MEAN FIELD LQG HIERARCHICAL GAMES WITH MULTIPLICATIVE NOISES: A DIRECT APPROACH*

BING-CHANG WANG †, HUANSHUI ZHANG ‡, AND JI-FENG ZHANG §

Abstract. This paper studies open-loop and feedback solutions to linear-quadratic-Gaussian mean field hierarchical games with multiplicative noises by a direct approach. The hierarchical game involves a leader and many followers, where the state and control weight matrices in their costs are not limited to be definite. From variational analysis with mean field approximations, we obtain a set of open-loop controls in terms of solutions to mean field forward-backward stochastic differential equations. By applying the matrix maximum principle, a set of decentralized feedback strategies is constructed. Different from traditional works, a cross term has appeared in derivation due to the appearance of mean field terms. For open-loop and feedback solutions, the corresponding optimal costs of all players are explicitly given in terms of the solutions to two Riccati equations, respectively.

Key words. Stackelberg game, mean field team, social control, forward-backward stochastic differential equation

MSC codes. 49N80, 91A16, 93E03, 93E20

1. Introduction.

1.1. Background and Motivation.

Mean field (MF) games have drawn much attention from various disciplines including control theory, applied mathematics and economics [29], [9], [11], [15]. In an MF game, the impact of each individual is negligible while the effect of the population is significant. The main methodology of MF games is to replace the interactions among agents by population aggregation effect, which structurally models the MF interactions in large population systems. Thus, the high-dimensional multi-agent optimization problem can be transformed into a low-dimensional local optimal control problem for a representative agent [29], [11]. Wide applications have been found in many fields, such as economics [54], [48], smart grid [44], engineering [28] and social sciences [3], [13]. As a classical type of MF models, mean field linear quadratic Gaussian (MF-LQG) games are intensively studied due to their analytical tractability and close connection to practical applications. For works on such kind of problems, readers can refer to [6], [18], [23], [30], [45], [50], [53]. The pioneering work [22] studied ϵ-Nash equilibrium strategies for MF-LQG games with discounted costs based on the Nash certainty equivalence. This approach was then applied to the cases with long run average costs [30] and with Markov jump parameters [50], respectively. For MF games with major players, the works [21], [12] considered continuous-time LQG games with complete and partial information; [51] investigated discrete-time LQG games with random parameters; [10] and [41] focused on the nonlinear case.

In contrast to the above models, the hierarchical (Stackelberg) game involves a leader-follower structure. Consider a hierarchical game with two layers. One layer of players are defined as leaders with a dominant position and the other players is defined as followers with a subordinate position. The leader has the priority to give a strategy first and then followers seek strategies to minimize their costs with response to the strategies of leaders. According to followers' optimal response, leaders will choose strategies to minimize their costs. Hierarchical games have been widely investigated in the literature (see e.g. [42], [57], [7], [56], [19]). Recently, MF hierarchical games have attracted great research interest [8], [52], [33], [5], [55]. The work [8] considered MF Stackelberg games with delayed instructions. [52] studied discrete-time hierarchical MF games with tracking-type costs and gave the ε-Stackelberg equilibrium. Authors in [33] investigated continuous-time MF-LQG Stackelberg games by the fixed-point method, and they asserted that "complexity brought by coupling among leader and

*Submitted to the editors DATE.

Funding: This work was funded by the National Natural Science Foundation of China under Grants 62122043 and 62433020, and the Innovative Research Groups of National Natural Science Foundation of China under Grant 61821004.

†School of Control Science and Engineering, Shandong University, Jinan, China (bcwang@sdu.edu.cn).

‡College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao, China (hszhang@sdu.edu.cn).

§Corresponding author. School of Automation and Electrical Engineering, Zhongyuan University of Technology, Zhengzhou 450007, Henan Province, China; and Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (jif@iss.ac.cn).

followers makes the use of direct approach almost impossible". This work is further generalized to the jump diffusion model [32]. Besides, [55] investigated feedback strategies of MF Stackelberg games by solving the master equations.

Different from noncooperative games, social optimization is a joint decision problem where all players work cooperatively to optimize the social cost. This is a typical class of team decision problem [17]. Authors in [23] studied social optima in the MF-LQG control, and provided an asymptotic team-optimal solution, which is extended to the case of mixed games in [24]. The work [53] investigated the MF social optimal problem where the jump parameter appears as a common source of randomness. More investigation can be found in [2] for team-optimal control with finite population and partial information, [39] for dynamic collective choice by finding social optima, [40] for stochastic dynamic teams and their MF limit, [46], [20] for MF teams with uncertainty in drift and volatility, and [34] for social control applications in economics. Besides, see [47] for value-iteration learning in ergodic MF-LQG social control, and [25] for online policy iteration in MF Pareto optimal control.

Normally, there are two routes to solve MF games and teams. One is called the fixed-point approach [22, 23, 9, 15], which starts by applying MF approximation and constructing a fixed-point equation. A set of decentralized strategies can be designed by tackling the fixed-point equation together with the optimal response of a representative player. In general, the fixed-point equation is difficult to solve. In addition, when solving the team problem by the fixed-point approach, an additional variable (called social impact [23, 53]) needs to be introduced. This leads to a drastic increase of computational complexity for MF teams with *multiplicative noises* [38], [16]. Another route is called the direct approach [26, 29, 49], which takes a path from finite-population to infinite-population systems. By decoupling the Hamiltonian system for N-player, one can obtain a centralized strategy which explicitly relies on the state of a player and population state average. Applying MF approximations, the decentralized control can be constructed. By the direct approach, the resulting control is neat and less computation is required, particularly for team problems [49].

1.2. Contribution and Novelty. This paper considers MF-LQG hierarchical games with a leader and many followers, where the state and control weight matrices in their costs are allowed to not be definite. The leader first give his strategy and then all the followers cooperate to optimize the *social cost*, the sum of their individual costs. For instance, consider an example of macroeconomic regulation, where the regulator/government is the leader, and local authorities are followers [36]. The state of the leader appears in both dynamics and cost of each follower. It shows that the dynamics and costs of followers are directly influenced by the behavior of the leader. Different from [24] and [33], our model involves population state average $x^{(N)}$ in both drift and diffusion terms in followers' dynamics. Until now, most previous works focused on open-loop solutions of MF leader-follower games, and only a few works were on feedback solutions. Furthermore, the relationship between open-loop and feedback solutions is still unclear.

In this paper, we study systematically open-loop and feedback solutions to MF hierarchical games by the direct approach. The open-loop solution starts with solving a centralized social control problem for followers, and obtaining a system of high-dimensional forward-backward stochastic differential equations (FBSDEs). By MF approximations, a set of open-loop controls of followers is designed in terms of an MF FBSDE. After applying followers' strategies, we derive necessary and sufficient conditions for the solvability of the leader's problem, and then obtain the feedback representation of the open-loop control by decoupling an FBSDE. From perturbation analysis, the proposed strategy is shown to be an $(\varepsilon_1, \varepsilon_2)$-Stackelberg equilibrium. Furthermore, we obtain the optimal costs of players in terms of the solutions of Riccati equations. Next, the feedback solution is investigated for MF Stackelberg games. Different from the open-loop solution, we presume that the leader has a strategy with the feedback form. With leader's feedback gain fixed, we obtain the feedback strategies of followers by decoupling high-dimensional FBSDEs. Applying the matrix maximum principle with MF approximations, we solve the optimal control problem for the leader, and then construct a set of decentralized feedback strategies for all players. By the technique of completing the square, we show that the proposed decentralized strategy is a feedback $(\varepsilon_1, \varepsilon_2)$-Stackelberg equilibrium and give an explicit form of the corresponding costs of players.

The main contributions of the paper are listed as follows.

- By adopting a direct approach, the open-loop and feedback solutions to leader-follower MF

games with multiplicative noises are obtained. Different from the fixed-point approach, *no additional terms* need to be introduced when MF social control problem is solved for followers.
- By variational analysis with MF approximations, an open-loop asymptotic Stackelberg equilibrium is given in terms of MF FBSDEs, which can be implemented offline.
- By decoupling high-dimensional FBSDEs and applying the matrix maximum principle, a set of decentralized feedback strategies is constructed. Different from traditional works, a cross term is introduced for deriving feedback strategies due to the appearance of MF coupling.

1.3. Organization and Notation. The paper is organized as follows. In Section 2, we formulate the problem of MF-LQG leader-follower games with multiplicative noises. In Section 3, we first obtain a set of open-loop control laws in terms of MF FBSDEs, and give its feedback representation by virtue of Riccati equations. In Section 4, we design the feedback strategies of MF Stakelberg games and provide the corresponding costs of all players. In Section 5, we give a numerical example to demonstrate the performance of different solutions. Section 6 concludes the paper.

Notation: Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete filtered probability space augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. $|\cdot|$ is the standard Euclidean norm and $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product. For a vector z and a matrix Q, $\|z\|_Q^2 = z^TQz$; $Q > 0$ ($Q \geq 0$) means that the matrix Q is positive definite (positive semi-definite). $Q^\dagger$ is the Moore-Penrose pseudoinverse[1] of the matrix Q, $\mathcal{R}(Q)$ denotes the range of a matrix (or an operator) Q. Let $C(0, T; \mathbb{R}^{m\times n})$ be the set of $\mathbb{R}^{m\times n}$-valued continuous function and $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ be the set of all $\{\mathcal{F}\}_{t\geq 0}$-adapted $\mathbb{R}^m$-valued processes $x(\cdot)$ such that $\|x(t)\|^2_{L^2} =: \mathbb{E}\int_0^T \|x(t)\|^2dt < \infty$. For a symmetric matrix $S \geq 0$, the quadratic form x^TSx is defined as $\|x\|^2_S$, where x^T is the transpose of x.

2. Problem Formulation. Consider a large-population system with a leader and N followers. The state processes of a leader and N followers satisfy the following stochastic differential equations:

$$
\begin{cases}
dx_0(t) = [A_0x_0(t) + B_0u_0(t)]dt + [C_0x_0(t) + D_0u_0(t)]dW_0(t),\\
dx_i(t) = [Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + Fx_0(t)]dt + [Cx_i(t) + Du_i(t) + \bar{G}x^{(N)}(t) + \bar{F}x_0(t)]dW_i(t),\\
x_0(0) = \xi_0, \quad x_i(0) = \xi_i, \quad i = 1, 2, \cdots, N,
\end{cases}
\tag{2.1}
$$

where $x_0 \in \mathbb{R}^{n_0}, u_0 \in \mathbb{R}^{m_0}$ are the state and input of the leader, and $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input of the ith follower, $i = 1, \cdots, N$, respectively. $x^{(N)}(t) \triangleq \frac{1}{N}\sum_{i=1}^N x_i(t)$ is the state average of all the followers. $\{W_0(\cdot), W_1(\cdot), \cdots, W_N(\cdot)\}$ are a sequence of independent d-dimensional standard Brownian motions defined on the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. Let $\mathcal{F}_t = \sigma(\xi_0, \xi_i, W_0(s), W_i(s), 0 \leq s \leq t, i = 1, \cdots, N))$. Denote $\mathcal{F}^0_t = \sigma(\xi_0, W_0(s), 0 \leq s \leq t)$ and $\mathcal{F}^i_t = \sigma(\xi_0, \xi_i, W_0(s), W_i(s), 0 \leq s \leq t)$ for $i = 1, \cdots, N$. The admissible control set for the leader is defined as follows: $\mathcal{U}_0 = \big\{u_0|u_0(t) \in L^2_{\mathcal{F}^0_t}(0, T; \mathbb{R}^m)\big\}$. The admissible decentralized control set for all the followers is defined by

$$
\mathcal{U}_d = \Big\{(u_1, \cdots, u_N)|u_i(t) \in L^2_{\mathcal{F}^i_t}(0, T; \mathbb{R}^m), i = 1, \cdots, N\Big\}.
$$

Also, the centralized control set for followers is given by

$$
\mathcal{U}_c = \Big\{(u_1, \cdots, u_N)|u_i(t) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}^m), i = 1, \cdots, N\Big\}.
$$

For the leader, the cost functional is defined by

$$
J_0(u_0, u) = \mathbb{E}\int_0^T \big[|x_0(t) - \Gamma_0x^{(N)}(t)|^2_{Q_0} + |u_0(t)|^2_{R_0}\big]dt + \mathbb{E}\big[|x_0(T) - \hat{\Gamma}_0x^{(N)}(T)|^2_{H_0}\big],
\tag{2.2}
$$

where Q_0, R_0 and H_0 are symmetric matrices with proper dimensions, and $u = (u_1, \cdots, u_N)$. For the ith follower, the cost functional is defined by

$$
J_i(u_0, u) = \mathbb{E}\int_0^T \big[|x_i(t) - \Gamma x^{(N)}(t) - \Gamma_1x_0(t)|^2_Q + |u_i(t)|^2_R\big]dt + \mathbb{E}\big[|x_i(T) - \hat{\Gamma}x^{(N)}(T) - \hat{\Gamma}_1x_0(T)|^2_H\big],
\tag{2.3}
$$

[1] $Q^\dagger$ is a unique matrix satisfying $QQ^\dagger Q = Q^\dagger, Q^\dagger QQ^\dagger = Q, (Q^\dagger Q)^T = Q^\dagger Q$, and $(QQ^\dagger)^T = QQ^\dagger$. See [35] for more properties of pseudoinverse.

where Q, R and H are symmetric matrices with proper dimensions. All the followers cooperate to minimize their social cost functional, denoted by

$$J_{\rm soc}^{(N)}(u_0, u) = \frac{1}{N}\sum_{i=1}^{N} J_i(u_0, u). \tag{2.4}$$

Now we make the following assumption.

(**A1**) $\{x_i(0)\}$ and $W_i(t), i = 1, 2, \cdots, N$ are independent of each other. $\mathbb{E}x_0(0) = \bar{\xi}_0$ and $\mathbb{E}x_i(0) = \bar{\xi}$, $i = 1, \cdots, N$. There exists a constant c_0 such that $\sup_{i=1,2,\cdots,N} \mathbb{E}|x_i(0)|^2 \le c_0$, where c_0 is independent of N.

We next discuss the decision hierarchy of the Stackelberg game. The leader holds a dominant position in the sense that it first announces its strategy u_0, and enforces on followers. The N followers then respond by cooperatively optimizing their social cost (2.4) under the leader's strategy. In this process, the leader takes into account of the rational reactions of followers.

Due to accessible information restriction and high computational complexity, one generally is not able to attain centralized Stackelberg equilibria, but only achieve asymptotic Stackelberg equilibria under decentralized information patterns.

We now introduce the definition of the open-loop (ϵ_1, ϵ_2)-Stackelberg equilibrium. From now on, the notation of time t may be suppressed if necessary.

Definition 2.1. *A set of control laws $(u_0^*, u_1^*, \cdots, u_N^*)$ is an open-loop (ϵ_1, ϵ_2)-Stackelberg equilibrium if the following hold:*

(i) When the leader announces a strategy $u_0^*(\cdot) \in \mathcal{U}_0$ over $[0, T]$, $u^* = (u_1^*, \cdots, u_N^*)$ attains an ϵ_1-optimal response, i.e.,

$$J_{\rm soc}^{(N)}(u_0^*, u^*) \le J_{\rm soc}^{(N)}(u_0^*, u) + \epsilon_1, \text{ for any } u \in \mathcal{U}_c,$$

(ii) For any $u_0 \in \mathcal{U}_0$, $J_0(u_0^*, u^*(u_0^*)) \le J_0(u_0, u(u_0)) + \epsilon_2$, where u^* and u are ϵ_1-optimal responses to strategies u_0^* and u_0, respectively.

Inspired by [33, 26, 49], we consider feedback strategies with the following form:

$$\begin{cases} u_0 = P_0 x_0 + \bar{P}\bar{x}, \\ u_i = \hat{K} x_i + \bar{K}\bar{x} + K_0 x_0, \ i = 1, \cdots, N \end{cases} \tag{2.5}$$

where $P_0, \bar{P}, \hat{K}, \bar{K}, K_0 \in L_2(0, T; \mathbb{R}^{n\times n})$; x_0, x_i and $\bar{x}$ satisfy

$$\begin{cases} dx_0 = [A_0 x_0 + B_0(P_0 x_0 + \bar{P}\bar{x})]dt + [C_0 x_0 + D_0(P_0 x_0 + \bar{P}\bar{x})]dW_0, \\ dx_i = [A x_i + B(\hat{K} x_i + \bar{K}\bar{x} + K_0 x_0) + G x^{(N)} + F x_0]dt \\ \qquad + [C x_i + D(\hat{K} x_i + \bar{K}\bar{x} + K_0 x_0) + \bar{G} x^{(N)} + \bar{F} x_0]dW_i, \\ d\bar{x} = \big\{[A + G + B(\hat{K} + \bar{K})]\bar{x} + (F + BK_0)x_0\big\}dt, \\ x_0(0) = \xi_0, \ x_i(0) = \xi_i, \ i = 1, 2, \cdots, N, \ \bar{x}(0) = \bar{\xi}. \end{cases} \tag{2.6}$$

In the above, $\bar{x} = \mathbb{E}[x_i|\mathcal{F}_t^0]$ is an approximation of $x^{(N)}$ for sufficiently large N.

We now introduce the definition of the feedback (ϵ_1, ϵ_2)-Stackelberg equilibrium.

Definition 2.2. *A set of strategies $(\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_N)$ is a feedback (ϵ_1, ϵ_2)-Stackelberg equilibrium if the following hold:*

(i) When the leader announces a strategy $\hat{u}_0 = P_0 x_0 + \bar{P}\bar{x}$ at time t, $\hat{u} = (\hat{u}_1, \cdots, \hat{u}_N)$ attains an ϵ_1-optimal feedback response, i.e.,

$$J_{\rm soc}^{(N)}(\hat{u}_0, \hat{u}) \le J_{\rm soc}^{(N)}(\hat{u}_0, u) + \epsilon_1, \text{ for any } u \in \mathcal{U}_c,$$

where both $\hat{u}_i$ and u_i have the form $\hat{K} x_i + \bar{K}\bar{x} + K_0 x_0$, $i = 1, \ldots N$;

(ii) For any $u_0 \in \mathcal{U}_0$, $J_0(\hat{u}_0(\hat{u}_0), \hat{u}) \le J_0(u_0, u(u_0)) + \epsilon_2$, where u_0 has the form $P_0 x_0 + \bar{P}\bar{x}$; $\hat{u}$ and u are ϵ_1-optimal feedback responses to strategies $\hat{u}_0$ and u_0, respectively.

3. Open-loop Solutions to Leader-Follower MF Games.

3.1. The MF Social Control Problem for N Followers.

Denote

$$Q_\Gamma \triangleq Q\Gamma + \Gamma^T Q - \Gamma^T Q\Gamma,\ H_{\hat\Gamma} \triangleq H\hat\Gamma + \hat\Gamma^T H - \hat\Gamma^T H\hat\Gamma,$$
$$Q_{\Gamma_1} \triangleq (I-\Gamma)^T Q\Gamma_1,\ H_{\hat\Gamma_1} \triangleq (I-\hat\Gamma)^T H\hat\Gamma_1.$$

Suppose u_0 is fixed. We now consider the following social control problem for N followers.

(P1): minimize J_{soc} over $u \in \mathcal{U}_c$, where

$$J_{\text{soc}}^{(N)}(u) = \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \Big[\big|x_i - \Gamma x^{(N)} - \Gamma_1 x_0\big|_Q^2 + |u_i|_R^2\Big]dt + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\big[|x_i(T) - \hat\Gamma x^{(N)}(T) - \hat\Gamma_1 x_0(T)|_H^2\big]$$

By examining the social cost variation, we obtain the optimal control laws for N followers. The proof is similar to that of Theorem [49], and hence omitted here.

THEOREM 3.1. *Problem (P1) admits an optimal control if and only if J_{soc}^N is convex in u and the following system of FBSDEs admits a set of adapted solutions $\{x_i, p_i, q_i^j, i, j = 1, \cdots, N\}$:*

$$(3.1)\quad \begin{cases} dx_i = (Ax_i + B\check u_i + Gx^{(N)} + Fx_0)dt + (Cx_i + D\check u_i + \bar G x^{(N)} + \bar F x_0)dW_i, \\ dp_i = -\big(A^T p_i + G^T p^{(N)} + C^T q_i^i + \bar G^T q^{(N)} + Qx_i - Q_\Gamma x^{(N)} - Q_{\Gamma_1} x_0\big)dt + \displaystyle\sum_{j=0}^N q_i^j dW_j, \\ x_i(0) = \xi_i,\ i = 1, \cdots, N,\ p_i(T) = Hx_i(T) - H_{\hat\Gamma} x^{(N)}(T) - H_{\hat\Gamma_1} x_0(T), \end{cases}$$

where $p^{(N)} = \frac{1}{N}\sum_{j=1}^N p_j$, $q^{(N)} = \frac{1}{N}\sum_{j=1}^N q_j^j$, and the optimal control laws of followers $\check u_i$ satisfy $R\check u_i + B^T p_i + D^T q_i^i = 0,\ i = 1, \cdots, N.$

Indeed, if $J_{\text{soc}}^{(N)}$ is uniformly convex in u, then Problem (P1) admits an optimal control necessarily [59]. For further existence analysis, we assume

(**A2**) $J_{\text{soc}}^{(N)}$ is uniformly convex in u.

Denote $\mathbb{E}_{\mathcal{F}^0}[\cdot] \triangleq \mathbb{E}[\cdot|\mathcal{F}_t^0]$. Letting $N \to \infty$, by the law of large numbers, we can approximate $\check x_i$, $\check p_i$ in (3.1) by $\bar x_i$, $\bar p_i$, $i = 1, \cdots, N$, which satisfy

$$(3.2)\quad \begin{cases} d\bar x_i = (A\bar x_i + Bu_i^* + G\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + Fx_0)dt + (C\bar x_i + Du_i^* + \bar G\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \bar F x_0)dW_i, \\ d\bar p_i = -\big(A^T\bar p_i + G^T\mathbb{E}_{\mathcal{F}^0}[\bar p_i] + C^T\bar q_i^i + \bar G^T\mathbb{E}_{\mathcal{F}^0}[\bar q_i^i] + Q\bar x_i - Q_\Gamma\mathbb{E}_{\mathcal{F}^0}[\bar x_i] - Q_{\Gamma_1}x_0\big)dt \\ \qquad + \bar q_i^i dW_i + \bar q_i^0 dW_0, \\ \bar x_i(0) = \xi_i,\ i = 1, \cdots, N,\ \bar p_i(T) = H\bar x_i(T) - H_{\hat\Gamma}\mathbb{E}_{\mathcal{F}^0}[\bar x_i(T)] - H_{\hat\Gamma_1}x_0(T), \end{cases}$$

with the decentralized control u_i^* satisfying

$$(3.3)\quad Ru_i^* + B^T\bar p_i + D^T\bar q_i^i = 0,\ i = 1, \cdots, N.$$

We now use the idea inspired by [59], [49], [37] to decouple the FBSDE (3.2). Let $\bar p_i = P\bar x_i + K\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \varphi,\ i = 1, \cdots, N$. Then, we have

$$(3.4)\quad \begin{aligned} d\bar p_i =& \dot P\bar x_i dt + d\varphi + P\Big[\big(A\bar x_i + B\bar u_i + G\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + Fx_0\big)dt + (C\bar x_i + D\bar u_i + \bar G\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \bar F x_0)dW_i\Big] \\ &+ \dot K\mathbb{E}_{\mathcal{F}^0}[\bar x_i]dt + K\big[(A+G)\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + B\mathbb{E}_{\mathcal{F}^0}[\bar u_i] + Fx_0\big]dt \\ =& -\Big[A^T(P\bar x_i + K\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \varphi) + C^T[\bar q_i^i] + G^T((P+K)\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \varphi) + \bar G^T\mathbb{E}_{\mathcal{F}^0}[\bar q_i^i] \\ &+ Q\bar x_i - Q_\Gamma\mathbb{E}_{\mathcal{F}^0}[\bar x_i] - Q_{\Gamma_1}x_0\Big]dt + \bar q_i^i dW_i + \bar q_i^0 dW_0, \end{aligned}$$

which implies

$$(3.5)\quad \bar q_i^i = P(C\bar x_i + D\bar u_i + \bar G\mathbb{E}_{\mathcal{F}^0}[\bar x_i] + \bar F x_0),\ i = 1, \cdots, N.$$

This together with (3.3) leads to

$$Ru_i^* + B^T(P\bar{x}_i + K\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i] + \varphi) + D^T P(C\bar{x}_i + D\bar{u}_i + \bar{G}\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i] + \bar{F}x_0) = 0.$$

Let $\Upsilon \triangleq R + D^T PD$. If $\mathcal{R}(B^T) \cup \mathcal{R}(D^T P) \subseteq \mathcal{R}(\Upsilon)$. Then, we have

$$u_i^* = -\Upsilon^\dagger\big[\big(B^T P + D^T PC\big)\bar{x}_i + \big(B^T K + D^T P\bar{G}\big)\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i] + B^T\varphi + D^T P\bar{F}x_0\big]. \tag{3.6}$$

This together with (3.4) gives

$$\dot{P} + A^T P + PA + C^T PC + Q - \big(B^T P + D^T PC\big)^T\Upsilon^\dagger\big(B^T P + D^T PC\big) = 0, P(T) = H, \tag{3.7}$$

$$\begin{aligned}\dot{K} &+ (A+G)^T K + K(A+G) + G^T P + PG - Q_\Gamma + C^T P\bar{G} + \bar{G}^T P(C+\bar{G})\\ &- (B^T P + D^T PC)^T\Upsilon^\dagger(B^T K + D^T P\bar{G}) - (B^T K + D^T PG)^T\Upsilon^\dagger(B^T P + D^T PC)\\ &- (B^T K + D^T P\bar{G})^T\Upsilon^\dagger(B^T K + D^T P\bar{G}) = 0,\ K(T) = -H_{\hat{\Gamma}},\end{aligned} \tag{3.8}$$

$$\begin{aligned} d\varphi &+ \Big\{\big[A + G - B\Upsilon^\dagger\big(B^T(P+K) + D^T P(C+\bar{G})\big)\big]^T\varphi + \big[(P+K)F_B\\ &+ (C+\bar{G})^T P\bar{F}_D - Q_{\Gamma_1}\big]x_0\Big\}dt - q_i^0 dW_0 = 0,\ \varphi(T) = -H_{\hat{\Gamma}_1}^T x_0(T),\end{aligned} \tag{3.9}$$

where $F_B \triangleq F - B\Upsilon^\dagger D^T P\bar{F}$ and $\bar{F}_D \triangleq \bar{F} - D\Upsilon^\dagger D^T P\bar{F}$. We assume

(A3) Equations (3.7)-(3.9) admit a set of solution (P, K, φ) such that $\Upsilon \geq 0$, and

$$\mathcal{R}(B^T) \cup \mathcal{R}(D^T P) \subseteq \mathcal{R}(\Upsilon). \tag{3.10}$$

Let $\Pi = P + K$. Then, Π satisfies

$$\begin{aligned}\dot{\Pi} &+ (A+G)^T\Pi + \Pi(A+G) - \big[B^T\Pi + D^T P(C+G)\big]^T\Upsilon^\dagger\big[B^T\Pi + D^T P(C+G)\big]\\ &+ (C+G)^T P(C+G) + Q - Q_\Gamma = 0,\ \Pi(T) = H - H_{\hat{\Gamma}}.\end{aligned} \tag{3.11}$$

Note that if $Q \geq 0$ and $H \geq 0$, then $Q - Q_\Gamma = (I-\Gamma)^T Q(I-\Gamma) \geq 0$ and $H - H_{\hat{\Gamma}} \geq 0$. Thus, when $Q \geq 0$, $R > 0$ and $H \geq 0$, (3.7) and (3.11) admit a unique solution, respectively. This implies (3.8) has a unique solution, which further gives (A3).

From the above discussion, we have the following result.

PROPOSITION 3.2. *Under (A3), the decentralized control given by (3.3) has a feedback representation (3.6).*

Applying (3.6) into (3.2), we obtain that $\bar{x} = \mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]$ satisfies

$$d\bar{x} = \big[\big(A + G - B\Upsilon^\dagger B^T\Pi - B\Upsilon^\dagger D^T P(C+\bar{G})\big)\bar{x} - B\Upsilon^\dagger B^T\varphi + (F - B\Upsilon^\dagger DP\bar{F})x_0\big]dt. \tag{3.12}$$

3.2. Optimization for the Leader. Denote $\bar{A} \triangleq A - B\Upsilon^\dagger(B^T P + D^T PC)$, and $\bar{C} \triangleq C - D\Upsilon^\dagger(B^T P + D^T PC)$. After applying the control laws of followers in (3.6), we have the following optimal control problem for the leader.

(P2): minimize $J_0(u_0, u^*(u_0))$ over $u_0 \in L^2_{\mathcal{F}_t}(0,T;\mathbb{R}^m)$, where

$$J_0(u_0, u^*(u_0)) = \mathbb{E}\int_0^T \big[|x_0 - \Gamma_0 x_*^{(N)}|^2_{Q_0} + |u_0|^2_{R_0}\big]dt + \mathbb{E}\big[|x_0(T) - \hat{\Gamma}_0 x_*^{(N)}(T)|^2_{H_0}\big],$$

$$dx_0 = (A_0x_0 + B_0u_0)dt + (C_0x_0 + D_0u_0)dW_0, 1x_0(0) = \xi_0, \tag{3.13}$$

$$\begin{aligned} dx_i^* &= \big[Ax_i^* + Gx_*^{(N)} - B\Upsilon^\dagger\big((B^T P + D^T PC)\bar{x}_i + (B^T K + D^T P\bar{G})\bar{x} + B^T\varphi\big) + F_B x_0\big]dt\\ &\quad + \big[Cx_i^* + \bar{G}x_*^{(N)} - D\Upsilon^\dagger\big((B^T P + D^T PC)\bar{x}_i + (B^T K + D^T P\bar{G})\bar{x} + B\varphi\big) + \bar{F}_D x_0\big]dW_i,\\ &\quad x_i^*(0) = \xi_i,\end{aligned}$$

$$\begin{aligned} d\varphi &= -\Big\{\big[\bar{A} + G - B\Upsilon^\dagger\big(B^T K + D^T P\bar{G}\big)\big]^T\varphi + \big[(P+K)F_B + (C+\bar{G})^T P\bar{F}_D + (\Gamma - I)^T Q\Gamma_1\big]x_0\Big\}dt\\ &\quad + q_i^0 dW_0, \varphi(T) = (\hat{\Gamma} - I)^T H\hat{\Gamma}_1 x_0(T),\end{aligned} \tag{3.14}$$

where x_i^* is the realized state under the control $u_i^*, i=1,\cdots,N$, and $x_*^{(N)}=\frac{1}{N}\sum_{i=1}^N x_i^*$. From (3.14), we have

$$dx_*^{(N)} =\big[(A+G)x_*^{(N)} - B\Upsilon^\dagger\big((B^TP+D^TPC)\bar{x}^{(N)}+(B^TK+D^TP\bar{G})\bar{x}+B^T\varphi\big)+F_Bx_0\big]dt$$
$$+\frac{1}{N}\sum_{i=1}^N\big[Cx_i^*+\bar{G}x_*^{(N)}-D\Upsilon^\dagger\big((B^TP+D^TPC)\bar{x}_i+(B^TK+D^TP\bar{G})\bar{x}+B\varphi\big)+\bar{F}_Dx_0\big]dW_i,$$
$$x_*^{(N)}(0)=\frac{1}{N}\sum_{i=1}^N\xi_i,$$

where $\bar{x}^{(N)}=\frac{1}{N}\sum_{i=1}^N\bar{x}_i$. Note that $\{W_i\}$ are independent Wiener processes and $\{x_i(0)\}$ are independent r.v.s. For the large population case, it is plausible to replace $\bar{x}^{(N)}$, $x_*^{(N)}$ by $\bar{x}$, which evolves from (3.12). Then we have the limiting optimal control problem for the leader.

(P2′): minimize $\bar{J}_0(u_0,u^*(u_0))$ over $u_0\in\mathcal{U}_0$, where

$$\bar{J}_0(u_0,u^*(u_0)) =\mathbb{E}\int_0^T\big[|x_0-\Gamma_0\bar{x}|^2_{Q_0}+|u_0|^2_{R_0}\big]dt+\mathbb{E}\big[|x_0-\hat{\Gamma}_0\bar{x}(T)|^2_{H_0}\big], \tag{3.15}$$

subject to

$$\begin{cases} dx_0 =(A_0x_0+B_0u_0)dt+(C_0x_0+D_0u_0)dW_0,\ x_0(0)=\xi_0,\\ d\bar{x} =\big[(\bar{A}+\hat{G})\bar{x}-B\Upsilon^\dagger B^T\varphi+(F-B\Upsilon^\dagger D^TP\bar{F})x_0\big]dt,\ \bar{x}(0)=\bar{\xi},\\ d\varphi =-\Big\{\big(\bar{A}+\hat{G}\big)^T\varphi+\big[(P+K)F_B+(C+\bar{G})^TP\bar{F}_D+(\Gamma-I)^TQ\Gamma_1\big]x_0\Big\}dt\\ \qquad +q_i^0dW_0,\ \varphi(T)=(\hat{\Gamma}-I)^TH\hat{\Gamma}_1x_0(T). \end{cases} \tag{3.16}$$

with $\hat{G}\triangleq G-B\Upsilon^\dagger(B^TK+D^TP\bar{G})$.

We first provide the condition under which Problem (P2′) is convex. The proof is similar to [18], [49], and so omitted here.

Lemma 3.3. *$\bar{J}_0(u_0,u^*(u_0))$ is convex in u_0 if and only if $\bar{J}_0^0(u_0,u^*(u_0))\geq 0$, where*

$$\bar{J}_0^0(u_0,u^*) =\mathbb{E}\int_0^T\big[|x_0^0-\Gamma_0\bar{x}^0|^2_{Q_0}+|u_0|^2_{R_0}\big]dt+\mathbb{E}\big[|x_0^0(T)-\hat{\Gamma}_0\bar{x}^0(T)|^2_{H_0}\big],$$

subject to

$$\begin{cases} dx_0^0 =(A_0x_0^0+B_0u_0)dt+(C_0x_0^0+D_0u_0)dW_0,\ x_0^0(0)=0,\\ d\bar{x}^0 =[(\bar{A}+\hat{G})\bar{x}^0-B\Upsilon^\dagger B^T\varphi^0+(F-B\Upsilon^\dagger D^TP\bar{F})x_0^0]dt,\ \bar{x}^0(0)=0,\\ d\varphi^0 =-\Big\{\big(\bar{A}+\hat{G}\big)^T\varphi^0+\big[(P+K)F_B+(C+\bar{G})^TP\bar{F}_D+(\Gamma-I)^TQ\Gamma_1\big]x_0^0\Big\}dt\\ \qquad +q_i^{0,0}dW_0,\ \varphi^0(T)=(\hat{\Gamma}-I)^TH\hat{\Gamma}_1x_0^0(T). \end{cases} \tag{3.17}$$

We now give the following maximum principle for (P2′).

Theorem 3.4. *Under (A1)-(A3), Problem (P2′) admits an optimal control u_0^* if and only if $\bar{J}_0(u_0,u^*(u_0))$ is convex in u_0, and the following FBSDE*

$$\begin{cases} dy_0 =-\big\{A_0^Ty_0+C_0^T\beta_0+(F-B\Upsilon^\dagger D^TP\bar{F})^T\bar{y}+\big[(P+K)F_B+(C+\bar{G})^TP\bar{F}_D-Q_{\Gamma_1}\big]^T\psi\\ \qquad +Q_0(x_0^*-\Gamma_0\bar{x}^*)\big\}dt+\beta_0dW_0, y_0(T)=H_0(x_0(T)-\hat{\Gamma}_0\bar{x}^*(T))-H_{\hat{\Gamma}_1}^T\psi(T),\\ d\bar{y} =-[(\bar{A}+\hat{G})^T\bar{y}-\Gamma_0^TQ_0(x_0^*-\Gamma_0\bar{x}^*)]+\bar{\beta}dW_0, \bar{y}(T)=-\hat{\Gamma}_0^TH_0(x_0^*(T)-\hat{\Gamma}_0\bar{x}^*(T)),\\ d\psi =\big[(\bar{A}+\hat{G})\psi-B\Upsilon^\dagger B^T\bar{y}\big]dt,\ \psi(0)=0 \end{cases} \tag{3.18}$$

has a solution such that u_0^* satisfies $R_0u_0^*+B_0^Ty_0+D_0^T\beta_0=0$.

Proof. Suppose $\{u_0^*\}$ is a candidate of the optimal control of Problem (P2′). Let x_0^* and $\bar{x}^*$ be the leader's state and followers' average effect under the control $\{u_0^*\}$. Note that

$$\bar{J}_0(u_0^*+\theta u_0, u(u_0^*+\theta u_0)) - \bar{J}_0(u_0^*, u^*(u_0^*)) = 2\theta I_1 + \theta^2 I_2, \tag{3.19}$$

where

$$I_1 = \mathbb{E}\int_0^T \big[\langle Q_0(x_0^*-\Gamma_0\bar{x}_*), x_0^0-\Gamma_0\bar{x}^0\rangle + \langle u_0^*, R_0u_0\rangle\big]dt + \mathbb{E}\big[\langle H_0(x_0^*(T)-\hat{\Gamma}_0\bar{x}^*(T)), x_0^0(T)-\hat{\Gamma}_0\bar{x}^0(T)\rangle\big], \tag{3.20}$$

$$I_2 = \mathbb{E}\int_0^T \big[|x_0^0-\Gamma_0\bar{x}^0|^2_{Q_0} + |u_0|^2_{R_0}\big]dt + \mathbb{E}\big[|x_0^0(T)-\hat{\Gamma}_0\bar{x}^0(T)|^2_{H_0}\big]. \tag{3.21}$$

Considering that for the given x_0^* and $\bar{x}^*$, FBSDE (3.18) admits a unique solution (One can solve BSDE for $(\bar{y},\bar{\beta})$ first, then solve FSDE for ψ and finally solve BSDE for (y_0,β_0)), by (3.17) and (3.18) and applying Itô's formula, we obtain

$$\begin{aligned}&\mathbb{E}[\langle H_0(x_0^*-\hat{\Gamma}_0\bar{x}^*)+\hat{\Gamma}_1^TH(\hat{\Gamma}-I)\psi(T), x_0^0(T)\rangle] = \mathbb{E}[\langle y_0(T), x_0^0(T)\rangle - \langle y_0(0), x_0^0(0)\rangle]\\ =&\mathbb{E}\int_0^T \Big\{\Big\langle -\big[(F-B\Upsilon^\dagger D^TP\bar{F})^T\bar{y} + \langle B_0^Ty_0+D_0^T\beta_0, u_0\rangle + \big[(P+K)F_B+(C+\bar{G})^TP\bar{F}_D\\ &+(\Gamma-I)^TQ\Gamma_1\big]^T\psi + Q_0(x_0^*-\Gamma_0\bar{x}^*)\big], x_0^0\Big\rangle\Big\}dt,\end{aligned} \tag{3.22}$$

$$\begin{aligned}&-\mathbb{E}[\langle\hat{\Gamma}_0^TH_0(x_0^*-\hat{\Gamma}_0\bar{x}^*), \bar{x}^0(T)\rangle] = \mathbb{E}[\langle\bar{y}(T), \bar{x}^0(T)\rangle - \langle\bar{y}(0), \bar{x}^0(0)\rangle]\\ =&\mathbb{E}\int_0^T \big[\langle\Gamma_0^TQ_0(x_0-\Gamma_0\bar{x}), \bar{x}^0\rangle - \langle B\Upsilon^\dagger B^T\bar{y}, \varphi^0\rangle + \langle(F-B\Upsilon^\dagger D^TP\bar{F})^T\bar{y}, x_0^0\rangle\big]dt.\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\mathbb{E}[\langle(\hat{\Gamma}-I)^TH\hat{\Gamma}_1x_0^0(T), \psi(T)\rangle] = \mathbb{E}[\langle\varphi^0(T), \psi(T)\rangle - \langle\varphi^0(0), \psi(0)\rangle]\\ =&\mathbb{E}\int_0^T\Big[\langle -B\Upsilon^\dagger B^T\bar{y}, \varphi^0\rangle - \Big\langle\big[(P+K)F_B+(C+\bar{G})^TP\bar{F}_D+(\Gamma-I)^TQ\Gamma_1\big]^T\psi, x_0^0\Big\rangle\Big]dt.\end{aligned} \tag{3.24}$$

From (3.20) and (3.22)-(3.24), it follows that $I_1 = \mathbb{E}\int_0^T \langle B_0^Ty_0 + D_0^T\beta_0 + Ru_0^*, u_0\rangle dt$. Note that θ is arbitrary. Then, by (3.19), u_0^* is a minimizer of (P2′) if and only if $I_1 = 0$ and $I_2 \geq 0$. Thus, by Lemma 3.3, u_0^* is an optimal control of (P2′) if and only if $Ru_0^* + B_0^Ty_0 + D_0^T\beta_0 = 0$ and $\bar{J}_0(u_0, u(u_0))$ is convex in u_0. □

Let $X = [x_0^T, \bar{x}^T, \psi^T]^T, Y = [y_0^T, \bar{y}^T, \varphi^T]^T$, $Z = [\beta_0^T, \bar{\beta}^T, (q_i^0)^T]^T, \mathcal{B}_0 = [B_0^T, 0, 0]^T, \mathcal{D}_0 = [D_0^T, 0, 0]^T$, and

$$\mathcal{A} = \begin{bmatrix} A_0 & 0 & 0\\ F-B\Upsilon^\dagger D^TP\bar{F} & \bar{A}+\hat{G} & 0\\ 0 & 0 & \bar{A}+\hat{G}\end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & B\Upsilon^\dagger B^T\\ 0 & B\Upsilon^\dagger B^T & 0\end{bmatrix},$$

$$\mathcal{C}_0 = \begin{bmatrix} C_0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \mathcal{H}_0 = \begin{bmatrix} H_0 & -H_0\hat{\Gamma}_0 & \hat{\Gamma}_1^TH(\hat{\Gamma}-I)\\ -\hat{\Gamma}_0^TH_0 & \Gamma_0^TH_0\Gamma_0 & 0\\ (\hat{\Gamma}-I)^TH\hat{\Gamma}_1 & 0 & 0\end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} -Q_0 & Q_0\Gamma_0 & \begin{array}{c}\Gamma_1^TQ(I-\Gamma)-F_B^T\Pi\\ -\bar{F}_D^TP(C+\bar{G})\end{array}\\ \Gamma_0^TQ_0 & -\Gamma_0^TQ_0\Gamma_0 & 0\\ \begin{array}{c}(I-\Gamma)^TQ\Gamma_1-\Pi F_B\\ -(C+\bar{G})P\bar{F}_D\end{array} & 0 & 0\end{bmatrix}.$$

Then, we can rewrite (3.16) and (3.18) as

$$\begin{cases} dX = (\mathcal{A}X - \mathcal{B}Y + \mathcal{B}_0 u_0^*)dt + (\mathcal{C}_0 X + \mathcal{D}_0 u_0^*)dW_0,\ X(0) = [\xi_0^T, \bar{\xi}^T, 0]^T \\ dY = (\mathcal{Q}X - \mathcal{A}^T Y - \mathcal{C}_0^T Z)dt + ZdW_0,\ Y(T) = \mathcal{H}_0 X(T), \end{cases} \tag{3.25}$$

together with the condition

$$R_0 u_0^* + \mathcal{B}_0^T Y + \mathcal{D}_0^T Z = 0. \tag{3.26}$$

We now provide a sufficient condition to guarantee the solvability of (3.25).

PROPOSITION 3.5. *Denote $\Upsilon_0 = R_0 + \mathcal{D}_0^T \mathcal{P} \mathcal{D}_0$. If the equation*

$$\dot{\mathcal{P}} + \mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} + \mathcal{C}_0^T\mathcal{P}\mathcal{C}_0 - \mathcal{Q} - \mathcal{P}\mathcal{B}\mathcal{P} - (\mathcal{B}_0^T\mathcal{P} + \mathcal{D}_0^T\mathcal{P}\mathcal{C}_0)^T \Upsilon_0^\dagger (\mathcal{B}_0^T\mathcal{P} + \mathcal{D}_0^T\mathcal{P}\mathcal{C}_0) = 0, \tag{3.27}$$

with $\mathcal{P}(T) = \mathcal{H}_0$ has a solution in $[0,T]$, then FBSDE (3.25) is solvable.

Proof. Suppose we have the relation $Y(t) = \mathcal{P}(t)X(t)$, $t \in [0,T]$. Then, it follows by Itô's formula that

$$\begin{aligned} dY =& \dot{\mathcal{P}}Xdt + \mathcal{P}[(\mathcal{A}X - \mathcal{B}\mathcal{P}X + \mathcal{B}_0 u_0^*)dt + (\mathcal{C}_0 X + \mathcal{D}_0 u_0^*)dW_0] \\ =& (\mathcal{Q}X - \mathcal{A}^T\mathcal{P}X - \mathcal{C}_0^T Z)dt + ZdW_0, \end{aligned} \tag{3.28}$$

which leads to $Z = \mathcal{P}(\mathcal{C}_0 X + \mathcal{D}_0 u_0^*)$. Plugging this into (3.26), we have $u_0^* = -\Upsilon_0^\dagger(\mathcal{B}_0^T\mathcal{P} + \mathcal{D}_0^T\mathcal{P}\mathcal{C}_0)X$, which further implies

$$Z = \mathcal{P}\big[\mathcal{C}_0 - \mathcal{D}_0^T \Upsilon_0^\dagger (\mathcal{B}_0^T\mathcal{P} + \mathcal{D}_0^T\mathcal{P}\mathcal{C}_0)\big]X.$$

Applying this into (3.28), we obtain (3.27). If the Riccati-like equation (3.27) has a solution in $[0,T]$, then by [31], FBSDE (3.25) admits an adapted solution. ☐

Remark 3.6. Noting that $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{H}_0$ are symmetric matrices, one can see that (3.27) is a symmetric Riccati equation. The existence condition of its solution can be referred to [1], [31].

For further analysis, assume

(A4) Equation (3.27) admits a solution in $C[0,T;\mathbb{R}^{3n}]$.

Under (A4), we construct the following decentralized control laws

$$\begin{cases} u_0^* =& -\Upsilon_0^\dagger(\mathcal{B}_0^T\mathcal{P} + \mathcal{D}_0^T\mathcal{P}\mathcal{C}_0)X, \\ u_i^* =& -\Upsilon^\dagger\big[(B^TP + D^TPC)\bar{x}_i + B^T\varphi + D^TP\bar{F}x_0^* + \big(B^TK + D^TP\bar{G}\big)\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]\big] \end{cases} \tag{3.29}$$

where $X, \bar{x}_i$ is given by (3.25), (3.2), and x_0^* is the realized state under the control u_0^*.

THEOREM 3.7. *Assume that (A1)-(A4) hold. Then $(u_0^*, u_1^*, \cdots, \hat{u}^*)$ given in (3.29) is an open-loop $(\varepsilon_1, \varepsilon_2)$-Stackelberg equilibrium, where $\varepsilon_i = O(1/\sqrt{N})$, $i = 1, 2$.*

Proof. See Appendix A. ☐

THEOREM 3.8. *For Problem (2.1)-(2.4), assume (A1)-(A4) hold, and $\xi_i, i = 1, \cdots, N$ have the same variance. Under the control (3.29), the corresponding social cost is given by*

$$J_{\text{soc}}^{(N)}(u^*, u_0^*) = \mathbb{E}[|\xi_i|^2_{P(0)} + |\bar{\xi}_0|^2_{K(0)} + 2\varphi^T(0)\bar{x}_0] + s_T, \tag{3.30}$$

and the asymptotic cost of the leader is $\lim_{N\to\infty} J_0(u_0^*, u^*) = \mathbb{E}\big[\xi_0^T y_0(0) + \bar{\xi}^T\bar{y}(0)\big]$, where

$$s_T = \mathbb{E}\int_0^T \big[|\bar{F}x_0|_P^2 - |B^T\varphi + D^TP\bar{F}x_0|^2_{\Upsilon^\dagger} + 2\varphi^TFx_0 + |\Gamma_1 x_0|_Q^2\big]dt. \tag{3.31}$$

Proof. See Appendix B. ☐

4. Feedback Solutions to MF Leader-Follower Games. In this section, we consider the feedback solution to the MF Stackelberg game (2.1)-(2.4). For simplicity, we consider the case that $Q \geq 0,\ Q_0 \geq 0,\ R > 0,\ R_0 > 0,\ H \geq 0$ and $H_0 \geq 0$.

4.1. The MF Social Control Problem for N Followers. Note that the leader plays against all followers. Assume that the leader admits a feedback control of the following form

$$u_0 = P_0 x_0 + \bar{P} x^{(N)}, \tag{4.1}$$

where P_0 and $\bar{P}$ are fixed. Then, we have the following social control problem for N followers.

(P3): minimize $J_{\rm soc}^{(N)}(u)$ over $u \in \mathcal{U}_c$, where $u_0 = P_0 x_0 + \bar{P} x^{(N)}$ and

$$J_{\rm soc}^{(N)}(u) = \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \Big\{ \big|x_i - \Gamma x^{(N)} - \Gamma_1 x_0\big|_Q^2 + |u_i|_R^2 \Big\} dt + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\big[|x_i(T) - \hat{\Gamma} x^{(N)}(T) - \hat{\Gamma}_1 x_0(T)|_H^2\big]. \tag{4.2}$$

By examining the social cost variation, we obtain the optimal control laws for N followers.

THEOREM 4.1. *Suppose $Q \geq 0$, $R > 0$ and $H \geq 0$. Assume the leader has the feedback control (4.1). Then, Problem (P3) has an optimal control in $\mathcal{U}_c$ if and only if the following system of FBSDEs admits a set of adapted solutions $\{x_i, p_i, q_i^j, i, j = 0, 1, \cdots, N\}$:*

$$\begin{cases} dx_0 = \big[A_0 x_0 + B_0(P_0 x_0 + \bar{P} x^{(N)})\big] dt + \big[C_0 x_0 + D_0(P_0 x_0 + \bar{P} x^{(N)})\big] dW_0, \\ dx_i = (A x_i + B \breve{u}_i + G x^{(N)} + F x_0) dt + (C x_i + D \breve{u}_i + \bar{G} x^{(N)} + \bar{F} x_0) dW_i, \\ dp_0 = -\,\big[(A_0 + B_0 P_0)^T p_0 + F^T p^{(N)} + (C_0 + D_0 P_0)^T q_0^0 + \bar{F}^T q^{(N)} \\ \qquad - Q_{\Gamma_1}^T x^{(N)} + \Gamma_1^T Q \Gamma_1 x_0)\big] + \sum_{j=0}^N q_0^j dW_j, \\ dp_i = -\,\big[A^T p_i + G^T p^{(N)} + \bar{P}^T B_0^T p_0 + C^T q_i^i + \bar{G}^T q^{(N)} + \bar{P}^T D_0^T q_0^0 \\ \qquad + Q x_i - Q_\Gamma x^{(N)} - Q_{\Gamma_1} \Gamma_1 x_0\big] dt + \sum_{j=0}^N q_i^j dW_j, \\ x_0(0) = \xi_0,\ x_i(0) = \xi_i,\ p_0(T) = -H_{\hat{\Gamma}_1}^T x^{(N)}(T) + \hat{\Gamma}_1^T H \hat{\Gamma}_1 x_0(T), \\ p_i(T) = H x_i(T) - H_{\hat{\Gamma}} x^{(N)}(T) - H_{\hat{\Gamma}_1} x_0(T),\ i = 1, \cdots, N. \end{cases} \tag{4.3}$$

Furthermore, the optimal controls of followers are given by

$$\breve{u}_i = -R^{-1}(B^T p_i + D^T q_i^i),\ i = 1, \cdots, N. \tag{4.4}$$

Proof. See Appendix C. □

Remark 4.2. For the feedback solution case, the term $x^{(N)}$ appears in leader's dynamics. Different from the open-loop case, an additional costate p_0 is needed. Indeed, as u_i is perturbed with δu_i, the changing magnitude of $x^{(N)}$ is $O(\|\delta u_i\|/N)$, which causes the perturbation $O(\|\delta u_i\|)$ of $J_{\rm soc}(u)$. This is evidently different from the game problem.

Let $p_0 = \Lambda_N^0 x_0 + \bar{\Lambda}_N x^{(N)}$, and $p_i = M_N x_i + \bar{M}_N x^{(N)} + M_N^0 x_0,\ i = 1, \cdots, N$. Denote $\breve{u}^{(N)} = \frac{1}{N}\sum_{i=1}^N \breve{u}_i$. Then, by applying Itô's formula to p_i, we have

$$\begin{aligned} dp_i =& \dot{M}_N x_i dt + M_N \big[(A x_i + B \breve{u}_i + G x^{(N)} + F x_0) dt + (C x_i + D \breve{u}_i + \bar{G} x^{(N)} + \bar{F} x_0) dW_i\big] \\ &+ \dot{\bar{M}}_N x^{(N)} + \bar{M}_N \Big[(A + G) x^{(N)} + B \breve{u}^{(N)} + F x_0) dt + \frac{1}{N}\sum_{j=1}^N (C x_j + D \breve{u}_j + \bar{G} x^{(N)} + \bar{F} x_0) dW_j\Big] \\ &+ \dot{M}_N^0 x_0 dt + M_N^0 \big[\big(A_0 x_0 + B_0(P_0 x_0 + \bar{P} x^{(N)})\big) dt + \big(C_0 x_0 + D_0(P_0 x_0 + \bar{P} x^{(N)})\big) dW_0\big] \\ =& -\,\big[A^T(M_N x_i + \bar{M}_N x^{(N)} + M_N^0 x_0) + G^T\big((M_N + \bar{M}_N) x^{(N)} + M_N^0 x_0\big) + \bar{P}^T B_0^T p_0 \\ &+ C^T q_i^i + \bar{G}^T q^{(N)} + \bar{P}^T D_0^T q_0^0 + Q x_i - Q_\Gamma x^{(N)} + (\Gamma - I)^T Q \Gamma_1 x_0\big] dt + \sum_{j=0}^N q_i^j dW_j, \end{aligned} \tag{4.5}$$

which together with (4.3) implies

$$q_i^i = \big(M_N + \frac{1}{N}\bar{M}_N\big)(Cx_i + D\check{u}_i + \bar{G}x^{(N)} + \bar{F}x_0), \qquad q_i^j = \frac{1}{N}\bar{M}_N(Cx_j + D\check{u}_j + \bar{G}x^{(N)} + \bar{F}x_0),\ j \neq i. \tag{4.6}$$

By (4.4), we have for any $i = 1, \cdots, N$,

$$R\check{u}_i + B^T(M_N x_i + \bar{M}_N x^{(N)} + M_N^0 x_0) + D^T\big(M_N + \frac{1}{N}\bar{M}_N\big)(Cx_i + D\check{u}_i + \bar{G}x^{(N)} + \bar{F}x_0) = 0.$$

This leads to

$$\check{u}_i = -\Upsilon_N^{-1}\big[(B^T M_N + D^T\check{M}_N C)x_i + (B^T\bar{M}_N + D^T\check{M}_N\bar{G})x^{(N)} + (B^T M_N^0 + D^T\check{M}_N\bar{F})x_0\big], \tag{4.7}$$

where $\check{M}_N \triangleq M + \frac{1}{N}\bar{M}_N$ and $\Upsilon_N \triangleq R + D^T\check{M}_N D$. Denote $\check{\Lambda}_N^0 \triangleq \Lambda_N^0 + \frac{1}{N}\bar{\Lambda}_N$. Applying Itô's formula to p_0, we obtain

$$\begin{aligned} dp_0 = &-\big[(A_0 + B_0P_0)^T(\Lambda_N^0 x_0 + \bar{\Lambda}_N x^{(N)}) + F^T\big((M + \bar{M}_N)x^{(N)} + M_N^0 x_0\big) \\ &+ (C_0 + D_0P_0)^T q_0^0 + \bar{F}^T q^{(N)} - \Gamma_1^T Q((I-\Gamma)x^{(N)} - \Gamma_1 x_0)\big]dt + \sum_{j=0}^N q_0^j dW_j, \end{aligned} \tag{4.8}$$

which together with (4.3) implies

$$q_0^0 = \check{\Lambda}_N^0\big(C_0x_0 + D_0(P_0x_0 + \bar{P}x^{(N)})\big), \qquad q_0^j = \frac{1}{N}\bar{\Lambda}\big(C_0x_0 + D_0(P_0x_0 + \bar{P}x^{(N)})\big),\ j > 0. \tag{4.9}$$

Applying (4.6), (4.7) and (4.9) into (4.5), we obtain

$$\begin{cases} \dot{M}_N + A^T M_N + M_N^T A + C^T M_N C + Q - (B^T M_N + D^T\check{M}_N C)^T\Upsilon_N^{-1} \\ \quad \times (B^T M_N + D^T\check{M}_N C) = 0,\ M_N(T) = H, \\ \dot{\bar{M}}_N + (A+G)^T\bar{M}_N + \bar{M}_N(A+G) + G^T M_N + M_N G + C^T\check{M}_N\bar{G} \\ \quad + \bar{G}^T\check{M}_N(C+\bar{G}) - Q_\Gamma + \bar{P}^T D_0^T\check{\Lambda}_N^0 D_0\bar{P} + M_N^0 B_0\bar{P} + \bar{P}^T B_0^T\bar{\Lambda}_N \\ \quad - \big(B^T M_N + D^T\check{M}_N C\big)^T\Upsilon_N^{-1}(B^T\bar{M}_N + D^T\check{M}_N\bar{G}) \\ \quad - \big(B\bar{M}_N + D^T\check{M}_N\bar{G}\big)^T\Upsilon_N^{-1}\big(B^T M_N + D^T\check{M}_N C\big) \\ \quad - \big(B\bar{M}_N + D^T\check{M}_N\bar{G}\big)^T\Upsilon_N^{-1}\big(B^T\bar{M}_N + D^T\check{M}_N\bar{G}\big) = 0, \bar{M}_N(T) = -H_{\hat{\Gamma}}, \\ \dot{M}_N^0 + (A+G)^T M_N^0 + M_N^0(A_0 + B_0P_0) + (M_N + \bar{M}_N)F + \bar{P}^T B_0^T\Lambda_N^0 \\ \quad - [B^T(M_N + \bar{M}_N) + D^T\check{M}_N(C+\bar{G})]^T\Upsilon_N^{-1}(B^T M_N^0 + D^T\check{M}\bar{F}) \\ \quad + (C+\bar{G})^T\check{M}_N\bar{F} + \bar{P}^T D_0^T\check{\Lambda}_N^0(C_0 + D_0P_0) + (\Gamma - I)^T Q\Gamma_1 = 0, \\ \quad M_N^0(T) = (\hat{\Gamma} - I)^T H\hat{\Gamma}_1. \end{cases} \tag{4.10}$$

Applying (4.6), (4.7) and (4.9) into (4.8), we have

$$\begin{cases} \dot{\Lambda}_N^0 + \Lambda_N^0(A_0 + B_0P_0) + (A_0 + B_0P_0)^T\Lambda_N^0 + (C_0 + D_0P_0)^T\check{\Lambda}_N^0(C_0 + D_0P_0) \\ \quad - (B^T\bar{\Lambda}_N^T + D^T\check{M}_N\bar{F})^T\Upsilon_N^{-1}(B^T M_N^0 + D^T\check{M}_N\bar{F}) \\ \quad + \bar{\Lambda}_N F + F^T M_N^0 + \bar{F}^T\check{M}\bar{F} + \Gamma_1^T Q\Gamma_1 = 0,\ \Lambda_N^0(T) = \hat{\Gamma}_1^T H\hat{\Gamma}_1, \\ \dot{\bar{\Lambda}}_N + \bar{\Lambda}_N(A+G) + (A_0 + B_0P_0)^T\bar{\Lambda}_N + F^T(M_N + \bar{M}_N) + \Lambda_N^0 B_0\bar{P} \\ \quad - (B^T\bar{\Lambda}_N^T + D^T\check{M}_N\bar{F})^T\Upsilon_N^{-1}[B^T(M_N + \bar{M}_N) + D^T\check{M}_N(C+\bar{G})] \\ \quad + \bar{F}^T\check{M}_N(C+\bar{G}) + \Gamma_1^T Q(\Gamma - I) = 0,\ \bar{\Lambda}_N(T) = \hat{\Gamma}_1^T H(\hat{\Gamma} - I). \end{cases} \tag{4.11}$$

Based on Theorem 4.1 and the above discussion, we have the following result.

PROPOSITION 4.3. *Assume (A1) holds, and (4.10)-(4.11) admit solutions, respectively. Then, Problem (P3) admits a feedback solution (4.7).*

Remark 4.4. Note that the social control problem (P3) is essentially an optimal control problem. Then, the feedback solution to Problem (P3) is equivalent to the feedback representation of its open-loop solution.

We now introduce the following set of equations:

$$
\begin{cases}
\dot{M}+A^TM+M^TA+C^TMC+Q-(B^TM+D^TMC)^T\Upsilon^{-1}\\
\quad\times(B^TM+D^TMC)=0,\ M(T)=H,\\
\dot{\bar{M}}+(A+G)^T\bar{M}+\bar{M}(A+G)+G^TM+MG+C^TM\bar{G}+\bar{G}^TM(C+\bar{G})\\
\quad-\big(B^TM+D^TMC\big)^T\Upsilon^{-1}(B^T\bar{M}+D^TM\bar{G})+\bar{P}^TD_0^T\Lambda^0D_0\bar{P}+\bar{P}^TB_0^T\bar{\Lambda}\\
\quad-\big(B\bar{M}+D^TM\bar{G}\big)^T\Upsilon^{-1}\big(B^TM+D^TMC\big)-Q_\Gamma+M^0B_0\bar{P}\\
\quad-\big(B\bar{M}+D^TM\bar{G}\big)^T\Upsilon^{-1}\big(B^T\bar{M}+D^TM\bar{G}\big)=0,\ \bar{M}(T)=-H_{\hat{\Gamma}},\\
\dot{M}^0+(A+G)^TM^0+M^0(A_0+B_0P_0)+(M+\bar{M})F+\bar{P}^TB_0^T\Lambda^0\\
\quad-[B^T(M+\bar{M})+D^TM(C+\bar{G})]^T\Upsilon^{-1}(B^TM^0+D^TM\bar{F})+(C+\bar{G})^TM\bar{F}\\
\quad+\bar{P}^TD_0^T\Lambda^0(C_0+D_0P_0))+(\Gamma-I)^TQ\Gamma_1=0,\ M^0(T)=(\hat{\Gamma}-I)^TH\hat{\Gamma}_1,\\
\dot{\Lambda}^0+\Lambda^0(A_0+B_0P_0)+(A_0+B_0P_0)^T\Lambda^0+(C_0+D_0P_0)^T\Lambda^0(C_0+D_0P_0)\\
\quad-(B^T\bar{\Lambda}^T+D^TM\bar{F})^T\Upsilon^{-1}(B^TM^0+D^TM\bar{F})+\bar{\Lambda}F+F^TM^0+\bar{F}^TM\bar{F}\\
\quad+\Gamma_1^TQ\Gamma_1=0,\ \Lambda^0(T)=\hat{\Gamma}_1^TH\hat{\Gamma}_1,\\
\dot{\bar{\Lambda}}+\bar{\Lambda}(A+G)+(A_0+B_0P_0)^T\bar{\Lambda}+F^T(M+\bar{M})+\Lambda^0B_0\bar{P}\\
\quad-(B^T\bar{\Lambda}^T+D^TM\bar{F})^T\Upsilon^{-1}[B^T(M+\bar{M})+D^TM(C+\bar{G})]+\bar{F}^TM(C+\bar{G})\\
\quad+\Gamma_1^TQ(\Gamma-I)=0,\ \bar{\Lambda}(T)=\hat{\Gamma}_1^TH(\hat{\Gamma}-I),
\end{cases}
\tag{4.12}
$$

where $\Upsilon\triangleq R+D^TMD$. From observation, we find that $M,\bar{M},\Lambda^0$ are symmetric and $M^0=\bar{\Lambda}^T$. For further analysis, we assume

(A5) (4.12) admits a solution $(M,\bar{M},M^0,\Lambda^0,\bar{\Lambda})$.

Remark 4.5. If (A5) holds, then by the continuous dependence of solutions on the parameter (see e.g. [27, Theorem 3.5] or [26, Theorem 4]), we obtain that for sufficiently large N, (4.10) and (4.11) admit solutions, respectively.

After applying the strategies of followers (4.7), we have

$$
\begin{aligned}
dx_i=&\big[(A-B\Upsilon_N^{-1}\Psi_N)x_i+(G-B\Upsilon_N^{-1}\bar{\Psi}_N)x^{(N)}+(F-B\Upsilon_N^{-1}\Psi_N^0)x_0\big]dt\\
&+\big[(C-D\Upsilon_N^{-1}\Psi_N)x_i+(\bar{G}-D\Upsilon_N^{-1}\bar{\Psi}_N)x^{(N)}+(\bar{F}-D\Upsilon_N^{-1}\Psi_N^0)x_0\big]dW_i,
\end{aligned}
\tag{4.13}
$$

where $\Psi_N\triangleq B^TM_N+D^T\check{M}_NC$, $\bar{\Psi}_N=B^T\bar{M}_N+D^T\check{M}_N\bar{G}$, and $\Psi_N^0=B^TM_N^0+D^T\check{M}_N\bar{F}$. This leads to

$$
\begin{aligned}
dx^{(N)}=&\big[\big(A+G-B\Upsilon_N^{-1}(\Psi_N+\bar{\Psi}_N)\big)x^{(N)}+(F-B\Upsilon_N^{-1}\Psi_N^0)x_0\big]dt\\
&+\frac{1}{N}\sum_{i=1}^N\big[(C-D\Upsilon_N^{-1}\Psi_N)x_i+(\bar{G}-D\Upsilon_N^{-1}\bar{\Psi}_N)x^{(N)}+(\bar{F}-D\Upsilon_N^{-1}\Psi_N^0)x_0\big]dW_i.
\end{aligned}
$$

For a sufficiently large N, by Remark 4.5 and the law of large numbers, $x^{(N)}$ can be approximated by the MF function $\bar{x}$, which satisfies

$$
d\bar{x}=\big[\big(A+G-B\Upsilon^{-1}(\Psi+\bar{\Psi})\big)\bar{x}+(F-B\Upsilon^{-1}\Psi^0)x_0\big]dt,
\tag{4.14}
$$

with

$$\Psi \triangleq B^T M + D^T MC,\ \bar{\Psi} \triangleq B^T \bar{M} + D^T M\bar{G}, \tag{4.15}$$
$$\Psi^0 \triangleq B^T M^0 + D^T M\bar{F}.$$

Based on Proposition 4.3, one can construct the decentralized feedback strategies for followers:

$$\hat{u}_i = -\Upsilon^{-1}(\Psi x_i + \bar{\Psi}\bar{x} + \Psi^0 x_0). \tag{4.16}$$

4.2. Optimization for the Leader. After applying the strategies (4.16) of followers, we have the optimal control problem for the leader.

(P4): minimize $J_0(u_0, \hat{u}(u_0))$ over $u_0 \in \mathcal{U}_d^0$, where

$$J_0(u_0, \hat{u}(u_0)) = \mathbb{E}\int_0^T \big[|x_0 - \Gamma_0 \hat{x}^{(N)}|^2_{Q_0} + |u_0|^2_{R_0}\big]dt + \mathbb{E}\big[|x_0(T) - \hat{\Gamma}_0 x^{(N)}(T)|^2_{H_0}\big],$$
$$dx_0 = \big(A_0 x_0 + B_0 u_0\big)dt + \big(C_0 x_0 + D_0 u_0\big)dW_0,\ x_0(0) = \xi_0,$$
$$d\hat{x}_i = \big[(A - B\Upsilon^{-1}\Psi)\hat{x}_i + G\hat{x}^{(N)} - B\Upsilon^{-1}\bar{\Psi}\bar{x} + (F - B\Upsilon^{-1}\Psi^0)x_0\big]dt$$
$$+ \big[(C - D\Upsilon^{-1}\Psi)\hat{x}_i + \bar{G}\hat{x}^{(N)} - D\Upsilon^{-1}\bar{\Psi}\bar{x} + (\bar{F} - D\Upsilon^{-1}\Psi^0)x_0\big]dW_i,\ \hat{x}_i(0) = \xi_i.$$

Since $\{W_i(t)\}$ and $\{x_i(0)\}$ are independent, for a sufficiently large N, it is plausible to replace $\hat{x}^{(N)}$ by $\bar{x}$, which evolves from (4.14). In view of (4.1), suppose that the decentralized feedback solution for the leader has the following form $u_0(t) = P_0(t)x_0 + \bar{P}(t)\bar{x},\ 0 \le t \le T$. Then, we have the following optimal control problem for the leader.

(P4′): minimize $\bar{J}_0(P_0, \bar{P})$ over $P_0, \bar{P} \in C(0, T; \mathbb{R}^{m\times n})$, where

$$\begin{cases} \bar{J}_0(P_0, \bar{P}) = \mathbb{E}\int_0^T \big[|x_0 - \Gamma_0 \bar{x}|^2_{Q_0} + |P_0 x_0 + \bar{P}\bar{x}|^2_{R_0}\big]dt + \mathbb{E}\big[|x_0(T) - \hat{\Gamma}_0 \bar{x}(T)|^2_{H_0}\big], \\ dx_0 = \big[(A_0 + B_0 P_0)x_0 + B_0 \bar{P}\bar{x}\big]dt + \big[\big(C_0 + D_0 P_0\big)x_0 + D_0 \bar{P}\bar{x}\big]dW_0,\ x_0(0) = \xi_0, \\ d\bar{x} = \big[\big(A + G - B\Upsilon^{-1}(\Psi + \bar{\Psi})\big)\bar{x} + (F - B\Upsilon^{-1}\Psi^0)x_0\big]dt,\ \bar{x}(0) = \bar{\xi}. \end{cases}$$

Let $X_0 = \mathbb{E}[x_0 x_0^T]$, $\bar{X} = \mathbb{E}[\bar{x}\bar{x}^T]$ and $Y = \mathbb{E}[\bar{x}x_0^T]$. Then, by Itô's formula [59], we obtain

$$\frac{dX_0}{dt} = (A_0 + B_0 P_0)X_0 + X_0(A_0 + B_0 P_0)^T + B_0 \bar{P} Y + Y^T (B_0 \bar{P})^T \tag{4.17}$$
$$+ (C_0 + D_0 P_0)X_0(C_0 + D_0 P_0)^T + (C_0 + D_0 P_0)Y^T (D_0 \bar{P})^T$$
$$+ D_0 \bar{P} Y (C_0 + D_0 P_0)^T + D_0 \bar{P}\bar{X}(D_0 \bar{P})^T,$$

$$\frac{d\bar{X}}{dt} = (A + G - B\Upsilon^{-1}(\Psi + \bar{\Psi}))\bar{X} + \bar{X}(A + G - B\Upsilon^{-1}(\Psi + \bar{\Psi}))^T \tag{4.18}$$
$$+ (F - B\Upsilon^{-1}\Psi^0)Y^T + Y(F - B\Upsilon^{-1}\Psi^0)^T,$$

$$\frac{dY}{dt} = (A + G - B\Upsilon^{-1}(\Psi + \bar{\Psi}))Y + (F - B\Upsilon^{-1}\Psi_0)X_0$$
$$+ Y(A_0 + B_0 P_0)^T + \bar{X}(B_0 \bar{P})^T. \tag{4.19}$$

Meanwhile, the cost function of the leader can be rewritten as

$$\bar{J}_0(P_0, \bar{P}) = \int_0^T tr\big(Q_0 X_0 - Q_0 \Gamma_0 Y - \Gamma_0^T Q_0 Y^T + \Gamma_0^T Q_0 \Gamma_0 \bar{X}$$
$$+ P_0^T R_0 P_0 X_0 + \bar{P}^T R_0 P_0 Y^T + P_0^T R_0 \bar{P} Y + \bar{P}^T R_0 \bar{P}\bar{X}\big)dt$$
$$+ tr\big[H_0 X_0(T) - H_0 \hat{\Gamma}_0 Y(T) - \hat{\Gamma}_0^T H_0 Y^T(T) + \hat{\Gamma}_0^T H_0 \hat{\Gamma}_0 \bar{X}(T)\big].$$

Denote $\hat{A}_0 \triangleq A_0 + B_0 P_0,\ \hat{C}_0 \triangleq C_0 + D_0 P_0,\ \hat{F} \triangleq F - B\Upsilon^{-1}\Psi^0,\ \hat{A} \triangleq A + G - B\Upsilon^{-1}(\Psi + \bar{\Psi})$. Define the Hamiltonian function of the leader as follow:

$$
\begin{aligned}
&H(P_0,\bar P,\Theta_1,\Theta_2,\Theta_3)\\
=&tr\Big(Q_0X_0-Q_0\Gamma_0Y-\Gamma_0^TQ_0Y^T+\Gamma_0^TQ_0\Gamma_0\bar X+P_0^TR_0P_0X_0+\bar P^TR_0P_0Y^T\\
&+P_0^TR_0\bar PY+\bar P^TR_0\bar P\bar X+[\hat A_0X_0+X_0\hat A_0^T+B_0\bar PY+Y^T(B_0\bar P)^T+\hat C_0X_0\hat C_0^T\\
&+\hat C_0Y^T(D_0\bar P)^T+D_0\bar PY\hat C_0^T+D_0\bar P\bar X(D_0\bar P)^T]\Theta_1^T+[\hat A\bar X+\bar X\hat A^T+\hat FY^T+Y\hat F^T]\Theta_2^T\\
&+\big[\hat AY+\hat FX_0+Y\hat A_0^T+\bar X(B_0\bar P)^T\big]\Theta_3^T+\big[\hat AY+\hat FX_0+Y\hat A_0^T+\bar X(B_0\bar P)^T\big]^T\Theta_3\Big).
\end{aligned}
$$

By the matrix maximum principle [4], we obtain the following adjoint equations:

$$
\begin{cases}
-\dot\Theta_1=\dfrac{\partial H}{\partial X_0}=Q_0+P_0^TR_0P_0+\hat A_0^T\Theta_1+\Theta_1\hat A_0+\hat C_0^T\Theta_1\hat C_0+\hat F^T\Theta_3+\Theta_3^T\hat F,\\
-\dot\Theta_2=\dfrac{\partial H}{\partial \bar X}=\Gamma_0^TQ\Gamma_0+\bar P^TR_0\bar P+\hat A^T\Theta_2+\Theta_2\hat A+\Theta_3B_0\bar P+(\Theta_3B_0\bar P)^T,\\
-\dot\Theta_3=\dfrac{\partial H}{\partial Y}=\bar P^TR_0P_0-\Gamma_0^TQ_0+(B_0\bar P)^T\Theta_1^T+\Theta_2\hat F+(D_0\bar P)^T\Theta_1\hat C_0+\hat A^T\Theta_3+\Theta_3\hat A_0,
\end{cases}
\tag{4.20}
$$

with the stationarity conditions

$$
0=\frac{\partial H}{\partial P_0}=2(R_0P_0X_0+R_0\bar PY+B_0^T\Theta_1X_0+D_0^T\Theta_1\hat C_0X_0+D_0^T\Theta_1D_0\bar PY+B_0^T\Theta_3^TY),
\tag{4.21}
$$

$$
0=\frac{\partial H}{\partial \bar P}=2(R_0P_0Y^T+R_0\bar P\bar X+B_0^T\Theta_1Y^T+D_0^T\Theta_1\hat C_0Y^T+D_0^T\Theta_1D_0\bar P\bar X+B_0^T\Theta_3^T\bar X).
\tag{4.22}
$$

Note that Θ_1 and Θ_2 are symmetric matrices. Then, from (4.21) and (4.22), we obtain

$$
\begin{cases}
P_0=-R_0^{-1}(B_0^T\Theta_1+D_0^T\Theta_1C_0),\\
\bar P=-\Upsilon_0^{-1}B_0^T\Theta_3^T,
\end{cases}
\tag{4.23}
$$

where $\Upsilon_0{=}R_0+D_0^T\Theta_1D_0$. After applying this into (4.20), we have

$$
\begin{cases}
\dot\Theta_1+A_0^T\Theta_1+\Theta_1A_0+C_0^T\Theta_1C_0-(B_0^T\Theta_1+D_0^T\Theta_1C_0)^T\Upsilon_0^{-1}(B_0^T\Theta_1+D_0^T\Theta_1C_0)\\
\qquad+\hat F^T\Theta_3+\Theta_3^T\hat F+Q_0=0,\ \Theta_1(T)=H_0,\\
\dot\Theta_2+\hat A^T\Theta_2+\Theta_2\hat A-\Theta_3B_0\Upsilon_0^{-1}B_0^T\Theta_3^T+\Gamma_0^TQ\Gamma_0=0,\ \Theta_2(T)=\hat\Gamma_0^TH_0\hat\Gamma_0,\\
\dot\Theta_3+\hat A^T\Theta_3+\Theta_3A_0-\Theta_3B_0\Upsilon_0^{-1}(B_0^T\Theta_1+D_0^T\Theta_1C_0)+\Theta_2^T\hat F+\Gamma_0^TQ_0=0,\\
\qquad\Theta_3(T)=-\hat\Gamma_0^TH_0.
\end{cases}
\tag{4.24}
$$

Based on the above discussions, we can get the following feedback strategies:

$$
\begin{cases}
\hat u_0=-\Upsilon_0^{-1}\big[(B_0^T\Theta_1+D_0^T\Theta_1C_0)x_0+B_0^T\Theta_3^T\bar x)\big],\\
\hat u_i=-\Upsilon^{-1}(\Psi x_i+\bar\Psi\bar x+\Psi^0x_0),\ i=1,\cdots,N,
\end{cases}
\tag{4.25}
$$

where $\bar x$ satisfies (4.14), and Ψ, $\bar\Psi$ and Ψ^0 are given by (4.15).

THEOREM 4.6. *Assume (A1) holds; (4.12) and (4.24) admit a set of solutions. Then, the strategy (4.25) is a feedback (ϵ_1,ϵ_2)-Stackelberg equilibrium, where $\epsilon_1=\epsilon_2=O(\frac{1}{\sqrt N})$. Furthermore, assume that $\xi_i, i=1,\cdots,N$ have the same variance. Then, the asymptotic average social cost of followers is given by*

$$
\lim_{N\to\infty}\frac1N J_{\rm soc}(\hat u,\hat u_0)=\mathbb E[|\xi_i|^2_{M(0)}+|\bar\xi|^2_{\bar M(0)}+2\xi_0^T\bar\Lambda(0)\xi_i+|\xi_0|^2_{\Lambda_0(0)}],
$$

and

$$
\lim_{N\to\infty}J_0(\hat u,\hat u_0)=\mathbb E[\xi_0^T\Theta_1(0)\xi_0+\bar\xi^T\Theta_2(0)\bar\xi+\bar\xi^T\Theta_3(0)\xi_0].
$$

Proof. See Appendix C. ∎

5. Simulation. In this section, we give a numerical example to compare the performances of the open-loop and feedback solutions. The simulation parameters are listed in Table 1.

TABLE 1
Simulation parameters

		A_0	B_0	C_0	D_0	Γ_0	Q_0	R_0	$\hat{\Gamma}_0$	H_0				
		-10	1	-0.5	0.5	1	1	1	1	2				
A	B	G	F	C	D	$\bar{G}$	$\bar{F}$	Γ	Γ_1	Q	R	$\hat{\Gamma}$	$\hat{\Gamma}_1$	H
-2	1	1	1	-0.2	0.2	0.2	0.2	1	1	1	1	1	1	2

Consider a multi-agent system with 1 leader and 100 followers. The initial distributions of states for the leader and followers satisfy normal distributions $N(10, 2)$ and $N(5, 1)$, respectively. The decentralized open-loop control (3.29) is given by solving (3.7), (3.8), (3.12) and (3.27). The solution to the Riccati equation (3.27) is shown in Fig. 1. The decentralized feedback strategy (4.25) is obtained by solving (4.12) and (4.24). The solutions to (4.12) and (4.24) are shown in Fig. 2. Fig. 3 gives the curves of followers' state averages and MF effects under open-loop and feedback solutions. Fig. 4 shows the state trajectories of the leader under the two solutions. It can be seen that state averages approximate MF effects well under both solutions, and the state average under open-loop control is larger than the one under feedback control.

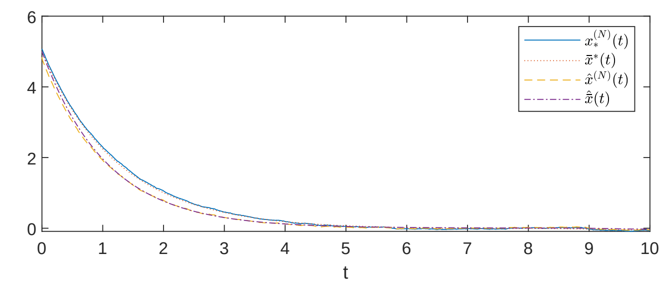

FIG. 1. *The solution to the Riccati equation* (3.27)*, and $P_{i,j}$ is the entry in ith row jth column of $\mathcal{P}$.*

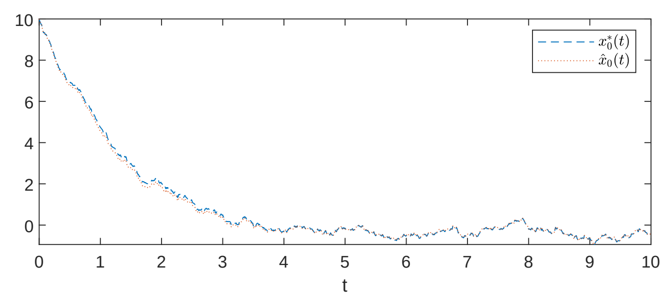

FIG. 2. *The solutions to* (4.12) *and* (4.24).

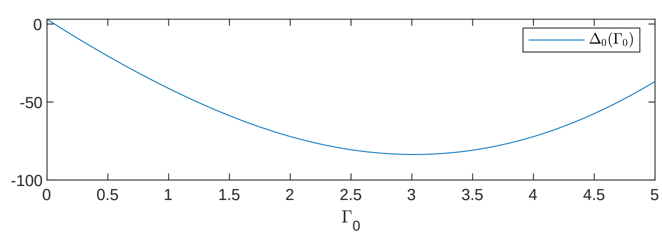

FIG. 3. *Followers' state averages and MF effects under open-loop and feedback controls.*

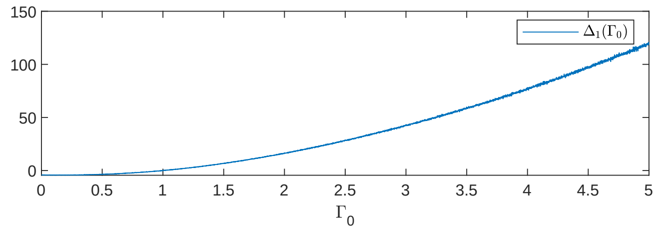

FIG. 4. *States of the leader under open-loop and feedback controls.*

6. Concluding Remarks. This paper studies open-loop and feedback solutions of MF-LQG Stackelberg games with multiplicative noises. By decoupling MF FBSDEs and applying MF approximations, we obtain a set of open-loop controls of players and a set of decentralized feedback strategies, respectively. Furthermore, the corresponding optimal costs of all players are explicitly given in terms of the solutions to two Riccati equations, respectively. A challenge is computing the system of Riccati equations for feedback strategies. A possible approach is resorting to reinforcement learning even if dynamics are partially unknown.

Appendix A. Proof of Theorem 3.7.

To prove Theorem 3.7, we provide two lemmas.

LEMMA A.1. *Assume that (A1)-(A4) hold. Then, the following holds:*

$$\sup_{0\le t\le T}\mathbb{E}\big[|\bar{x}^{(N)}-\bar{x}|^2+|\bar{p}^{(N)}-\mathbb{E}_{\mathcal{F}^0}[\bar{p}_i]|^2+|\bar{q}^{(N)}-\mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i]|^2\big]=O(\frac{1}{N}), \tag{A.1}$$

where $\bar{p}^{(N)}=\frac{1}{N}\sum_{i=1}^N\bar{p}_i$ and $\bar{q}^{(N)}=\frac{1}{N}\sum_{i=1}^N\bar{q}_i^i$.

Proof. After applying u_i^*, $i=0,\cdots,N$, we have

$$\begin{aligned} d\bar{x}_i =&\big(\bar{A}\bar{x}_i+\hat{G}\bar{x}-B\Upsilon^\dagger B^T\varphi+F_Bx_0^*\big)dt\\ &+\big[\bar{C}\bar{x}_i+\big(\bar{G}-D\Upsilon^\dagger(B^TK+D^TP\bar{G})\big)\bar{x}-D\Upsilon^\dagger B\varphi+\bar{F}_Dx_0^*\big]dW_i. \end{aligned}\tag{A.2}$$

By (A4), $\mathbb{E}\int_0^T|u_0^*|^2dt\le c_1$. Then, it leads to $\mathbb{E}\int_0^T|x_0^*|^2dt\le c_2$. By (3.12), $\max_{0\le t\le T}\mathbb{E}[|\bar{x}(t)|^2]\le c_3$. This further gives that $\sup_{0\le t\le T}\mathbb{E}[|\bar{x}_i(t)|^2]\le c_4$. By (A.2) and (3.12), we obtain

$$\begin{aligned} d(\bar{x}^{(N)}-\bar{x})=&\bar{A}(\bar{x}^{(N)}-\bar{x})dt\\ &+\frac{1}{N}\sum_{i=1}^N\big[\bar{C}\bar{x}_i+\big(\bar{G}-D\Upsilon^\dagger(B^TK+D^TP\bar{G})\big)\bar{x}-D\Upsilon^\dagger B\varphi+\bar{F}_Dx_0^*\big]dW_i, \end{aligned}$$

which gives

$$\begin{aligned} \bar{x}^{(N)}(t)-\bar{x}(t)=&\Phi(t,0)[\bar{x}^{(N)}(0)-\bar{x}(0)]\\ &+\frac{1}{N}\sum_{i=1}^N\int_0^t\Phi(t,s)\big[\bar{C}\bar{x}_i+\big(\bar{G}-D\Upsilon^\dagger(B^TK+D^TP\bar{G})\big)\bar{x}-D\Upsilon^\dagger B\varphi+\bar{F}_Dx_0^*\big]dW_i(s). \end{aligned}$$

Here, $\Phi(t,s)$ satisfies $\frac{d\Phi(t,s)}{dt}=\bar{A}\Phi(t,s),\ \Phi(s,s)=I$. By (A1), we further have

$$\begin{aligned} &\mathbb{E}|\bar{x}^{(N)}(t)-\bar{x}(t)|^2\\ \le&\big|\Phi(t,0)\big|^2\mathbb{E}|\bar{x}^{(N)}(0)-\bar{x}(0)|^2+\frac{1}{N^2}\sum_{i=1}^N\int_0^tc_1\big|\Phi(t,s)\big|^2\max_{1\le i\le N}\mathbb{E}\big(|\bar{x}_i|^2+|\bar{x}|^2+|\varphi|^2+|x_0^*|^2\big)ds\\ \le&\frac{1}{N}\Big\{\big|\Phi(t,0)\big|^2\max_{1\le i\le N}\big[\mathbb{E}|x_{i0}|^2+c_2\sup_{0\le t\le T}\mathbb{E}\big(|\bar{x}_i|^2+|\bar{x}|^2+|\varphi|^2+|x_0^*|^2\big)\big]\Big\}=O(\frac{1}{N}). \end{aligned}\tag{A.3}$$

Note that $\bar{p}_i=P\bar{x}_i+K\bar{x}+\varphi$. Then, we have

$$\sup_{0\le t\le T}\mathbb{E}\big[|\bar{p}^{(N)}(t)-\mathbb{E}_{\mathcal{F}^0}[\bar{p}_i(t)]|^2\big]=\sup_{0\le t\le T}\mathbb{E}\big[|P(\bar{x}^{(N)}(t)-\bar{x}(t))|^2\big]=O(1/N).$$

From (3.5), (3.6) and (A.3), we obtain

$$\sup_{0\le t\le T}\mathbb{E}\big[|\bar{q}^{(N)}(t)-\mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i(t)]|^2=\sup_{0\le t\le T}\mathbb{E}\big[|P\bar{C}(\bar{x}^{(N)}(t)-\bar{x}(t))|^2\big]=O(1/N).$$

□

Lemma A.2. *Assume that (A1)-(A4) hold. Then, the following holds:*

$$\text{(A.4)}\qquad \begin{aligned}\sup_{0\le t\le T}\mathbb{E}|x_*^{(N)}(t)-\bar{x}(t)|^2=O(\frac{1}{N}),\\ \sup_{0\le t\le T}\mathbb{E}|x_i^*(t)-\bar{x}_i(t)|^2=O(\frac{1}{N}),\end{aligned}$$

where $x_i^*, i=1,\cdots,N$ is the realized state under the control $u_i^*, i=1,\cdots,N$.

Proof. By (3.14) and (3.2), it can be verified that $\max_{1\le i\le N}\mathbb{E}\int_0^T(|x_i^*|^2+|u_i^*|^2)dt\le c_3$. From (3.12), we have

$$d(x_*^{(N)}-\bar{x})=(\bar{A}+G)(x_*^{(N)}-\bar{x})dt+\frac{1}{N}\sum_{j=1}^N(Cx_j^*+Du_j^*+\bar{G}x_*^{(N)}+\bar{F}x_0^*)dW_j.$$

Similar to (A.3), we have

$$\text{(A.5)}\qquad \mathbb{E}|x_*^{(N)}-\bar{x}|^2=O(1/N).$$

From (3.14) and (A.2),

$$d(x_i^*-\bar{x}_i)=[A(x_i^*-\bar{x}_i)+G(x_*^{(N)}-\bar{x})]dt+[C(x_i^*-\bar{x}_i)+\bar{G}(x_*^{(N)}-\bar{x})]dW_i,$$

with $x_i^*(0)-\bar{x}_i(0)=0$. Let $\Phi_i(t)$ be the solution to the following SDE:

$$d\Phi_i(t)=A\Phi_i(t)dt+C\Phi_i(t)dW_i(t),\ \Phi_i(0)=I.$$

Then, one can obtain

$$x_i^*-\bar{x}_i=\int_0^t\Phi_i(t)\Phi_i^\dagger(s)G(x_*^{(N)}(s)-\bar{x}(s))ds+\int_0^t\Phi_i(t)\Phi_i^\dagger(s)\bar{G}(x_*^{(N)}(s)-\bar{x}(s))dW_i(s).$$

Note that $\mathbb{E}\int_0^T|\Phi_i^T(t)\Phi_i(t)|dt<c$. Then, from (A.5) we have

$$\begin{aligned}\mathbb{E}|x_i^*-\bar{x}_i|^2\le&2T\mathbb{E}\int_0^t\big|\Phi_i(t)\Phi_i^\dagger(s)\big|^2\big|G(x_*^{(N)}(s)-\bar{x}(s))\big|^2ds\\&+2\mathbb{E}\int_0^t\big|\Phi_i(t)\Phi_i^\dagger(s)\big|^2\big|\bar{G}(x_*^{(N)}(s)-\bar{x}(s))\big|^2ds=O(\frac{1}{N}).\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.7. (For followers). We first prove that for $u\in\mathcal{U}_c$, $J_{\rm soc}(u)<\infty$ implies that $\mathbb{E}\int_0^T(|x_i|^2+|u_i|^2)dt<\infty$, for all $i=1,\cdots,N$. In views of (A2), by [43] we have

$$\delta_0\sum_{i=1}^N\mathbb{E}\int_0^T|u_i|^2dt-c_0\le J_{\rm soc}(u)<\infty,$$

which implies $\sum_{i=1}^N\mathbb{E}\int_0^T|u_i|^2dt<c_1$. By (2.1) and Schwarz's inequality [59],

$$\begin{aligned}\mathbb{E}|x_i(t)|^2\le&c_2\mathbb{E}\int_0^t|x^{(N)}(\tau)|^2d\tau+c_3\\ \le&\frac{c_2}{N}\mathbb{E}\int_0^t\sum_{j=1}^N|x_j(\tau)|^2d\tau+c_3.\end{aligned}$$

By Gronwall's inequality, we have $\sum_{j=1}^N \mathbb{E}|x_j(t)|^2 \leq Nc_3e^{c_2t} \leq Nc_3e^{c_2T}$.

Let $\tilde{x}_i = x_i - x_i^*$, $\tilde{u}_i = u_i - u_i^*$ and $\tilde{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \tilde{x}_i$. Then, by (2.1) and (3.14), we get

$$\text{(A.6)} \qquad d\tilde{x}_i =(A\tilde{x}_i + G\tilde{x}^{(N)} + B\tilde{u}_i)dt + (C\tilde{x}_i + \bar{G}\tilde{x}^{(N)} + D\tilde{u}_i)dW_i, \tilde{x}_i(0) = 0.$$

By (3.1) we have $J_{\rm soc}^{(N)}(u_0^*, u) = \frac{1}{N}\sum_{i=1}^N (J_i(u_0^*, u^*) + \tilde{J}_i(u_0^*, \tilde{u}) + \mathcal{I}_i)$, where

$$\begin{aligned}
\tilde{J}_i(u_0^*, \tilde{u}) &\triangleq \mathbb{E}\int_0^T \big[|\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \Gamma_1\tilde{x}_0|_Q^2 + |\tilde{u}_i|_R^2\big]dt \\
&\quad + \mathbb{E}|\tilde{x}_i(T) - \hat{\Gamma}\tilde{x}^{(N)}(T) - \hat{\Gamma}_1\tilde{x}_0(T)|_H^2, \\
\mathcal{I}_i &= 2\mathbb{E}\int_0^T \big[\big(x_i^* - \Gamma x_*^{(N)} - \Gamma_1 x_0^*\big)^T Q\big(\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \Gamma_1\tilde{x}_0\big) + \tilde{u}_i^T L u_0^* + \tilde{u}_i^T R u_i^*\big]dt \\
&\quad + \mathbb{E}\big[\big(x_i^*(T) - \hat{\Gamma}x_*^{(N)}(T) - \hat{\Gamma}_1 x_0^*(T)\big)^T H\big(\tilde{x}_i(T) - \hat{\Gamma}\tilde{x}^{(N)}(T) - \hat{\Gamma}_1\tilde{x}_0(T)\big)\big].
\end{aligned}$$

From (A.6) and Itô's formula it follows

$$\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\big[\tilde{x}_i^T(T)\big(H\bar{x}_i(T) - H_{\hat{\Gamma}}\bar{x}(T) - H_{\hat{\Gamma}_1}x_0^*(T)\big)\big] = \sum_{i=1}^N \mathbb{E}\big[\tilde{x}_i^T(T)\bar{p}_i(T))\big] \\
=&\sum_{i=1}^N \mathbb{E}\int_0^T \Big\{ - \tilde{x}_i^T\big[A^T\bar{p}_i + G^T\mathbb{E}_{\mathcal{F}^0}[\bar{p}_i] + C^T\bar{q}_i^i + \bar{G}^T\mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i] + Q\bar{x}_i - Q_\Gamma\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i] \\
&+ (\Gamma - I)^T Q\Gamma_1 x_0^*\big] + [A\tilde{x}_i + G\tilde{x}^{(N)} + B\tilde{u}_i]^T\bar{p}_i + [C\tilde{x}_i + \bar{G}\tilde{x}^{(N)} + D\tilde{u}_i]^T\bar{q}_i^i\Big\}dt \\
=&\mathbb{E}\int_0^T \sum_{i=1}^N \Big\{ - \tilde{x}_i^T\big[Q\bar{x}_i - Q_\Gamma\bar{x} + (\Gamma - I)^T Q\Gamma_1 x_0^*\big] - \tilde{u}_i^T R u_i^*\Big\}dt \\
&+ \sum_{i=1}^N \mathbb{E}\int_0^T \tilde{x}_i^T\big[G^T(\bar{p}^{(N)} - \mathbb{E}_{\mathcal{F}^0}[\bar{p}_i])dt + \bar{G}^T(\bar{q}^{(N)} - \mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i])\big]dt.
\end{aligned}$$

From this and direct computations, one can obtain

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \mathcal{I}_i =&\frac{1}{N}\sum_{i=1}^N 2\mathbb{E}\Big\{\int_0^T \tilde{x}_i^T\big[Q(x_i^* - \bar{x}_i)) + Q_\Gamma(x_*^{(N)} - \bar{x}) + G^T(\bar{p}^{(N)} - \mathbb{E}_{\mathcal{F}^0}\big[\bar{p}_i\big]) \\
&+ \bar{G}^T(\bar{q}^{(N)} - \mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i])\big]dt + \big[\tilde{x}_i^T(T)\big(H(x_i^*(T) - \bar{x}_i(T)) - H_{\hat{\Gamma}}(x_*^{(N)}(T) - \bar{x}(T))\big]\Big\} \\
\leq&\frac{c}{N}\sum_{i=1}^N \Big[\mathbb{E}\int_0^T |\tilde{x}_i|^2dt\Big]^{1/2} \cdot \Big[\mathbb{E}\int_0^T \big(|x_i^* - \bar{x}_i|^2 + |x_*^{(N)} - \bar{x}|^2 + |\bar{p}^{(N)} - \mathbb{E}_{\mathcal{F}^0}\big[\bar{p}_i\big]|^2 \\
&+ |\bar{q}^{(N)} - \mathbb{E}_{\mathcal{F}^0}[\bar{q}_i^i|^2]\big)dt\Big]^{1/2} + O\big(\frac{1}{\sqrt{N}}\big) \\
\leq&O\big(\frac{1}{\sqrt{N}}\big) = \epsilon_1.
\end{aligned}$$

Note that by (A2), $\sum_{i=1}^N \tilde{J}_i(\tilde{u}, u_0^*) \geq 0$. Then, we have $J_{\rm soc}(u^*, u_0^*) \leq J_{\rm soc}(u, , u_0^*) + \epsilon_1$.

(*For the leader*). By (3.15) and Schwarz's inequality, we have

$$\text{(A.7)} \qquad \begin{aligned}
J_0(u_0^*, u^*) = \mathbb{E}&\int_0^T \big[|\bar{x}_0^* - \Gamma_0\bar{x} + \Gamma_0(x_*^{(N)} - \bar{x})|_{Q_0}^2 + |u_0^*|_{R_0}^2\big]dt \\
&+\mathbb{E}\big[|\bar{x}_0^*(T) - \hat{\Gamma}_0\bar{x}(T) + \hat{\Gamma}_0(x_*^{(N)}(T) - \bar{x}(T))|_{H_0}^2\big]dt \\
&\leq \bar{J}_0(u_0^*, u^*) + \int_0^T \big[2\big(\mathbb{E}|x_0^* - \Gamma_0\bar{x}|^2 \cdot \mathbb{E}|Q_0\Gamma_0(x_*^{(N)} - \bar{x})|^2\big)^{1/2}
\end{aligned}$$

$$+\mathbb{E}|\Gamma_0(x_*^{(N)}-\bar{x})|^2_{Q_0}\big]dt+\mathbb{E}\big[|\hat{\Gamma}_0(x_*^{(N)}(T)-\bar{x}(T))|^2_{H_0}\big]$$
$$+2\big(\mathbb{E}|x_0^*(T)-\hat{\Gamma}_0\bar{x}(T)|^2\cdot\mathbb{E}|H_0\hat{\Gamma}_0(x_*^{(N)}(T)-\bar{x}(T))|^2\big)^{1/2}$$
$$\le\bar{J}_0(u_0^*,u^*)+O(1/\sqrt{N}).$$

It follows from Theorem 3.4 that $\bar{J}_0(u_0^*,u^*)\le\bar{J}_0(u_0,u^*)$. This together with (A.7) implies

$$J_0(u_0^*,u^*(u_0^*))\le\bar{J}_0(u_0,u(u_0))+O(1/\sqrt{N}), \tag{A.8}$$

for any $u_0\in\mathcal{U}_0$. From (3.15), we obtain

$$\begin{aligned}\bar{J}_0(u_0,u)=&\mathbb{E}\int_0^T\big[|x_0-\Gamma_0x_*^{(N)}+\Gamma_0(x_*^{(N)}-\bar{x})|^2_{Q_0}+|u_0|^2_{R_0}\big]dt\\&+\mathbb{E}\big[|x_0^*(T)-\bar{\Gamma}_0x_*^{(N)}(T)+\bar{\Gamma}_0(x_*^{(N)}(T)-\bar{x}(T))|^2_{H_0}\big]dt\\\le&J_0(u_0,u)+O(1/\sqrt{N}),\end{aligned}$$

which with (A.8) gives $J_0(u_0^*,u^*(u_0^*))\le J_0(u_0,u(u_0))+\varepsilon_2$, where $\varepsilon_2=O(1/\sqrt{N})$. □

Appendix B. Proof of Theorems 3.8 and 4.6. To prove Theorem 3.8, we first give a lemma. Consider an MF-type problem: optimize the cost functional

$$\mathcal{J}_i(u_i)=\mathbb{E}\int_0^T\big(|\bar{x}_i-\Gamma\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]-\Gamma_1x_0|^2_Q+|u_i|^2_R\big)dt+\mathbb{E}\big[|\bar{x}_i(T)-\hat{\Gamma}\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i(T)]-\hat{\Gamma}_1x_0(T)|^2_H\big] \tag{B.1}$$

subject to ($\bar{x}_i(0)=\xi_i$)

$$d\bar{x}_i=(A\bar{x}_i+Bu_i+G\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]+Fx_0)dt+(Cx_i+Du_i+\bar{G}\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]+\bar{F}x_0)dW_i. \tag{B.2}$$

LEMMA B.1. *Assume (A1) and (A4) hold. For Problem (B.1)-(B.2), the optimal control u_i^* is given by (3.6), and the corresponding optimal cost is $\mathbb{E}[|\xi_i|^2_{P(0)}+|\bar{\xi}_0|^2_{K(0)}+2\varphi^T(0)\bar{x}_0]+s_T$.*

Proof. Note that $\mathbb{E}_{\mathcal{F}^0}[\bar{x}_i]=\bar{x}$ satisfies

$$d\bar{x}=\big[(A+G)\bar{x}+B\bar{u}+Fx_0\big]dt, \tag{B.3}$$

where $\bar{u}=\mathbb{E}_{\mathcal{F}^0}[\bar{u}_i]$. By a similar proof to [58], [49], we obtain

$$\begin{aligned}\mathcal{J}_i(u_i)=&\mathbb{E}[|x_{i0}-\bar{x}_0|^2_{P(0)}+\bar{x}_0^T(P(0)+K(0))\bar{x}_0+2\varphi^T(0)\bar{x}_0]+s_T\\&+\mathbb{E}\int_0^T\big[|u_i-\bar{u}+\Upsilon^\dagger(B^TP+D^TPC)(\bar{x}_i-\bar{x})|^2_\Upsilon\\&+|\bar{u}+\Upsilon^\dagger(B^T(P+K)+D^TP(C+\bar{G}))\bar{x}+B^T\varphi+D^TP\bar{F}x_0|^2_\Upsilon\big]dt\\\ge&\mathbb{E}[|\xi_i|^2_{P(0)}+|\bar{\xi}_0|^2_{K(0)}+2\varphi^T(0)\bar{x}_0]+s_T.\end{aligned}$$

□

Proof of Theorem 3.8. Applying the control (3.29) into the social cost, it follows that

$$\begin{aligned}&J_{\rm soc}^{(N)}(u^*,u_0^*)\\=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Big[\int_0^T\big(|x_i^*-\Gamma x_*^{(N)}-\Gamma_1x_0^*|^2_Q+|u_i^*|^2_R\big)dt+|x_i^*(T)-\hat{\Gamma}x_*^{(N)}(T)-\hat{\Gamma}_1x_0^*(T)|^2_H\Big]\\=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Big\{\int_0^T\big[|\bar{x}_i-\Gamma\bar{x}-\Gamma_1\bar{x}_0+x_i^*-\bar{x}_i-\Gamma(x_*^{(N)}-\bar{x})-\Gamma_1(x_0^*-\bar{x}_0)|^2_Q\\&+|\Upsilon^\dagger(B^TP+D^TPC)\bar{x}_i+(B^TK+D^TP\bar{G})\bar{x}+B^T\varphi+D^TP\bar{F}\bar{x}_0)|^2_R\big]dt\\&+|\bar{x}_i(T)-\hat{\Gamma}\bar{x}(T)-\hat{\Gamma}_1\bar{x}_0(T)+x_i^*(T)-\bar{x}_i(T)-\hat{\Gamma}(x_*^{(N)}(T)-\bar{x}(T))-\Gamma_1(x_0^*(T)-\bar{x}_0(T))|^2_H\Big\}.\end{aligned}$$

By Lemma A.2 and Schwarz's inequality, one can obtain

$$
\begin{aligned}
&\big|J_{soc}^{(N)}(u^*,u_0^*)-\sum_{i=1}^N\mathcal{J}_i(u_i^*)\big|\\
\le&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\big[|x_i^*-\bar{x}_i|_Q^2+|\Gamma(x_*^{(N)}-\bar{x})|_Q^2+|\Gamma_1(x_0^*-\bar{x}_0)|_Q^2\big]dt+\frac{c}{N}\sum_{i=1}^N\sup_{0\le t\le T}\big(\mathbb{E}|x_i^*-\bar{x}_i|_Q^2\big)^{1/2}\\
&+\frac{C}{N}\sum_{i=1}^N\sup_{0\le t\le T}\big(\mathbb{E}|\Gamma(x_*^{(N)}-\bar{x})|_Q^2\big)^{1/2}+\frac{c}{N}\sum_{i=1}^N\sup_{0\le t\le T}\big(\mathbb{E}|\Gamma_1(x_0^*-\bar{x}_0)|_Q^2\big)^{1/2}\\
\le&O(\frac{1}{\sqrt{N}}).
\end{aligned}
$$

This together with Lemma B.1 leads to (3.30).

(For the leader) By a similar argument with the proof of Theorem 3.4, one can obtain

$$\bar{J}_0(u_0^*,u^*)=\mathbb{E}\Big\{\xi_0^Ty_0(0)+\bar{\xi}^T\bar{y}(0)+\int_0^T\big[\langle R_0u_0^*+B_0^Ty_0+\bar{B}_1^T\bar{y},u_0^*\rangle\big]dt\Big\}.$$

By (3.26), we have $\lim_{N\to\infty}J_0(u_0^*,u^*)=\mathbb{E}\big[\xi_0^Ty_0(0)+\bar{\xi}^T\bar{y}(0)\big]$. Thus, the theorem follows. □

Appendix C. Proofs of Theorems 4.1 and 4.6.

Proof of Theorem 4.1. Suppose that $\{\breve{u}_i, i=1,\cdots,N\}$ is an optimal control of Problem (P3). Denote by $\breve{x}_i$ the state of player i under the optimal control $\breve{u}_i$. For any $u_i\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^r)$ and $\lambda\in\mathbb{R}$ ($\lambda\neq 0$), let $u_i^\lambda=\breve{u}_i+\lambda u_i$, $i=1,\cdots,N$. Denote by x_0^λ, x_i^λ the solution to the following perturbed equation:

$$
\begin{cases}
dx_0^\lambda=\big[A_0x_0^\lambda+B_0(P_0x_0^\lambda+\bar{P}x_\lambda^{(N)})\big]dt+\big[C_0x_0^\lambda+D_0(P_0x_0^\lambda+\bar{P}x_\lambda^{(N)})\big]dW_0,\\
dx_i^\lambda=\big(Ax_i^\lambda+B(\breve{u}_i+\lambda u_i)+Gx_\lambda^{(N)}+Fx_0^\lambda\big)dt+\big(Cx_i^\lambda+Du_i^\lambda+\bar{G}x_\lambda^{(N)}+\bar{F}x_0^\lambda\big)dW_i,\\
x_0^\lambda(0)=\xi_0,\ x_i^\lambda(0)=\xi_i,\ i=1,2,\cdots,N,
\end{cases}
$$

with $x_\lambda^{(N)}=\frac{1}{N}\sum_{i=1}^N x_i^\lambda$. Let $z_i=(x_i^\lambda-\breve{x}_i)/\lambda$. It can be verified that z_i satisfies

$$
\begin{cases}
dz_0=\big[(A_0+B_0P_0)z_0+B_0\bar{P}z^{(N)}\big]dt+\big[(C_0+D_0P_0)z_0+D_0\bar{P}z^{(N)}\big]dW_0,\ z_0(0)=0,\\
dz_i=[Az_i+Bu_i+Gz^{(N)}+Fz_0]dt+[Cz_i+Du_i+\bar{G}z^{(N)}+\bar{F}z_0]dW_i,\ z_i(0)=0,
\end{cases}
$$

where $i=1,2,\cdots,N$, and $z^{(N)}=\frac{1}{N}\sum_{i=1}^N z_i$. From (4.2), we have

$$J_{\rm soc}^{(N)}(\breve{u}+\lambda u)-J_{\rm soc}^{(N)}(\breve{u})=2\lambda I_1+\lambda^2I_2, \tag{C.1}$$

where

$$
\begin{aligned}
I_1=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\big[\breve{x}_i^TQz_i-(\breve{x}^{(N)})^TQ_\Gamma z^{(N)}-\breve{x}_0^TQ_{\Gamma_1}\hat{\Gamma}_1^Tz^{(N)}-(\breve{x}^{(N)})^TQ_{\Gamma_1}z_0\\
&+\breve{x}_0^T\Gamma_1^TQ\Gamma_1z_0+\breve{u}_iRu_i\big]dt+\sum_{i=1}^N\mathbb{E}\big[\breve{x}_i^T(T)Hz_i(T)-(\breve{x}^{(N)}(T))^TH_{\hat{\Gamma}}z^{(N)}(T)\\
&-\breve{x}_0^T(T)H_{\hat{\Gamma}_1}^Tz^{(N)}(T)-[\breve{x}^{(N)}(T)]^TH_{\hat{\Gamma}_1}z_0(T)+\breve{x}_0^T(T)\hat{\Gamma}_1^TH\hat{\Gamma}_1z_0(T)\big],
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
I_2=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\big[|z_i|_Q^2-|z^{(N)}|^2_{Q_\Gamma}-2\Gamma z_0^TQ_{\Gamma_1}^Tz^{(N)}+z_0^T\Gamma_1^TQ\Gamma_1z_0+|u_i|_R^2\big]dt\\
&+\sum_{i=1}^N\mathbb{E}\big[|z_i(T)|_H^2-|z^{(N)}(T)|^2_{H_{\hat{\Gamma}}}-2(z_0(T))^TH_{\hat{\Gamma}_1}^Tz^{(N)}(T)+|z_0(T)|^2_{\hat{\Gamma}_1^TH\hat{\Gamma}_1}\big].
\end{aligned}
\tag{C.3}
$$

Let $\{\breve{p}_i, \breve{q}_i^j, i, j = 0, 1, \cdots, N\}$ be a set of solutions to (4.3). Then, by Itô's formula, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\big[\langle \hat{\Gamma}_1^T H(\hat{\Gamma} - I)\breve{x}^{(N)}(T) + \hat{\Gamma}_1^T H \hat{\Gamma}_1 \breve{x}_0^T(T), z_0(T)\rangle\big] \\
&= \sum_{i=1}^N \mathbb{E}[\langle \breve{p}_0(T), z_0(T)\rangle - \langle \breve{p}_0(0), z_0(0)\rangle] \\
&= \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{\Big\langle - \big[(A_0 + B_0 P_0)^T \breve{p}_0 + F^T \breve{p}^{(N)} + (C_0 + D_0 P_0)^T \breve{q}_0^0 + \bar{F}^T \breve{q}^{(N)} \\
&\quad - \Gamma_1^T Q((I - \Gamma)\breve{x}^{(N)} - \Gamma_1 \breve{x}_0)\big], z_0\Big\rangle + \langle \breve{p}_0, (A_0 + B_0 P_0) z_0 + B_0 \bar{P} z^{(N)}\rangle \\
&\quad + \langle \breve{q}_0^0, (C_0 + D_0 P_0) z_0 + D_0 \bar{P} z^{(N)}\rangle\Big\} dt \\
&= \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{\Big\langle - [F\breve{p}^{(N)} + \bar{F}\breve{q}^{(N)} - \Gamma_1^T Q((I - \Gamma)\breve{x}^{(N)} - \Gamma_1 \breve{x}_0)], z_0\Big\rangle \\
&\quad + \langle \bar{P}^T B_0^T \breve{p}_0 + \bar{P}^T D_0^T \breve{q}_0^0, z_i\rangle\Big\} dt,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^N \mathbb{E}[\langle H\breve{x}_i(T) - H_{\hat{\Gamma}}\breve{x}^{(N)}(T) + (\hat{\Gamma} - I)^T H \hat{\Gamma}_1 \breve{x}_0(T), z_i(T)\rangle] \\
&= \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{\Big\langle - \big[Q\breve{x}_i - Q_\Gamma \breve{x}^{(N)} + (\Gamma - I)^T Q \Gamma_1 \breve{x}_0 + \bar{P}^T B_0^T \breve{p}_0 + \bar{P}^T D_0^T \breve{q}_0^0\big], z_i\Big\rangle \\
&\quad + \langle F\breve{p}^{(N)} + \bar{F}\breve{q}^{(N)}, z_0\rangle + \langle B^T \breve{p}_i + D^T \breve{q}_i^i, u_i\rangle\Big\} dt,
\end{aligned}$$

where the second equation holds since $\sum_{i=1}^N \mathbb{E}\langle G^T \breve{p}^{(N)}, z_i\rangle = \sum_{i=1}^N \mathbb{E}\langle \breve{p}_i, G z^{(N)}\rangle$ and $\sum_{i=1}^N \mathbb{E}\langle \bar{G}^T \breve{q}^{(N)}, z_i\rangle = \sum_{i=1}^N \mathbb{E}\langle \breve{q}_i^i, \bar{G} z^{(N)}\rangle$. From the above equations and (C.2),

$$\begin{aligned}
I_1 =& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \big[\big\langle Q\breve{x}_i - Q_\Gamma \breve{x}^{(N)} + (\Gamma - I)^T Q \Gamma \breve{x}_0, z_i\big\rangle + \langle \Gamma_1^T Q(\Gamma - I)\breve{x}^{(N)} + \Gamma_1^T Q \Gamma_1 \breve{x}_0, z_0\rangle \\
&+ \langle R\breve{u}_i, u_i\rangle\big] dt + \sum_{i=1}^N \mathbb{E}\big[\big\langle H\breve{x}_i(T) - H_{\hat{\Gamma}}\breve{x}^{(N)}(T) + (\hat{\Gamma} - I)^T H \hat{\Gamma} x_0(T), z_i(T)\big\rangle \\
&+ \big\langle \hat{\Gamma}_1^T H(\hat{\Gamma} - I)\breve{x}^{(N)}(T) + \hat{\Gamma}_1^T H \hat{\Gamma}_1 \breve{x}_0^T(T), z_0(T)\big\rangle\big] \\
=& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T [\langle R\breve{u}_i + B^T \breve{p}_i + D^T \breve{q}_i^i, u_i\rangle] dt. \qquad \text{(C.4)}
\end{aligned}$$

Note that $Q - Q_\Gamma = (I - \Gamma)^T Q (I - \Gamma)$ and $H - H_{\hat{\Gamma}} = (I - \hat{\Gamma})^T H (I - \hat{\Gamma})$. Then, we have

$$\begin{aligned}
I_2 =& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \big[|z_i - z^{(N)}|_Q^2 + |z^{(N)}|_{Q - Q_\Gamma}^2 + 2(\Gamma z_0)^T Q(\Gamma - I) z^{(N)} + |\Gamma_1 z_0|_Q^2 + |u_i|_R^2\big] dt \\
&+ \sum_{i=1}^N \mathbb{E}\big[|z_i(T) - z^{(N)}(T)|_H^2 + |z^{(N)}(T)|_{H - H_{\hat{\Gamma}}}^2 - 2 z_0^T(T) H_{\hat{\Gamma}_1}^T z^{(N)}(T) + |\hat{\Gamma}_1 z_0(T)|_H^2\big] \\
=& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \big[|z_i - z^{(N)}|_Q^2 + |(I - \Gamma) z^{(N)} - \Gamma_1 z_0|_Q^2 + |u_i|_R^2\big] dt \\
&+ \sum_{i=1}^N \mathbb{E}\big[|z_i(T) - z^{(N)}(T)|_H^2 + |(I - \hat{\Gamma}) z^{(N)}(T) - \hat{\Gamma}_1 z_0(T)|_H^2\big].
\end{aligned}$$

Since $Q \geq 0$, $R > 0$, and $H \geq 0$, we obtain $I_2 \geq 0$. From (C.1), $\breve{u}$ is a minimizer to (P1) if and only if $I_1 = 0$, which is equivalent to $R\breve{u}_i + B^T\breve{p}_i + D^T\breve{q}_i^i = 0,\ i = 1, \cdots, N$. Thus, we have the optimality system (4.3). This implies that (4.3) admits a solution $(\breve{x}_i, \breve{p}_i, \breve{q}_i^j, i, j = 1, \cdots, N)$. □

Proof of Theorem 4.6. (For followers). By (2.6), it can be verified that under feedback strategies (2.5), $\mathbb{E}\int_0^T(|x_0|^2 + |\bar{x}|^2)dt < c$. This further gives $\mathbb{E}\int_0^T(|x_i|^2 + |x^{(N)}|^2)dt < c_1$. Besides, from (2.6), we have

$$
\begin{aligned}
d(x^{(N)} - \bar{x}) =&(A + G + B\hat{K})(x^{(N)} - \bar{x})dt\\
&+ \frac{1}{N}\sum_{j=1}^{N}[(C + D\hat{K})x_i + \bar{G}x^{(N)} + D\hat{K}\bar{x} + (\bar{F} + DK_0)x_0]dW_j,
\end{aligned}
$$

Similar to (A.3), we have for any $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}|x^{(N)}(t) - \bar{x}(t)|^2 \leq \big|\bar{\Phi}(t, 0)\big|^2\mathbb{E}|x^{(N)}(0) - \bar{x}(0)|^2\\
&+ \frac{1}{N^2}\sum_{i=1}^{N}\int_0^t c\big|\bar{\Phi}(t, s)\big| \max_{1\leq i\leq N}\mathbb{E}\big(|x_i|^2 + |x^{(N)}|^2 + |\bar{x}|^2 + |x_0|^2\big)ds = O(\frac{1}{N}),
\end{aligned}
\tag{C.5}
$$

where $\bar{\Phi}(t, s)$ satisfies $\frac{d\bar{\Phi}(t,s)}{dt} = (A+G+B\hat{K})\bar{\Phi}(t, s)$, $\bar{\Phi}(s, s) = I$. Note that $\bar{x} = \mathbb{E}[x_i|\mathcal{F}^0] = \mathbb{E}[x^{(N)}|\mathcal{F}^0]$ (which follows from (2.6)). Then, we have

$$
\mathbb{E}[\bar{x}^T(x^{(N)} - \bar{x})] = \mathbb{E}\big[\bar{x}^T\mathbb{E}[x^{(N)} - \bar{x}|\mathcal{F}^0]\big] = 0.
\tag{C.6}
$$

From (2.3) and (C.5), we have

$$
\begin{aligned}
J_{\rm soc}^{(N)}(u_0, u) =&\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\int_0^T\big[|x_i|_Q^2 - |x^{(N)}|^2_{Q_\Gamma} - 2x_0^TQ_{\hat{\Gamma}_1}^Tx^{(N)} + |\Gamma_1x_0|_Q^2 + |u_i|_R^2\big]dt\\
&+ \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[|x_i(T)|_H^2 - |x^{(N)}(T)|^2_{H_{\hat{\Gamma}}} - 2(H_{\hat{\Gamma}_1}x_0(T))^T\bar{x}(T) + |\Gamma_1x_0(T)|_H^2\big]\\
\leq&\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\int_0^T\big[|x_i|_Q^2 - |\bar{x}|^2_{Q_\Gamma} - 2x_0^TQ_{\hat{\Gamma}_1}^T\bar{x} + |\Gamma_1x_0|_Q^2 + |u_i|_R^2\big]dt\\
&+ \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[|x_i(T)|_H^2 - |\bar{x}(T)|^2_{H_{\hat{\Gamma}}} - 2(H_{\hat{\Gamma}_1}x_0(T))^T\bar{x}(T) + |\Gamma_1x_0(T)|_H^2\big] + \epsilon_1\\
\triangleq&\bar{J}_{\rm soc}^{(N)}(u_0, u) + \epsilon_1.
\end{aligned}
\tag{C.7}
$$

We now deform $\bar{J}_{\rm soc}^{(N)}(u_0, u)$ by the method of completing squares. Note that $\bar{x} = \mathbb{E}[x_i|\mathcal{F}^0]$ satisfies

$$
d\bar{x} = [(A + G)\bar{x} + B\bar{u} + Fx_0]dt,
\tag{C.8}
$$

where $\bar{u} = \mathbb{E}[u_i|\mathcal{F}^0]$. Then, it follows that

$$
d(x_i - \bar{x}) =[A(x_i - \bar{x}) + B(u_i - \bar{u}) + G(x^{(N)} - \bar{x})]dt + (Cx_i + Du_i + \bar{G}x^{(N)} + \bar{F}x_0)dW_i.
$$

From (C.6), applying Itô's formula to $|x_i - \bar{x}|_M^2$, we obtain

$$
\begin{aligned}
&\mathbb{E}\big[|x_i(T) - \bar{x}(T)|_H^2 - |x_i(0) - \bar{x}(0)|^2_{M(0)}\big]\\
=&\mathbb{E}\int_0^T\Big\{(x_i - \bar{x})^T(\dot{M} + A^TM + MA + C^TMC)(x_i - \bar{x}) + (u_i - \bar{u})^TD^TMD(u_i - \bar{u})\\
&+ 2(u_i - \bar{u})^T(B^TM + D^TMC)(x_i - \bar{x}) + \bar{u}^TD^TMD\bar{u} + x_0^T\bar{F}^TM\bar{F}x_0\\
&+ \bar{x}^T(C + G)^TM[(C + \bar{G})\bar{x} + 2\bar{F}x_0] + 2\bar{u}^TD^TM[(C + \bar{G})\bar{x} + \bar{F}x_0]\\
&+ 2(x^{(N)} - \bar{x})^T[(\bar{G}^TMC + G^TM)(x_i - \bar{x}) + \bar{G}^TMD(u_i - \bar{u})]\Big\}dt.
\end{aligned}
\tag{C.9}
$$

It follows by (C.8) that

$$
\begin{aligned}
\text{(C.10)}\qquad &\mathbb{E}\big[\bar{x}^T(T)(H-H_{\hat{\Gamma}})\bar{x}(T)-\bar{x}^T(0)(M(0)+\bar{M}(0))\bar{x}(0)\big]\\
=&\mathbb{E}\int_0^T\big\{\bar{x}^T[\dot{M}+\dot{\bar{M}}+(A+G)^T(M+\bar{M})+(M+\bar{M})(A+G)]\bar{x}\\
&+2\bar{x}^T(M+\bar{M})B\bar{u}+2\bar{x}^T(M+\bar{M})Fx_0\big\}dt.
\end{aligned}
$$

By (2.6) and Itô's formula,

$$
\begin{aligned}
\text{(C.11)}\qquad &\mathbb{E}\big[x_0^T(T)\hat{\Gamma}_1^T H\hat{\Gamma}_1 x_0(T)-x_0^T(0)\Lambda^0(0)x_0(0)\big]\\
=&\mathbb{E}\int_0^T\big\{x_0^T[\dot{\Lambda}^0+(A_0+B_0P_0)^T\Lambda^0+\Lambda^0(A_0+B_0P_0)+(C_0+D_0P_0)^T\Lambda^0(C_0+D_0P_0)]x_0\\
&+2x_0^T[\Lambda^0B_0\bar{P}+(C_0+D_0P_0)^T\Lambda^0D_0\bar{P}]\bar{x}+2\bar{x}^T\bar{P}^TD_0^T\Lambda^0D_0\bar{P}\bar{x}\big\}dt.
\end{aligned}
$$

Applying Itô's formula to $x_0^T\bar{\Lambda}\bar{x}$ and $\bar{x}^TM^0x_0$, we have

$$
\begin{aligned}
\text{(C.12)}\qquad &\mathbb{E}\big[-x_0^T(T)H_{\hat{\Gamma}_1}^T\bar{x}(T)-x_0^T(0)\bar{\Lambda}(0)\bar{x}(0)\big]\\
=&\mathbb{E}\int_0^T\big\{x_0^T[\dot{\bar{\Lambda}}+\bar{\Lambda}(A+G)+(A_0+B_0P_0)^T\bar{\Lambda}]\bar{x}+x_0^T\bar{\Lambda}(B\bar{u}+Fx_0)+\bar{x}^T\bar{P}^TB_0^T\bar{\Lambda}\bar{x}\big\}dt,
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(C.13)}\quad &\mathbb{E}\big[-\bar{x}^T(T)H_{\hat{\Gamma}_1}x_0(T)-\bar{x}^T(0)M^0(0)x_0(0)\big]\\
=&\mathbb{E}\int_0^T\big\{\bar{x}^T[\dot{M}^0+(A+G)^TM^0+M^0(A_0+B_0P_0)]\bar{x}+(B\bar{u}+Fx_0)^TM^0x_0+\bar{x}^TM^0B_0\bar{P}\bar{x}\big\}dt.
\end{aligned}
$$

From (4.12), (C.9)-(C.13), one can obtain

$$
\begin{aligned}
&\bar{J}_{\mathrm{soc}}^{(N)}(u_0,u)\\
=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\big[|x_i-\bar{x}|_Q^2+|\bar{x}|^2_{Q-Q_\Gamma}+2[(\Gamma-I)^TQ\Gamma_1x_0]^T\bar{x}+|\Gamma_1x_0|_Q^2+|u_i-\bar{u}|_R^2+|\bar{u}|_R^2\big]dt\\
&+\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[|x_i(T)-\bar{x}(T)|_H^2+|\bar{x}(T)|^2_{H-H_{\hat{\Gamma}}}+2[(\hat{\Gamma}-I)^TH\hat{\Gamma}_1x_0(T)]^T\bar{x}(T)+|\Gamma_1x_0(T)|_H^2\big]\\
=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[|x_i(0)-\bar{x}(0)|^2_{M(0)}+|\bar{x}(0)|^2_{M(0)+\bar{M}(0)}+2x_0^T(0)\bar{\Lambda}(0)x^{(N)}(0)+|x_0(0)|^2_{\Lambda_0(0)}\big]\\
&+\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{(x_i-\bar{x})^T\Psi^T\Upsilon^{-1}\Psi(x_i-\bar{x})+(u_i-\bar{u})^T\Upsilon(u_i-\bar{u})+2(u_i-\bar{u})^T\Psi(x_i-\bar{x})\\
&+\bar{u}^T\Upsilon\bar{u}+\bar{x}^T(\Psi+\bar{\Psi})^T\Upsilon^{-1}(\Psi+\bar{\Psi})\bar{x}+2\bar{u}^T[(\Psi+\bar{\Psi})\bar{x}+\Psi^0x_0]+(\Psi^0x_0)^T\Upsilon^{-1}\Psi^0x_0\\
&+2\bar{x}^T(\Psi+\bar{\Psi})^T\Upsilon^{-1}\Psi^0x_0+2(x^{(N)}-\bar{x})^T[(\bar{G}^TMC+G^TM)(x_i-\bar{x})+\bar{G}^TMD(u_i-\bar{u})]\Big\}dt\\
=&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[|\xi_i|^2_{M(0)}+|\bar{\xi}|^2_{\bar{M}(0)}+2\xi_0^T\bar{\Lambda}(0)\xi_i+|\xi_0|^2_{\Lambda_0(0)}\big]\\
&+\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{|u_i-\bar{u}+\Upsilon^{-1}\Psi(x_i-\bar{x})|^2_\Upsilon+|\bar{u}+\Upsilon^{-1}[(\Psi+\bar{\Psi})\bar{x}+\Psi^0x_0]|^2_\Upsilon\\
&+2(x^{(N)}-\bar{x})^T[\bar{G}^TMC+G^TM](x_i-\bar{x})+\bar{G}^TMD(u_i-\bar{u})]\Big\}dt\\
\geq&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[|\xi_i|^2_{M(0)}+|\bar{\xi}|^2_{\bar{M}(0)}+2\xi_0^T\bar{\Lambda}(0)\xi_i+|\xi_0|^2_{\Lambda_0(0)}\big]\\
&+\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T2(x^{(N)}-\bar{x})^T[(\bar{G}^TMC+G^TM)(x_i-\bar{x})+\bar{G}^TMD(u_i-\bar{u})]dt.
\end{aligned}
$$

Note that $\hat{u}_i = -\Upsilon^{-1}(\Psi x_i + \bar{\Psi}\bar{x} + \Psi^0 x_0)$. Then, from (C.5) and (C.7), we have $J_{\rm soc}^{(N)}(\hat{u}_0, \hat{u}) \leq J_{\rm soc}^{(N)}(\hat{u}_0, u) + \epsilon_1$, where $\epsilon_1 = O(1/\sqrt{N})$.

(For the leader). From (2.2), we have

$$\begin{aligned} \text{(C.14)}\qquad J_0(\hat{u}_0, \hat{u}(\hat{u}_0)) \leq & \bar{J}_0(\hat{u}_0, \hat{u}(\hat{u}_0)) + \mathbb{E}\int_0^T \Big[2\big(|x_0(t) - \Gamma_0\bar{x}(t)|^2 |Q_0\Gamma_0(\hat{x}^{(N)}(t) - \bar{x}(t))|^2\big)^{1/2} \\ & + |\Gamma_0(\hat{x}^{(N)}(t) - \bar{x}(t))|^2_{Q_0}\Big] dt + |\hat{\Gamma}_0(\hat{x}^{(N)}(T) - \bar{x}(T))|^2_{H_0}\Big] \\ & + 2\mathbb{E}\Big[\big(|x_0(T) - \hat{\Gamma}_0\bar{x}(T)|^2 |H_0\hat{\Gamma}_0(\hat{x}^{(N)}(T) - \bar{x}(T))|^2\big)^{1/2} \\ \leq & \bar{J}_0(\hat{u}_0, \hat{u}(\hat{u}_0)) + O(1/\sqrt{N}). \end{aligned}$$

By Itô's formula, one can obtain

$$\begin{aligned} \text{(C.15)}\qquad & \mathbb{E}[x_0^T(T)H_0x_0(T)] - \mathbb{E}[x_0^T(0)\Theta_1(0)x_0(0)] \\ = & \mathbb{E}\int_0^T \big[x_0^T(\dot{\Theta}_1 + A_0^T\Theta_1 + \Theta_1A_0 + C_0^T\Theta_1C_0)x_0 + 2u_0^T(B_0^T\Theta_1 + D_0^T\Theta_1C_0)x_0\big] dt, \end{aligned}$$

$$\begin{aligned} \text{(C.16)}\qquad & \mathbb{E}[\bar{x}^T(T)\hat{\Gamma}_0^T H_0\hat{\Gamma}_0\bar{x}(T)] - \mathbb{E}[\bar{x}^T(0)\Theta_2(0)\bar{x}(0)] \\ = & \mathbb{E}\int_0^T \big[\bar{x}^T(\dot{\Theta}_2 + \hat{A}^T\Theta_2 + \Theta_2\hat{A})\bar{x} + 2x_0^T\hat{F}^T\Theta_2\bar{x}\big] dt, \end{aligned}$$

and

$$\begin{aligned} \text{(C.17)}\qquad & \mathbb{E}[\bar{x}^T(T)(-\hat{\Gamma}_0^T H_0)x_0(T)] - \mathbb{E}[\bar{x}^T(0)\Theta_3(0)x_0(0)] \\ = & \mathbb{E}\int_0^T \big[\bar{x}^T(\dot{\Theta}_3 + \hat{A}^T\Theta_3 + \Theta_3A_0)x_0 + \bar{x}^T\Theta_3B_0u_0 + x_0^T\hat{F}^T\Theta_3x_0\big] dt. \end{aligned}$$

It follows from (C.15)-(C.17) that

$$\begin{aligned} \text{(C.18)}\qquad \bar{J}_0(u_0, u(u_0)) = & \mathbb{E}[x_0^T(0)\Theta_1(0)x_0(0) + \bar{x}^T(0)\Theta_2(0)\bar{x}(0) + \bar{x}^T(0)\Theta_3(0)x_0(0)] \\ & + \mathbb{E}\int_0^T \Big[x_0^T(B_0^T\Theta_1 + D_0^T\Theta_1C_0)^T\Xi^{-1}(B_0^T\Theta_1 + D_0^T\Theta_1C_0)x_0 \\ & + \bar{x}^T\Theta_3B_0\Xi^{-1}B_0^T\Theta_3\bar{x} + 2\bar{x}^T\Theta_3B_0\Xi^{-1}(B_0^T\Theta_1 + D_0^T\Theta_1C_0)x_0 \\ & + 2u_0^T[(B_0^T\Theta_1 + D_0^T\Theta_1C_0)x_0 + B_0^T\Theta_3\bar{x}] + u_0^T\Xi u_0\Big] dt \\ = & \mathbb{E}[\xi_0^T\Theta_1(0)\xi_0 + \bar{\xi}^T\Theta_2(0)\bar{\xi} + \bar{\xi}^T\Theta_3(0)\xi_0] + \mathbb{E}\int_0^T \Big[\Big|u_0 \\ & + \Xi^{-1}(B_0^T\Theta_1 + D_0^T\Theta_1C_0)x_0 + \Xi^{-1}B_0^T\Theta_3\bar{x}\Big|^2_{\Xi}\Big] dt \\ \geq & \mathbb{E}[\xi_0^T\Theta_1(0)\xi_0 + \bar{\xi}^T\Theta_2(0)\bar{\xi} + \bar{\xi}^T\Theta_3(0)\xi_0] = \bar{J}_0(\hat{u}_0, \hat{u}(\hat{u}_0)). \end{aligned}$$

This together with (C.14) leads to $J_0(\hat{u}_0, \hat{u}(\hat{u}_0)) \leq \bar{J}_0(u_0, u(u_0)) + O(1/\sqrt{N})$. The reminder of the proof is similar to that of Theorem 3.7. $\square$

REFERENCES

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory.* Basel, Switzerland: Birkhäuser Verlag, 2003.

[2] J. Arabneydi and A. Mahajan, *Team-optimal solution of finite number of mean-field coupled LQG subsystems,* Proc. 54th IEEE Conference on Decision and Control (CDC), Osaka, Japan, 2015, pp. 5308-5313.

[3] M. R. Arefin, T. Masaki, K. M. A. Kabir, and J. Tanimoto, *Interplay between cost and effectiveness in influenza vaccine uptake: a vaccination game approach,* Proc. R. Soc. A, 475(2019), pp. 1-19.

[4] M. Athans, *The matrix minimum principle*, Inform. and Control, 11(1968), pp. 592-606.

[5] M. M. Baharloo, J. Arabneydi, and A. G. Aghdam, *Minmax mean-field team approach for a leader-follower network: A saddle-point strategy*, IEEE Contr. Syst. Lett, 4(2020), pp. 121-126.

[6] M. Bardi and F. S. Priuli, *Linear-quadratic n-person and mean-field games with ergodic cost*, SIAM J. Control Optim., 52(2014), pp. 3022-3052.
[7] T. Başar, A. Bensoussan and S. P. Sethi, *Differential games with mixed leadership: the open-loop solution*, Appl. Math. Comput, 217(2010), pp. 972-979.
[8] A. Bensoussan, M. H. M. Chau and S. C. P. Yam, *Mean field Stackelberg games:Aggregation of delayed instructions*, SIAM J. Control Optim., 53(2015), pp. 2237-2266.
[9] A. Bensoussan, J. Frehse and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer, New York, 2013.
[10] R. Buckdahn, J. Li, and S. Peng, *Nonlinear stochastic differential games involving a major player and a large number of collectively acting minor agents*, SIAM J. Control Optim., 52(2014), pp. 451-492.
[11] P. E. Caines, M. Huang, and R. P. Malhame, *Mean field games*, in Handbook of Dynamic Game Theory, T. Basar and G. Zaccour Eds, Springer, Berlin, 2017, pp. 345-372.
[12] P. E. Caines and A. C. Kizilkale, *ε-Nash equilibria for partially observed LQG mean field games with a major player*, IEEE Trans. Automat. Control, 62(2017), pp. 3225-3234.
[13] R. Carmona, G. Dayanikl, and M. Laurière. Mean field models to regulate carbon emissions in electricity production. *Dynamic Games and Applications*, 12(2022), pp. 897-928.
[14] R. Carmona, F. Delarue and D. Lacker, *Mean field games with common noise*, Ann Probab, 44(2014), pp. 3740-3803.
[15] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications: I and II*. Springer-Verlag, 2018.
[16] X. Feng, J. Huang and Z. Qiu, *Mixed social optima and Nash equilibrium in linear-quadratic-Gaussian mean-field system*, IEEE Trans. Automat. Control, 67(2022), pp. 6858-6865.
[17] Y. C. Ho and K. C. Chu, *Team decision theory and information structures in optimal control Part I*, IEEE Trans. Automat. Control, 17(1972), pp. 15-22.
[18] J. Huang and M. Huang, *Robust mean field linear-quadratic-Gaussian games with unknown l^2-disturbance*, SIAM J. Control Optim., 55(2017), pp. 2811-2840.
[19] J. Huang, B.-C. Wang and T. Xie, *Social optima in leader-follower mean field linear quadratic control*, ESAIM, COCV, 27(2021).
[20] J. Huang, B.-C. Wang, and J. Yong, *Social optima in mean field linear-quadratic-Gaussian control with volatility uncertainty*, SIAM J. Control Optim., 59(2021), pp. 825-856.
[21] M. Huang, *Large-population LQG games involving a major player: the Nash certainty equivalence principle*, SIAM J. Control Optim., 48(2010), pp. 3318-3353.
[22] M. Huang, P. E. Caines and R. P. Malhamé, *Large-population cost-coupled LQG problems with non-uniform agents: individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Automat. Control, 52(2007), pp. 1560-1571.
[23] M. Huang, P. E. Caines and R. P. Malhamé, *Social optima in mean-field LQG control: centralized and decentralized strategies*, IEEE Trans. Automat. Control, 57(2012), pp. 1736-1751.
[24] M. Huang and S. L. Nguyen, *Linear-quadratic mean field teams with a major agent*, in Proceedings of the 55th IEEE Conference on Decision and Control, 2016, pp. 6958-6963.
[25] X. Jiang, Y. Wang, D. Zhao and L. Shi, Online Pareto optimal control of mean-field stochastic multi-player systems using policy iteration, Science China Information Sciences, 67(2024), 140202.
[26] M. Huang and M. Zhou, *Linear quadratic mean field games: Asymptotic solvability and relation to the fixed point approach*, IEEE Trans. Automat. Control, 65(2020), pp. 1397-1412.
[27] H. K. Khalil. *Nonlinear Systems*, 3rd edition. Prentice Hall, Inc, 2002.
[28] M. Larranaga, J. Denis, M. Assaad, and K. D. Turck, *Energy-efficient distributed transmission scheme for MTC in dense wireless networks: a mean-field approach*, IEEE Internet Things J., 7(2020), pp. 477-490.
[29] J. Lasry and P. Lions, *Mean field games*, Japanese Journal of Mathematic, 2(2007), pp. 229-260.
[30] T. Li and J. Zhang, *Asymptotically optimal decentralized control for large population stochastic multiagent systems*, IEEE Trans. Automat. Control, 53(2008), pp. 1643-1660.
[31] J. Ma and J. Yong, *Forward-backward stochastic differential equations and their applications*, Lecture Notes in Math, Springer-Verlag, 1999.
[32] J. Moon, *Linear-quadratic stochastic leader-follower differential games for Markov jump-diffusion models*, Automatica, 147(2023), 110713.
[33] J. Moon and T. Başar, *Linear quadratic mean field Stackelberg differential games*, Automatica, 97(2018), pp. 200-213.
[34] G. Nuño, & B. Moll, *Social optima in economies with heterogeneous agents*, Rev Econ Dynam, 28(2018), pp. 150-180.
[35] R. Penrose, *A generalized inverse of matrices*, Proc. Cambridge Philos. Soc., 52(1955), pp. 17-19.
[36] R. S. Pindyck, *Optimal economic stabilization policies under decentralized control and conflicting objectives*, IEEE Trans. Automat. Control, 22(1977), pp. 517-530.
[37] Q. Qi, H. Zhang, and Z. Wu, *Stabilization control for linear continuous-time mean-field systems*, IEEE Trans. Automat. Control, 64(2019), pp. 3461-3468.
[38] Z. Qiu, J. Huang and T. Xie, *Linear quadratic Gaussian mean-field controls of social optima*, Math. Control. Relat. Fields, 13(2023), pp. 1-34.
[39] R. Salhab, J. L. Ny and R. P. Malhame, *Dynamic collective choice: Social optima*, IEEE Trans. Automat. Control, 63(2018), pp. 3487-3494.
[40] S. Sanjari and S. Yuksel. *Optimal solutions to infinite-player stochastic teams and mean-field teams.* IEEE Trans. Automat. Control, 66(2021), pp. 1071-1086.

[41] N. Sen and P. E. Caines, *Mean field game theory with a partially observed major agent*, SIAM J. Control Optim., 54(2016), pp. 3174-3224.

[42] M. Simaan and J. Cruz, *A Stackelberg solution for games with many players*, IEEE Trans. Automat. Control, 18(1973), pp. 322-324.

[43] J. Sun, X. Li, and J. Yong, *Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems*, SIAM J. Control Optim., 54(2016), pp. 2274-2308.

[44] M. A. Tajeddini, H. Kebriaei, L. Glielmo, *Decentralized hierarchical planning of PEVs based on mean-field reverse Stackelberg game*, IEEE T Autom Sci Eng, 17(2020), pp. 2014-2024.

[45] H. Tembine, Q. Zhu and T. Başar, *Risk-sensitive mean-field games*, IEEE Trans. Automat. Control, 59(2014), pp. 835-850.

[46] B.-C. Wang, J. Huang and J.-F. Zhang, *Social optima in robust mean field LQG control: From finite to infinite horizon*, IEEE Trans. Automat. Control, 66(2021), pp. 1529-1544.

[47] B.-C. Wang, S. Li, and Y. Cao. *An online value iteration method for linear-quadratic mean field social control with unknown dynamics*, Science China Information Sciences, 67(2024), 140203.

[48] B.-C. Wang and M. Huang, *Mean field production output control with sticky prices: Nash and social solutions*, Automatica, 100(2019), pp. 90-98.

[49] B. -C. Wang and H. Zhang, *Indefinite linear quadratic mean field social control problems with multiplicative noise*, IEEE Trans. Automat. Control, 66(2021), pp. 5221-5236.

[50] B.-C. Wang and J.-F. Zhang, *Mean field games for large population multiagent systems with Markov jump parameters*, SIAM J. Control Optim., 50(2012), pp. 2308-2334.

[51] B.-C. Wang and J.-F. Zhang, *Distributed control of multi-agent systems with random parameters and a major agent*, Automatica, 48(2012), pp. 2093-2106.

[52] B.-C. Wang and J.-F. Zhang, *Hierarchical mean field games for multiagent systems with tracking-type costs: distributed ε-Stackelberg equilibria*, IEEE Trans. Automat. Control, 59(2014), pp. 2241-2247.

[53] B.-C. Wang and J.-F. Zhang, *Social optima in mean field linear-quadratic-Gaussian models with Markov jump parameters*, SIAM J. Control Optim., 55(2017), pp. 429-456.

[54] G. Y. Weintraub, C. L. Benkard and B. V. Roy, *Markov perfect industry dynamics with many firms*, Econometrica, 76(2008), pp. 1375-1411.

[55] X. Yang and M. Huang, *Linear quadratic mean field Stackelberg games: Master equations and time consistent feedback strategies*, Proc. 60th IEEE Conference on Decision and Control (CDC), Austin, TX, USA, 2021, pp. 171-176.

[56] J. Xu, H. Zhang, and T. Chai, *Necessary and sufficient condition for two-player Stackelberg strategy*, IEEE Trans. Automat. Control, 60(2015), pp. 1356-1361.

[57] J. Yong, *A leader-follower stochastic linear quadratic differential game*, SIAM J. Control Optim., 41(2002), pp. 1015-1041.

[58] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*, SIAM J. Control Optim., 51(2013), pp. 2809-2838.

[59] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.