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Brief Paper

The Optimal Regulation of Generalized State-space Systems with Quadratic Cost* †

CHENG ZHAOLIN‡, HONG HUIMIN§ and ZHANG JIFENG||

Key Words—Linear optimal regulator; regulator theory; stability; linear systems; system theory; linear algebra.

Abstract—In this paper, the optimal feedback control for regulating generalized state-space systems with quadratic cost is presented by using an algebraic method, and the weighting matrix Q in the cost is allowed to be positive semi-definite.

1. *Introduction*

SINCE ROSENBROCK (1974) introduced the restricted system equivalence (RSE) for the generalized state-space system, i.e. for the linear system

$$E\dot{x} = Ax + Bu \tag{1}$$

with E being singular and $\det(sE - A) \not\equiv 0$, there has been a lot of research into various problems such as controllability and observability of the system (Campbell, 1980; Yip and Sincovec, 1981; Verghese *et al.*, 1981; Cobb, 1984; also see Lewis, 1986), poles assignment and the elimination of impulsive behavior of the system by state feedback (Cobb, 1981), and optimal regulators with quadratic cost, i.e. the LQ problem (Pandolfi, 1981; Cobb, 1983a; Bender and Laub, 1987a, b), and so on. However, the weighting matrices in the LQ problem investigated by Cobb (1983a) are positive definite. In this paper, the LQ problem is treated algebraically for the general case where Q is allowed to be positive semi-definite, and the problem is transformed into the LQ problem for a regular state-space system by invoking strong stabilizability and strong detectability of system (1). We believe that the algebraic method adopted here is easier than the geometric one used by Cobb (1983a) for engineeers to comprehend. The same problem has also been considered by Bender and Laub (1987a, b). They used Hamiltonian minimization. However, for the infinite-horizon case, stronger conditions are required there than that required in this paper.

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‡ The Department of Mathematics, Shandong University, Jinan, P.R.C.

§ The Department of Control Engineering, Shandong University, Weihai, P.R.C.

|| The Institute of Systems Science, Academia Sinica, Beijing, P.R.C.

2. *Optimal regulator with quadratic cost*

For system (1), with initial state $x(0^-)$ which can be consistent or not (Cobb, 1983b), consider the cost functional

$$J(u, x(0^-)) = \int_{0^-}^{\infty} (x^\tau Qx + u^\tau Ru)\, dt \tag{2}$$

where u is the input, $u \in R^r$, x is the state, $x \in R^n$, Q and R are constant matrices, Q is positive semi-definite and R is positive definite. Let $\mathcal{U}$ be the set of admissible controls in which any admissible control is piecewise sufficiently smooth and makes $J(u, x(0^-))$ finite. Then the LQ problem is to find the optimal control $u^* \in \mathcal{U}$, such that

$$J(u^*, x(0^-)) = \min_{u \in \mathcal{U}} J(u, x(0^-)). \tag{3}$$

Definition 1. System (1) is strongly stabilizable if

$$\operatorname{rank}[sE - A \quad B] = n \tag{4}$$

for any complex s with non-negative real part and $s = \infty$.

Lemma 1. Let T be a non-singular matrix such that

$$T[sE - A \quad B] = \begin{bmatrix} sE_1 - A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{5}$$

where E_1 has full row rank; then equation (4) holds at $s = \infty$ if and only if

$$\operatorname{rank}\begin{bmatrix} E_1 & 0 \\ A_2 & B_2 \end{bmatrix} = n. \tag{6}$$

The proof of the lemma was given by Verghese *et al.* (1979). Equation (4), holding at $s = \infty$, is a necessary and sufficient condition for eliminating impulsive behaviour of system (1) (Cobb, 1981).

Lemma 2. For system (1), there exists a state feedback matrix $K \in R^{r \times n}$ such that the system

$$E\dot{x} = (A + BK)x + Bv \tag{7}$$

is RSE to the system

$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 v \\ 0 = x_2 + B_2 v \end{cases} \tag{8}$$

if and only if equation (4) holds at $s = \infty$, where $x_1 \in R^{n_E}$, $n_E = \operatorname{rank} E$, $x_2 \in R^{n-n_E}$, $A_1 \in R^{n_E \times n_E}$, $B_1 \in R^{n_E \times r}$, $B_2 \in R^{(n-n_E) \times r}$.

By saying that system (7) is RSE to system (8), we mean that there exist two non-singular matrices $F, G \in R^{n \times n}$ such that

$$F[sE - (A + BK) \quad B]\begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} sI_{n_E} - A_1 & 0 & B_1 \\ 0 & -I_{n-n_E} & B_2 \end{bmatrix}.$$

Definition 2. The system

$$\Sigma_0: \begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{9}$$

is strongly detectable if

$$\text{rank}\begin{bmatrix} C \\ sE - A \end{bmatrix} = n \tag{10}$$

for any complex s with non-negative real part and $s = \infty$.

A criterion of testing that equation (10) holds at $s = \infty$ was given by Verghese *et al.* (1981); see Lemma 3.

Lemma 3. Equation (10) holds at $s = \infty$ for system Σ_0 if and only if there are no constant vectors α, $\beta \in R^n$, $\beta \neq 0$, such that

$$\begin{bmatrix} C \\ sE - A \end{bmatrix}\beta = \begin{bmatrix} 0 \\ E\alpha \end{bmatrix}, \quad \text{for any } s \in C. \tag{11}$$

Decompose matrix Q of cost (2) in different ways: $Q = C^\tau C = C_1^\tau C_1$, and denote

$$\Sigma: \begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{12}$$

$$\Sigma_1: \begin{cases} E\dot{x} = AX + Bu \\ y = C_1 x. \end{cases} \tag{13}$$

Then we have the following lemma.

Lemma 4. Σ is strongly detectable if and only if Σ_1 is.

The lemma shows that the strong detectability for system Σ is independent of the decomposition of matrix Q. We omit its proof and proceed directly to the LQ problem for system (1).

Theorem. If system Σ is strongly stabilizable and strongly detectable, then the LQ problem for system (1) has a solution, the optimal control is realized by a linear state feedback when $t > 0$, and the optimal closed loop system is asymptotically stable.

Proof. The basic idea of the proof is to transfer the LQ problem of a generalized state-space system into that of a regular system by means of conceptions of strong stabilizability and strong detectability.

(a) *Reforming* $J(u, x(0^-))$

In view of the strong stabilizability of system Σ, from Lemma 2 we can find a matrix $K_1 \in R^{r \times n}$ such that the system

$$E\dot{x} = (A + BK_1)x + BV \tag{14}$$

is RSE to the system

$$\dot{x}_1 = A_1 x_1 + B_1 v \tag{15}$$

$$0 = x_2 + B_2 v \tag{16}$$

i.e. there exist non-singular matrices $F, G \in R^{n \times n}$ such that

$$x = G\bar{x} \tag{17}$$

$$FEG = \begin{bmatrix} I_{n_E} & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

$$F(A + BK_1)G = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-n_E} \end{bmatrix} \tag{19}$$

$$FB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{20}$$

where $n_E = \text{rank } E$, $B_1 \in R^{n_E \times r}$, $B_2 \in R^{(n-n_E)\times r}$, $A_1 \in R^{n_E \times n_E}$, $\bar{x} = [x_1^\tau x_2^\tau]^\tau$, $x_1 \in R^{n_E}$, $x_2 \in R^{n-n_E}$. Let $x(t)$ be the solution of system (1) driven by $u \in \mathcal{U}$ with initial state $x(0^-)$. Set

$$v(t) = u(t) - K_1 x(t). \tag{21}$$

Then $x(t)$ satisfies (14), since $\bar{x}(t) = G^{-1}x(t)$, then components $x_1(t)$ and $x_2(t)$ of $\bar{x}(t)$ satisfy (15) and (16), respectively. Rewriting $J(u, x(0^-))$ via (16)–(21), we have

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ v \end{bmatrix}^\tau \begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix}^\tau \begin{bmatrix} G^\tau & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_n & K_1^\tau \\ 0 & I_r \end{bmatrix}$$

$$\times \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K_1 & I_r \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} x_1 \\ v \end{bmatrix} \mathrm{d}t. \tag{22}$$

Representing the weighting matrix in (22) by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\tau & M_{22} \end{bmatrix} \tag{23}$$

where $M_{11} \in R^{n_E \times n_E}$, $M_{12} \in R^{n_E \times r}$, $M_{22} \in R^{r \times r}$, we have

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ v \end{bmatrix}^\tau \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\tau & M_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ v \end{bmatrix} \mathrm{d}t. \tag{24}$$

(b) *Showing* M_{22} *positive definite and rewriting cost* (24).

Assume the contrary, i.e. that M_{22} degenerate, then there exists $\alpha_0 \in R^r$, $\alpha_0 \neq 0$, such that $\alpha_0^\tau M_{22}\alpha_0 = 0$. Thus we have

$$[0 \quad \alpha_0^\tau] M [0 \quad \alpha_0^\tau]^\tau = 0 \tag{25}$$

that is,

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I_n & 0 \\ K_1 & I_r \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} = 0.$$

It follows that

$$QG\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} = 0, \quad \alpha_0 = K_1 G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix}. \tag{26}$$

Letting

$$\beta = G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} \tag{27}$$

and noticing that $Q = C^\tau C$, then we obtain

$$C\beta = 0. \tag{28}$$

From (26)–(27), (18)–(20), it follows that

$$F(sE - A)\beta = F(sE - (A + BK_1))G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} + FBK_1\beta$$

$$= \begin{bmatrix} sI_{n_E} - A_1 & 0 \\ 0 & -I_{n-n_E} \end{bmatrix} \begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} + FB\alpha_0 = \begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}. \tag{29}$$

Letting

$$\alpha = G\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix} \tag{30}$$

and noticing (29) and (18), we have

$$(sE - A)\beta = F^{-1}\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}$$

$$= F^{-1}\left\{FEG + \begin{bmatrix} 0 & 0 \\ 0 & I_{n-n_E} \end{bmatrix}\right\} \begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}$$

$$= EG\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix} = E\alpha. \tag{31}$$

Combining (28) with (31) gives

$$\begin{bmatrix} C \\ sE - A \end{bmatrix}\beta = \begin{bmatrix} 0 \\ E\alpha \end{bmatrix}, \quad \text{for any } s \in C. \tag{32}$$

Since system Σ is strongly detectable by assumption, applying Lemma 3 to (32), we obtain $\beta = 0$. Therefore, $\alpha_0 = 0$ from (26)–(27). This contradicts the hypothesis. It follows that M_{22} is positive definite. Now cost (24) can be rewritten as the form

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ M_{22}^{-1}M_{12}^{\tau}x_1 + v \end{bmatrix}^{\tau} \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{12}^{\tau} & 0 \\ 0 & M_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ M_{22}^{-1}M_{12}^{\tau}x_1 + v \end{bmatrix} \mathrm{d}t = \int_{0^-}^{\infty} (x_1^{\tau}M_{11}^{*}x_1 + w^{\tau}M_{22}w)\,\mathrm{d}t \quad (33)$$

where

$$w(t) = M_{22}^{-1}M_{12}^{\tau}x_1(t) + v(t) \quad (34)$$

$$M_{11}^{*} = M_{11} - M_{12}M_{22}^{-1}M_{12}^{\tau} \quad (35)$$

and M_{11}^{*} is positive semi-definite as M is positive semi-definite. Since $J(u, x(0^-)) < \infty$ for any $u \in \mathscr{U}$ and M_{22} is positive definite, we see that $w(t) \in L^2$. Hence $w(t)$ is neither a Dirac-δ function nor its derivatives. Furthermore, since the trajectory $x(t)$ is a linear combination of a Dirac-δ function, its derivatives and piecewise sufficiently smooth functions (Cobb, 1981), we can see that $w(t)$ is piecewise sufficiently smooth. Paying attention to (15)–(17) and (34), we have

$$\dot{x}_1 = (A_1 - B_1M_{22}^{-1}M_{12}^{\tau})x_1 + B_1w, \quad t \geq 0 \quad (36)$$

$$0 = x_2 - B_2M_{22}^{-1}M_{12}^{\tau}x_1 + B_2w, \quad t > 0. \quad (37)$$

Hence $\bar{x}(t)$, then $x(t)$, are also piecewise sufficiently smooth.

(c) *Equivalence of the LQ problems*

Denote by $\mathscr{P}_1$ and $\mathscr{P}_2$ the LQ problems, respectively, for system (1) with cost (2) and for linear system (36) with cost

$$J(w, x_1(0)) = \int_{0}^{\infty} (x_1^{\tau}M_{11}^{*}x_1 + w^{\tau}M_{22}w)\,\mathrm{d}t \quad (38)$$

where $w \in \mathscr{W}$, and $\mathscr{W}$ is the set of piecewise sufficiently smooth functions which make cost (38) finite. Consider $\mathscr{P}_2$ with the initial condition

$$x_1(0) = [I_{n_E} \quad 0]G^{-1}x(0^-). \quad (39)$$

In view of (21), (34) and

$$x_1(t) = [I_{n_E} \quad 0]G^{-1}x(t), \quad (40)$$

the solution $x(t)$ of system (1) driven by an admissible control $u \in \mathscr{U}$ with initial state $x(0^-)$ gives an admissible control $w \in \mathscr{W}$ and a solution $x_1(t)$ of system (36) driven by w with initial state (39). Combining (33) with (38) and noticing that $w(t)$ is piecewise sufficiently smooth, we conclude that

$$J(w, x_1(0)) = J(u, x(0^-)). \quad (41)$$

Hence

$$\min_{u \in \mathscr{U}} J(u, x(0^-)) \geq \min_{w \in \mathscr{W}} J(w, x_1(0)). \quad (42)$$

Inversely, if the solution $x_1(t)$ of system (36) driven by $w \in \mathscr{W}$ with initial state (39) is given, then we can obtain a control function $u(t)$, and a solution $x(t)$ of system (1) driven by this control function with initial state $x(0^-)$ by using the following relations:

$$x(t) = G\bar{x}(t) = G\begin{bmatrix} x_1(t) \\ -B_2(w(t) - M_{22}^{-1}M_{12}^{\tau}x_1(t)) \end{bmatrix}, \quad t > 0 \quad (43)$$

and

$$u(t) = w(t) - M_{22}^{-1}M_{12}^{\tau}x_1(t) + K_1x(t). \quad (44)$$

Here

$$x(0^+) = G\begin{bmatrix} x_1(0) \\ -B_2(w(0) - M_{22}^{-1}M_{12}^{\tau}x_1(0)) \end{bmatrix}. \quad (45)$$

$x(0^+)$ does not equal $x(0^-)$ in general, and

$$x(0^+) - x(0^-) = G\begin{bmatrix} 0 \\ -B_2(w(0) - M_{22}^{-1}M_{12}^{\tau}x_1(0)) - [0 \quad I_{n-n_E}]G^{-1}x(0^-) \end{bmatrix}. \quad (46)$$

A jump may occur to $x(t)$ when t moves from $t = 0^-$ to 0^+; this is because $x(0^-)$ may be inconsistent with $u(t)$. However, $x(t)$ does not have any impulsive behaviour. Hence

$$\int_{0^-}^{0^+} (x^{\tau}Qx + u^{\tau}Ru)\,\mathrm{d}t = 0 \quad (47)$$

and

$$J(u, x(0^-)) = J(w, x_1(0)). \quad (48)$$

Equation (48) shows that $u(t)$ is an admissible control, i.e. $u(t) \in \mathscr{U}$. Thus we have

$$\min_{w \in \mathscr{W}} J(w, x_1(0)) \geq \min_{u \in \mathscr{U}} J(u, x(0^-)). \quad (49)$$

The equivalence of $\mathscr{P}_1$ and $\mathscr{P}_2$ now has been proved by (42) and (49), i.e.

$$\min_{u \in \mathscr{U}} J(u, x(0^-)) = \min_{w \in \mathscr{W}} J(w, x_1(0)) \quad (50)$$

where $x_1(0)$ is defined as in (39). Obviously, problem $\mathscr{P}_2$ is much easier to solve.

(d) *Solution of problem $\mathscr{P}_2$*

Let

$$\Sigma_2 : \begin{cases} \dot{x}_1 = (A_1 - B_1M_{22}^{-1}M_{12}^{\tau})x_1 + B_1w, \\ y = C_1x_1 \end{cases} \quad (51)$$

where C_1 is defined as in $M_{11}^{*} = C_1^{\tau}C_1$. It is clear that stabilizability of system Σ implies that of Σ_2. We show that system Σ_2 is detectable below.

Assume the contrary, i.e. that Σ_2 is not detectable, then there exist a complex s_0 with a non-negative real part, and a nonzero vector $\alpha_1 \in R^{n_E}$ such that

$$\begin{bmatrix} C_1 \\ s_0I_{n_E} - (A_1 - B_1M_{22}^{-1}M_{12}^{\tau}) \end{bmatrix}\alpha_1 = 0. \quad (52)$$

Thus

$$\begin{bmatrix} M_{11}^{*} & 0 \\ 0 & M_{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} = 0 \quad (53)$$

and

$$(s_0I_{n_E} - A_1)\alpha_1 = -B_1M_{22}^{-1}M_{12}^{\tau}\alpha_1. \quad (54)$$

Putting

$$\beta_1 = G\begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 \quad (55)$$

from (53), (35) and (22)–(23), we have

$$Q\beta_1 = 0 \quad (56)$$

and

$$K_1\beta_1 = M_{22}^{-1}M_{12}^{\tau}\alpha_1. \quad (57)$$

From (18)–(20), (54)–(55) and (57), we obtain

$$(s_0E - A)\beta_1 = F^{-1}\left\{F(s_0E - (A + BK_1))G \times \begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 + FBK_1\beta_1\right\} = F^{-1}\left\{\begin{bmatrix} s_0I_{n_E} - A_1 & 0 \\ 0 & -I_{n-n_E} \end{bmatrix} \times \begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}K_1\beta_1\right\} = 0. \quad (58)$$

Combining (58) with (56), and noticing that $Q = C^{\tau}C$, we can see that

$$\begin{bmatrix} C \\ s_0E - A \end{bmatrix}\beta_1 = 0.$$

Since system Σ is assumed to be detectable, $\beta_1 = 0$, and then equation (55) gives $\alpha_1 = 0$. This contradicts the hypothesis. Hence system Σ_2 is detectable.