

# An LMI Approach to Static Output Feedback Stabilization of Linear Systems \*

WANG Jinzhi and ZHANG Jifeng

(Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences · Beijing, 100080, P.R. China)

**Abstract:** In this paper, a linear matrix inequality (LMI) approach without constraints is developed to solve W- and P-problems, which are related to designing static output feedback (SOF) stabilization controls for linear systems. As an application, the SOF stabilization problem of uncertain systems is considered. Some LMI-based sufficient and necessary conditions and design approaches are presented.

**Key words:** LMI; static output feedback; uncertainty

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## 线性系统静态输出反馈镇定的 LMI 方法

王金枝 张纪峰

(中国科学院数学与系统科学研究院系统科学研究所·北京, 100080)

**摘要:** 利用无约束条件的线性矩阵不等式(LMI)研究了 W-问题和 P-问题, 而后的解可用来设计静态输出反馈(SOF)镇定控制. 作为一个应用考虑了不确定系统的静态输出反馈问题, 给出了依赖于 LMI 条件的 SOF 设计方法.

**关键词:** 线性矩阵不等式; 静态输出反馈; 不确定性

### 1 Introduction

The design of static output feedback (SOF) stabilization controllers for either certain or uncertain systems is one of the most basic and important control problems. Although various approaches have been proposed, the analytical or numerical solution is still hard to get in general. The difficulty is that designing an SOF stabilization controller is actually equivalent to solving a set of bilinear matrix inequalities (BMIs), which are non-convex in general, and therefore, difficult to deal with<sup>[1,2]</sup>. Recently, some efforts have been made to identify some special cases in which the solvability of the BMIs can be transferred into that of a set of linear matrix inequalities (LMIs) which can numerically be solved by LMI Toolbox<sup>[3]</sup>. For instance, under some sufficient conditions, it is shown<sup>[4]</sup> that the SOF stabilization problem is solvable if so is the so-called W-or P-problems expressed in terms LMIs and linear matrix equality (LME) constraints. However, it is still an open prob-

lem either to find the feasible conditions of or to construct a solution for a set of LMIs with an LME constraint.

The purpose of this paper is to provide some sufficient and necessary conditions for the feasibilities of both W- and P-problems, and formulate the conditions in terms of pure LMIs (i. e., without LME or any other constraint). In addition, an effective method for designing SOF stabilization controllers is given. As an application of the ideas and methods, we reconsider the SOF stabilization problem of systems with polytopic uncertainties, and essentially generalize the results of Reference [5].

Throughout this paper, the following notations are used.  $A^T$  is the transpose of a matrix  $A$ .  $A > 0$  ( $A < 0$ ) means  $A$  is positive (negative) definite. And for an  $n \times m$  full column rank matrix  $A$ ,  $A^\perp$  denotes an  $(n - m) \times n$  matrix with the following properties:  $A^\perp A = 0_{(n-m) \times m}$ ,  $[A \ A^\perp] \in \mathbb{R}^{n \times n}$  is invertible and  $A^\perp A^\perp{}^T = I_{(n-m) \times (n-m)}$ .

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## 2 LMI approaches to the W- and P-problems

Consider linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  are system state, input and measured output, and  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$  are state, input and output matrices, respectively.

By SOF controls of system (1) we mean the controls of the form  $u = -Ly(t)$  with  $L$  an  $m \times p$  constant matrix. And by SOF stabilization controls of system (1) we mean such SOF controls that stabilize the system or, equivalently, such that  $A - BLC$  is stable.

Related to this problem is the following W- and P-problems<sup>[4]</sup>:

**Definition 1** W-Problem: Given matrices  $A, B, C$ , with  $C$  full row rank the W-problem includes finding, if possible, matrices  $W, M, N$  such that

$$\begin{cases} AW + WA' - BNC - C'N'B' < 0, \\ W > 0, MC = CW. \end{cases} \quad (2)$$

From [4], if  $(W, M, N)$  is a solution of the W-problem, then  $u = -NM^{-1}y$  is an SOF stabilization control of system (1).

**Definition 2** P-problem: Given matrices  $A, B, C$ , with  $B$  full column rank, the P-problem includes finding, if possible, matrices  $P, M, N$  such that

$$\begin{cases} PA + A'P - BNC - C'N'B' < 0, \\ P > 0, BM = PB. \end{cases} \quad (3)$$

Similarly, if  $(P, M, N)$  is a solution of the P-problem, then  $u = -M^{-1}Ny$  is also an SOF stabilization control of system (1).

For the convenience of citation, we introduce the following lemma.

**Lemma 1**<sup>[6]</sup> Suppose  $D \in \mathbb{R}^{n \times q}$  and  $E \in \mathbb{R}^{n \times q}$  are full column rank. Then there exists an  $n \times n$  positive definite matrix  $P$  such that  $PD = E$  if and only if  $D^T E = E^T D > 0$ . Furthermore, all solutions of  $PD = E$  can be expressed as

$$P = E(D^T E)^{-1} E^T + D^{\perp T} X D^{\perp}, \quad (4)$$

where  $X \in \mathbb{R}^{(n-q) \times (n-q)}$  is an arbitrary positive definite matrix.

**Theorem 1** Suppose  $C$  is full row rank. Then W-problem is feasible if and only if there exist matrices  $X > 0, V > 0$  and  $N \in \mathbb{R}^{m \times p}$  such that

$$AC_0 V C_0^T + C_0 V C_0^T A^T + AC^{\perp T} X C^{\perp} +$$

$$C^{\perp T} X C^{\perp} A^T - BNC - C^T N^T B^T < 0, \quad (5)$$

where  $C_0 = C^T (CC^T)^{-1}$ . Furthermore, when (5) holds, system (1) is stabilized by SOF control law  $u = -NCC^T V^{-1}y$ .

**Proof** Suppose that there exist matrices  $X > 0, V > 0$  and  $N \in \mathbb{R}^{m \times p}$  satisfying (5). Let  $M = V(CC^T)^{-1}$ . Then we have

$$CC^T M^T = MCC^T > 0, \quad (6)$$

and

$$AC_0 M C + C^T M^T C_0^T A^T + AC^{\perp T} X C^{\perp} + C^{\perp T} X C^{\perp} A^T - BNC - C^T N^T B^T < 0. \quad (7)$$

From (6) and Lemma 1,  $WC^T = C^T M^T$  has a positive definite matrix solution  $W$ , which can be expressed in the form of

$$W = C_0 M C + C^{\perp T} X C^{\perp}, \quad (8)$$

where  $X$  is a positive definite matrix. Substituting (8) into (7) leads to the first inequality of (2). Thus, the W-problem is feasible.

Conversely, assume that W-problem is feasible. Let  $V = M(CC^T)$ . Then the result can be easily implied in terms of Lemma 1.

This completes the proof.

**Remark 1** The same approach as Theorem 1 can also easily be used to discuss P-problem and similar results can be obtained.

**Remark 2** The presented approach can also easily be used to design SOF stabilization controls for discrete-time linear systems, decentralized output feedback control, output feedback  $H_{\infty}$  control, and other related control problems described, for instance, in [4].

## 3 Application to uncertain systems

In this section, by using the approaches developed in Section 2 we consider the SOF stabilization problem of systems with polytopic uncertainties:

$$\begin{cases} \dot{x} = Ax + Bu \text{ with } (A, B) \in \Omega \triangleq \\ \quad \{(A, B) : (A_1, B_1), \dots, (A_k, B_k)\}, \\ y = Cx, \end{cases} \quad (9)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}^m$  the input and  $y \in \mathbb{R}^p$  the output.

The system (9) actually represents a family of multi-input and multi-output systems, and is very different from the matched uncertain ones, which admits a nominal system and matched uncertainties. To our knowledge, the first progress on the SOF stabilization of system (9) was made by Kar in [5], the following three

