Design of satisfaction output feedback controls for stochastic nonlinear systems under quadratic tracking risk-sensitive index

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Abstract In this paper, the design problem of satisfaction output feedback controls for stochastic nonlinear systems in strict feedback form under long-term tracking risk-sensitive index is investigated. The index function adopted here is of quadratic form usually encountered in practice, rather than of quartic one used to beg the essential difficulty on controller design and performance analysis of the closed-loop systems. For any given risk-sensitive parameter and desired index value, by using the integrator backstepping method, an output feedback control is constructively designed so that the closed-loop system is bounded in probability and the risk-sensitive index is upper bounded by the desired value.

Keywords: integrator backstepping, nonlinear system, stochastic disturbance, risk-sensitive index, output feedback.

The design of global stabilization controls for nonlinear systems has been a research topic under intensive investigation. After the celebrated characterization of the feedback linearizable systems[1], a breakthrough is achieved with the introduction of the integrator backstepping design methodology[2], which provides a general constructive tool for designing global stabilization controls for nonlinear systems in or feedback equivalent to strict-feedback (SF) form. Since the early 1990s, a series of research results on SF systems have been obtained (e.g. refs. [3—10] and the references therein).

Stochastic risk-sensitive (RS) control is more general than H_∞ and H_2 control. It is closely related to differential game problems[11−19]. For example, when disturbance vanishes, the large deviation limit of the stochastic RS control is nothing but a deterministic differential game problem. These connections have stimulated the research on stochastic RS controls.

A big progress on control design of SF stochastic nonlinear systems has been made recently[20−22]. Under the assumption that the disturbance vector field vanishes at the origin, refs. [20, 21] study the problem of designing a control to asymptotically stabilize the closed-loop systems in probability. Ref. [20] considers full state-feedback control design, while ref. [21] considers output-feedback control design. In ref. [22], with a quadratic regulation RS index, design of satisfaction state-feedback control is studied, and the assumption that the disturbance vector field vanishes at the origin is removed.

Design of output-feedback control is more challenging than that of full state-feedback control. In the 1990s, a general approach to output-feedback control was developed. The key thought is to introduce the so-called information states first, and then, by a measure transformation, change the output-feedback control design problem into a full state-feedback one of an augmented system. However, generally speaking, the equality (or inequality) of the information state satisfied is infinite-dimensional, to which an explicit finite-dimensional solution exists only for linear, bilinear or some special nonlinear systems. By using quartic value function, ref. [26] investigated the design problem of satisfaction output-feedback control under a quartic RS index. Different from the information state method, in ref. [26], the output-feedback controller is explicitly obtained, and no strict assumption except smoothness is imposed on the nonlinearities of functions involved (for instance, the noise gain functions).

This paper is a further research of ref. [26]. Different from ref. [26], this paper adopts a quadratic tracking RS index, and the value function used for control design is quadratic. While ref. [26] studies the quartic regulation RS index, which is a special case of a quartic tracking RS index when the reference equals zero. Compared with quartic one, quadratic index is more practical. This is not only because many physical quantities (for instance, the energy function) can be characterized by quadratic indices, even in optimal control theory, quadratic indices are also used a lot. Besides, the quadratic RS control is closely related to H_2 and H_∞ control: when $\theta \to 0$, quadratic RS control leads to H_2 control; when $\theta \to \infty$ and the disturbance vanishes, quadratic RS control leads to H_∞ control[11,14]. Another characteristic of this paper is the use of a weighted quadratic value function. In refs. [20, 21, 26], quartic value functions are used, and the motivation is to restrain the stochastic disturbance by enlarging the power of feedback control and increasing the power of the state variables in control laws. In addition, the value range of the characteristic parameter of the value function is enlarged from $\frac{2}{3}$ (see ref. [26]) to set $(\frac{1}{2}, 1)$. This provides control designers with a freedom in choosing value function.

The main results of this paper indicate that: for any given RS parameter and desired tracking RS index value, under certain conditions, a dynamic output-feedback control can always be constructively designed so that the closed-loop system is bounded in probability and the RS index is upper bounded by the desired value.

1 Problem formulation

Consider the following stochastic nonlinear systems in strict feedback form[21,26]:

$$
\begin{aligned}
dx_1 &= x_2dt + f_1(y)dt + \varphi_1(y)dw, \\
dx_2 &= x_3dt + f_2(y)dt + \varphi_2(y)dw, \\
&\ \vdots \\
dx_{n-1} &= x_ndt + f_{n-1}(y)dt + \varphi_{n-1}(y)dw, \\
dx_n &= udt + f_n(y)dt + \varphi_n(y)dw, \\
y &= x_1,
\end{aligned}
\tag{1}
$$

where $x=(x_1,x_2,\cdots,x_n)^\tau\in\mathbb{R}^n$ is system state; $u\in\mathbb{R}$ is control input; $y\in\mathbb{R}$ is measurable output; $f_i(y)\in\mathbb{R}$ $(i=1,...,n)$ is system nonlinearity depending only on the output y; $\varphi_i(y)\in\mathbb{R}^s$ $(i=1,\cdots,n)$ is the gain function of system disturbance, which also depends only on system output y; $w\in\mathbb{R}^s$ is system disturbance.

In this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ denotes the real n-dimensional space; for any given matrix W, W^τ denotes its transpose; if W is square, then $\mathrm{tr}(W)$ denotes its trace, i.e. the sum of all elements on the main diagonal line; for any $x\in\mathbb{R}^n$, x_i denotes its ith component, $x_{[i]}$ denotes the vector consisting of the first i components of x in the original order, i.e. $x_{[i]}=(x_1,\cdots,x_i)^\tau$; $\mathcal{C}^i$ denotes the set of all functions with ith continuous derivative; for any given ith continuously differentiable function $y_d(t)$, $y_d^{(i)}(t)$ denotes the ith derivative with respect to its variable t , the first and second derivatives are denoted by $\dot{y}_d$ and $\ddot{y}_d$ respectively, and $y_d^{[i]}$ denotes the $i+1$-dimensional vector consisting of $y_d,\dot{y}_d,\cdots,y_d^{(i)}$, i.e. $y_d^{[i]}=\left(y_d,\dot{y}_d,\ddot{y}_d,\cdots,y_d^{(i)}\right)^\tau$. Obviously, $x_{[1]}=x_1$, $x_{[n]}=x$, $y_d^{[0]}=y_d$. Besides, when a function shows up at the first time, we will clearly write out the variables, then for simplicity of expression, we often omit the variables in later use if there is no confusion caused.

The main results of this paper are based on the following assumptions:

A1. $w\in\mathbb{R}^s$ is a standard independent Brownian motion, defined on a probability space $(\Omega,\mathcal{F},P)$, with Ω the sample space, $\mathcal{F}$ the σ-algebra, P the probability measure.

A2. $f_i(\cdot)$ and $\varphi_i(\cdot)$ $(i=1,\cdots,n)$ are smooth functions, i.e. $f_i(\cdot)\in\mathcal{C}^\infty$ and $\varphi_i(\cdot)\in\mathcal{C}^\infty$.

A3. Desired system output y_d is deterministic, it and its derivatives $\dot{y}_d,\cdots,y_d^{(n+1)}$ are known, uniformly bounded, and $y_d^{(n+1)}=G(y_d^{[n]})$, where G: $\mathbb{R}^{n+1}\to\mathbb{R}$ is a continuous function.

Assumption A2 ensures that $f_i(\cdot)$ and $\varphi_i(\cdot)$ $(i=1,\cdots,n)$ are local Lipschitz functions, and with Assumption A3, ensures the global boundedness of $\varphi(y_d)$. From Assumption A3 it is easy to see that $y_d^{[n]}$ is the state of the following deterministic autonomous system:

$$\frac{dy_d^{[n]}}{dt}=\begin{bmatrix}[0\ I_n]y_d^{[n]}\\ G(y_d^{[n]})\end{bmatrix},\tag{2}$$

where I_n is $n\times n$ identity matrix. This condition will play a key role in showing the existence and uniqueness of the solution to the closed-loop systems.

The goal of control design is to make the following quadratic tracking RS index function achieve a pre-defined long-term index value:

$$J_\theta(u)=\lim_{T\to\infty}\frac{1}{T}\cdot\frac{2}{\theta}\ln\left\{E\exp\left(\frac{\theta}{2}\int_0^T(y-y_d)^2dt\right)\right\},\tag{3}$$

i.e. for any given $R_l>0$, to ensure that the RS index $J_\theta(u)$ is not larger than R_l, where θ is called RS parameter, $y-y_d$ is the output tracking error. When $y_d\equiv 0$, $J_\theta(u)$ becomes an exponential quadratic output index function. When $\theta>0$, the index function weights heavily the large deviation of $y-y_d$ (from a base zero) through the exponential operator, which leads to a risk-averse control design problem. The larger θ, the more conservative the controller. When $\theta\to\infty$ and the disturbance converges to zero, the optimization will lead to a differential-game problem[22]. Actually, by the value range of θ, the RS problem can be classified[16,18] as follows:

(i) when $\theta > 0$, it is a risk-averse problem; (ii) when $\theta < 0$, it is a risk-seeking problem; (iii) when $\theta \to 0$, the index function converges to a standard integral cost, and so it is known as a risk-neutral problem.

In this paper, we will study only the case where θ is positive. The risk-seeking case (i.e. with $\theta < 0$) can be studied analogously but will not be covered here.

For convenience of expression, we list the following two definitions.

Definition 1[22]**.** For a given RS parameter θ, a controller u is said to achieve a guaranteed risk-sensitivity index $R_l \geqslant 0$, if the following inequality holds for the output of the closed-loop system:

$$J_\theta(u) \leqslant R_l. \tag{4}$$

Definition 2[22,27]**.** If the solution $\{x(t),\, t \geqslant 0\}$ of system (1) satisfies

$$\lim_{c\to\infty} \sup_{t\in[0,\infty)} P\{|x(t)| > c\} = 0,$$

then system (1) is said to be bounded in probability.

With respect to system $dx = F(x)dt + G(x)dw$, define a differential operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}F(x) + \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}G(x)G^\tau(x)\right].$$

Consider a compact form of system (1):

$$dx = f(x)dt + g(x)udt + h(x)dw.$$

As in refs. [22, 26], under the long-term RS index function $J_\theta(u)$, a satisfaction RS output feedback control is designed:

$$\begin{cases} \dot{\xi} = \alpha\left(\xi, y, y_d^{[n]}\right), \\ u = \mu\left(\xi, y, y_d^{[n]}\right), \end{cases} \tag{5}$$

so that with this control, there exists a positive, radially unbounded $\mathcal{C}^2$ value function $V(x,\xi)$ satisfying the following Hamilton-Jacobi-Bellman (HJB) inequality:

$$\begin{bmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial \xi} \end{bmatrix} \begin{bmatrix} f + g\mu \\ \alpha \end{bmatrix} + \frac{\theta}{4}\frac{\partial V}{\partial x}hh^\tau\left(\frac{\partial V}{\partial x}\right)^\tau + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 V}{\partial x^2}hh^\tau\right) + (y - y_d)^2 \leqslant R_l, \tag{6}$$

where $\alpha(\cdot) \in \mathcal{C}^1$, $\mu(\cdot) \in \mathcal{C}^1$, $R_l \geqslant 0$ is the desired index value.

From (6) it is easy to see that the difference between the stochastic HJB and deterministic HJB is that the former has one more term $\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 V}{\partial x^2}hh^\tau\right)$. This is due to the application of Itô formula. How to deal with this term is the key to the control design and performance analysis.

The following theorem can be directly derived from Theorems 5 and 6 of ref. [22], and will be the base of the coming design and analysis.

Theorem 1. Consider stochastic nonlinear system (1) and RS index (3). Suppose system (1) satisfies Assumptions A1—A3. For any given $\theta > 0$ and $R_l \geqslant 0$, if there exists a positive, radially unbounded $\mathcal{C}^2$ function $V(x, \xi, y_d^{[n-1]})$, a $\mathcal{C}^1$ function $\sigma(x, \xi, y_d^{[n]})$, a nonnegative function

$l(x,\xi,y_d^{[n]})$ and a controller in form (5) such that

$$dV(x,\xi,y_d^{[n-1]}) \leqslant \sigma(x,\xi,y_d^{[n]})dw - \frac{\theta}{4}\sigma(x,\xi,y_d^{[n]})\sigma^\tau(x,\xi,y_d^{[n]})dt - l(x,\xi,y_d^{[n]}) - (y-y_d)^2dt + R_l dt, \tag{7}$$

then (i) the closed-loop system has a unique solution almost surely; (ii) the RS index $J_\theta(u)$ is upper bounded by R_l; (iii) in addition, if there are positive constants c_1 and c_2 such that

$$\mathcal{L}V(x,\xi,y_d^{[n-1]}) \leqslant -c_1V(x,\xi,y_d^{[n-1]}) + c_2, \tag{8}$$

then the closed-loop system is bounded in probability.

Proof. Let $Y = \left(x^\tau,\xi^\tau,\left(y_d^{[n]}\right)^\tau\right)^\tau$. Then by Assumption A3 and (2) we know that Y is the state of an autonomous system, and there exists a constant $c_0 > 0$ such that $c_0 \geqslant 2\left|y_d^{(n)}y_d^{(n+1)}\right|$. Set $X = \left(x^\tau,\xi^\tau,\left(y_d^{[n-1]}\right)^\tau\right)^\tau$ and define

$$\tilde{V}(Y) = V(X) + \left(y_d^{(n)}\right)^2 + R_l + c_0.$$

Then, by (7) we have

$$\mathcal{L}\tilde{V} = -\frac{\theta}{4}\sigma\sigma^\tau - l(x,\xi,y_d^{[n]}) - (y-y_d)^2 + R_l + 2y_d^{(n)}y_d^{(n+1)} \leqslant \tilde{V}.$$

Thus, item (i) follows from the radial unboundedness of $\tilde{V}$ and Theorem 4.1 of chapter 3 of ref. [27].

As for item (ii), by (7) we know that for any $T \geqslant 0$,

$$V(X(T)) + \int_0^T [(y-y_d)^2 + l(x,\xi,y_d^{[n]})]dt \leqslant V(X(0)) + \int_0^T \sigma(x,\xi,y_d^{[n]})dw - \frac{\theta}{4}\int_0^T \sigma(x,\xi,y_d^{[n]})\sigma^\tau(x,\xi,y_d^{[n]})dt + R_lT. \tag{9}$$

Furthermore,

$$\begin{aligned}&\frac{2}{T}\cdot\frac{1}{\theta}\ln\left\{E\exp\left(\frac{\theta}{2}\int_0^T(y-y_d)^2dt\right)\right\}\\ \leqslant{}& \frac{2}{T}\cdot\frac{1}{\theta}\ln\left\{E\exp\left\{\frac{\theta}{2}\left[V(X(T)) + \int_0^T[(y-y_d)^2 + l(x,\xi,y_d^{[n]})]dt\right]\right\}\right\}\\ \leqslant{}& \frac{V(X(0))}{T} + \frac{2}{T}\cdot\frac{1}{\theta}\ln\left\{E\exp\left\{\frac{\theta}{2}\left[\int_0^T\sigma dw - \frac{\theta}{4}\int_0^T\sigma\sigma^\tau dt\right]\right\}\right\} + R_l.\end{aligned} \tag{10}$$

Let

$$\zeta(T) \triangleq \exp\left\{\int_0^T\left(\frac{\theta}{2}\sigma(x,\xi,y_d^{[n]})dw - \frac{1}{2}\cdot\frac{\theta^2}{4}\sigma(x,\xi,y_d^{[n]})\sigma^\tau(x,\xi,y_d^{[n]})dt\right)\right\}.$$

Then $\zeta(T)$ is a supermartingale[22] and $E\zeta(T) \leqslant E\zeta(0) = 1$ ($\forall T \geqslant 0$). So, we have

$$\frac{2}{T}\cdot\frac{1}{\theta}\ln\left\{E\exp\left\{\frac{\theta}{2}\left[\int_0^T\sigma dw - \frac{\theta}{4}\int_0^T\sigma\sigma^\tau dt\right]\right\}\right\} \leqslant 0. \tag{11}$$

And hence,

$$J_\theta \leqslant \varlimsup_{T\to\infty}\left(\frac{V(X(0))}{T} + \frac{2}{T}\cdot\frac{1}{\theta}\ln\left\{E\left[\zeta(T)\right]\right\} + R_l\right) = R_l.$$

As for item (iii), because V is positive definite and radially unbounded, there exists a continuous, strictly increasing scalar function $\alpha:\ [0,\infty)\to[0,\infty)$ with $\lim_{s\to\infty}\alpha(s)=\infty$ such that $V(\cdot)\geqslant\alpha(|\cdot|)$. This together with (8) gives

$$\begin{aligned}\sup_{t\in[0,\infty)} P\{|X(t)|>c\} &\leqslant \sup_{t\in[0,\infty)} P\{V(X(t))>\alpha(c)\}\\ &\leqslant \frac{V(X(0))+c_2/c_1}{\alpha(c)}.\end{aligned}$$

Therefore,

$$0\leqslant \lim_{c\to\infty}\sup_{t\in[0,\infty)} P\{|X(t)|>c\}\leqslant \lim_{c\to\infty}\left\{\frac{V(X(0))+c_2/c_1}{\alpha(c)}\right\}=0,$$

i.e. the closed-loop system is bounded in probability.

In the following section, we will design a control law $u(x,\xi,y_d^{[n]})$, and at the same time, constructively give a value function $V(X)$, a nonnegative function $l(x,\xi,y_d^{[n]})$ and a $\mathcal{C}^1$ function $\sigma(x,\xi,y_d^{[n]})$ such that (7)—(8) hold, and further, such that the closed-loop systems are bounded in probability, and the RS index is not greater than the desired value.

2 Preliminary results

Since the states of system (1) are unknown and unavailable for control design, a state observer based on the system output y is needed:

$$\begin{aligned}\dot{\widehat{x}}_1 &= \widehat{x}_2+k_1(y-\widehat{x}_1)+f_1(y),\\ &\ \ \vdots\\ \dot{\widehat{x}}_{n-1} &= \widehat{x}_n+k_{n-1}(y-\widehat{x}_1)+f_{n-1}(y),\\ \dot{\widehat{x}}_n &= u+k_n(y-\widehat{x}_1)+f_n(y),\end{aligned}\tag{12}$$

where parameters $k_1,\cdots,k_n$ are constants and such that all the roots of polynomial $s^n+k_1s^{n-1}$ $+\cdots+k_n$ have negative real parts.

Let $\widehat{x}=(\widehat{x}_1,\cdots,\widehat{x}_n)^\tau$. Denote the state estimation error as $\tilde{x}=x-\widehat{x}$. Then we have

$$d\tilde{x}=\begin{bmatrix}-k_1 & & & \\ \vdots & & I_{n-1} & \\ -k_{n-1} & & & \\ -k_n & 0 & \cdots & 0\end{bmatrix}\tilde{x}dt+\varphi(y)dw\triangleq A\tilde{x}dt+\varphi(y)dw,\tag{13}$$

where $\varphi(y)=(\varphi_1^\tau,...,\varphi_n^\tau)^\tau$.

Since parameters k_i $(i=1,\cdots,n)$ are such that A is strictly stable, there is a unique positive definite matrix P satisfying Riccati equation:

$$A^\tau P+PA=-I.\tag{14}$$

Up to now, the whole system with observer (12) is

$$
\begin{aligned}
d\tilde{x} &= A\tilde{x}dt + \varphi(y)dw, \\
dy &= (\widehat{x}_2 + \tilde{x}_2)dt + f_1(y)dt + \varphi_1(y)dw, \\
d\widehat{x}_1 &= (\widehat{x}_2 + k_1(y - \widehat{x}_1) + f_1(y))dt, \\
&\vdots \\
d\widehat{x}_{n-1} &= (\widehat{x}_n + k_{n-1}(y - \widehat{x}_1) + f_{n-1}(y))\,dt, \\
d\widehat{x}_n &= (u + k_n(y - \widehat{x}_1) + f_n(y))\,dt.
\end{aligned} \tag{15}
$$

System (15) has a low triangular structure (except the subsystem on $\tilde{x}$), so the integrator backstepping method can be used to design output feedback controls.

In the next section, a design procedure of satisfaction output-feedback controls under quadratic tracking RS index will be given.

3 Output feedback control design

In this section, by using the integrator backstepping method we will present a design procedure of satisfaction output feedback controls under RS index for SF systems.

In the case where the disturbance disappears, dynamic $\tilde{x}$ converges to zero asymptotically, and so, can be regarded as zero-dynamic when we design tracking controllers by the integrator backstepping method.

Define state variable z as follows:

$$
z_1 = y - y_d,\ z_i = \widehat{x}_i - \alpha_{i-1}(\widehat{x}_{[i-1]}, y, y_d^{[i-1]}),\ 2 \leqslant i \leqslant n,
$$

where α_i $(1 \leqslant i \leqslant n-1)$ is virtual control, whose concrete expression will be given later. Obviously, z_1 is nothing but the tracking error.

With this state transformation, system (15) becomes

$$
\begin{aligned}
d\tilde{x} &= A\tilde{x}dt + \varphi(y)dw, \\
dz_1 &= \left(z_2 + \alpha_1(y, y_d^{[1]}) + \tilde{x}_2\right)dt + F_1(y, y_d^{[1]})dt + \varPsi_1(y)dw, \\
&\vdots \\
dz_i &= \left(z_{i+1} + \alpha_i(\widehat{x}_{[i]}, y, y_d^{[i]}) + F_i(\widehat{x}_{[i]}, y, y_d^{[i]})\right)dt \\
&\quad - \frac{\partial \alpha_{i-1}}{\partial y}\tilde{x}_2 dt + \varPsi_i(\widehat{x}_{[i]}, y, y_d^{[i-1]})dw, \\
&\vdots \\
dz_n &= \left(u + F_n(\widehat{x}, y, y_d^{[n]})\right)dt - \frac{\partial \alpha_{n-1}}{\partial y}\tilde{x}_2 dt + \varPsi_n(\widehat{x}, y, y_d^{[n-1]})dw,
\end{aligned} \tag{16}
$$

where $\varPsi_1 = \varphi_1$, $F_1 = f_1(y) - \dot{y}_d$; and for $i = 2, \cdots, n$, $\varPsi_i = -\dfrac{\partial \alpha_{i-1}}{\partial y}\varphi_1$ and

$$
F_i = (k_i(y - \widehat{x}_1) + f_i(y)) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \widehat{x}_j}(z_{j+1} + \alpha_j + k_j(y - \widehat{x}_1) + f_j(y))
$$

$$-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)}-\frac{\partial\alpha_{i-1}}{\partial y}(z_2+\alpha_1+f_1(y))-\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\varphi_1(y)\varphi_1^\tau(y)\right).$$

Unlike in refs. [20, 21, 26], in this paper quadratic weighted value functions are used for control design and convergence analysis:

$$V_n=\phi(\tilde{x}^\tau P\tilde{x})+\sum_{j=1}^{n}\varXi_j(\tilde{x}_1,z_{[j-1]},y_d^{[j-1]})z_j^2, \tag{17}$$

where $P>0$ is given by (14); $\phi(\xi)=(c+\xi)^\gamma-c^\gamma$, $\xi=\tilde{x}^\tau P\tilde{x}$, $\frac{1}{2}<\gamma<1$, and $c>0$ is a constant to be specified later; $\varXi_j(\tilde{x}_1,z_{[j-1]},y_d^{[j-1]})$ $(j=1,\cdots,n)$ are positive weighted functions. In this paper, γ is called characteristic parameter of the value function V_n. By the definition, it is easy to see that the value function is positive, radially unbounded, and $V_n(0)=0$, in other words, it possesses the necessary properties as a Lyapunov function.

Remark 1. When the gain functions of the disturbance are globally and uniformly bounded, similar to ref. [22], one can simply take $\phi=\tilde{x}^\tau P\tilde{x}$. Of course, in order to control the quadratic Itô term, in this case the Riccati equation (14) satisfied by P should be modified properly. Furthermore, when disturbance vanishes at the equilibrium point, a control law can be designed to reach zero RS index.

Let $\varXi_0=0$, $\alpha_0=0$,

$$\sigma_n=2\frac{\partial\phi}{\partial\xi}\tilde{x}^\tau P\varphi+\sum_{i=1}^{n}z_i^2\frac{\partial\varXi_i}{\partial\tilde{x}_1}\varphi_1+\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{\partial[\varXi_i z_i^2]}{\partial z_j}\varPsi_j,$$

$$M_1=2\varXi_1F_1,\quad M_i=2(\varXi_iF_i+\varXi_{i-1}z_{i-1})+\sum_{j=1}^{i-1}\frac{\partial\varXi_i}{\partial z_j}(z_{j+1}+\alpha_j+F_j)z_i,\ i=2,\cdots,n,$$

$$N_i=\frac{\partial\varXi_i}{\partial\tilde{x}_1}z_i-\sum_{j=1}^{i-1}\frac{\partial\varXi_i}{\partial z_j}\frac{\partial\alpha_{j-1}}{\partial y}z_i-2\varXi_i\frac{\partial\alpha_{i-1}}{\partial y},\ i=1,\cdots,n.$$

Then by (17) and Itô formula, we have

$$\begin{aligned}dV_n=&-\frac{\partial\phi}{\partial\xi}\|\tilde{x}\|^2dt+\sigma_n(\tilde{x},z,y_d^{[n]})dw+\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2}\varphi\varphi^\tau\right)dt\\&+\sum_{i=1}^{n}\sum_{j=0}^{i-1}\frac{\partial\varXi_i}{\partial y_d^{(j)}}y_d^{(j+1)}z_i^2dt+\sum_{i=1}^{n}N_i(\tilde{x}_1,z_{[i]},y_d^{[i-1]})\tilde{x}_2z_idt\\&+2\sum_{j=1}^{n-1}z_j\varXi_j(z_{j+1}+\alpha_j)dt+2z_n\varXi_nudt+\sum_{i=1}^{n}M_i(\tilde{x}_1,z_{[i]},y_d^{[i-1]})z_idt\\&+\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\frac{\partial^2(\varXi_iz_i^2)}{\partial(\tilde{x}_1,z_{[i]})^2}(\varphi_1^\tau,\varPsi_1^\tau,\cdots,\varPsi_i^\tau)^\tau(\varphi_1^\tau,\varPsi_1^\tau,\cdots,\varPsi_i^\tau)\right]dt.\end{aligned} \tag{18}$$

Notice that

$$\begin{aligned}\sigma_n\sigma_n^\tau \leqslant\; & 12\left(\frac{\partial\phi}{\partial\xi}\right)^2\tilde{x}^\tau P\varphi\varphi^\tau P\tilde{x} + 3\left(\sum_{i=1}^n z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\right)\left(\sum_{i=1}^n z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\right)^\tau \\ & + 3\left(\sum_{i=1}^n\sum_{j=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\right)\left(\sum_{i=1}^n\sum_{j=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\right)^\tau \\ \leqslant\; & 24\gamma^2(c+\xi)^{2\gamma-2}\|P\|^2\|\tilde{x}\|^2\left[\|\varphi(y_d)\|^2+\|\bar{\varphi}(z_1,y_d)\|^2 z_1^2\right] \\ & + 3\left(\sum_{i=1}^n z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\right)\left(\sum_{i=1}^n z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\right)^\tau \\ & + 3\left(\sum_{i=1}^n\sum_{j=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\right)\left(\sum_{i=1}^n\sum_{j=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\right)^\tau, \end{aligned} \tag{19}$$

where $\bar{\varphi}(z_1,\ y_d)$ satisfies $\varphi(y)=\varphi(z_1+y_d)=\varphi(y_d)+z_1\bar{\varphi}(z_1,y_d)$.

From $\gamma\in(\frac{1}{2},\ 1)$ it follows that $p=\frac{2\gamma}{2-2\gamma}>1$, $q=\frac{2\gamma}{4\gamma-2}>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $(2q-1)(\gamma-1)=1-q$. By Young's inequality

$$x^\tau y\leqslant\frac{\varepsilon^p}{p}\|x\|^p+\frac{1}{q\varepsilon^q}\|y\|^q,\quad \forall x,y\in\mathbb{R}^n,\ \varepsilon>0,\ p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \tag{20}$$

we have

$$\begin{aligned} & 24\gamma^2(c+\xi)^{2\gamma-2}\|\tilde{x}\|^2\|P\|^2\|\bar{\varphi}(z_1,y_d)\|^2 z_1^2 \\ \leqslant\; & 24\gamma^2\|P\|^2\left[\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p}+\frac{\varepsilon_1^q}{q}(c+\xi)^{2q(\gamma-1)}\|\tilde{x}\|^{2q}\right] \\ =\; & 24\gamma^2\|P\|^2\left[\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p}+\frac{\varepsilon_1^q}{q}\frac{\|\tilde{x}\|^{2(q-1)}}{(c+\xi)^{q-1}}(c+\xi)^{(\gamma-1)}\|\tilde{x}\|^2\right] \\ \leqslant\; & 24\gamma^2\|P\|^2\left[\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p}+\frac{\varepsilon_1^q}{q}\frac{1}{\lambda_{\min}^{q-1}(P)}(c+\xi)^{\gamma-1}\|\tilde{x}\|^2\right]. \end{aligned} \tag{21}$$

Here and hereafter $\varepsilon_1,\varepsilon_{2i},\varepsilon_3,\varepsilon_4,\varepsilon_5$ are constants to be specified.

Substituting (21) into (19) leads to

$$\begin{aligned}\sigma_n\sigma_n^\tau \leqslant\; & 24\gamma^2 c^{\gamma-1}\|P\|^2\|\varphi(y_d)\|^2(c+\xi)^{\gamma-1}\|\tilde{x}\|^2 \\ & + 24\gamma^2\|P\|^2\left[\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p}+\frac{\varepsilon_1^q}{q}\frac{1}{\lambda_{\min}^{q-1}(P)}(c+\xi)^{\gamma-1}\|\tilde{x}\|^2\right] \\ & + 3\sum_{i=1}^n z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\left(z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1+2\sum_{j=1}^{i-1}z_j^2\frac{\partial\Xi_j}{\partial\tilde{x}_1}\varphi_1\right)^\tau \\ & + 3\sum_{i=1}^n\sum_{j=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\left(\sum_{k=1}^i\frac{\partial[\Xi_i z_i^2]}{\partial z_k}\Psi_k+2\sum_{j=1}^{i-1}\sum_{k=1}^j\frac{\partial[\Xi_j z_j^2]}{\partial z_k}\Psi_k\right)^\tau. \end{aligned} \tag{22}$$

Again, by Young's inequality (20) we have

$$\sum_{i=1}^{n} N_i\tilde{x}_2 z_i \leqslant \sum_{i=1}^{n}\left(\frac{(2\gamma-1)}{2\gamma\varepsilon_{2i}^{\frac{2\gamma}{2\gamma-1}}}|N_i|^{\frac{2\gamma}{2\gamma-1}}|z_i|^{\frac{2\gamma}{2\gamma-1}}+\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}\|\tilde{x}\|^{2\gamma}\right), \tag{23}$$

$$\begin{aligned}\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2\phi}{\partial\tilde{x}^2}\varphi\varphi^\tau\right) &\leqslant \frac{\gamma\mathrm{tr}(P\varphi\varphi^\tau)}{(c+\xi)^{1-\gamma}}-\frac{2\gamma(1-\gamma)\mathrm{tr}(P\tilde{x}\tilde{x}^\tau P\varphi\varphi^\tau)}{(c+\xi)^{2-\gamma}}\\ &\leqslant \frac{\gamma}{c^{1-\gamma}}\left(\mathrm{tr}(\varphi^\tau(y_d)P\varphi(y_d))+2z_1\mathrm{tr}(\varphi^\tau(y_d)P\overline{\varphi}(z_1,y_d))\right.\\ &\quad\left.+z_1^2\mathrm{tr}(\overline{\varphi}^\tau(z_1,y_d)P\overline{\varphi}(z_1,y_d))\right). \end{aligned}\tag{24}$$

Due to the state diffeomorphism

$$z_{[i]}=\vartheta(\widehat{x}_{[i]},y,y_d^{[i-1]}),\quad i=1,\cdots,n,$$

$\Psi_i(\cdot)$ can be expressed as

$$\Psi_1(y)=\varphi_1(z_1+y_d)=\varphi_1(y_d)+z_1\overline{\varphi}_1(z_1,y_d),$$

$$\begin{aligned}&\Psi_i(\widehat{x}_{[i-1]},y,y_d^{[i-1]})=\overline{\Psi}_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})\\ &=\overline{\Psi}_i(0,0,y_d^{[i-1]})+\overline{\Psi}_i(\tilde{x}_1,0,y_d^{[i-1]})\tilde{x}_1+\sum_{j=1}^{i-1}z_j\overline{\Psi}_{ij}(\tilde{x}_1,z_{[j]},y_d^{[i-1]}),\end{aligned}$$

where $i=2,\cdots,n$.

It is easy to verify

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\frac{\partial^2[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})z_i^2]}{\partial(\tilde{x}_1,z_{[i]})^2}(\varphi_1^\tau,\Psi_1^\tau,\cdots,\Psi_i^\tau)^\tau(\varphi_1^\tau,\Psi_1^\tau,\cdots,\Psi_i^\tau)\right]\\ &=\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})^2}z_i^2 & 2\dfrac{\partial[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})}z_i\\ 2\left(\dfrac{\partial[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})}\right)^\tau z_i & 2\Xi_i\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}^\tau\right]\\ &=\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})^2}z_i & 2\dfrac{\partial[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})}\\ 2\left(\dfrac{\partial[\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})]}{\partial(\tilde{x}_1,z_{[i-1]})}\right)^\tau & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}^\tau\right]z_i\\ &\quad+\Xi_1(\tilde{x}_1,y_d)\Psi_1\Psi_1^\tau+\sum_{i=2}^{n}\Xi_i(\tilde{x}_1,z_{[i-1]},y_d^{[i-1]})\Psi_i\Psi_i^\tau.\end{aligned}\tag{25}$$

To this expression, we may cancel out the first term (with z_i as a factor) by properly choosing α_i $(i=1,\cdots,n)$. For the second term, we have

$$\begin{aligned}\Xi_1(\tilde{x}_1,y_d)\Psi_1\Psi_1^\tau&=\Xi_1(\tilde{x}_1,y_d)[\varphi_1(y_d)+z_1\overline{\varphi}_1(z_1,y_d)][\varphi_1(y_d)+z_1\overline{\varphi}_1(z_1,y_d)]^\tau\\ &\leqslant 2\Xi_1(\tilde{x}_1,y_d)\|\varphi_1(y_d)\|^2+2\Xi_1(\tilde{x}_1,y_d)\|\overline{\varphi}_1(z_1,y_d)\|^2z_1^2.\end{aligned}$$

By properly choosing weighted function Ξ_1, term $\Xi_1(\tilde{x}_1,y_d)\|\varphi_1(y_d)\|^2$ can be made sufficiently small. For instance, Ξ_1 can be taken as

$$\Xi_1(\tilde{x}_1, y_d) = \frac{\kappa_1}{1 + \|\varphi_1(y_d)\|^2},$$

where and whereafter $\kappa_1, \cdots, \kappa_n$ are constants to be specified. Then

$$\Xi_1(\tilde{x}_1, y_d)\Psi_1\Psi_1^\tau \leqslant 2\kappa_1 + 2\Xi_1(\tilde{x}_1, y_d)\|\overline{\varphi}_1(z_1, y_d)\|^2 z_1^2. \tag{26}$$

For the last term of (25), we have

$$\begin{aligned}
&\sum_{i=2}^{n} \Xi_i \Psi_i \Psi_i^\tau \\
&= \sum_{i=2}^{n} \Xi_i \left(\overline{\Psi}_i(0, 0, y_d^{[i-1]}) + \overline{\Psi}_i(\tilde{x}_1, 0, y_d^{[i-1]})\tilde{x}_1 + \sum_{j=1}^{i-1} z_j \overline{\Psi}_{ij}(\tilde{x}_1, z_{[j]}, y_d^{[i-1]}) \right) \\
&\quad \left(\overline{\Psi}_i(0, 0, y_d^{[i-1]}) + \overline{\Psi}_i(\tilde{x}_1, 0, y_d^{[i-1]})\tilde{x}_1 + \sum_{j=1}^{i-1} z_j \overline{\Psi}_{ij}(\tilde{x}_1, z_{[j]}, y_d^{[i-1]}) \right)^\tau \\
&\leqslant 3\sum_{i=2}^{n} \Xi_i \|\overline{\Psi}_i(0, 0, y_d^{[i-1]})\|^2 + 3\sum_{i=2}^{n} \Xi_i \|\overline{\Psi}_i(\tilde{x}_1, 0, y_d^{[i-1]})\|^2 \tilde{x}_1^2 \\
&\quad + 3\sum_{i=2}^{n} i\Xi_i \sum_{j=1}^{i-1} \Xi_j^{-1} \|\overline{\Psi}_{ij}(\tilde{x}_1, z_{[j]}, y_d^{[i-1]})\|^2 \Xi_j z_j^2.
\end{aligned}$$

Similarly, by properly choosing weighted function Ξ_i $(i = 2, \cdots, n)$, $\Xi_i \|\overline{\Psi}_i(0, 0, y_d^{[i-1]})\|^2$, $\Xi_i \|\overline{\Psi}_i(\tilde{x}_1, 0, y_d^{[i-1]})\|^2$ and $\Xi_i \Xi_j^{-1} \|\overline{\Psi}_{ij}(\tilde{x}_1, z_{[j]}, y_d^{[i-1]})\|^2$ $(i = 2, \cdots, n;\ j = 1, \cdots, i-1)$ can be made sufficiently small. For instance, Ξ_i can be taken as

$$\Xi_i = \frac{\kappa_i}{\Sigma_i}, \quad i = 2, \cdots, n,$$

where

$$\begin{aligned}
\Sigma_i = 1 &+ \|\overline{\Psi}_i(0, 0, y_d^{[i-1]})\|^2 + \|\overline{\Psi}_i(\tilde{x}_1, 0, y_d^{[i-1]})\|^2 |\tilde{x}_1|^{2-2\gamma} \\
&+ \sum_{j=1}^{i-1} \Xi_j^{-1} \|\overline{\Psi}_{ij}(\tilde{x}_1, z_{[j]}, y_d^{[i-1]})\|^2.
\end{aligned}$$

Then we have

$$\sum_{i=2}^{n} \Xi_i(\tilde{x}_1, z_{[i-1]}, y_d^{[i-1]})\Psi_i\Psi_i^\tau \leqslant 3\sum_{i=2}^{n} \kappa_i + 3\sum_{i=2}^{n} \kappa_i |\tilde{x}_1|^{2\gamma} + 3\sum_{i=2}^{n} i\kappa_i \sum_{j=1}^{i-1} \Xi_j z_j^2. \tag{27}$$

By subtracting from and add terms $\frac{\theta}{4}\sigma_n\sigma_n^\tau dt$, $(y - y_d)^2 dt$ and $\sum_{i=1}^{n} \Xi_i \beta_i z_i^2 dt$ to the right hand side of (18), and applying the estimates (19)—(27), we have

$$\begin{aligned}
dV_n \leqslant\ & -\frac{\partial\phi}{\partial\xi}\|\tilde{x}\|^2dt+\sigma_n dw-\frac{\theta}{4}\sigma_n\sigma_n^\tau dt-\sum_{i=1}^{n}\Xi_i\beta_i z_i^2dt-(y-y_d)^2dt\\
&+(y-y_d)^2dt+\frac{\gamma}{c^{1-\gamma}}\big(\mathrm{tr}\varphi^\tau(y_d)P\varphi(y_d)+2z_1\mathrm{tr}(\varphi^\tau(y_d)P\overline{\varphi}(z_1,y_d))\\
&+z_1^2\mathrm{tr}(\overline{\varphi}^\tau(z_1,y_d)P\overline{\varphi}(z_1,y_d))\big)dt+\sum_{i=1}^{n}\sum_{j=0}^{i-1}\frac{\partial\Xi_i}{\partial y_d^{(j)}}y_d^{(j+1)}z_i^2dt\\
&+6\theta\gamma^2\frac{1}{c^{1-\gamma}}\|P\|^2\|\varphi(y_d)\|^2(c+\xi)^{\gamma-1}\|\tilde{x}\|^2dt\\
&+6\theta\gamma^2\|P\|^2\left[\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p}+\frac{\varepsilon_1^q}{q}\frac{1}{\lambda_{\min}^{q-1}(P)}(c+\xi)^{\gamma-1}\|\tilde{x}\|^2\right]dt\\
&+\frac{3}{4}\theta\sum_{i=1}^{n}z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\left(z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1+2\sum_{j=1}^{i-1}z_j^2\frac{\partial\Xi_j}{\partial\tilde{x}_1}\varphi_1\right)^\tau dt\\
&+\frac{3}{4}\theta\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{\partial[\Xi_i z_i^2]}{\partial z_j}\Psi_j\left(\sum_{k=1}^{i}\frac{\partial[\Xi_i z_i^2]}{\partial z_k}\Psi_k+2\sum_{j=1}^{i-1}\sum_{k=1}^{j}\frac{\partial[\Xi_j z_j^2]}{\partial z_k}\Psi_k\right)^\tau dt\\
&+\sum_{i=1}^{n}\Xi_i\beta_i z_i^2dt+\sum_{i=1}^{n}M_iz_idt+2\Xi_1(\tilde{x}_1,y_d)\|\overline{\varphi}_1(z_1,y_d)\|^2z_1^2dt\\
&+\sum_{i=1}^{n}\left(\frac{(2\gamma-1)}{2\gamma\varepsilon_{2i}^{\frac{2\gamma}{2\gamma-1}}}|N_i|^{\frac{2\gamma}{2\gamma-1}}|z_i|^{\frac{2\gamma}{2\gamma-1}}+\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}\|\tilde{x}\|^{2\gamma}\right)dt\\
&+\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})^2}z_i & 2\dfrac{\partial\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})}\\ 2\left(\dfrac{\partial\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})}\right)^\tau & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}\begin{bmatrix}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{bmatrix}^\tau\right]z_idt\\
&+2\sum_{j=1}^{n-1}\Xi_j\alpha_jz_jdt+2\Xi_nz_nudt+2\kappa_1dt+2\kappa_1z_1^2dt\\
&+3\sum_{i=2}^{n}\kappa_idt+3\sum_{i=2}^{n}\kappa_i|\tilde{x}_1|^{2\gamma}dt+3\sum_{i=2}^{n}i\kappa_i\sum_{j=1}^{i-1}\Xi_jz_j^2dt, \qquad (28)
\end{aligned}$$

where $\beta_1, \cdots, \beta_n$ are constants to be specified.

We are now in a position to design the virtual and real control laws.

Design the first virtual control α_1 so as to cancel out all the terms with z_1 as a factor on the right hand side of (28) but the first and last lines, i.e.

$$\begin{aligned}
\alpha_1 =\ & \Bigg\{-\frac{z_1}{2\Xi_1}-\frac{M_1}{2\Xi_1}-\frac{1}{2}\beta_1z_1-\frac{1}{2\Xi_1}\frac{\gamma}{c^{1-\gamma}}[2\mathrm{tr}(\varphi^\tau(y_d)P\overline{\varphi}(z_1,y_d))\\
&+z_1\mathrm{tr}(\overline{\varphi}^\tau(z_1,y_d)P\overline{\varphi}(z_1,y_d))]-\frac{3}{2\Xi_1}\theta\gamma^2\|P\|^2\frac{1}{p\varepsilon_1^p}\|\bar{\varphi}(z_1,y_d)\|^{2p}|z_1|^{2p-2}z_1\\
&-\frac{3z_1}{2}\Xi_1\theta\Psi_1\Psi_1^\tau-\frac{z_1}{2\Xi_1}\frac{\partial\Xi_1}{\partial y_d}\dot{y}_d-\frac{3z_1^3}{8\Xi_1}\theta\left(\frac{\partial\Xi_1}{\partial\tilde{x}_1}\varphi_1\right)\left(\frac{\partial\Xi_1}{\partial\tilde{x}_1}\varphi_1\right)^\tau\\
&-\frac{1}{2\Xi_1}\frac{(2\gamma-1)}{2\gamma\varepsilon_{21}^{\frac{2\gamma}{2\gamma-1}}}|N_1|^{\frac{2\gamma}{2\gamma-1}}|z_1|^{\frac{2-2\gamma}{2\gamma-1}}z_1-\|\overline{\varphi}_1(z_1,y_d)\|^2z_1
\end{aligned}$$

$$-\frac{1}{4\Xi_1}\left[\left[\begin{array}{cc}\frac{\partial^2\Xi_1}{\partial\tilde{x}_1^2}z_1 & 2\frac{\partial\Xi_1}{\partial\tilde{x}_1}\\ 2\left(\frac{\partial\Xi_1}{\partial\tilde{x}_1}\right)^\tau & 0\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\end{array}\right]^\tau\right]\Bigg\}_{z_1=y-y_d}.$$

Similarly, design $\alpha_i,\ i=2,\cdots,n-1$ so as to cancel out all the terms with z_i as a factor on the right hand side of (28) but the first and last lines, i.e.

$$\begin{aligned}\alpha_i = &\left\{-\frac{\beta_i z_i}{2}-\frac{M_i}{2\Xi_i}-\frac{1}{2\Xi_i}\sum_{j=0}^{i-1}\frac{\partial\Xi_i}{\partial y_d^{(j)}}y_d^{(j+1)}z_i\right.\\ &-\frac{3\theta}{8\Xi_i}\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1\left(z_i^2\frac{\partial\Xi_i}{\partial\tilde{x}_1}\varphi_1+2\sum_{j=1}^{i-1}z_j^2\frac{\partial\Xi_j}{\partial\tilde{x}_1}\varphi_1\right)^\tau z_i\\ &-\frac{3\theta}{8\Xi_i}\left(\sum_{j=1}^{i-1}\frac{\partial\Xi_i}{\partial z_j}\Psi_j z_i+2\Xi_i\Psi_i\right)\left(\sum_{k=1}^{i}\frac{\partial[\Xi_i z_i^2]}{\partial z_k}\Psi_k+2\sum_{j=1}^{i-1}\sum_{k=1}^{j}\frac{\partial[\Xi_j z_j^2]}{\partial z_k}\Psi_k\right)^\tau\\ &-\frac{1}{2\Xi_i}\frac{(2\gamma-1)}{2\gamma\varepsilon_{2i}^{\frac{2\gamma}{2\gamma-1}}}|N_i|^{\frac{2\gamma}{2\gamma-1}}|z_i|^{\frac{2-2\gamma}{2\gamma-1}}z_i\\ &\left.-\frac{1}{4\Xi_i}\mathrm{tr}\left[\left[\begin{array}{cc}\frac{\partial^2\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})^2}z_i & 2\frac{\partial\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})}\\ 2\left(\frac{\partial\Xi_i}{\partial(\tilde{x}_1,z_{[i-1]})}\right)^\tau & 0\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_i\end{array}\right]^\tau\right]\right\}_{z_{[i]}=\vartheta(\hat{x}_{[i]},y,y_d^{[i-1]})}.\end{aligned}$$

Finally, design control u as

$$\begin{aligned}u = &\left\{-\frac{\beta_n z_n}{2}-\frac{M_i}{2\Xi_i}-\frac{1}{2\Xi_n}\sum_{j=0}^{n-1}\frac{\partial\Xi_n}{\partial y_d^{(j)}}y_d^{(j+1)}z_n\right.\\ &-\frac{3\theta}{8\Xi_n}\frac{\partial\Xi_n}{\partial\tilde{x}_1}\varphi_1\left(z_n^2\frac{\partial\Xi_n}{\partial\tilde{x}_1}\varphi_1+2\sum_{j=1}^{n-1}z_j^2\frac{\partial\Xi_j}{\partial\tilde{x}_1}\varphi_1\right)^\tau z_n\\ &-\frac{3\theta}{8\Xi_n}\left(\sum_{j=1}^{n-1}\frac{\partial\Xi_n}{\partial z_j}\Psi_j z_n+2\Xi_n\Psi_n\right)\left(\sum_{k=1}^{n}\frac{\partial[\Xi_n z_n^2]}{\partial z_k}\Psi_k+2\sum_{j=1}^{n-1}\sum_{k=1}^{j}\frac{\partial[\Xi_j z_j^2]}{\partial z_k}\Psi_k\right)^\tau\\ &-\frac{1}{2\Xi_n}\frac{(2\gamma-1)}{2\gamma\varepsilon_{2n}^{\frac{2\gamma}{2\gamma-1}}}|N_n|^{\frac{2\gamma}{2\gamma-1}}|z_n|^{\frac{2-2\gamma}{2\gamma-1}}z_n\\ &\left.-\frac{1}{4\Xi_n}\mathrm{tr}\left[\left[\begin{array}{cc}\frac{\partial^2\Xi_n}{\partial(\tilde{x}_1,z_{[n-1]})^2}z_n & 2\frac{\partial\Xi_n}{\partial(\tilde{x}_1,z_{[n-1]})}\\ 2\left(\frac{\partial\Xi_n}{\partial(\tilde{x}_1,z_{[n-1]})}\right)^\tau & 0\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_n\end{array}\right]\left[\begin{array}{c}\varphi_1\\ \Psi_1\\ \vdots\\ \Psi_n\end{array}\right]^\tau\right]\right\}_{z_{[n]}=\vartheta(\hat{x}_{[n]},y,y_d^{[n-1]})}.\end{aligned} \tag{29}$$

Substituting $\alpha_1,\ \cdots,\ \alpha_{n-1},\ u$ defined above into (28) gives

$$dV_n \leqslant \sigma_n dw - \frac{\theta}{4}\sigma_n\sigma_n^\tau dt - (y-y_d)^2dt - l'(\tilde{x})dt - \sum_{i=1}^n \delta_i z_i^2 dt + R'dt, \tag{30}$$

where

$$l'(\tilde{x}) = \delta' \frac{\|\tilde{x}\|^2}{(c+\xi)^{1-\gamma}} - \sum_{i=2}^n 3\kappa_i|\tilde{x}_1|^{2\gamma} - \sum_{i=1}^n \frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}\|\tilde{x}\|^{2\gamma}, \tag{31}$$

$$\delta' = \gamma - 6\theta\gamma^2 c^{\gamma-1}\|P\|^2\|\varphi(y_d)\|^2 - \frac{6\theta\gamma^2\|P\|^2\varepsilon_1^q}{q\lambda_{\min}^{q-1}(P)}, \tag{32}$$

$$R' = \frac{\gamma}{c^{1-\gamma}}\mathrm{tr}\left(\varphi^\tau(y_d)P\varphi(y_d)\right) + 2\kappa_1 + 3\sum_{i=2}^n \kappa_i, \tag{33}$$

$$\delta_1 = \varXi_1\beta_1 - 2\kappa_1 - 3\varXi_1\sum_{j=2}^n j\kappa_j, \cdots, \delta_i = \varXi_i\beta_i - 3\varXi_i\sum_{j=i+1}^n j\kappa_j, \cdots, \delta_n = \varXi_n\beta_n. \tag{34}$$

By Lemma A1 in the Appendix we see

$$\|\tilde{x}\|^{2\gamma} \leqslant \lambda_{\max}^{1-\gamma}(P)\left(M(c) + (c+\xi)^{1-\gamma}\|\tilde{x}\|^2\right).$$

Substituting this inequality into (31) renders

$$l'(\tilde{x}) \geqslant \delta\frac{\|\tilde{x}\|^2}{(c+\xi)^{1-\gamma}} - R'', \tag{35}$$

where

$$\delta = \delta' - \lambda_{\max}^{1-\gamma}(P)\sum_{i=2}^n 3\kappa_i - \lambda_{\max}^{1-\gamma}(P)\sum_{i=1}^n\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma},$$

$$R'' = \lambda_{\max}^{1-\gamma}(P)\sum_{i=2}^n 3\kappa_i M(c) + \lambda_{\max}^{1-\gamma}(P)\sum_{i=1}^n\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}M(c).$$

Substituting (35) into (30) and letting

$$l(\tilde{x}, z) = \delta\frac{\|\tilde{x}\|^2}{(c+\xi)^{1-\gamma}} + \sum_{i=1}^n \delta_i z_i^2, \tag{36}$$

$$R = R' + R'', \tag{37}$$

we have

$$dV_n \leqslant \sigma_n dw - \frac{\theta}{4}\sigma_n\sigma_n^\tau dt - (y-y_d)^2dt - l(\tilde{x},\ z)dt + Rdt. \tag{38}$$

By Lemma A2 in the Appendix we see that for any given RS index $R_l > 0$, there exist positive constants c, κ_i, ε_1, ε_{2i}, β_i such that

$$\delta > 0, \quad \delta_i > 0, \quad R \leqslant R_l. \tag{39}$$

In summary, we have the following theorem.

Theorem 2. Consider system (1) and the quadratic tracking RS index (3). Suppose Assumptions A1—A3 hold. For any given RS parameter $\theta > 0$ and desired index value $R_l > 0$, if the output feedback controller is given by (29), and the design parameters satisfy (39), then (i) the closed-loop system has a unique solution almost surely; (ii) the RS index is not larger than the desired value R_l; (iii) the closed-loop system is bounded in probability.

Proof. By (38) and Theorem 1 it is easy to get item (i) and item (ii). As for item (iii), by the definition of value function V_n and (38), it is easy to see that there are positive constants c_1 and c_2 such that

$$\mathcal{L}V \leqslant -c_1 V + c_2.$$

Then, by Theorem 1 we get item (iii).

Remark 2. As for the value range of the characteristic parameter of value function (17), the following two factors should be considered. Firstly, since $\tilde{x}$ is unknown, in order to guarantee the stability of the closed-loop system, the fifth term $\sum_{i=1}^{n} N_i(\tilde{x}_1, z_{[i]}, y_d^{[i-1]})\tilde{x}_2 z_i dt$ on the right hand side of (18) should be dominated by the first term $-\frac{\partial \phi}{\partial \xi}\|\tilde{x}\|^2 dt$. In other words, the power 2γ of $\tilde{x}$ in the first term on the right hand side of (18) should be greater than 1, the power of $\tilde{x}$ in the fifth term, or $\gamma > 1/2$. Secondly, to deal with the second term on the right hand of (18), a method often used is to subtract from and add $\frac{\theta}{4}\sigma_n\sigma_n^\tau dt$ to the right hand side of (18). Here, the negative term $-\frac{\theta}{4}\sigma_n\sigma_n^\tau dt$ is used to control the second term $\sigma_n dw$, while the positive term $\frac{\theta}{4}\sigma_n\sigma_n^\tau dt$ is dominated by the first term $-\frac{\partial \phi}{\partial \xi}\|\tilde{x}\|^2 dt$ and the system input. Thus, it is natural to require that the power 2γ of $\tilde{x}$ in $\frac{\partial \phi}{\partial \xi}\|\tilde{x}\|^2$ is greater than $4\gamma - 2$, the power of $\tilde{x}$ in $\sigma_n\sigma_n^\tau$, i.e. $2\gamma > 4\gamma - 2$, or equivalently, $\gamma < 1$.

Remark 3. Due to the special structure of the value function and the limitation of the value range of the characteristic parameter, the controller designed here cannot realize zero RS index. The reason is that: in order to get zero RS index, R in (38) should be zero, which requires $M(c) = 0$, or equivalently, $c = 0$. In this case, when $\varphi\varphi^\tau|_{\tilde{x}=0} \neq 0$, by $\gamma < 1$ we would have $\frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \phi}{\partial \tilde{x}^2}\varphi\varphi^\tau\right)\Big|_{\tilde{x}=0} = \infty$. This may lead to larger error.

4 Conclusion

In this paper, the satisfaction output feedback control design problem of stochastic nonlinear systems in strict feedback form under a long-term tracking RS index is investigated. The index function adopted here is of quadratic form usually encountered in practice, rather than of quartic one used to beg the essential difficulty on controller design and performance analysis of the closed-loop systems. For any given RS parameter and desired index value, by using the integrator backstepping method, an output feedback control is constructively designed so that the closed-loop system is bounded in probability and the RS index is upper bounded by the desired value. Problems needing further study are: (i) output feedback control of stochastic nonlinear systems with zero-dynamics under a quadratic RS index, (ii) control design based on the information states method.

Appendix

For any given positive matrix P, use $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ to denote its maximum and minimum eigenvalue, respectively.

Lemma A1. For any given positive matrix P and constant $\gamma \in (\frac{1}{2}, 1)$, define function

$$\Pi(x,c) = \|x\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + x^\tau Px)^{\gamma-1}\|x\|^2. \tag{A1}$$

Let

$$M(c) = \sup_{x\in\mathbb{R}^n} \Pi(x,c). \tag{A2}$$

Then when c is a finite positive constant, so is $M(c)$ and it satisfies

$$\lim_{c\to 0^+} M(c) = 0 \quad \text{and} \quad \lim_{c\to +\infty} M(c) = +\infty.$$

Proof. From the definition of $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ it follows that for any nonzero vector $x \in \mathbb{R}^n$,

$$\lambda_{\min}(P) \leqslant \frac{x^\tau Px}{\|x\|^2} \leqslant \lambda_{\max}(P). \tag{A3}$$

Further, by $\gamma < 1$ we have for any nonzero vector $x \in \mathbb{R}^n$,

$$\begin{aligned}\Pi(x,c) &\leqslant \|x\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\|x\|^2)^{\gamma-1}\|x\|^2 \\ &= \lambda_{\max}^{\gamma-1}(P)\|x\|^{2\gamma}\left[1 - \left(1 + \frac{c}{\lambda_{\max}(P)\|x\|^2}\right)^{\gamma-1}\right].\end{aligned}$$

Hence, when $\|x\| \to \infty$, we have

$$\Pi(x,c) \leqslant c(1-\gamma)\lambda_{\max}^{\gamma-2}(P)\|x\|^{2(\gamma-1)}\left[1 + o(1)\right] \xrightarrow[\|x\| \to \infty]{} 0.$$

Let

$$\mathcal{X}_0 = \{x \in \mathbb{R}^n : Px = \lambda_{\max}(P)x\}.$$

Then it can be shown that for any constant $c > 0$,

$$M(c) \geqslant \sup_{x\in\mathcal{X}_0, \|x\|=1} \Pi(x,c) = \lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P))^{\gamma-1} > 0.$$

Therefore, for any constant $c > 0$, if x_1 is the maximum of $\Pi(x,c)$, then there must be $x_1 \neq 0$ and $\|x_1\| < \infty$. Furthermore, we can show that $x_1 \in \mathcal{X}_0$, since otherwise, there would be $x_1^\tau Px_1 < \lambda_{\max}(P)\|x_1\|^2$. Take x_0 in $\mathcal{X}_0$ such that $\|x_0\| = \|x_1\|$. Then,

$$\begin{aligned}\Pi(x_0,c) &= \|x_0\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + x_0^\tau Px_0)^{\gamma-1}\|x_0\|^2 \\ &= \|x_0\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\|x_0\|^2)^{\gamma-1}\|x_0\|^2 \\ &= \|x_1\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\|x_1\|^2)^{\gamma-1}\|x_1\|^2 \\ &> \|x_1\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + x_1^\tau Px_1)^{\gamma-1}\|x_1\|^2.\end{aligned}$$

This contradicts the fact that x_1 is the maximum of $\Pi(x,c)$.

Thus, there must be

$$\begin{aligned}M(c) &= \sup_{x\in\mathbb{R}^n} \Pi(x,c) = \sup_{x\in\mathcal{X}_0} \Pi(x,c) \\ &= \sup_{x\in\mathcal{X}_0} \left(\|x\|^{2\gamma}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\|x\|^2)^{\gamma-1}\|x\|^2\right).\end{aligned}$$

In other words, the vector maximization problem has been transformed to a scalar one on α of the following two-variable function $f(\alpha, c)$:

$$f(\alpha,c) = \alpha^\gamma\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\alpha)^{\gamma-1}\alpha.$$

That is, for any given constant $c \geqslant 0$, we have

$$M(c) = \sup_{\alpha \geqslant 0} f(\alpha, c).$$

Since

$$\frac{\partial f(\alpha, c)}{\partial c} = (1-\gamma)(c + \lambda_{\max}(P)\alpha)^{\gamma-2}\alpha > 0, \quad \forall \alpha > 0,$$

when $\alpha > 0$, $f(\alpha, c)$ is a strictly increasing function of c. It can also be shown: $M(c)$ is strictly increasing, i.e. for any $0 \leqslant c_1 < c_2$, there is always $0 \leqslant M(c_1) < M(c_2)$. This is because if α_{c_1} maximizes $f(\alpha, c_1)$, or $M(c_1) = f(\alpha_{c_1}, c_1) = \sup_{\alpha \geqslant 0} f(\alpha, c_1)$, then from the strict increase property of $f(\cdot, c)$ we have

$$M(c_1) = f(\alpha_{c_1}, c_1) < f(\alpha_{c_1}, c_2) \leqslant \sup_{\alpha \geqslant 0} f(\alpha, c_2) = M(c_2).$$

If α_c maximizes $f(\alpha, c)$, then it satisfies

$$\alpha_c^{\gamma-1}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\alpha_c)^{\gamma-1} = \frac{1-\gamma}{\gamma}c(c + \lambda_{\max}(P)\alpha_c)^{\gamma-2}.$$

Thus,

$$\begin{aligned} M(c) &= f(\alpha_c, c) = \alpha_c(\alpha_c^{\gamma-1}\lambda_{\max}^{\gamma-1}(P) - (c + \lambda_{\max}(P)\alpha_c)^{\gamma-1}) \\ &= \frac{(1-\gamma)c}{\gamma}\frac{\alpha_c}{(c + \lambda_{\max}(P)\alpha_c)^{2-\gamma}} \leqslant \frac{(1-\gamma)c^{\gamma}}{\gamma\lambda_{\max}(P)}. \end{aligned}$$

This together with $M(0) = 0$ and the fact that $M(c)$ is strictly increasing in $[0, \infty)$ gives

$$\lim_{c \to 0^+} M(c) = M(0) = 0.$$

We now show

$$\lim_{c \to +\infty} M(c) = +\infty. \tag{A4}$$

Since otherwise, from the fact that $M(c)$ is strictly increasing in $[0, \infty)$ it would follow that $\lim_{c \to +\infty} M(c)$ is existent and finite. Let $\Upsilon = \lim_{c \to +\infty} M(c)$. Then for $c = \left(1 - \frac{1}{2^{1-\gamma}}\right)^{-\frac{1}{\gamma}} \times (2\lambda_{\max}(P)\Upsilon)^{\frac{1}{\gamma}}$, we would have $M(c) \geqslant f(c\lambda_{\max}^{-1}(P), c) = 2\Upsilon > \Upsilon$. This contradicts $\Upsilon = \lim_{c \to +\infty} M(c)$ and the fact that $M(c)$ is strictly increasing in $[0, \infty)$. Thus, (A4) holds.

Lemma A2. For any given RS index $R_l > 0$, there exist positive constants c, κ_i, ε_1, ε_{2i} and β_i, such that inequality (39) holds.

Proof. By the following expression on δ and R:

$$\begin{aligned} \delta = \gamma &- 6\theta\gamma^2 c^{\gamma-1}\|P\|^2\|\varphi(y_d)\|^2 - \frac{6\theta\gamma^2\|P\|^2\varepsilon_1^q}{q\lambda_{\max}^{q-1}(P)} \\ &- \lambda_{\max}^{1-\gamma}(P)\sum_{i=2}^{n}3\kappa_i - \lambda_{\max}^{1-\gamma}(P)\sum_{i=1}^{n}\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}, \end{aligned} \tag{A5}$$

$$\begin{aligned} R = \frac{\gamma}{c^{1-\gamma}}&\mathrm{tr}\,(\varphi^{\tau}(y_d)P\varphi(y_d)) + 2\kappa_1 + 3\sum_{i=2}^{n}\kappa_i \\ &+ \lambda_{\max}^{1-\gamma}(P)\sum_{i=2}^{n}3\kappa_i M(c) + \lambda_{\max}^{1-\gamma}(P)\sum_{i=1}^{n}\frac{\varepsilon_{2i}^{2\gamma}}{2\gamma}M(c), \end{aligned} \tag{A6}$$

we see that if

$$\varepsilon_1 < \left(\frac{q\lambda_{\max}^{q-1}(P)}{24\theta\|P\|^2}\right)^{\frac{1}{q}},$$

$$c > \left[\max\left\{3R_l^{-1}\mathrm{tr}(\varphi^\tau(y_d)P\varphi(y_d)),\ \ 24\theta\|P\|^2\|\varphi(y_d)\|^2\right\}\right]^{\frac{1}{1-\gamma}},$$

$$\kappa_i < \max\left\{\frac{1}{12n\lambda_{\max}^{1-\gamma}(P)},\ \ \frac{R_l}{9n\left(1+\lambda_{\max}^{1-\gamma}(P)M(c)\right)}\right\},\quad i=1,\cdots,n,$$

$$\varepsilon_{2i} < \left(\frac{1}{4n\lambda_{\max}^{1-\gamma}(P)}\right)^{\frac{1}{2\gamma}}\max\left\{1,\ \ \left(\frac{R_l}{3M(c)}\right)^{\frac{1}{2\gamma}}\right\},\quad i=1,\cdots,n,$$

then it will be $\delta > 0$ and $R \leqslant R_l$.

For κ_i chosen as above, if we simply take

$$\begin{cases} \beta_1 > \dfrac{2\kappa_1}{\Xi_1} + 3\displaystyle\sum_{j=2}^{n} j\kappa_j; \quad \beta_n > 0; \\ \beta_i > 3\displaystyle\sum_{j=i+1}^{n} j\kappa_j, \quad i = 2,\cdots,n-1, \end{cases}$$

then $\delta_i > 0$.

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