Optimality Analysis of Adaptive Sampled Control of Hybrid Systems With Quadratic Index

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Abstract—**This note is concerned with the sampled-data based linear quadratic (LQ) adaptive control of continuous-time systems with unknown Markov jump parameters. A parameter estimator and a control design method are given. It is shown that when the sample step size is small, the sampled-data based adaptive control is suboptimal under LQ index. The result is illustrated by a simulation example.**

Index Terms—**Adaptive control, linear quadratic (LQ) index, Markov jump parameter, sampled-data based control, stochastic system.**

I. Introduction

We consider continuous-time systems with unknown Markov jump parameters

$$\dot{x_t} = A(\theta_t)x_t + B(\theta_t)u_t \tag{1}$$

where $x_t \in I\!\!R^n$ and $u_t \in I\!\!R^m$ are the system state and input, respectively; $A(\theta_t)$ and $B(\theta_t)$ are real valued matrices; θ_t is the unknown Markov jump parameter process taking values in a finite set $\mathcal{S} = \{1, 2, \ldots, N\}$ with transition probability matrix

$$P(\tau) = [P_{ij}(\tau)] = [P(\theta_{t+\tau} = j|\theta_t = i)] = e^{\Lambda\tau}. \tag{2}$$

Here, $\Lambda = (\lambda_{ij})$, $\lambda_{ij} \geq 0$, $j \neq i$, and

$$\lambda_i = -\lambda_{ii} = \sum_{j=1, j\neq i}^{N} \lambda_{ij}. \tag{3}$$

For simplicity, we assume that the initial time is $t_0 = 0$, and the initial time values $x_0 = x(0)$ and $\theta_0 = \theta(0)$ are deterministic. Since almost all sample paths of $\theta(\cdot)$ are constant except for a finite number of simple jumps in any finite time interval of $[0, \infty)$, we define the paths of $x(\cdot)$ in an obvious way, joining solution arcs of (1) at jump points of θ. The $x(t)$ sample paths so determined are then continuous with probability one [1].

In this note, we first design sampled-data (SD) based adaptive control for stochastic systems (1) under the following quadratic index:

$$J(x_0, \theta_0, u) = E\left[\int_0^\infty \left(x_t^T Q(\theta_t)x_t + u_t^T R(\theta_t)u_t\right) dt\right] \tag{4}$$

where $Q(\theta_t) \geq 0$, $R(\theta_t) > 0$, and x^T denotes the transpose of x. Then, we investigate the optimality of the SD-based adaptive control, especially, the index difference between the SD-based adaptive control and conventional optimal linear quadratic (LQ) control.

Systems with Markov jump parameters belong to the category of "hybrid systems," which are emerging as a convenient mathematical framework for the formulation of various design problems in the fields such as target tracking, fault tolerant control, and manufacturing processes, etc. [2]. The study on the LQ optimal control problem of such systems can be traced back at least to the work of Krasovskii and Lidskii [3]. When the Markov jump parameter process θ_t and the system state process x_t are known for all t, stochastic stabilizability and controllability of the systems are investigated and the jump LQ optimal control problem is solved in [4]. Later, more progress has been made on such as output feedback control, optimal control (with infinite Markov jump parameters), and almost-sure and moment stabilization (with finite-state Markov jump parameters) [5]–[7]. Recently, some results on adaptive stabilization control of Markov jump parameter systems are presented in [8] and [9] for the case where the system state process x_t is known for all t.

This note is devoted to the case where the Markov jump process θ_t is unknown and only the sampled-data (i.e. the information measured at sample time instances) rather than the complete process of the system state is available for control design. Since digital technology offers many benefits, modern control systems usually employ digital technology for controllers and sometimes sensors [10]. The SD-based LQ control problem of stochastic linear continuous-time systems with known parameters is studied in [11]. In addition to the stability analysis of the closed-loop systems, the index difference between SD-based LQ control and conventional LQ control is investigated. The purpose of this note is to: 1) give an SD-based parameter estimator to estimate the unknown Markov jump process θ_t, 2) design a suboptimal adaptive control based on the sampled-data of the state process, the parameter estimates and the index (4), and 3) analyze the impact of the sample size on the optimality of the SD-based adaptive control.

The note is organized as follows. In Section II, some preliminary results and notations are listed. In Section III, sampled-data based parameter estimate and adaptive control are designed, and the optimality of the control is analyzed. In Section IV, an example is presented to illustrate the result of Section III. Section V gives some concluding remarks.

II. Preliminaries and Notations

First, we introduce the following definition [4].

Definition 2.1: Systems (1)–(3) are said to be stochastically stabilizable if, for all finite $x_0 \in I\!\!R^n$ and $\theta_0 \in \mathcal{S}$, there exists a linear feedback control law

$$u_t = -L(\theta_t)x(t)$$

with $\|L(\theta_t)\| < \infty$ such that

$$E\left[\int_0^\infty x^T(t, x_0, \theta_0, u)x(t, x_0, \theta_0, u)dt|x_0, \theta_0\right] \leq x_0^T \widetilde{M} x_0$$

where $\widetilde{M}$ is a symmetric positive–definite matrix. For short, we will simply say that $[A(\theta_t), B(\theta_t)]$ is stochastically stabilizable.

Here and hereafter, $\|x\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ denotes the Euclidean norm of vector x, and $\|A\|$ denotes the corresponding induced matrix norm of A, which is equal to the largest eigenvalue of $(A^T A)^{1/2}$. For simplicity, in the sequel, we sometimes write $A(i)$, $B(i)$, $Q(i)$, $R(i)$, $K(i)$, $L(i)$, $M(i)$, etc., by A_i, B_i, Q_i, R_i, K_i, L_i, and M_i, respectively.

Assume that $[A_i, Q_i^{1/2}]$ is observable for each $i \in \mathcal{S}$, and $[A(\theta_t), B(\theta_t)]$ is stochastically stabilizable. Then, from [4] we know

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that when the values of θ_t are completely available at all time instance t, the N-coupled algebraic Riccati equation set

$$A_i^T M_i+M_iA_i-M_iB_iR_i^{-1}B_i^TM_i+\sum_{j=1}^{N}\lambda_{ij}M_j+Q_i=0,\ i\in\mathcal{S} \quad (5)$$

has a unique set of positive–definite matrices $\{M_i, i=1,2,\ldots,N\}$; the optimal control is

$$u_t^* = -L(\theta_t)x_t \triangleq -R^{-1}(\theta_t)B^T(\theta_t)M(\theta_t)x_t \quad (6)$$

the closed-loop system is stable in the sense that

$$\lim_{t\to\infty} E\left[\|x_t\|^2\right]=0$$

and the optimal index is

$$J(u^*) = x_0^T M(\theta_0)x_0.$$

Here, R_i, Q_i and M_i are defined by (4) and (5), respectively.

III. Main Results

Let

$$L_i = R_i^{-1}B_i^TM_i$$

where B_i, R_i are defined in (1) and (4), respectively; $\{M_i, i\in\mathcal{S}\}$ is the set of unique, positive solutions of (5). Then, the SD-based adaptive control is designed as

$$u_t = -L(\hat{\theta}_{kh})x_{kh}, \qquad t\in[kh,(k+1)h) \quad (7)$$

where h is the sample step size, and

$$\hat{\theta}_{kh} = \arg\min_{i\in\mathcal{S}} \|x_{kh}-\hat{x}_{kh}(i)\|.$$

Here, $\hat{x}_{kh}(i)$ is the solution of

$$\dot{x}_t = A_ix_t + B_iu_t$$

with initial value $x_{(k-1)h}$, or equivalently

$$\hat{x}_{kh}(i) = e^{A_ih}x_{(k-1)h} - \int_{(k-1)h}^{kh} e^{A_i(kh-\tau)}d\tau \times B_iL\left(\hat{\theta}_{(k-1)h}\right)x_{(k-1)h}.$$

Theorem 3.1: Consider system (1)–(3), and assume $[A(\theta_t), B(\theta_t)]$ is stochastically stabilizable; and for $\forall i\in\mathcal{S}$, $[A_i, Q_i^{1/2}]$ is observable. Let

$$\mathcal{D}_{ij}(h) = \frac{e^{A_ih}-e^{A_jh}}{h} - \frac{1}{h}\left(\int_0^h e^{A_i(h-\tau)}d\tau B_i - \int_0^h e^{A_j(h-\tau)}d\tau B_j\right)L_j, \qquad i,j\in\mathcal{S}.$$

If for some $\varepsilon_0\in(0,1)$ and $h_0\in(0,1]$

$$d_0<1,\quad \frac{ld_1}{1-d_1}+3\lambda hl_1\frac{(1+d_0)^2}{(1-d_0)^2}\le\frac{1-\varepsilon_0}{2} \qquad \forall h\le h_0 \quad (8)$$

$$\det\mathcal{D}_{ij}(h)\neq 0 \qquad \forall i\neq j \quad \forall h\le h_0 \quad (9)$$

then, under the SD-based adaptive control (7) with sample size $h\le\min\{1,h_0\}$, we have

$$E\left[\int_0^\infty\left(\|x_s\|^2+\|u_s\|^2\right)ds\right]<\infty$$

$$E\left[\int_0^\infty\left(x_s^TQ(\theta_s)x_s+u_s^TR(\theta_s)u_s\right)ds\right] \le x_0^TM(\theta_0)x_0+4l_2\lambda x_0^T\widetilde{K}x_0h+O(h^2) \quad (10)$$

where

$$d_0 = 2c_1he^{2ch},\ d_1=c_1he^{ch} \quad (11)$$

$$\lambda=(N-1)\left(e^{\|\Lambda\|}-1\right),\ c_1=\max_{i,j}\|A_i-B_iL_j\| \quad (12)$$

$$c=\max_i\|A_i\|,\ l=\max_{i,j}\left\|M_iB_iR_i^{-1}B_j^TK_j\right\| \quad (13)$$

$$l_1=\max_{i,j,k}\left\|(L_i-L_j)^TB_k^TK_k\right\| \quad (14)$$

$$l_2=\max_{i,j,k,m,n}\left\|(L_i-L_j)^TR_k(L_m-L_n)\right\| \quad (15)$$

$$\widetilde{K}=\max_{i\in\mathcal{S}}\frac{x_0^TK_ix_0}{\alpha_0\|K_i\|},\ \alpha_0=\min_{i\in\mathcal{S}}\frac{\varepsilon_0}{\lambda_{\max}(K_i)} \quad (16)$$

and $\{K_j, j\in\mathcal{S}\}$ is the (unique) symmetric solution set of

$$(A_i-B_iL_i)^TK_i+K_i(A_i-B_iL_i)+\sum_{j=1}^{N}\lambda_{ij}K_j=-I, \qquad i\in\mathcal{S}. \quad (17)$$

Remark 3.1: By simple calculations, we can see that

$$d_1\le d_0\le\frac{1}{5},\ \frac{(1+d_0)^2}{(1-d_0)^2}\le 2 \qquad \forall h\le\min\left\{1,\ \frac{3-2\sqrt{2}}{2c_1e^{2c}}\right\}. \quad (18)$$

Thus, (8) holds for all $h\le 2(1-\varepsilon_0)/(5lc_1e^c+24\lambda l_1)$. As for (9), noticing that

$$\lim_{h\to 0}D_{ij}(h)=A_i-A_j-(B_i-B_j)L_j$$

we know that if the distinguishable condition (similar to that of [12])

$$\det(A_i-A_j-(B_i-B_j)L_j)\neq 0 \qquad \forall i,j\in\mathcal{S}$$

holds, then by the continuity of $D_{ij}(h)$ in terms of h there must be a sufficient small h_0 such that (9) is satisfied for all $h\in(0,h_0]$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1: Suppose θ_t is a Markov process taking values in a finite set $\mathcal{S}=\{1,2,\ldots,N\}$, and subject to (2) and (3); and $f(\theta_t):\mathbb{R}^1\to\mathbb{R}^{n\times m}$, $g(\theta_t):\mathbb{R}^1\to\mathbb{R}^{m\times n}$, are measurable functions with respect to $\sigma\{\theta_s, s \le t\}$. Then, in the case where k is any nonnegative integer and $h \le 1$

$$E\left[\int_{kh}^{(k+1)h} \|(f(\theta_\mu) - f(\theta_{kh}))\, g(\theta_\mu)\|\, d\mu | \theta_{kh} = i_0\right] \le \lambda h^2 \max_{l\in\mathcal{S}} \|(f(l) - f(i_0))\, g(l)\|$$

where λ is given in (12).

Proof: Since almost all sample paths of $\theta(t)$ are constant except for a finite number of simple jumps in any finite time interval of $[0,\infty)$, we suppose the series of finite number of jump instants are $s_1, s_2, \ldots, s_m$, for one sample path of $\theta(t)$ during time interval $[kh, (k+1)h)$, satisfying

$$kh = s_0 \le s_1 < s_2 < \cdots < s_m < s_{m+1} = (k+1)h.$$

Noticing that

$$\sum_{j\ne i_0} P_{i_0 j}(h) = h \sum_{j\ne i_0} \left(\frac{e^{\Lambda h} - I}{h}\right)_{i_0 j} \le h \sum_{j\ne i_0} \left\|\frac{e^{\Lambda h} - I}{h}\right\| \le \lambda h \tag{19}$$

we have

$$\begin{aligned} &E\left[\int_{kh}^{(k+1)h} \|(f(\theta_\mu) - f(\theta_{s_0}))\, g(\theta_\mu)\|\, d\mu | \theta_{s_0} = i_0\right] \\ &= \sum_{\substack{\theta_{s_i}\in\mathcal{S}, \theta_{s_i}\ne i_0\\ i=1,\ldots,m}} \sum_{i=0}^{m} \left\|\left(f\left(\theta_{s_{i+1}}\right) - f(i_0)\right) g\left(\theta_{s_{i+1}}\right)\right\| \\ &\quad\times (s_{i+1} - s_i) \prod_{i=0}^{m} P_{\theta_{s_i}, \theta_{s_{i+1}}}(s_{i+1} - s_i) \\ &\le \left(\max_{l\in\mathcal{S}} \|(f(l) - f(i_0))\, g(l)\|\, h\right) \\ &\quad\times \sum_{\substack{\theta_{s_i}\in\mathcal{S}, \theta_{s_i}\ne i_0\\ i=1,\cdots,m}} \prod_{i=0}^{m} P_{\theta_{s_i}, \theta_{s_{i+1}}}(s_{i+1} - s_i) \\ &\le \max_{l\in\mathcal{S}} \|(f(l) - f(i_0))\, g(l)\|\, h \sum_{\theta_{(k+1)h} = j, j\ne i_0} P_{i_0 j}(h) \\ &\le \lambda h^2 \max_{l\in\mathcal{S}} \|(f(l) - f(i_0))\, g(l)\|. \end{aligned}$$

∎

Lemma 3.2: For system (1)–(3), suppose (9) holds. Then, under adaptive control (7) with sample size $h \le \min\{1, h_0\}$, we have

$$P(\hat{\theta}_{kh} = \theta_{kh}) \ge 1 - \lambda h.$$

Proof: When $h \le \min\{1, h_0\}$, by (9) we know that during the period $(k-1)h \le t < kh$, if θ_t does not jump, then the estimation $\hat{\theta}_{kh}$ of θ_{kh} is precisely correct. So

$$\{\hat{\theta}_{kh} = \theta_{kh}\} \supset \left\{\theta_t = \theta_{(k-1)h} | (k-1)h \le t < kh\right\}.$$

That is

$$P(\hat{\theta}_{kh} = \theta_{kh}) \ge 1 - \sum_{j\ne\theta_{(k-1)h}} P_{\theta_{(k-1)h} j}(h)$$

which together with (19) completes the proof. ∎

Lemma 3.3: Under the conditions of Theorem 3.1, we have for any $t \in [t', t' + h)$

$$\|x_t\| \le (1 + d_1)\|x_{t'}\| \quad \|x_{t'}\| \le \frac{1}{1 - d_1}\|x_t\| \tag{20}$$

and for any $t \in [t' - h, t' + h)$

$$\|x_t\| \le (1 + d_0)\|x_{t'-h}\| \quad \|x_{t'-h}\| \le \frac{1}{1 - d_0}\|x_t\| \tag{21}$$

where d_0 and d_1 are given by (11), and

$$t' = \left\lfloor \frac{t}{h} \right\rfloor h$$

with $\lfloor x \rfloor$ being the maximal integer less than or equal to x.

Proof: Substituting the SD-based adaptive control (7) into (1), we get the following closed-loop form:

$$\dot{x}_t = A(\theta_t)x_t - B(\theta_t)L(\hat{\theta}_{t'})x_{t'}. \tag{22}$$

From this, we have for any $t \in [t', t' + h)$

$$x_t - x_{t'} = \int_{t'}^{t} \left[A(\theta_s) - B(\theta_s)L(\hat{\theta}_{t'})\right] ds \cdot x_{t'} + \int_{t'}^{t} A(\theta_s)(x_s - x_{t'})ds.$$

Hence, for any $t \in [t', t' + h)$

$$\|x_t - x_{t'}\| \le c \int_{t'}^{t} \|x_s - x_{t'}\| ds + c_1 h \|x_{t'}\|$$

which together with the Gronwall lemma [13] gives

$$\|x_t - x_{t'}\| \le c_1 h e^{c(t-t')} \|x_{t'}\| \le d_1 \|x_{t'}\| \tag{23}$$

where c_1 and c are defined by (12) and (13). Thus, from

$$\|x_t\| \le \|x_{t'}\| + \|x_t - x_{t'}\| \le (1 + d_1)\|x_{t'}\|$$

we get the first inequality of (20), and from

$$\|x_{t'}\| \le \|x_t\| + \|x_t - x_{t'}\| \le \|x_t\| + d_1\|x_{t'}\|$$

we get the second inequality of (20). Here, the condition $d_1 \le d_0 < 1$ has been used [see (8) and (11)].

By (22) we have, for all $t \in [t' - h, t' + h)$

$$x_t - x_{t'-h} = \int_{t'-h}^{t} \left[A(\theta_s) - B(\theta_s)L(\hat{\theta}_{t'-h})\right] ds \cdot x_{t'-h} + \int_{t'-h}^{t} A(\theta_s)(x_s - x_{t'-h})ds.$$

Hence, similar to (20), we can get (21). ■

Proof of Theorem 3.1: Let

$$A_1(\theta_t) = A(\theta_t) - B(\theta_t)L(\theta_t).$$

Then, (1) with SD-based adaptive control (7) has the following closed-loop form:

$$\dot{x_t} = A_1(\theta_t)x_t + B(\theta_t)\left(L(\theta_t) - L(\hat{\theta}_{t'})\right)x_t + B(\theta_t)L(\hat{\theta}_{t'})(x_t - x_{t'}). \quad (24)$$

Noticing that $[A(\theta_t), B(\theta_t)]$ is stochastically stabilizable, by [4] we know that the symmetric solutions K_i $(i \in \mathcal{S})$ of the equation set

$$A_1^T(i)K_i + K_iA_1(i) + \sum_{j=1}^{N}\lambda_{ij}K_j = -I$$

are positive definite. Construct $K(\theta_t)$ such that $K(\theta_t) = K_i$ when $\theta_t = i$. Then, similar to [1, eq. (2.26)], by (24), we have

$$\begin{aligned}\widetilde{\mathcal{A}}\left(x_t^TK(\theta_t)x_t\right) &\triangleq \lim_{\Delta\to 0}\frac{1}{\Delta}\left(E\left[x_{t+\Delta}^TK(\theta_{t+\Delta})x_{t+\Delta}|\theta_t\right] - x_t^TK(\theta_t)x_t\right)\\ &= x_t^T\left[A_1^T(\theta_t)K(\theta_t) + K(\theta_t)A_1(\theta_t) + \sum_j\lambda_{\theta_tj}K_j\right]x_t\\ &\quad + 2x_t^T\left(L(\theta_t) - L(\hat{\theta}_{t'})\right)^TB^T(\theta_t)K(\theta_t)x_t\\ &\quad + 2(x_t - x_{t'})^TL^T(\hat{\theta}_{t'})B^T(\theta_t)K(\theta_t)x_t\\ &= -\|x_t\|^2 + 2x_t^T\left(L(\theta_t) - L(\theta_{t'})\right)^TB^T(\theta_t)K(\theta_t)x_t\\ &\quad + 2x_t^T\left(L(\theta_{t'}) - L(\hat{\theta}_{t'})\right)^TB^T(\theta_t)K(\theta_t)x_t\\ &\quad + 2(x_t - x_{t'})^TL^T(\hat{\theta}_{t'})B^T(\theta_t)K(\theta_t)x_t \end{aligned} \quad (25)$$

where $\mathcal{A}$ is the infinitesimal operator of the joint process $\{\theta_t, x_t\}$.

For any given positive integer K, by Lemma 3.1, we have

$$\begin{aligned} E\int_0^{Kh} & x_t^T\left(L(\theta_t) - L(\theta_{t'})\right)^TB^T(\theta_t)K(\theta_t)x_tdt\\ &\le E\left[\int_0^{Kh}\left\|\left(L(\theta_t) - L(\theta_{t'})\right)^TB^T(\theta_t)K(\theta_t)\right\| \times (1+d_1)^2\|x_{t'}\|^2dt\right]\\ &\le \sum_{k=0}^{K-1}\sum_{i=1}^{N}E\left[\int_{kh}^{(k+1)h}\left\|\left(L(\theta_s)-L(\theta_{s'})\right)^TB^T(\theta_s)K(\theta_s)\right\| \times (1+d_1)^2\|x_{kh}\|^2ds|\theta_{kh} = i\right]\\ &\quad \times P(\theta_{kh} = i)\\ &\le l_1(1+d_1)^2\lambda hE\int_0^{Kh}\|x_{t'}\|^2dt\\ &\le l_1\frac{(1+d_1)^2}{(1-d_1)^2}\lambda hE\int_0^{Kh}\|x_t\|^2dt. \end{aligned} \quad (26)$$

Here, l_1 is given by (14), and the inequalities (20) have been used.

Similarly, for any $t > 0$, we can get

$$E\int_{t'}^{t}x_s^T\left(L(\theta_s) - L(\theta_{t'})\right)^TB^T(\theta_s)K(\theta_s)x_sds \le l_1\frac{(1+d_1)^2}{(1-d_1)^2}\lambda hE\int_{t'}^{t}\|x_s\|^2ds. \quad (27)$$

This, together with (26), leads to

$$E\int_0^{t}x_s^T\left(L(\theta_s) - L(\theta_{s'})\right)^TB^T(\theta_s)K(\theta_s)x_sds \le l_1\frac{(1+d_1)^2}{(1-d_1)^2}\lambda hE\int_0^{t}\|x_s\|^2ds. \quad (28)$$

Similar to (26) and (27), by (21) and Lemma 3.2 we have

$$\begin{aligned} E\int_0^{Kh} & x_t^T\left(L(\theta_{t'}) - L(\hat{\theta}_{t'})\right)^TB^T(\theta_t)K(\theta_t)x_tdt\\ &\le E\left[\int_0^{Kh}\left\|\left(L(\theta_{t'}) - L(\hat{\theta}_{t'})\right)^TB^T(\theta_t)K(\theta_t)\right\| \times(1+d_0)^2\|x_{t'-h}\|^2dt\right]\\ &\le \sum_{k=0}^{K-1}\sum_{i=1}^{N}E\left[\int_{kh}^{(k+1)h}\left\|\left(L(\theta_{s'})-L(\hat{\theta}_{s'})\right)^TB^T(\theta_s)K(\theta_s)\right\| \times(1+d_0)^2\|x_{s'-h}\|^2ds|\theta_{s'-h} = i\right]\\ &\quad \times P(\theta_{s'-h} = i)\\ &\le 2l_1\frac{(1+d_0)^2}{(1-d_0)^2}\left(1-(1-\lambda h)\right)E\int_0^{Kh}\|x_t\|^2dt\\ &\le 2l_1\frac{(1+d_0)^2}{(1-d_0)^2}\lambda hE\int_0^{Kh}\|x_t\|^2dt \end{aligned}$$

and

$$E\int_{t'}^{t}x_s^T\left(L(\theta_{s'}) - L(\hat{\theta}_{s'})\right)^TB^T(\theta_s)K(\theta_s)x_sds \le 2l_1\frac{(1+d_0)^2}{(1-d_0)^2}\lambda hE\int_{t'}^{t}\|x_s\|^2ds.$$

That is

$$E\int_0^{t}x_s^T\left(L(\theta_{s'}) - L(\hat{\theta}_{s'})\right)^TB^T(\theta_s)K(\theta_s)x_sds \le 2l_1\frac{(1+d_0)^2}{(1-d_0)^2}\lambda hE\int_0^{t}\|x_s\|^2ds. \quad (29)$$

Notice that by (13), (20), and (23)

$$(x_t - x_{t'})^T L^T(\hat{\theta}_{t'}) B^T(\theta_t) K(\theta_t) x_t \le \frac{l d_1}{1-d_1}\|x_t\|^2$$

and by $0 < d_1 \le d_0 < 1$

$$\frac{(1+d_1)^2}{(1-d_1)^2} \le \frac{(1+d_0)^2}{(1-d_0)^2}.$$

Then, it follows from (25), (28), and (29) that

$$\begin{aligned} E\left[x_t^T K(\theta_t) x_t\right] &- x_0^T K(\theta_0) x_0 \\ &\le -\left[1 - \frac{2ld_1}{1-d_1} - 6l_1\lambda h \frac{(1+d_0)^2}{(1-d_0)^2}\right] E\left[\int_0^t \|x_s\|^2 ds\right] \\ &\le -\left[1 - \frac{2ld_1}{1-d_1} - 6l_1\lambda h \frac{(1+d_0)^2}{(1-d_0)^2}\right] \min_{i\in\mathcal{S}} \frac{1}{\lambda_{\max}(K_i)} \\ &\quad \times E\int_0^t x_s^T K(\theta_s) x_s ds. \end{aligned} \tag{30}$$

Noticing that the second inequality of (8) is equivalent to

$$1 - \frac{2ld_1}{1-d_1} - 6\lambda h l_1 \frac{(1+d_0)^2}{(1-d_0)^2} \ge \varepsilon_0 > 0$$

by (30) and the second equality of (16) we have

$$E\left[x_t^T K(\theta_t) x_t\right] \le x_0^T K(\theta_0) x_0 - \alpha_0 E\left[\int_0^t x_s^T K(\theta_s) x_s ds\right]$$

which together with the Gronwall lemma gives

$$E\left[x_t^T K(\theta_t) x_t\right] \le e^{-\alpha_0 t} x_0^T K(\theta_0) x_0. \tag{31}$$

Thus

$$\lim_{t\to\infty} E\left[\|x_t\|^2\right] = 0 \quad E\left[\int_0^\infty x_t^T x_t dt\right] \le x_0^T \widetilde{K} x_0 \tag{32}$$

where $\widetilde{K}$ is given by (16).

We now study the index. Define

$$u_t^* = -L(\theta_t) x_t.$$

By (20) and (31), similar to (28) and (29), we get

$$\begin{aligned} E\int_0^\infty &(u_s - u_s^*)^T R(\theta_s)(u_s - u_s^*) ds \\ = E\int_0^\infty \Big[&x_s^T \left(L(\theta_s) - L(\hat{\theta}_{s'})\right)^T R(\theta_s) \left(L(\theta_s) - L(\hat{\theta}_{s'})\right) x_s \\ &+ 2x_s^T \left(L(\theta_s) - L(\hat{\theta}_{s'})\right)^T R(\theta_s) L(\hat{\theta}_{s'})(x_s - x_{s'}) \\ &+ (x_s - x_{s'})^T L^T(\hat{\theta}_{s'}) R(\theta_s) L(\hat{\theta}_{s'})(x_s - x_{s'})\Big] ds \\ \le\ & l_2\lambda h \frac{(1+d_0)^2}{(1-d_0)^2}(2 + 2\lambda h) E\int_0^\infty \|x_s\|^2 ds \\ &+ l_4 \frac{d_1^2}{(1-d_1)^2} E\int_0^\infty \|x_s\|^2 ds \\ &+ 4l_3\lambda h \frac{d_1(1+d_0)^2}{(1-d_1)(1-d_0)^2} E\int_0^\infty \|x_s\|^2 ds \\ =\ & \left[2l_2\lambda h \frac{(1+d_0)^2}{(1-d_0)^2} + f(h)h^2\right] E\int_0^\infty \|x_s\|^2 ds \\ =\ & \left[2l_2\lambda h \frac{(1+d_0)^2}{(1-d_0)^2} + O(h^2)\right] E\int_0^\infty \|x_s\|^2 ds \end{aligned} \tag{33}$$

where l_2 is given by (15), and

$$l_3 = \max_{i,j,k,m} \|(L_i - L_j)^\tau R_k L_m\|,\ l_4 = \max_{i,j} \left\|L_j^\tau R_i L_j\right\|$$

$$\begin{aligned} f(h) = 2l_2\lambda^2 \frac{(1+d_0)^2}{(1-d_0)^2} &+ l_4 \frac{c_1^2 e^{2ch}}{(1-d_1)^2} \\ &+ 4l_3\lambda \frac{(1+d_0)^2}{(1-d_1)(1-d_0)^2} c_1 e^{ch}. \end{aligned}$$

Similar to [1, eq. (2.29)], by (1) we obtain

$$\begin{aligned} \widetilde{\mathcal{A}}&\left(x_t^T M(\theta_t) x_t\right) \\ &\triangleq \lim_{\Delta\to 0} \frac{1}{\Delta}\left(E\left[x_{t+\Delta}^T M(\theta_{t+\Delta}) x_{t+\Delta} | \theta_t\right] - x_t^T M(\theta_t) x_t\right) \\ &= x_t^T \left(A^T(\theta_t) M(\theta_t) + M(\theta_t) A(\theta_t) + \sum_{j=1}^N \lambda_{\theta_t j} M_j\right) x_t \\ &\quad + u_t^T B^T(\theta_t) M(\theta_t) x_t + x_t^T M(\theta_t) B(\theta_t) u_t. \end{aligned}$$

Here, $\widetilde{\mathcal{A}}$ is the infinitesimal operator of the joint process $\{\theta_t, x_t\}$. Then, by Dynkin's formula, we have

$$\begin{aligned} E\left[x_t^T M(\theta_t) x_t\right] = x_0^T M(\theta_0) x_0 &+ E\int_0^t \Big(u_s^T B^T(\theta_s) M(\theta_s) x_s \\ &\qquad + x_s^T M(\theta_s) B(\theta_s) u_s\Big) ds \\ &+ E\int_0^t x_s^T \Big(A^T(\theta_s) M(\theta_s) + M(\theta_s) A(\theta_s) \\ &\qquad + \sum_{j=1}^N \lambda_{\theta_s j} M_j\Big) x_s ds. \end{aligned}$$

Thus, according to (5)

$$E\left[x_t^T M(\theta_t)x_t + \int_0^t \left(x^T(s)Q(\theta_s)x(s) + u_s^T R(\theta_s)u_s\right)ds\right]$$
$$= x_0^T M(\theta_0)x_0 + E\int_0^t \left(u_s + R^{-1}(\theta_s)B^T(\theta_s)M(\theta_s)x_s\right)^T R(\theta_s) \times \left(u_s + R^{-1}(\theta_s)B^T(\theta_s)M(\theta_s)x_s\right)ds.$$

This together with (32) and (33) leads to

$$\begin{aligned} J(u) &= E\left[\int_0^\infty \left(x_s^T Q(\theta_s)x_s + u_s^T R(\theta_s)u_s\right)ds\right] \\ &= x_0^T M(\theta_0)x_0 - \lim_{t\to\infty} E\left[x_t^T M(\theta_t)x_t\right] + E\left[\int_0^\infty (u_s - u_s^*)^T R(\theta_s)(u_s - u_s^*)\,ds\right] \\ &= x_0^T M(\theta_0)x_0 + E\left[\int_0^\infty (u_s - u_s^*)^T R(\theta_s)(u_s - u_s^*)\,ds\right] \\ &\le x_0^T M(\theta_0)x_0 + \left(2l_2\lambda \frac{(1+d_0)^2}{(1-d_0)^2} x_0^T \widetilde{K} x_0\right)h + O(h^2) \\ &\le x_0^T M(\theta_0)x_0 + 4l_2\lambda x_0^T \widetilde{K} x_0 h + O(h^2). \end{aligned}$$

Here, the second inequality of (18), i.e. $((1+d_0)^2/(1-d_0)^2) \le 2$, has been used.

Remark 3.2: Here, we consider only the case where the system is of noise-free and the state $x(t)$ is assumed to be measured precisely at the sampling instants. Heuristically, this is a good starting point to study the more general cases where there are, for instance, unknown random disturbances, measurement noises, and the parameter set $\mathcal{S}$ has infinite (countable or uncountable) elements. It is worth noticing that if the system has Brownian motion as its disturbance and the coefficient of the disturbance is a nonzero constant, then minimum value of index (4) will be infinity, and so, not suitable for optimal control synthesis. In this case, a good choice might be the following averaged index function:

$$J(u) = \limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left(x_s^T Q(\theta_s)x_s + u_s^T R(\theta_s)u_s\right)ds.$$

IV. Example

Example 1: Consider a system of the form (1) with $N = 2$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In the index (4), we have $Q_i = I$ and $R_i = I$ for $i = 1, 2$.

Note that $[A_i, Q_i^{1/2}]$ is observable for $i = 1, 2$. By [4], $[A(\theta_t), B(\theta_t)]$ is stochastically stabilizable. The solution set of the two coupled algebraic Riccati equations is

$$M_1 = \begin{bmatrix} 1.732\,05 & 1 \\ 1 & 1.732\,05 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} 1.732\,05 & 1 \\ 1 & 1.732\,05 \end{bmatrix}$$

the feedback gain matrices are

$$L_1 = [1 \quad 1.732\,05] \quad L_2 = [1.732\,05 \quad 1]$$

and the solution set of (17) is

$$K_1 = \begin{bmatrix} 1.0415 & 0.2679 \\ 0.2679 & 0.5774 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} 0.5774 & 0.2679 \\ 0.2679 & 1.0415 \end{bmatrix}.$$

It is easy to show that $\det \mathcal{D}_{12}(h) \neq 0$ and $\det \mathcal{D}_{21}(h) \neq 0$ for all $0 \le h \le 2.3$, and (8) holds for all $h \le 0.01$ and $\varepsilon_0 \in (0, 0.3]$.

Let us denote the index difference between the SD-based adaptive LQ control and the conventional LQ control as

$$\begin{aligned} \Delta J(t) &= J(x_0, \theta_0, u, t) - J(x_0, \theta_0, u^*, t) \\ &= E\left[\int_0^t \left(x^T(t)Q(\theta_t)x(t) + u_t^T R(\theta_t)u_t\right)dt\right] \\ &\quad - E\left[\int_0^t \left(x^T(t)Q(\theta_t)x(t) + (u_t^*)^T R(\theta_t)u_t^*\right)dt\right] \end{aligned}$$

where u_t and u_t^* are defined by (6) and (7), respectively. By (10), we have

$$\Delta J \triangleq \limsup_{t\to\infty} \|\Delta J(t)\| \le 4l_2\lambda x_0^T \widetilde{K} x_0 h + O(h^2).$$

The simulation results of Example 1 are shown in Figs. 1 and 2 with $h = 0.01$ and 0.0001, respectively, where the first part of each figure represents $\Delta J(t)$, the index difference between the SD-based adaptive LQ control and the conventional LQ control, the second part represents the jump system parameter $\theta(t)$, the third and the fourth parts represent the state process $x(t)$, and the last part represents the control $u(t)$. To fully display the differences, 30 simulations with a fixed initial state values are given for h =0.01 and 0.0001, respectively. To observe the jump properties clearly, only three of the 30 $\theta(t)$ processes are depicted in each of the two figures. It can be seen that the smaller the step size h is, the smaller the index difference $\Delta J(t)$ is.

V. Concluding Remarks

In this note, the SD-based LQ adaptive control problem of continuous-time linear systems with Markov jump parameters is investigated. For the case where the Markov jump process θ_t is unknown and only the sampled-data rather than the complete process of the system state is available, a parameter estimator and a control design method are

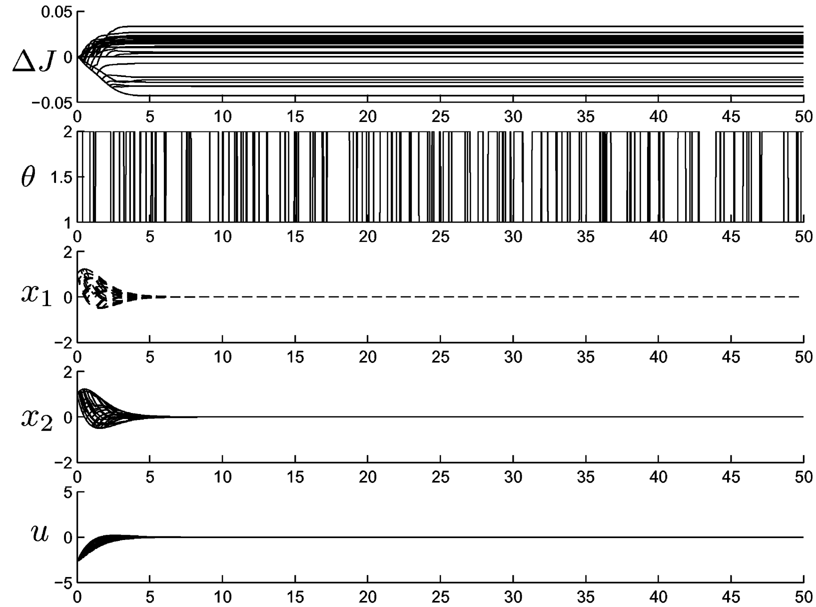

Fig. 1. Curves of ΔJ, θ, x_1, x_2, and u, when $h = 0.01$.

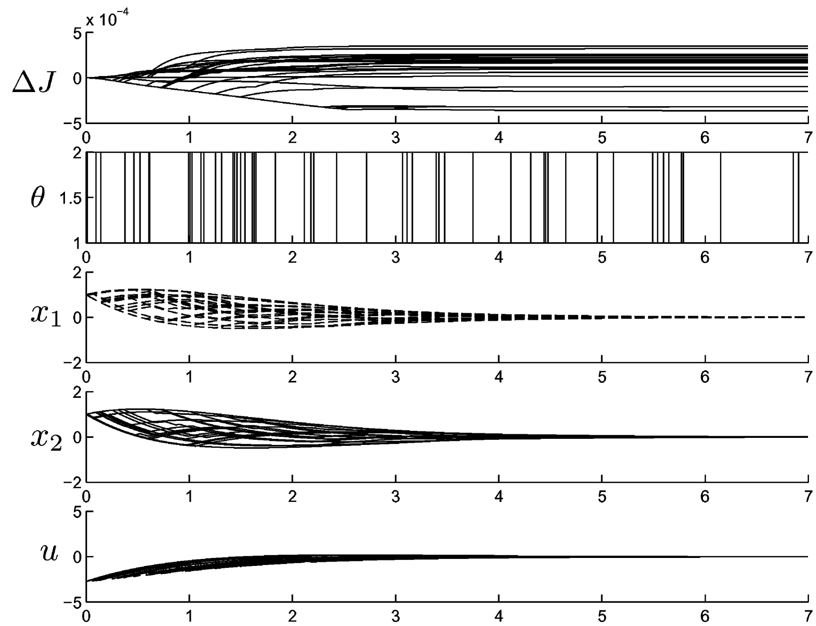

Fig. 2. Curves of ΔJ, θ, x_1, x_2, and u, when $h = 0.0001$.

given. It is shown that when the sample step size is small, the sampled-data based adaptive control is suboptimal under LQ index. It is worth mentioning that our controller is designed directly based on the original continuous system and the original continuous performance index, without involving any discretized models and discretized indexes. As for how to figure out the maximal range of h, how to choose h optimally, and how to obtain an explicit expression describing the relationship between the sample step size and system structure and parameters, it is very difficult and complex, and needs further study.

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Remarks on the L^p-Input Converging-State Property

E. P. Ryan

***Abstract*—Let $\mathbb{X} \subset \mathbb{R}^N$ and consider a system $\dot{x} = f(x,u)$, $f : \mathbb{X} \times \mathbb{R}^M \to \mathbb{R}^N$, with the property that the associated autonomous system $\dot{x} = f(x,0)$ has an asymptotically stable compactum C with region of attraction A. Assume that x is a solution of the former, defined on $[0,\infty)$, corresponding to an input function u. Assume further that, for each compact $K \subset \mathbb{X}$, there exists $k > 0$ such that $|f(z,v) - f(z,0)| \le k|v|$ for all $(z,v) \in K \times \mathbb{R}^M$. A simple proof is given of the following L^p-input converging-state property: if $u \in L^p$ for some $p \in [1,\infty)$ and x has an ω-limit point in A, then x approaches C.**

***Index Terms*—Asymptotic stability, converse Lyapunov theory, domain of attraction.**

I. Introduction

For a linear system $\dot{x} = Fx + Gu$, with F Hurwitz, the following properties are elementary: P1) if x is a solution on $\mathbb{R}_+ := [0,\infty)$ corresponding to an input $u \in L^\infty$ with $u(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$; and P2) if x is a solution on $\mathbb{R}_+$ corresponding to an input $u \in L^p$ for some $p \in [1,\infty)$, then $x(t) \to 0$ as $t \to \infty$. Exploitation of these properties is widespread in the literature (on, for example, adaptive control, robustness to disturbances, and interconnected/cascaded systems). The question of nonlinear counterparts arises: to what extent do properties P1) and P2) persist in the context of a finite-dimensional nonlinear system $\dot{x} = f(x,u)$ under the hypothesis that 0 is an asymptotically stable equilibrium of the associated autonomous system $\dot{z} = f^*(z)$, where $f^*(\cdot) := f(\cdot,0)$? Even in the simplest of nonlinear systems satisfying the latter hypothesis, properties P1) and P2) may fail to hold. One such scalar system is given by $\dot{x} = -x + x^2 u$ which, with initial data $x(0) = 1$ and input $u : t \mapsto 2e^{-t}$, has unbounded solution $x : t \mapsto e^t$.

In [1] (under the assumption that f is continuous and is locally Lipschitz in its first argument, uniformly with respect to its second argument in compact sets), a proof is provided of the following "well-known but hard-to-cite fact" [a nonlinear counterpart of the converging-input converging-state property P1)]. If a) x is a solution of the system $\dot{x} = f(x,u)$, defined on $\mathbb{R}_+$, corresponding to an input $u \in L^\infty$ with $u(t) \to 0$ as $t \to \infty$, b) 0 is an asymptotically stable equilibrium of the associated autonomous system $\dot{z} = f^*(z)$ with domain of attraction A, and c) x is K-recurrent for some compact $K \subset A$, then $x(t) \to 0$ as $t \to \infty$. Here, K-recurrence is the property that, for each $T > 0$, there exists $t > T$ such that $x(t) \in K$. The recurrence hypothesis c) is equivalent to positing that x has an ω-limit point in A. The nonlinear converging-input converging-state property in [1] has a closely-related antecedent in [2], Theorem 2 of which contains the essence of the result.

The purpose of the present note is to provide a nonlinear counterpart of the L^p-input converging-state property P2) (we use the term "input" in the general sense of either a control input or disturbance input): the essence is to identify conditions under which an input of bounded energy generates a converging state.[1] The main result subsumes the following: if a) x is a solution of the system $\dot{x} = f^*(x) + g(x)u$ (f^* and g locally Lipschitz), defined on $\mathbb{R}_+$, corresponding to an input $u \in L^p$ for some $p \in [1,\infty)$, b) the associated autonomous system $\dot{z} = f^*(z)$ has an asymptotically stable compactum C with domain of attraction A, and c) x has an ω-limit point in A, then x approaches C (in a sense made precise later).

II. Preliminaries

The Euclidean norm and inner product on $\mathbb{R}^N$ (or $\mathbb{R}^M$) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. For $G \in \mathbb{R}^{N\times M}$, $|G| := \min_{|u|=1} |Gu|$. For a nonempty set $C \subset \mathbb{R}^N$, the function $d_C : \mathbb{R}^N \to \mathbb{R}_+$, given by $d_C(y) := \inf_{c\in C} |y - c|$, is the distance function for C and a function $x : \mathbb{R}_+ \to \mathbb{R}^N$ is said to approach C if $d_C(x(t)) \to 0$ as $t \to \infty$. Let $I \subset \mathbb{R}$ be such that $\mathbb{R}_+ \subset I$, let $\mathbb{X} \subset \mathbb{R}^N$ be nonempty and open, and let $x : I \to \mathbb{X}$. A point $z \in \mathrm{cl}(\mathbb{X})$ is an ω-limit point of x if there exists an unbounded sequence (t_n) in $\mathbb{R}_+$ with $x(t_n) \to z$ as $n \to \infty$; the ω-limit set of x is the set of all ω-limit points of x, this set is is denoted by $\Omega(x)$. For later convenience, we record some well-known properties of ω-limit sets (see, for example, [4]).

Proposition 2.1: For every function $x : \mathbb{R}_+ \to \mathbb{X}$ the following hold.

i) $\Omega(x)$ is closed.
ii) $\Omega(x) = \emptyset$ if, and only if, $|x(t)| \to \infty$ as $t \to \infty$.
iii) If x is continuous and $\Omega(x)$ is nonempty and compact, then x is bounded.
iv) If x is continuous and bounded, then $\Omega(x)$ is nonempty, compact, connected, and is approached by x.

III. The System

Denote, by $\mathcal{U} := L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$, the space of locally integrable functions $u : \mathbb{R}_+ \to \mathbb{R}^M$. Let $\mathbb{X} \subset \mathbb{R}^N$ be nonempty and open. Let $f : \mathbb{X} \times \mathbb{R}^M \to \mathbb{R}^N$ be continuous and such that

$$\forall \text{ compact } K \subset \mathbb{X} \ \exists k > 0 : |f(x,u) - f(x,0)| \le k|u| \quad \forall (x,u) \in K \times \mathbb{R}^M \tag{1}$$

a canonical case being that wherein f is affine in the input, viz. $f(x,u) = f^*(x) + g(x)u$. We assume further that

$$f^*(\cdot) := f(\cdot,0) \text{ is locally Lipschitz.} \tag{2}$$

[1] In a context different from that of this note, asymptotic properties of solutions of systems with inputs of bounded energy (viz. the "bounded-energy weakly-converging-state," and the "bounded-energy frequently-bounded-state" properties) play a role in asymptotic characterizations of integral-input-to-state stability in [3].

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