INT. J. CONTROL, 1989, VOL. 49, NO. 6, 1915–1935

Convergence rates in stochastic adaptive tracking†

HAN-FU CHEN‡ and JI-FENG ZHANG‡

For stochastic control systems described by the ARMAX model with unknown matrix coefficients, the stochastic adaptive control is designed so that the parameter estimates converge to the true values with a rate of convergence $O((\log n)(\log\log n)^c/n^\alpha)$ with $\alpha > 0$, $c > 1$ and the tracking error tends to its minimum value at a speed of $O(n^{-1/2\varepsilon})$ with $\varepsilon > 0$.

1. Introduction

Since Åström and Wittenmark introduced the self-tuning regulator in 1973, the stochastic adapative tracking problem has drawn much attention from control scientists (Goodwin *et al.* 1981, Goodwin and Sin 1979, Chen and Caines 1985, Caines and Lafortune 1984, Chen and Guo 1987 a, b, Chen 1984). The difficult and important problem of simultaneously identifying unknown parameters and tracking a reference signal was first considered by Caines and Lafortune (1984), where consistent estimation and suboptimal tracking were simultaneously obtained. Subsequently, Chen and Guo (1987 a, b, 1986 a), introducing an attenuating excitation to the control, achieved the consistent parameter estimate and the minimal tracking error simultaneously. However, these results are all based on the stochastic gradient algorithm for parameter estimation which is not as good with convergence properties as the estimate of least squares (ELS) algorithm. The essential difficulty of applying the ELS algorithm to the stochastic adaptive tracking problem consists in that *a posteriori* rather than the *a priori* information is used in the ELS algorithm, as pointed out by Kumar (1985).

The ELS-based adaptive tracker for the unit delay case is designed by Lai and Wei (1986) for single-input single-output systems with bounded noise and by Guo and Chen (1987) for stable multi-input multi-output systems but without boundedness restriction for noise. In this paper the multi-delay case is treated and the stability assumption on the system required by Guo and Chen (1987) has been removed so that the same convergence rates as those obtained by Guo and Chen (1987) are established.

2. Statement of the problem

Consider the stochastic control system

$$A(z)y_n = B(z)u_{n-d} + C(z)w_n, \quad d \geqslant 1; \quad y_i = w_i = 0, \quad u_i = 0, \quad i < 0 \tag{1}$$

Received 5 February 1988.

† Work supported by the National Natural Science Foundation of China and the TWAS Research Grant No. 87–43.

‡ Institute of Systems Science, Academia Sinica, Beijing, People's Republic of China.

with l inputs u_n, m outputs y_n and m-dimensional driven noise w_n, where

$$A(z) = I + A_1 z + \ldots + A_p z^p, \quad p \geqslant 0 \tag{2}$$

$$B(z) = B_1 + B_2 z + \ldots + B_q z^{q-1}, \quad q \geqslant 1 \tag{3}$$

$$C(z) = I + C_1 z + \ldots + C_r z^r, \quad r \geqslant 0 \tag{4}$$

are matrix polynomials in a shift-back operator with unknown coefficients denoted by

$$\theta^\tau = [-A_1 \quad \ldots \quad -A_p \quad B_1 \quad \ldots \quad B_q \quad C_1 \quad \ldots \quad C_r] \tag{5}$$

and $\{w_n, \mathscr{F}_n\}$ is a martingale difference sequence with

$$\sup_n E[\|w_{n+1}\|^2/\mathscr{F}_n] < \gamma < \infty, \quad \text{a.s.} \tag{6}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} w_i w_i^\tau = R > 0 \tag{7}$$

Problem A

In the adaptive tracking problem the $\mathscr{F}_n$-measurable control u_n is designed to force the output y_{n+d} to follow a given $\mathscr{F}_n$-measurable reference signal y^*_{n+d}. In addition, $\{y^*_n\}$ is mutually independent of $\{w_n\}$.

It is not difficult to show (see Appendix A) that

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n} \|y_i - y_i^*\|^2 \geqslant \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} (F(z)w_i)^\tau \cdot (F(z)w_i) = \operatorname{tr} \sum_{j=0}^{d-1} F_j R F_j^\tau \tag{8}$$

where $G(z)$ and $F(z) = F_0 + F_1 z + \ldots + F_{d-1} z^{d-1}$ with $F_0 = I$ are the unique solution (see Appendix A) of the Diophantine equation

$$(\det C(z))I = F(z)(\operatorname{Adj} C(z))A(z) + z^d G(z) \tag{9}$$

So it is natural to call u_n leading to

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n} (y_i - y_i^*)(y_i - y_i^*)^\tau = \sum_{j=0}^{d-1} F_j R F_j^\tau$$

the optimal control and the convergence rate of

$$\left\| \frac{1}{n} \sum_{i=0}^{n} (y_i - y_i^*)(y_i - y_i^*)^\tau - \frac{1}{n} \sum_{i=0}^{n} (F(z)w_i)(F(z)w_i)^\tau \right\|$$

the tracking speed.

Problem B

In the stochastic model reference adaptive control (MRAC) problem the $\mathscr{F}_n$-measurable control u_n is designed in order to reduce system (1) to

$$A^0(z) y_n = B^0(z) u^0_{n-d} \tag{10}$$

where $A^0(z)$ and $B^0(z)$ are given matrix polynomials of orders $\bar{p}$ and $\bar{q}$ respectively, $A^0(z)$ is stable and u^0_n is $\mathscr{F}_n$-measurable external input. Obviously, (1) can be written in the form

$$A^0(z) y_n = B^0(z) u^0_{n-d} + \varepsilon_n \tag{11}$$

with

$$\varepsilon_n = (A^0(z) - A(z))y_n + B(z)u_{n-d} - B^0(z)u^0_{n-d} + C(z)w_n \tag{12}$$

It is not difficult to show (see Appendix B) that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \varepsilon_i^\tau \varepsilon_i \geqslant \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} (\bar{F}(z)w_i)^\tau \cdot (\bar{F}(z)w_i) = \operatorname{tr} \sum_{j=0}^{d-1} \bar{F}_j R \bar{F}_j^\tau \tag{13}$$

where

$$\bar{F}(z) = \bar{F}_0 + \bar{F}_1 z + \ldots + \bar{F}_{d-1} z^{d-1} \tag{14}$$

and

$$A^0(z)F(z) = \bar{F}_0 + \bar{F}_1 z + \ldots + \bar{F}_{d-1} z^{d-1} + z^d \bar{N}(z) \tag{15}$$

Therefore, the adaptive control u_n leading to

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \varepsilon_i \varepsilon_i^\tau = \sum_{j=0}^{d-1} \bar{F}_j R \bar{F}_j^\tau$$

is optimal and the speed of

$$\left\| \frac{1}{n}\sum_{i=0}^{n} \varepsilon_i \varepsilon_i^\tau - \frac{1}{n}\sum_{i=0}^{n} (\bar{F}(z)w_i)(\bar{F}(z)w_i)^\tau \right\|$$

is the convergence rate of MRAC.

In the present paper we shall give optimal stochastic adaptive controls based on the ELS algorithm for both problems of A and B and characterize their convergence rates.

For the unknown matrix θ the ELS estimate θ_n is defined as follows:

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y^\tau_{n+1} - \varphi^\tau_n \theta_n) \tag{16}$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^\tau P_n, \quad a_n = (1 + \varphi_n^\tau P_n \varphi_n)^{-1} \tag{17}$$

$$P_o = sI, \quad s = m(p+r) + lq$$

$$\begin{aligned} \varphi_n^\tau = [y_n^\tau \ \ \ldots \ \ y^\tau_{n-p+1} \ \ u^\tau_{n-d+1} \ \ \ldots \ \ u^\tau_{n-d-q+2} \\ y_n^\tau - \varphi^\tau_{n-1}\theta_n \ \ \ldots \ \ y^\tau_{n-r+1} - \varphi^\tau_{n-r}\theta_{n-r+1}] \end{aligned} \tag{18}$$

with an arbitrary initial value

$$\theta_0^\tau = [-A_{10} \ldots -A_{p0} \quad B_{10} \ldots B_{q0} \quad C_{10} \ldots C_{r0}]$$

3. Adaptive control laws for $m = l$

Problem A

Set

$$A_n(z) = I + A_{1n} z + \ldots + A_{pn} z^p \tag{19}$$

$$B_n(z) = B_{1n} + B_{2n} z + \ldots + B_{qn} z^{q-1} \tag{20}$$

$$C_n(z) = \mathrm{I} + C_{1n} z + \ldots + C_{rn} z^r \tag{21}$$

$$\det C(z) = 1 + \bar{c}_1 z + \ldots + \bar{c}_{mr} z^{mr} \tag{22}$$

$$G(z) = G_0 + G_1 z + \dots + G_{p_1} z^{p_1} \tag{23}$$

$$F(z)(\text{Adj } C(z))B(z) = D_0 + D_1 z + \dots + D_{p_2} z^{p_2} \tag{24}$$

$$\theta^* = [G_0 G_1 \dots G_{p_1} \quad D_0 D_1 \dots D_{p_2} \quad \bar{c}_1 I \dots \bar{c}_{mr} I]^\tau$$

where $D_0 = B_1$ is non-degenerate, $p_2 = (m-1)r + q + d - 1$ and $p_1 = \max(mr - d, (m-1)r + p - 1)$.

In Guo and Chen (1987), for an ELS-based adaptive tracker the desired control $u_n^{(1)}$ is defined by

$$B_{1n} u_n^{(1)} = y_{n+1}^* - \theta_n^\tau \varphi_n + B_{1n} u_n \tag{25}$$

where B_{1n} denotes the estimate given by θ_n for B_1 and u_n is the attenuately excited version of $u_n^{(1)}$. Clearly, (25) is reduced to the well-known equation defining adaptive tracking control if the dither is removed, i.e. $u_n \equiv u_n^{(1)}$ (Goodwin *et al.* 1981, Kumar 1985).

In this paper the delay may be greater than one and in order to define an appropriate adaptive control we should find the predicted value $\hat{y}_{n+d}$ for y_{n+d} on the basis of $\{u_i, y_i, i \leqslant n\}$. Using (1) and (9) we have

$$\begin{aligned}(\det C(z))(y_n - F(z)w_n) &= G(z)y_{n-d} - (\det C(z))F(z)w_n \\ &\quad + F(z)(\text{Adj } C(z))A(z)y_n \\ &= G(z)y_{n-d} - (\det C(z))F(z)w_n \\ &\quad + F(z)(\text{Adj } C(z))B(z)u_{n-d} + (\det C(z))F(z)w_n \\ &= G(z)y_{n-d} + F(z)(\text{Adj } C(z))B(z)u_{n-d}\end{aligned} \tag{26}$$

Hence

$$\hat{y}_{n+d} = (\det C(z))^{-1}(G(z)y_n + F(z)(\text{Adj } C(z))B(z)u_n)$$

and the optimal tracking control should be defined from

$$(\det C(z))y_{n+d}^* = G(z)y_n + F(z)(\text{Adj } C(z))B(z)u_n \tag{27}$$

This leads us to define the undisturbed adaptive control $u_n^{(1)}$ from the following equation

$$B_{1n}u_n^{(1)} = (\det C_n(z))y_{n+d}^* - G_n(z)y_n - (F_n(z)(\text{Adj } C_n(z))B_n(z))u_n + B_{1n}u_n \tag{28}$$

instead of (25), where $F_n(z)$ and $G_n(z)$ satisfy the Diophantine equation

$$(\det C_n(z))I = F_n(z)(\text{Adj } C_n(z))A_n(z) + z^d G_n(z) \tag{29}$$

which is tantamount to (9) and is solvable as shown in Appendix A.

If the growth rates of the input and output are not too fast, then we take $u_n^{(1)}$ as the desired control with the hope of getting better parameter estimates because the ELS is implemented. If the output grows up too fast, then we switch the parameter estimate from the ELS algorithm to the stochastic gradient (SG) algorithm and the minimum-phase property guarantees y_n tracking y_n^*. Finally, if $u_n^{(1)}$ itself grows too fast, we then simply take zero as the desired control. To be precise, we specify the random intervals on which we apply one or other desired control. Let the stopping times $\{\tau_i\}$ and $\{\sigma_i\}$