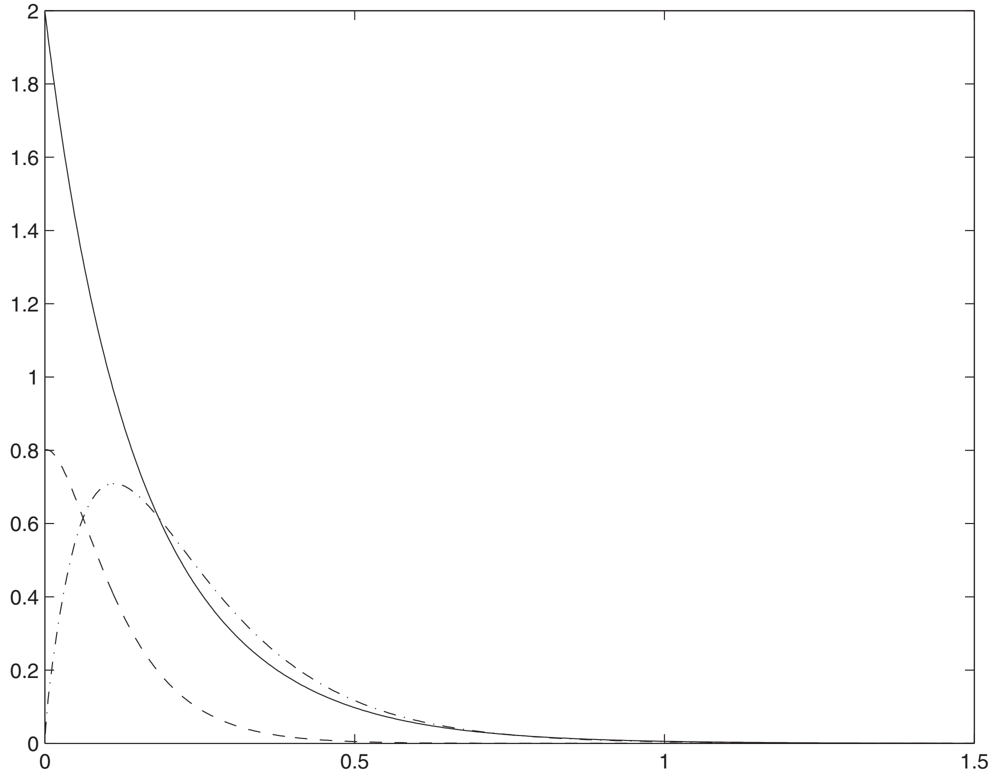

Figure 1. System states: ——, x_0; – – – –, x_1 and – · – · –, x_2.

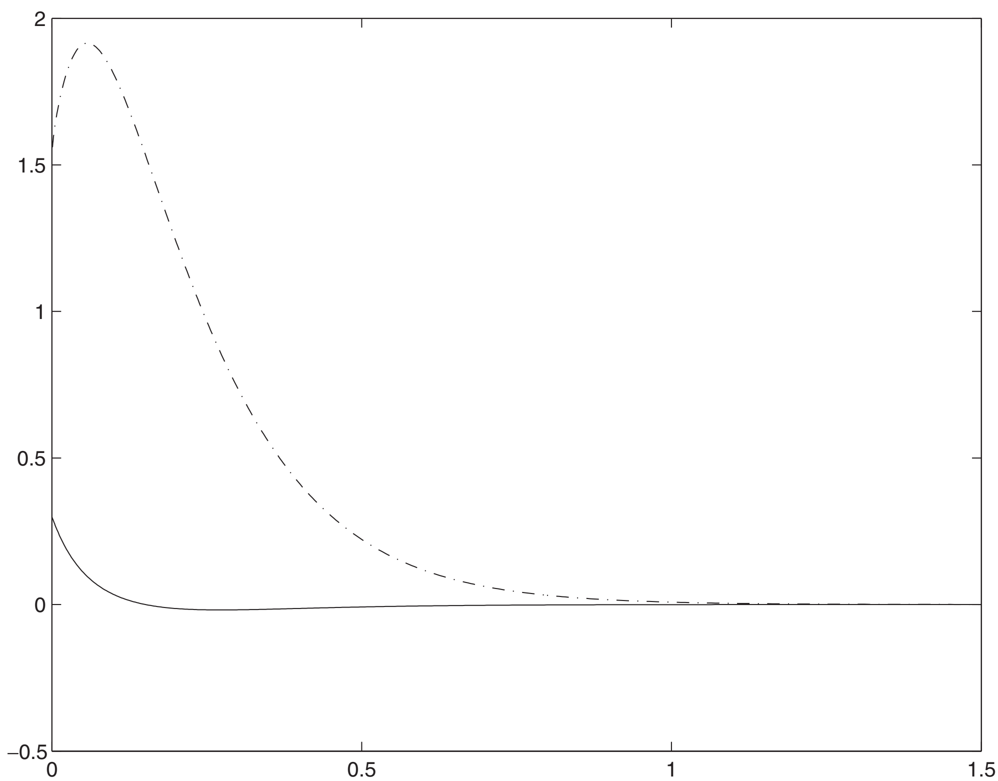

Figure 2. Observer states: ——, $\widehat{\chi}_1$; and – · – · –, $\widehat{\chi}_2$.

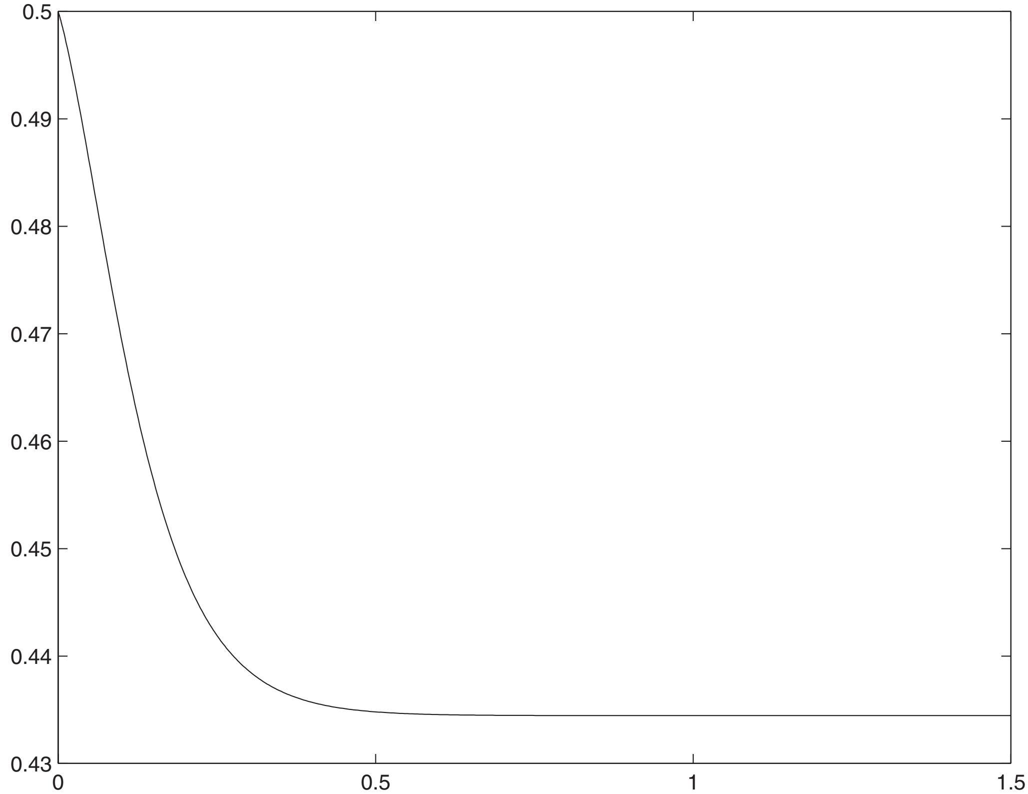

Figure 3. Parameter estimation $\hat{\theta}$.

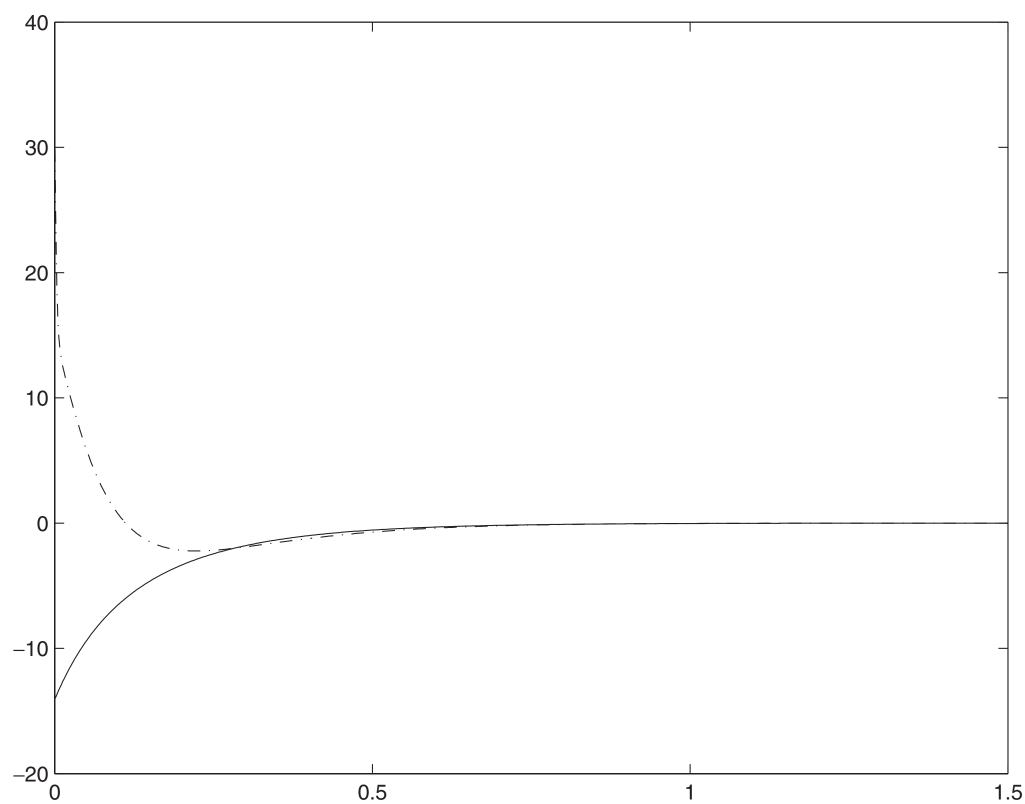

Figure 4. Control inputs: ——, u_0 and $- \cdot - \cdot -$, u_1.

Pan (2002), only one estimator is used to estimate the unknown parameters. This avoids the undesirable overparametrization estimate. By using the integrator backstepping approach and based on the observer and parameter estimator given, a constructive design procedure of output-feedback adaptive stabilization control is presented. It is shown that, under some conditions, the control designed ensures the closed-loop system is globally asymptotically stable when there is no nonlinear drift in the first subsystem, and semiglobally asymptotically stable, otherwise.

Acknowledgements

The work has been supported in part by the National Natural Science Foundation of China under Grants 60221301, 60274021, 60304002 and 60334040, and the Science and Technical Development Plan of Shandong Province under Grant 2004GG4204014.

References

A. Astolfi, "Discontinuous control of nonholonomic systems", *Systems & Control Letters*, 27, pp. 37–45, 1996.

S. Battilotti, "An unifying framework for the semiglobal stabilization of nonlinear uncertain systems via measurement feedback", *IEEE Trans. on Automatic Control*, 46, pp. 3–16, 2001.

A.M. Bloch, M. Reyhanoglu and N.H. McClamroch, "Control and stabilization of nonholonomic systems", *IEEE Trans. on Automatic Control*, 37, pp. 1747–1757, 1992.

K.D. Do and J. Pan, "Adaptive global stabilization of nonholonomic systems with strong nonlinear drifts", *Systems & Control Letters*, 46, pp. 195–205, 2002.

L. Huang, "Mechanics and control sciences", *Acta Automatica Sinica*, 28, pp. 23–29, 2002.

S.S. Ge, Z. Sun, T.H. Lee and M.W. Spong, "Feedback linearization and stabilization of second-order nonholonomic chained systems" *Int. J. of Control*, 74, pp. 1383–1392, 2001.

S.S. Ge, "Comments on 'Adaptive global stabilization of nonholonomic systems with strong nonlinear drifts' [*Systems & Control Letters*, 46, 195–205, 2002], *System & Control Letters*, 49, pp. 401–403, 2003.

Z.P. Jiang and H. Nijmeijer, "A recursive technique for tracking control of nonholonomic systems in chained form", *IEEE Trans. on Automatic Control*, 44, pp. 265–279, 1999.

I. Kanellakopoulos, "Adaptive control of nonlinear systems: a tutorial", *Adaptive Control, Filtering, and Signal Processing*, K.J. Åström, G.C. Goodwin, P.R. Kumar Eds, Vol. 74, New York: Springer-Verlag, pp. 89–133, 1995.

H.K. Khalil, *Nonlinear Systems*, 3rd ed., Uppper Saddle River, New Jersey: Prentice Hall, 2002.

I. Kolmanovsky and N.H. McClamroch, "Development in nonholonomic control problems", *IEEE Control System Magazine*, 15, pp. 20–36, 1995.

Z. Xi, G. Feng, Z.P. Jiang and D.Z. Cheng, "A switching algorithm for global exponential stabilization of uncertain chained systems", *IEEE Trans. on Automatic Control*, 48, pp. 1793–1798, 2003.