Consensus of multi-agent systems under binary-valued measurements: An event-triggered coordination approach ⋆

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Abstract

This paper investigates the consensus problem of multi-agent systems. Owing to the effect of bandwidth restriction, the communication information between two adjacent agents is based on the binary-valued observation of the imprecise sender's output, which is only 1-bit. Each agent employs the recursive projection algorithm to estimate the states of all neighbours. Then, an event-triggered coordination technique is proposed to address the consensus problem. The first characteristic of this technique is that the information data transmission between two adjacent agents is managed by an event-triggered scheduler. The second characteristic of this technique is that each agent adopts the event-triggered control protocol to achieve the ultimate target. By this technique, the estimation error and the consensus error can converge to zero in the mean square sense and in the almost sure sense with explicit convergence speeds. Moreover, the communication rate of the communication scheme between two adjacent agents does not exceed fifty percent. And the communication rate of the control scheme can almost surely converge to zero with an explicit rate. Compared with the existing results, these two communication rates can be extremely reduced when achieving the same convergence value. As a result, the communication resources can be saved in terms of quantization, communication and control. An example is presented to illustrate the advantage of the proposed technique.

Key words: Consensus, multi-agent systems, binary-valued observation, event-triggered coordination.

1 Introduction

Consensus is one of the key problems of multi-agent systems (MASs) and has been a hot topic for a long time [1–6]. As an effective tool, the event-triggered control strategy is extensively used to deal with the consensus problem of MASs [7–9]. The main characteristic of this strategy is that the designed controller of each agent is affected by an event. If this event is satisfied, then information data transmission inside the controller is achieved. Therefore, by the event-triggered control strategy, frequent data transmission can be avoided, and hence the communication resources can be saved. However, the majority of existing works concerning the event-triggered consensus problem acquiesce in sustained communication between two adjacent agents. In other words, the communication data is transmitted from one agent to its neighbours all the time. From the perspective of consensus achievement and further resource saving, an interesting question is whether the sustained communication scheme can be replaced by the event-triggered communication scheme.

In recent years, an event-triggered coordination technique has been proposed in [10–13], which can answer the above question. With this technique, event-triggered control and event-triggered communication perform simultaneously. To put it another way, when some event is triggered, each agent broadcasts its state information to all the neighbours, and updates its control signal at the same time. In this way, the communication between two adjacent agents is no longer continuous, and the communication frequency can be extremely reduced. As a result, the communication resources can be saved. However, in these works, the communication schemes are based on the exact state information, which is infinite-bit. In practical engineering, owing to the limitation of channel bandwidth, the transmitted information needs to be quantized to a finite number of bits.

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Consequently, the available information is rather finite and imprecise. Such communication condition will make the event-triggered coordination problem more challenging. Furthermore, the less the amount of exchanged information is, the harder this problem is.

During the past decades, quantized consensus problem of MASs has attracted considerable research attention [14–17]. Moreover, the idea of event-triggered control is successfully used to address this problem with multi-level quantization [18–20]. Speaking of quantized consensus, the authors consider that the most difficult situation may be that each agent can only receive 1-bit binary-valued information from its neighbours. In other words, each agent can only distinguish whether its neighbor's workload with noises is larger than a threshold or not [21–28]. Currently, there are mainly two approaches to deal with the consensus problem of MASs with binary-valued communication. One is offline [23, 26, 27], in which the estimation procedure is executed for some holding time, and then the control procedure is started at some skipping time. These two procedures are performed alternatively. Thus, they are separate and are not performed at every time instant. The other is online [24,25,28], in which these two procedures are coupled and can be performed at every time instant. However, [23,24] and their follow-up works [25–28], sustained control data transmission and sustained communication between two adjacent agents are both required, which may cause resource waste." For the sake of consensus achievement and resource saving, a natural question is whether the event-triggered coordination technique can be used for the consensus problem of MASs with binary-valued communication. And to the best of our knowledge, this question is still open, which motivates this paper.

Based on the above discussions, this paper concentrates on the consensus problem of MASs with binary-valued communication via an event-triggered coordination technique. The main contributions are threefold.

- The event-triggered coordination technique is first proposed to address the consensus problem of MASs with only binary-valued communication. With this technique, the mean-square consensus and the almost sure consensus with explicit convergence speeds can be realized. Furthermore, this technique allows that event-triggered control and event-triggered communication can perform asynchronously, which is different from those in [10–13]. In [10–13], the communication processes between two adjacent agents do not consider the effect of bandwidth restriction and require the exact state information of each agent, which is infinite-bit and can cause high power consumption. Compared with them, this paper considers the effect of bandwidth restriction. And the communication information is even binary-valued, i.e. 1-bit information. Thus, in terms of the power consumption, the communication resources can be saved by the technique in this paper in comparison to those in [10–13].
- The communication process between two adjacent agents is managed by an event-triggered scheduler depending on the estimation effect. The scheduler triggers communication only when the estimation effect is bad. Thus, the designed communication scheme can avoid frequent data transmission compared with the sustained ones in [23, 24]. Moreover, the communication rate of the communication scheme cannot exceed fifty percent, which is much lower than those in [23, 24]. In [23, 24], this communication rate is one hundred percent. Intuitively, the lower the communication rate is, the lower the frequency of data transmission is. Consequently, in terms of the transmission times of data, the communication resources can be saved by the communication scheme in this paper in comparison to those in [23, 24].
- The event-triggered control strategy is related to the consensus effect and the estimation effect. Data transmission in the process of controller design is triggered only when at least one effect is bad. Thus, the designed control scheme can avoid frequent data transmission in comparison to the state-feedback ones in [23, 24]. Even so, under some conditions, the convergence speed in this paper can reach the fastest case in [24] $\big(O(t^{-1})\big)$, and is faster than the fastest case in [23] $\big(O(t^{-\frac{1}{2}})\big)$. In addition, the communication rate related to the control scheme can almost surely converge to zero with a rate $o(t^{-\kappa})$ (κ is a positive constant depending on the triggering condition and the convergence speed), which can be extremely reduced compared with those in [23, 24] when achieving the same convergence value. In [23, 24], this communication rate is one hundred percent. Intuitively, the lower the communication rate is, the lower the frequency of data transmission is. As a result, in terms of the transmission times of data, the communication resources can be further saved by the control scheme in this paper in comparison to those in [23, 24].

The rest of this paper is organized as follows. In Section 2, the considered problem is formulated. In Section 3, communication schedulers are designed. In Section 4, dynamic event-triggered controllers are designed. Section 5 investigates the convergence and the convergence speeds of the consensus error and the estimation error. Section 6 shows the communication rates of the designed schemes. In Section 7, a simulation example is given to demonstrate the effectiveness of derived results. Finally, Section 8 summarizes the main results.

Notations. $\mathbb{R}$ represents the set of real numbers. $\mathbb{Z}$ represents the set of integer numbers. $\mathbb{E}$ represents the mathematical expectation operator. $\mathbb{P}$ represents the probability operator. $\|\cdot\|$ represents the Euclidean norm. A^T represents the transpose of matrix A. $exp(\cdot)$ represents the exponential function. $f \xrightarrow{a.s.} g$ means f converges to g almost surely or with probability 1.

2 Problem formulation

Consider the following MAS:

$$x_i(t+1) = x_i(t) + u_i(t), \tag{1}$$

where $t \in \mathbb{Z}$, $x_i(t) \in \mathbb{R}$ is the system state, $u_i(t) \in \mathbb{R}$ is the control of ith agent, $i = 1, \ldots, n$. The interaction topology among the agents is described by an undirected and connected graph $\mathcal{G}$. $(i, j) \in \mathcal{G}$ means that there is an edge between agent i and agent j. The adjacency matrix is defined by $A = [a_{ij}]_{n\times n}$, where $a_{ii} = 0$, $a_{ij} = 1$ if there exists an edge between agent i and agent j, and

$a_{ij} = 0$ otherwise. The set of neighbors of agent i is denoted as N_i. The degree of agent i is denoted as d_i. The Laplacian matrix is defined as $L = D - A$, where $D = diag\{d_i\}$, $i = 1, \dots, n$. The eigenvalues of L satisfy $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$. The initial state of each agent satisfies $|x_i(0)| \leq M_0$ for all $i = 1, 2, \dots, n$, where $M_0 > 0$ is a given constant.

In this paper, it is assumed that the information that each agent receives from its neighbour or neighbours is affected by some stochastic noises. And the observation is based on a binary-valued quantizer.

$$\begin{cases} y_{ij}(t) = x_j(t) + \omega_{ij}(t), \\ s_{ij}(t) = I_{\{y_{ij}(t) \leq C_{ij}\}}, \end{cases} \tag{2}$$

where $y_{ij}(t)$ is the output information that agent j intends to send to agent i, but is affected by the noise $\omega_{ij}(t)$. Thus, $x_j(t)$ cannot be measured well. The noises $\{\omega_{ij}(t), (i,j) \in \mathcal{G}\}$ are identically normally distributed random variables with mean 0, and are independent with respect to i, j, t. The distribution function and the associated density function are, respectively, $F(\cdot)$ and $f(x) = \frac{dF(x)}{dx} \neq 0$. The σ-algebra generated by $\{\omega_{ij}(t)\}$ is defined as $\mathcal{D}_t = \sigma\{\omega_{ij}(1), \dots, \omega_{ij}(t)\}$. The communication information that agent i actually receives from agent j is based on the binary-valued measurement $s_{ij}(t)$, which is decided by the indicator function $I_{\{\cdot\}}$. That is, $s_{ij}(t) = 1$ if $y_{ij}(t) \leq C_{ij}$, and $s_{ij}(t) = 0$ otherwise. Constant C_{ij} is a given threshold, $j \in N_i$.

Definition 2.1 *If there exist constants $c > 0$ and $t^* > 0$, such that the sequences $f(t)$ and $g(t)$ satisfy $|\frac{f(t)}{g(t)}| \leq c$ for any $t > t^*$, then $f(t) \doteq O(g(t))$. If $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, then $f(t) \doteq o(g(t))$.*

In [23, 24], the communication between two adjacent agents is sustained, i.e. the binary-valued information $s_{ij}(t)$ is transmitted from agent j to agent i all the time. Moreover, [23, 24] employ the state-feedback control method to achieve the consensus target. For the same target, this paper proposes an event-triggered coordination technique, by which event-triggered communication and event-triggered control can perform asynchronously. With this technique, the communication rates can be reduced and the communication resources can be saved. The system schematic is shown in Figure 1.

The target of this paper is to propose an event-triggered coordination technique for each agent to estimate its neighbour's state or neighbours' states, and make system (1) achieve consensus with explicit convergence speeds.

3 Scheduler design

In this section, schedulers with triggered events will be designed. According to Figure 1, schedulers decide when the binary-valued information $s_{\{\cdot\}}(t)$ are sent to estimators. For convenience, we define a scheduler switch $\beta_{ij} = I_{\{\cdot\}}$ for any i and $j \in N_i$. If $\beta_{ij} = 1$, then the scheduler j sends $s_{ij}(t)$ to the estimator i. If $\beta_{ij} = 0$, then the communication is denied. The communication

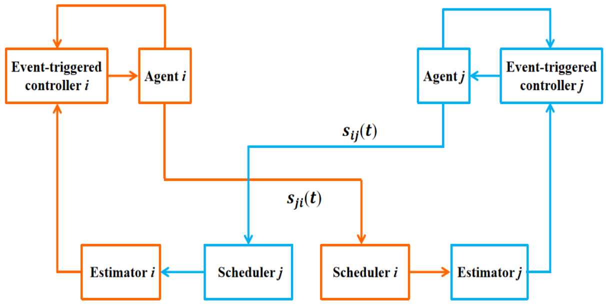

Fig. 1. The communication between two adjacent agents is based on a triggered event to be designed later. The agent j sends the binary-valued information $s_{ij}(t)$ ($s_{ij}(t) = 0$ or 1) to the scheduler j. And when the event is triggered, the scheduler j sends $s_{ij}(t)$ to the estimator i to estimate $x_j(t)$.

rate of the communication scheme between the scheduler j and the estimator i is defined as [31]

$$\bar{\beta}_{ij} = \lim_{t\to\infty} \frac{\sum_{k=1}^{t} \beta_{ij}(k)}{t}. \tag{3}$$

Now, we design three kinds of triggered events for each scheduler as follows.

Scenario I: $\beta_{ij}(t) = I_{\{s_{ij}(t)=1\}}$. In this scenario, only when $s_{ij}(t) = 1$, the communication is permissible. In other words, agent j only sends the information $s_{ij}(t) = 1$ to agent i. By noting $\mathbb{E}(\beta_{ij}(t)) = F(C_{ij} - x_j(t))$, and following the law of large numbers [33], we have

$$\frac{\sum_{k=1}^{t} \beta_{ij}(k)}{t} \xrightarrow{a.s.} \frac{\sum_{k=1}^{t} F(C_{ij} - x_j(k))}{t} \leq 1, \ t \to \infty. \tag{4}$$

Thus, by (3) and (4), $\bar{\beta}_{ij} \leq 1$. Thus, in this scenario, the communication rate related to the communication scheme between the scheduler j and the estimator i is not obviously reduced in comparison to those with continuous communication in [23, 24].

Scenario II: $\beta_{ij}(t) = I_{\{s_{ij}(t)=0\}}$. In this scenario, agent j only sends the information $s_{ij}(t) = 0$ to agent i. Again, by noting $\mathbb{E}(\beta_{ij}(t)) = 1 - F(C_{ij} - x_j(t))$, and following the law of large numbers [33], it holds

$$\frac{\sum_{k=1}^{t} \beta_{ij}(k)}{t} \xrightarrow{a.s.} 1 - \frac{\sum_{k=1}^{t} F(C_{ij} - x_j(k))}{t} \leq 1, \ t \to \infty. \tag{5}$$

Similar to the scenario I, by (3) and (5), $\bar{\beta}_{ij} \leq 1$, which implies the communication rate related to the communication scheme between the scheduler j and the estimator i is also not obviously reduced compared with those in [23, 24].

Scenario III: $\beta_{ij}(t) = I_{\{s_{ij}(t)=0, C_{ij}-x_j(t)>\chi\}} + I_{\{s_{ij}(t)=1, C_{ij}-x_j(t)\leq\chi\}}$, where $\chi = F^{-1}(\frac{1}{2})$. In this scenario, only when $s_{ij}(t) = 0$ and $C_{ij} - x_j(t) > \chi$ or $s_{ij}(t) = 1$ and $C_{ij} - x_j(t) \leq \chi$ are satisfied, the communication is permissible. Let $F_{\min}(x) = \min\{F(x), 1 - F(x)\}$ for any $x \in \mathbb{R}$. Then, $F_{\min}(x) \leq \frac{1}{2}$

and

$$F_{\min}(x) = \begin{cases} F(x), & x < \chi, \\ 1 - F(x), & x \geq \chi. \end{cases}$$

Moreover, it holds

$$\mathbb{E}(\beta_{ij}(t)) = F_{\min}(C_{ij} - x_j(t)). \tag{6}$$

From (6), and by the law of large numbers [33], it holds

$$\frac{\sum_{k=1}^{t} \beta_{ij}(k)}{t} \xrightarrow{a.s.} \frac{\sum_{k=1}^{t} F_{\min}(C_{ij} - x_j(k))}{t} \leq \frac{1}{2},\ t \to \infty. \tag{7}$$

By (3) and (7), $\bar{\beta}_{ij} \leq \frac{1}{2}$. Thus, the communication rate related to the communication scheme between the scheduler j and the estimator i can be extremely reduced compared with those in [23, 24]. However, this scenario is difficult to implement since $x_j(t)$ is unknown and needs to be estimated when designing the scheduler j. Thus, in the sequel, we redefine $\beta_{ij}(t)$ as

$$\begin{aligned} \beta_{ij}(t) = & I_{\{s_{ij}(t)=0, C_{ij}-\hat{x}_{ij}(t-1)>\chi\}} \\ & + I_{\{s_{ij}(t)=1, C_{ij}-\hat{x}_{ij}(t-1)\leq\chi\}}, \end{aligned} \tag{8}$$

where $\chi = F^{-1}(\frac{1}{2})$, and $\hat{x}_{ij}(t)$ is the i-th agent's estimate to the state $x_j(t)$ of agent j. In this case, the scheduler j only recognizes whether $\hat{x}_{ij}(t-1)$ satisfies $C_{ij} - \hat{x}_{ij}(t-1) > \chi$ rather than the exact $\hat{x}_{ij}(t-1)$. In addition, in this case, $\mathbb{E}(\beta_{ij}(t)) \neq F_{\min}(C_{ij} - x_j(t))$. Thus, (7) may not be satisfied. In Section 6, we will reanalyze the communication rate $\bar{\beta}_{ij}$.

In this paper, we will adopt the scenario III with (8) to achieve the target in Section 2.

Remark 3.1 *In the scenario III, an event-triggered communication scheme (8) is proposed. If $\beta_{ij}(t) = 1$, then the communication between two adjacent agents is triggered. In [10–13], the communication processes are also event-triggered. However, the communication processes in [10–13] require the exact state information of each agent, which is infinite-bit and can cause high power consumption. But in the communication scheme (8), only 1-bit binary-valued communication data is transmitted. Thus, in terms of the power consumption, the communication resources can be saved through the communication scheme (8).*

4 Controller design

In this section, event-triggered controllers will be designed to realize the target in Section 2. From Figure 1 we know that the designed controller of each agent depends on the estimate of its neighbours' states. Thus, before constructing controllers, we establish the following estimation process

$$\begin{aligned} \hat{x}_{ij}(t) = & \Pi_M\Big\{\hat{x}_{ij}(t-1) + \gamma\rho(t)(F(C_{ij} - \hat{x}_{ij}(t-1)) \\ & -\tilde{s}_{ij}(t))\Big\}, \end{aligned} \tag{9}$$

where constant $M > M_0$, constant γ is the step size for estimation updating, $\rho(t) \doteq (t + \hat{b})^{-\hat{\mu}}$ satisfies $\rho(t) \in (0, \frac{1}{d_i}]$, where $\hat{\mu} \in (0, 1]$ and $\hat{b} \geq 0$, $\Pi_M(z)$ is the recursive projection operator

$$\Pi_M(z) := \arg\min_{|a|\leq M} |z - a|,\ \forall z \in \mathbb{R},$$

and

$$\tilde{s}_{ij}(t) := \beta_{ij}(t)s_{ij}(t) + I_{\{C_{ij}-\hat{x}_{ij}(t-1)>\chi\}}(1 - \beta_{ij}(t)). \tag{10}$$

It should be stressed that the estimation process (9) is different from those in [23, 24]. The former relies on $\tilde{s}_{ij}(t)$, and the latter relies on $s_{ij}(t)$. Specifically, the estimation processes in [23, 24] rely on sustained communication between two adjacent agents, i.e. $s_{ij}(t)$ is transmitted to the estimator i all the time, and then the estimate $\hat{x}_{ij}(t)$ can be constructed. However, from Figure 1, only when an event is triggered, i.e. $\beta_{ij}(t) = 1$, the estimation process proceeds. For example, if $s_{ij}(t) = 0$ and $C_{ij} - \hat{x}_{ij}(t-1) \leq \chi$ for some t, then $\beta_{ij}(t) = 0$, which means the scheduler switch is open and hence the communication between the procedure and the estimator is denied. Therefore, $s_{ij}(t) = 0$ cannot be sent to the estimator i. From this perspective, the communication frequency between two adjacent agents in this paper can be reduced.

The following lemma shows that the value of $\tilde{s}_{ij}(t)$ is equal to that of $s_{ij}(t)$ for any $i = 1, \ldots, n$, $j \in N_i$ and $t \geq 1$. Thus, even if $s_{ij}(t)$ is not sent to the estimator i at some t, the estimator i can still judge the value of $s_{ij}(t)$.

Lemma 4.1 *For any $i = 1, \ldots, n$, $j \in N_i$ and $t \geq 1$, it holds that $\tilde{s}_{ij}(t) = s_{ij}(t)$.*

Proof. We distinguish the following two cases.

Case I: $\beta_{ij}(t) = 1$. In this case, the scheduler switch is closed and the estimator i can receive $s_{ij}(t)$. By (10), we have $\tilde{s}_{ij}(t) = s_{ij}(t)$.

Case II: $\beta_{ij}(t) = 0$. In this case, the scheduler switch is open and the estimator i cannot receive $s_{ij}(t)$.

If $C_{ij} - \hat{x}_{ij}(t-1) > \chi$, then $s_{ij}(t) = 1$ must be true. Otherwise, if $s_{ij}(t) = 0$, we can get $\beta_{ij}(t) = 1$, which is a contradiction. In this way, $\tilde{s}_{ij}(t) = 1 = s_{ij}(t)$.

If $C_{ij} - \hat{x}_{ij}(t-1) \leq \chi$, then $s_{ij}(t) = 0$ must be true. Thus, $\tilde{s}_{ij}(t) = 0 = s_{ij}(t)$.

Based on the above analysis, the proof is completed. ∎

By (9), we are in a position to construct the following event-triggered controllers

$$u_i(t) = -\rho(t) \sum_{j\in N_i} (x_i(t_k^i) - \hat{x}_{ij}(t_k^i)),\ t \in [t_k^i, t_{k+1}^i), \tag{11}$$

where $j \in N_i$, $i = 1, \ldots, n$, $k = 0, 1, \ldots$, and the triggering instant sequence $\{t_k^i\}$ with $t_0^i = 0$ $(i = 1, \ldots, n)$ arises from the following condition

$$\Gamma_i(t) > \frac{\Upsilon_i(t)}{\nu_i} M, \tag{12}$$

where $t > t_k^i$, $\nu_i > 0$ is a constant, and

$$\Gamma_i(t) = \Big| \sum_{j\in N_i} (e_i(t) - e_{ij}(t)) \Big| - \alpha_i(t)\phi_i(t) - g_i(t)M, \quad (13)$$

where $e_i(t) = x_i(t_k^i) - x_i(t)$, $e_{ij}(t) = \hat{x}_{ij}(t_k^i) - \hat{x}_{ij}(t)$, $\alpha_i(t) \in (0,1)$ with $\alpha_i(0) \le 1 - \frac{M_0}{M}$, $g_i(t) > 0$, and

$$\phi_i(t) = \min\Big\{ \big| \sum_{j\in N_i} (x_i(t) - \hat{x}_{ij}(t)) \big|, M \Big\}, \quad (14)$$

$\Upsilon_i(t)$ is an internal dynamic variable satisfying the following equation

$$\Upsilon_i(t+1) = \tilde{\nu}_i \Upsilon_i(t) - \Gamma_i(t), \quad (15)$$

where $\tilde{\nu}_i \in (0,1)$ satisfies $\tilde{\nu}_i \nu_i \ge M$ and $\Upsilon_i(0) \ge 0$.

Remark 4.2 *If ν_i tends to infinity, then the dynamic event-triggered condition (12) will become a static one. In [32], it has been shown that the next execution time given by a dynamic event-triggered condition is larger than that given by a static one. In this paper, this assertion can also be derived from Theorem 6.4. Moreover, Theorem 6.4 further shows a comparison of the communication rate between the dynamic event-triggered mechanism related to (12) and the corresponding static one. That is, in some cases, the communication rate via the former is much lower and can converge to zero faster than that via the latter. For more details, please see Remark 6.6.*

Remark 4.3 *The underlying motivation for designing the event-triggered controller (11)-(15) lies in saving resource while achieving consensus target. With the controller (11)-(15), frequent data transmission can be avoided in comparison to the state-feedback ones in [23, 24], and hence the communication resources can be saved. Moreover, as stated in Remark 4.2, the dynamic event-triggered condition (12) can further reduce the communication rate compared with the static one.*

The following lemma is in connection with the nonnegativity of $\Upsilon_i(t)$, which is necessary to guarantee (12) being a dynamic triggering condition [32].

Lemma 4.4 *For any $t \ge 0$, $\Upsilon_i(t) \ge 0$.*

Proof. From (12), we can get that for any $t \in [t_k^i, t_{k+1}^i)$, $\Gamma_i(t) \le \frac{\Upsilon_i(t)}{\nu_i} M$. Then, by (15), we can derive

$$\Upsilon_i(t+1) \ge (\tilde{\nu}_i - \frac{M}{\nu_i})\Upsilon_i(t), \ t \ge 0.$$

Since $\Upsilon_i(0) \ge 0$ and $\tilde{\nu}_i \nu_i \ge M$, it holds that $\Upsilon_i(t) \ge 0$ for any $t \ge 0$. ∎

The following lemmas are in connection with the convergence of $\Upsilon_i(t)$, which are useful for the convergence speed analysis as shown in Theorems 5.5, 5.7 and 5.8.

Lemma 4.5 *([34]) For any given c, $k_0 > 0$, $0 < p < 1$ and $q \in \mathbb{R}$, we have*

$$\sum_{l=1}^{k} \frac{e^{c(l+k_0)^p}}{(l+k_0)^q} = O\Big(\frac{e^{c(k+k_0)^p}}{(k+k_0)^{p+q-1}} \Big).$$

Actually, Lemma 4.5 also holds provided that $k_0 \ge 0$ and $0 < p \le 1$. The proof is almost the same as that of Lemma 3.3 in [34].

Lemma 4.6 *Let $\alpha_i(t) = O(t^{-\psi_1})$ and $g_i(t) = O(t^{-\psi_2})$, where $\psi_1 > 0$ and $\psi_2 > 0$ are constants. Then, $\Upsilon_i(t) = O(t^{-\psi})$, where $\psi = \min\{\psi_1, \psi_2\}$.*

Proof. From (13) to (15), we have

$$\Upsilon_i(t+1) \le \tilde{\nu}_i \Upsilon_i(t) + (\alpha_i(t) + g_i(t))M.$$

Then, through an iterative procedure, we can get

$$\Upsilon_i(t) \le \tilde{\nu}_i^t \Upsilon_i(0) + \sum_{k=0}^{t-1} (\alpha_i(k) + g_i(k)) \tilde{\nu}_i^{t-1-k} M. \quad (16)$$

By Lemma 4.5, it holds that

$$\sum_{k=0}^{t-1} (\alpha_i(k) + g_i(k)) \tilde{\nu}_i^{t-1-k} = O(t^{-\psi}),$$

which in conjunction with $\tilde{\nu}_i^t = o(t^{-\psi})$ yields $\Upsilon_i(t) = O(t^{-\psi})$. ∎

At the end of this section, the following transformation of system (1) is presented, which will be helpful for the main results in the subsequent sections.

Let the estimation error as $\varepsilon_{ij}(t) = \hat{x}_{ij}(t) - x_j(t)$. And let $\eta_{ij}(t) = e_i(t) - e_{ij}(t)$. Then, substituting (11) into (1) deduces that

$$\begin{aligned} x_i(t+1) &= x_i(t) - \rho(t) \sum_{j\in N_i} (x_i(t) - x_j(t) \\ &\quad + x_j(t) - \hat{x}_{ij}(t) + e_i(t) - e_{ij}(t)) \\ &= x_i(t) - \rho(t) \sum_{j\in N_i} (x_i(t) - x_j(t)) \\ &\quad + \rho(t) \sum_{j\in N_i} \varepsilon_{ij}(t) - \rho(t) \sum_{j\in N_i} \eta_{ij}(t). \end{aligned} \quad (17)$$

5 Convergence and convergence speed analysis

Based on the analysis in Section 3 and Section 4, this section will establish the convergence and the convergence speeds of the estimation error $\varepsilon_{ij}(t)$ and the consensus error $x_i(t) - x_j(t)$.

In Lemma 5.1, the boundedness test for system states of (1) will be presented. Then, in subsection 5.1, the mean-square convergence and convergence speed will be shown from Lemma 5.2 to Theorem 5.6. Finally, in subsection 5.2, we will present Theorems 5.7 and 5.8 for showing the almost sure convergence and convergence speed, respectively.

Lemma 5.1 *For any $t \geq 0$ and any $i = 1, \ldots, n$, if the following condition holds*

$$\alpha_i(t+1) \leq (1 - d_i\rho(t))\alpha_i(t) - \frac{\rho(t)}{\nu_i}\Big[\nu_i(\alpha_i(t) + g_i(t)) + M\sum_{k=0}^{t-1}(\alpha_i(k) + g_i(k))\tilde{\nu}_i^{t-1-k} + \tilde{\nu}_i^t\Upsilon_i(0)\Big], \tag{18}$$

then $|x_i(t)| \leq (1 - \alpha_i(t))M$.

Proof. The proof is given in Appendix, so it is omitted here. ∎

Next, the mean-square convergence and the almost sure convergence of $\varepsilon_{ij}(t)$ and $x_i(t) - x_j(t)$ $(j \neq i)$ are considered, respectively. Before proceeding with the main results, the following parameters are needed:

Symbol	Meaning
f_M	$f(C+M)$, $f(\cdot)$ is below (2)
C	$\max\{\lvert C_{ij}\rvert, i = 1, \ldots, n,\ j \in N_i\}$, C_{ij} is in (2)
d_M	$\max_i\{d_i\}$, d_i is below (1)
α_M	$\max_i \sup_t\{\alpha_i(t)\}$, $\alpha_i(t)$ is in (13)
$g(t)$	$\sum_{i=1}^n \sum_{j\in N_i} g_j^2(t)M^2$, $g_j(t)$ is in (13)
$\tilde{\Upsilon}(t)$	$\sum_{i=1}^n \Upsilon_i^2(t)$, $\Upsilon_i(t)$ is in (15)
γ	The step size given in (9)

5.1 Mean-square convergence

In this subsection, the mean-square convergence and convergence speeds of $\varepsilon_{ij}(t)$ and $x_i(t) - x_j(t)$ $(j \neq i)$ are investigated. Before proceeding with these results, the following lemmas are needed.

Lemma 5.2 *Let $R(t) = \mathbb{E}(\varepsilon^T(t)\varepsilon(t))$, where $\varepsilon(t) = (\varepsilon_{1r_1}(t), \ldots, \varepsilon_{1r_{d_1}}(t), \varepsilon_{2r_{d_1+1}}(t), \ldots, \varepsilon_{2r_{d_1+d_2}}(t), \ldots, \varepsilon_{nr_{d_1+\ldots+d_{n-1}+1}}(t), \ldots, \varepsilon_{nr_{d_1+\ldots+d_n}}(t))^T$. Then, the following inequality holds:*

$$\begin{aligned} &R(t) \\ &\leq (1 - h_1\rho(t))R(t-1) + h_2\rho(t)V(t-1) + \bar{c}\rho^2(t) \\ &\quad + 3\rho(t)g(t-1) + \tfrac{3M^2 d_M\rho(t)}{\nu_m^2}\tilde{\Upsilon}(t-1),\ t \geq 1, \end{aligned} \tag{19}$$

where $h_1 = 2\gamma f_M - \frac{\lambda_n d_M}{\delta} - 2d_M - (1 + 6\alpha_M^2 d_M^2)$, $h_2 = \delta + 12\alpha_M^2 d_M^2$, $\bar{c} > 0$ and $\delta > 0$ is an arbitrary constant, $\nu_m = \min_i\{\nu_i\}$.

Proof. The proof is given in Appendix, so it is omitted here. ∎

Lemma 5.3 *Let $V(t) = \mathbb{E}(x^T(t)Lx(t))$. Then, the following inequality holds:*

$$\begin{aligned} V(t) \leq\ &(1 - h_4\rho(t))V(t-1) + h_3\rho(t)R(t-1) \\ &+ \hat{c}\rho^2(t) + 3\tfrac{\lambda_n}{\tilde{c}}\rho(t)g(t-1) \\ &+ \tfrac{3M^2 d_M\lambda_n\rho(t)}{\tilde{c}\nu_m^2}\tilde{\Upsilon}(t-1),\ t > T_1, \end{aligned} \tag{20}$$

where $h_3 = \frac{2\lambda_n^2 d_M}{\lambda_2^2} + \frac{6\alpha_M^2\lambda_n d_M}{\tilde{c}}$, $h_4 = \frac{\lambda_2^2}{\lambda_n} - \tilde{c} - \frac{12\alpha_M^2\lambda_n d_M}{\tilde{c}}$, $T_1 = \max\{d_M, \frac{2\lambda_n^4}{\lambda_2^3}\}$, $\hat{c} > 0$ and $\tilde{c} > 0$ is an arbitrary constant.

Proof. The proof is given in Appendix, so it is omitted here. ∎

The following lemma is necessary for the mean-square convergence speed.

Lemma 5.4 *For any constant $\hat{\beta} > 0$ and $\hat{\alpha} > 0$, the following assertions hold as $k \to \infty$:*

(i)

$$\prod_{i=1}^k (1 - \hat{\beta}\rho(i)) = \begin{cases} O((k+\hat{b})^{-\hat{\beta}}), & \hat{\mu} = 1, \\ O\big(exp(\frac{\hat{\beta}}{\hat{\mu}-1}(k+\hat{b})^{1-\hat{\mu}})\big), & 0 < \hat{\mu} < 1. \end{cases}$$

(ii) If $\hat{\mu} = 1$, then

$$\sum_{i=1}^k \prod_{l=i+1}^k (1 - \hat{\beta}\rho(l))\rho^{\hat{\alpha}+1}(i) = \begin{cases} O((k+\hat{b})^{-\hat{\beta}}), & \hat{\alpha} > \hat{\beta}, \\ O((k+\hat{b})^{-\hat{\beta}}\ln(k+\hat{b})), & \hat{\alpha} = \hat{\beta}, \\ O((k+\hat{b})^{-\hat{\alpha}}), & \hat{\alpha} < \hat{\beta}. \end{cases}$$

If $0 < \hat{\mu} < 1$ and $\hat{\mu}(1+\hat{\alpha}) > 1$, then

$$\sum_{i=1}^k \prod_{l=i+1}^k (1 - \hat{\beta}\rho(l))\rho^{\hat{\alpha}+1}(i) = O((k+\hat{b})^{-\hat{\mu}(1+\hat{\alpha})}).$$

Proof. The proof is given in Appendix, so it is omitted here. ∎

From Lemma 5.2 to Lemma 5.4, the following theorem related to the mean-square convergence and convergence speeds of $\varepsilon_{ij}(t)$ and $x_i(t) - x_j(t)$ $(j \neq i)$ can be derived, respectively.

Theorem 5.5 *Assume the conditions of Lemma 4.6 hold. Let $U(t) = (R(t), V(t))^T$, $C = (\bar{c}, \hat{c})^T$, $C_1 = 3(1, \frac{\lambda_n}{\tilde{c}})^T$, $C_2 = \frac{3M^2 d_M}{\nu_m^2}(1, \frac{\lambda_n}{\tilde{c}})^T$, and $H = \begin{pmatrix} h_1 & -h_2 \\ -h_3 & h_4 \end{pmatrix}$.*

Then, the following assertions hold as $t \to \infty$:

(i) If $\hat{\mu} = 1$, then

$$\|U(t)\| = \begin{cases} O((t+\hat{b})^{-\lambda_{\min}(H)}), & \lambda_{\min}(H) < \tilde{\psi}, \\ O((t+\hat{b})^{-\lambda_{\min}(H)}\ln(t+\hat{b})), & \lambda_{\min}(H) = \tilde{\psi}, \\ O((t+\hat{b})^{-\tilde{\psi}}), & \lambda_{\min}(H) > \tilde{\psi}, \end{cases}$$

(ii) If $\hat{\mu} \in (0,1)$ and $\hat{\mu} + \tilde{\psi} > 1$, then

$$\|U(t)\| = O((t+\hat{b})^{-\hat{\mu}-\tilde{\psi}}),$$

where $\tilde{\psi} = \min\{\hat{\mu}, 2\psi\}$, ψ is defined in Lemma 4.6.

Proof. The proof is given in Appendix, so it is omitted here. ∎

In the following theorem, some sufficient conditions are provided to guarantee $0 < \lambda_{\min}(H) < \tilde{\psi}$, $\lambda_{\min}(H) = \tilde{\psi}$ and $\lambda_{\min}(H) > \tilde{\psi}$, where $\tilde{\psi} > 0$.

Theorem 5.6 *Assume α_M satisfies $0 < \alpha_M < \frac{\lambda_2}{2\sqrt{6}\lambda_n d_M}\sqrt{\frac{\lambda_2^2}{\lambda_n}(1-\frac{1}{2d_M}) - 2\tilde{\psi}}$. Then, the following assertions hold:*
(i) If $\frac{1}{2f_M}(\frac{h_2^2}{h_4}+\zeta_1) < \gamma < \zeta_2$, then $0 < \lambda_{\min}(H) < \tilde{\psi}$,
(ii) If $\gamma = \zeta_2$, then $\lambda_{\min}(H) = \tilde{\psi}$,
(iii) If $\gamma > \zeta_2$, then $\lambda_{\min}(H) > \tilde{\psi}$,
where $\zeta_1 = 1 + 2d_M + 6\alpha_M^2 d_M^2 + \frac{\lambda_n d_M}{\delta}$ and $\zeta_2 = \frac{1}{2f_M}(\zeta_1 + \frac{h_2^2}{h_4 - \tilde{\psi}} + \tilde{\psi})$.

Proof. The proof is given in Appendix, so it is omitted here. ∎

5.2 Almost sure convergence

In this subsection, based on the results in Subsection 5.1, the almost sure convergence and convergence speeds of $\varepsilon_{ij}(t)$ and $x_i(t) - x_j(t)$ $(j \neq i)$ are investigated.

Theorem 5.7 *Assume the conditions of Lemma 4.6, the conditions $\alpha_M \in (0, \frac{\lambda_2}{4\lambda_n d_M}\sqrt{\frac{\lambda_2^2}{\lambda_n}(1-\frac{1}{2d_M}) - 2\tilde{\psi}})$ and $\gamma > \frac{1}{2f_M}(\frac{h_2^2}{h_4}+\zeta_1)$ in Theorem 5.6 hold. If $h_1 \geq h_3$ and $h_4 \geq h_2$, where h_l $(l = 1, \ldots, 4)$ are given in Lemma 5.2 and Lemma 5.3, and $\hat{\mu} + \tilde{\psi} > 1$, then for any $i = 1, \ldots, n$ and $j \in N_i$, $\varepsilon_{ij}(t) \xrightarrow{a.s.} 0$ and $x_i(t) - x_j(t) \xrightarrow{a.s.} 0$ as $t \to \infty$.*

Proof. The proof is given in Appendix, so it is omitted here. ∎

The following theorem is related to the almost sure convergence speeds of $\varepsilon_{ij}(t)$ and $x_i(t) - x_j(t)$ $(j \neq i)$.

Theorem 5.8 *Assume $\sup_t\{\frac{1}{\rho(t)} - \frac{1}{\rho(t-1)}\} \leq \tilde{h}$ with $\tilde{h} = \min\{h_1 - h_3, h_4 - h_2\}$, and the rest of conditions of Theorem 5.7 hold. Then for any $i = 1, \ldots, n$ and $j \in N_i$, the following assertions hold as $t \to \infty$:*
(i) If $\hat{\mu} = 1$, then

$$\vartheta_{ij}(t) = \begin{cases} O(\frac{\ln t}{t}), & 2\psi \geq 1, \\ O(\frac{1}{t^{2\psi}}), & 0 < 2\psi < 1, \end{cases}$$

(ii) If $\hat{\mu} \in (\frac{1}{2}, 1)$, then

$$\vartheta_{ij}(t) = \begin{cases} O(\frac{1}{t^{2\hat{\mu}-1}}), & 2\psi \geq 1, \\ O(\frac{1}{t^{2\hat{\mu}-1}}), & 0 < 2\psi < 1 \ \& \ \hat{\mu} \leq 2\psi, \\ O(\frac{1}{t^{2\psi+\hat{\mu}-1}}), & 0 < 2\psi < 1 \ \& \ \hat{\mu} > 2\psi, \end{cases}$$

where $\vartheta_{ij}(t) \in \{\varepsilon_{ij}^2(t), (x_i(t) - x_j(t))^2\}$.

Proof. The proof is given in Appendix, so it is omitted here. ∎

6 Communication rate analysis

In this section, we successively analyze the communication rate of the event-triggered control scheme (ETCS) consisting of (11)-(12), and the communication rate of the communication scheme between schedulers and estimators (CSSE).

6.1 Communication rate of ETCS

Definition 6.1 *([29]) Let $K_i(t)$ represent the number of triggering times of agent i in $[0, t]$. Then the communication rate of the event-triggered control scheme consisting of* (11)-(12) *is denoted by $\chi_i(t) = \frac{K_i(t)}{t}$.*

Let $z_i(t) = \sum_{j \in N_i}(\hat{x}_{ij}(t) - x_i(t))$. Then, the triggering condition (12) can be rewritten as

$$|z_i(t) - z_i(t_k^i)| > \alpha_i(t)\phi_i(t) + \varsigma_i(t)M, \tag{21}$$

where $\varsigma_i(t) = g_i(t) + \frac{\Upsilon_i(t)}{\nu_i}$, and the parameters $\alpha_i(t)$, $g_i(t)$ and M are the same as those in (12).

To obtain the communication rate of ETCS, the following assumption is needed.

Assumption 6.2 *For any $i = 1, \ldots, n$, the following assertions are satisfied:*

$$(i) \lim_{t\to\infty}\inf \frac{\varsigma_i(t + t^{\mu})}{\varsigma_i(t)} > 0, \ (ii) \lim_{t\to\infty}\inf \frac{\varsigma_i(t)}{\varpi(t)t^{\mu}} > 0,$$

where $\mu \in (0, 1)$ is a constant, $\varpi(t) = \rho(t)\max\{2\gamma d_i, r^{\frac{1}{2}}(t), \varsigma_i(t)\}$, and $r(t) \in \{t^{-1}lnt, t^{-2\psi}, t^{1-2\hat{\mu}}, t^{1-\hat{\mu}-2\psi}\}$.

Remark 6.3 *(i) Let $\mu = 0.01$, $\gamma = 10$, $\nu_i = 10$, $\tilde{\nu}_i = 0.2$, $M = 1$, $\Upsilon_i(0) = 1$, $d_i = 2$, $\rho(t) = (t + 10)^{-0.9}$, $g_i(t) = 0.01(t+0.1)^{-0.5}$ and $\alpha_i(t) = 0.1(t+20)^{-1}$. Then, $r(t) = t^{-0.8}$ and both case (i) and case (ii) are satisfied.*
(ii) Actually, Assumption 6.2 means that the function $\varsigma_i(t)$ can be the polynomial decay function of t instead of the exponential decay function. In view of $\varsigma_i(t) = g_i(t) + \frac{\Upsilon_i(t)}{\nu_i}$ with $g_i(t)$ and $\frac{\Upsilon_i(t)}{\nu_i}$ being the triggering parameters in the triggering condition (12), (12) can be more easily triggered for the exponential decay function case than the polynomial decay function case. In terms of reducing data transmission for resource saving, the triggering condition (12) should be slowly triggered. Moreover, by Assumption 6.2, the communication rate $\chi_i(t)$ can be proved to converge to zero in the following theorem.

Now, we are in a position to propose the following theorem about communication rate of ETCS.

Theorem 6.4 *Assume Assumption 6.2 and the conditions of Theorem 5.8 hold. Then, the communication rate of ETCS satisfies*

$$\mathbb{P}\{\lim_{t\to\infty}\chi_i(t)t^{\kappa} = 0\} = 1, \tag{22}$$

where $0 < \kappa < \frac{\mu}{\mu+1}$.

Proof. The proof is given in Appendix, so it is omitted here. ∎

Remark 6.5 *Intuitively, there exists a tradeoff among the triggering condition (12), the convergence speeds and the communication rate of ETCS. Consider the case that $\psi_1 \geq \psi_2$, $\hat{\mu} = 1$ and $2\psi \leq \hat{\mu}$. In this case, for the given gain $\rho(t)$ and step size γ, by (13) and (15), based on Theorem 5.5, Theorem 5.8 and Theorem 6.4, if the decay speed of $g_i(t)$ is increasing, so are the decay speeds of*

$\Upsilon_i(t)$ and $\varsigma_i(t)$, then (12) will be easier to be satisfied, and the parameter ψ_2 will become larger. From the case (i) of Theorem 5.5 and the case (i) of Theorem 5.8, the convergence speeds will be improved. In addition, from the case (ii) of Assumption 6.2, the parameter μ will become smaller, which yields a smaller parameter κ. As a result, the communication rate of ETCS will be improved and its decay speed will be reduced.

Remark 6.6 *Remark 4.2 claims that the communication rate via the dynamic mechanism related to (12) can converge to zero faster than that via the static mechanism. Now, the detailed reason is offered. Consider the case that $\alpha_i(t) = O(t^{-\frac{1}{3}})$ and $g_i(t) = O(t^{-\frac{1}{2}})$, then $\Upsilon_i(t) = O(t^{-\frac{1}{3}})$. Assume $\alpha_i(t) = t^{-\frac{4}{5}}$, $g_i(t) = t^{-\frac{3}{4}}$, $\Upsilon_i(t) = t^{-\frac{2}{3}}$ and $\varpi(t) = t^{-1}$, then by the case (ii) of Assumption 6.2, $\mu \in (0, \frac{1}{3}]$. For the static mechanism, ν_i tends to infinity, then $\varsigma_i(t)$ becomes $g_i(t)$. Again, by the case (ii) of Assumption 6.2, $\mu \in (0, \frac{1}{4}]$. Obviously, the larger μ is, the larger κ may be. Consequently, the communication rate via the dynamic mechanism related to (12) can converge to zero faster than that via the static mechanism.*

6.2 Communication rate of CSSE

The communication rate of CSSE is defined in (3). In Section 3, we do not precisely analyze the communication rate $\bar{\beta}_{ij}$ since $\mathbb{E}(\beta_{ij}(t)) \neq F_{\min}(C_{ij} - x_j(t))$ by (8). In this subsection, based on Theorem 5.7, we are in a position to reanalyze $\bar{\beta}_{ij}$. Before proceeding with the main result, the following lemma is needed.

Lemma 6.7 *([33]) If $\{f_n, n \geq 1\}$ is a martingale satisfying $\mathbb{E}f_n^2 < \infty$ such that $\sum\limits_{n=1}^{\infty} \frac{\mathbb{E}(f_n - f_{n-1})^2}{n^2} < \infty$, then $\lim\limits_{n\to\infty} \frac{f_n}{n} = 0$ a.s.*

Now, by virtue of Theorem 5.7 and Lemma 6.7, the following result about $\bar{\beta}_{ij}$ can be derived.

Theorem 6.8 *Assume the conditions of Theorem 5.7 hold. Then, for any $i = 1, \ldots, n$ and $j \in N_i$,*

$$\bar{\beta}_{ij} = \lim_{t\to\infty} \frac{\sum_{k=1}^{t} F_{\min}(C_{ij} - x_j(k))}{t} \leq \frac{1}{2} \text{ a.s.}$$

Proof. The proof is given in Appendix, so it is omitted here. ∎

Remark 6.9 *In [23, 24], the communication between each scheduler and its adjacent estimator is sustained. In other words, each scheduler switch is always closed. Thus, $\beta_{ij}(t) \equiv 1$ for any $t \geq 1$ and hence $\bar{\beta}_{ij} \equiv 1$. However, as shown in Theorem 6.8, $\bar{\beta}_{ij} \leq \frac{1}{2}$, which means the communication rate of CSSE can be reduced to a great extent compared with those in [23, 24].*

7 An example

Consider system (1) with parameters $a_{12} = a_{13} = a_{16} = a_{21} = a_{24} = a_{31} = a_{35} = a_{42} = a_{46} = a_{53} = a_{56} = a_{61} = a_{64} = a_{65} = 1$, and $a_{ij} = 0$ otherwise. The network topology is shown in Fig.2. The initial value of system state is $x(0) = [2.5, -4.3, 6.8, 5.9, -1.3, -3.6]^T$. In this case, some system parameters are chosen as $\hat{x}_{ij}(0) = [2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, 1, 2]^T$, $i = 1, \ldots, 6$, $j \in N_i$, $M = 1$, $C_{ij} = 0$, $\mu = 0.4$, $\gamma = 10$, $\nu_i = 10$, $\tilde{\nu}_i = 0.2$, $\Upsilon_i(0) = 1$. The random noises $\{\omega_{ij}(t), (i, j) \in \mathcal{G}\}$ are assumed to have standard normal distributions.

7.1 Convergence analysis

To realize the control target, the event-triggered controller (11) is designed with the control gain $\rho(t) = (t + 10)^{-0.9}$, and the threshold parameters are $\alpha_i(t) = 0.1(t+20)^{-1}$ and $g_i(t) = 0.01(t+0.1)^{-0.3}$. Fig.3 describes the simulation result of system state. Fig.4 shows the estimation result. Fig.5 shows the simulation result of $V(t)$ with the algorithms in this paper and in [23], which demonstrates that the convergence speed of $V(t)$ in this paper is faster than that in [23]. Fig.6 shows the convergence speed of $V(t)$ compared with $O\big((\frac{1}{t})^{1.5}\big)$. Fig.7 shows the effect of the step size γ on the convergence speed of $V(t)$.

7.2 Communication rate analysis

By adopting the state-feedback control strategy in [24], i.e. $\alpha_i(t) \equiv 0$ and $g_i(t) \equiv 0$, the convergence speed is $O\big((\frac{1}{t})^{1.8}\big)$. And the convergence speed in this paper is $O\big((\frac{1}{t})^{1.5}\big)$. Thus, the convergence speed of the state-feedback control strategy is higher. However, its communication rate may also be higher. Let $t = 30$. Then, $(\frac{1}{t})^{1.8} = 0.0022$. In other words, when designing the state-feedback controllers, data transmission should happen 30 times to make the convergence value reach 0.0022. Let $t = 59$. Then, $(\frac{1}{t})^{1.5} = 0.0022$. From (67), the triggering times of agent i satisfies $K_i(t) = 18$. To put it another way, when designing the event-triggered controllers, data transmission only occurs 18 times to make the convergence value reach 0.0022. Moreover, from Theorem 6.4, $\chi_i(t)$ tends to zero as t goes to infinity. As a result, frequent data transmission can be avoided, and hence the communication resources can be saved.

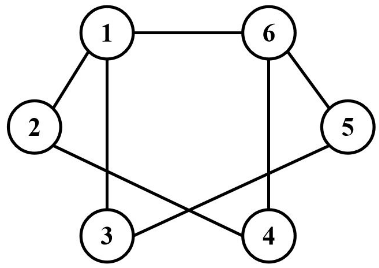

Fig. 2. Network topology.

8 Conclusion

This paper has proposed an event-triggered coordination technique to address the consensus problem of MASs with binary-valued measurements. With this technique, the estimation error and the state error can realize the mean-square convergence and the almost sure convergence with explicit convergence speeds. In addition, the communication rates of ETCS and CSSE have been analyzed. Future works will consider the consensus problem under external attacks, as well as the privacy preserving consensus problem.

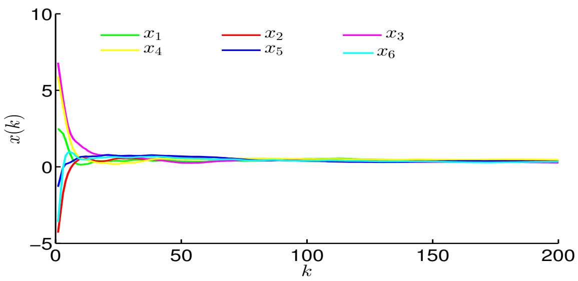

Fig. 3. Trajectory of system state $x(k)$.

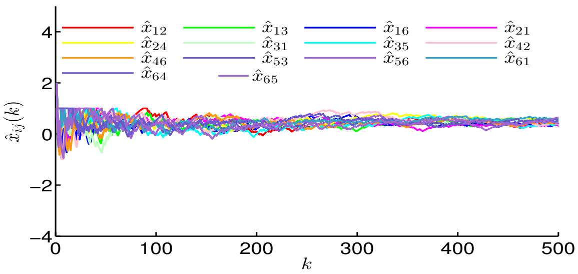

Fig. 4. Trajectory of estimation $\hat{x}_{ij}(k)$.

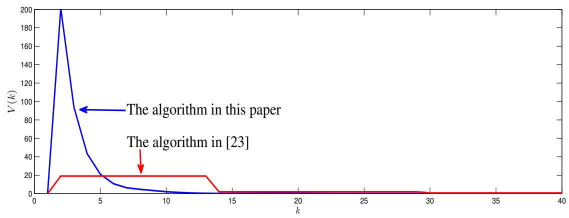

Fig. 5. Trajectory of $V(k)$.

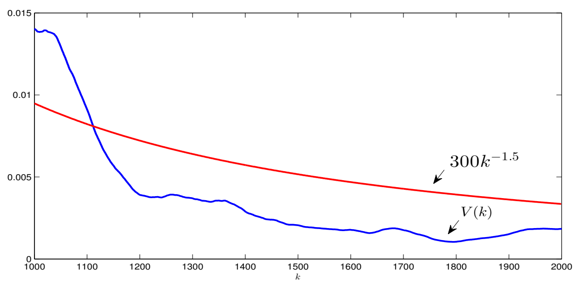

Fig. 6. Convergence speed of $V(k)$.

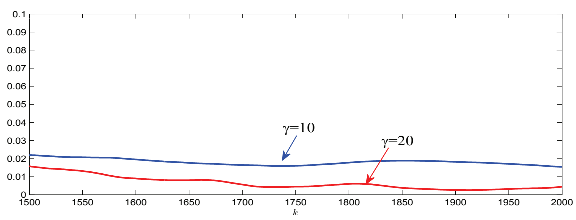

Fig. 7. The effect of the step size γ on $V(k)$.

Appendix

8.1 Proof of Lemma 5.1

From (1) and (11), it holds that

$$x_i(t+1) = (1 - d_i\rho(t))x_i(t) - \rho(t)\sum_{j\in N_i}(e_i(t) - e_{ij}(t)) + \rho(t)\sum_{j\in N_i}\hat{x}_{ij}(t),$$

which in conjunction with $0 < \rho(t) \le \frac{1}{d_i}$ yields

$$|x_i(t+1)| \le (1 - d_i\rho(t))|x_i(t)| + \rho(t)|\sum_{j\in N_i}(e_i(t) - e_{ij}(t))| + d_i\rho(t)M.$$

For any $t \ge 0$, it follows from (12)-(14) that

$$|\sum_{j\in N_i}(e_i(t) - e_{ij}(t))| \le (\alpha_i(t) + g_i(t) + \frac{\Upsilon_i(t)}{\nu_i})M.$$

Thus,

$$|x_i(t+1)| \le (1 - d_i\rho(t))|x_i(t)| + (\alpha_i(t) + g_i(t) + \tfrac{\Upsilon_i(t)}{\nu_i} + d_i)\rho(t)M.$$

Since $\alpha_i(0) \le 1 - \frac{M_0}{M}$, we have $|x_i(0)| \le M_0 \le (1 - \alpha_i(0))M$. Thus, it holds from (18) that

$$\begin{aligned}|x_i(1)| &\le (1 - d_i\rho(0))|x_i(0)| + (\alpha_i(0) + g_i(0) + \tfrac{\Upsilon_i(0)}{\nu_i} + d_i)\rho(0)M \\ &\le (1 - \alpha_i(1))M.\end{aligned}$$

Now, assume $|x_i(t)| \le (1-\alpha_i(t))M$ holds for some $t = k$. For $t = k+1$, we can get from (16) and (18) that

$$\begin{aligned}|x_i(k+1)| &\le (1 - d_i\rho(k))(1 - \alpha_i(k))M + (\alpha_i(k) + g_i(k) + \tfrac{\Upsilon_i(k)}{\nu_i} + d_i)\rho(k)M \\ &\le (1 - \alpha_i(k+1))M.\end{aligned}$$

By the mathematical induction, it holds $|x_i(t)| \le (1 - \alpha_i(t))M$ for any $t \ge 1$ and any $i = 1, \ldots, n$. The proof is completed. ∎

8.2 Proof of Lemma 5.2

By Lemma 5.1, we have $|x_i(t)| \le M$ for any $t \ge 1$ and any $i = 1, \ldots, n$. Thus, it holds that $x_j(t) = \Pi_M(x_j(t))$ for any $t \ge 1$ and any $i = 1, \ldots, n$. By (9),

(17) and Lemma 4.1, we have

$$
\begin{aligned}
&\mathbb{E}(\varepsilon_{ij}^2(t)) \\
&= \mathbb{E}(\hat{x}_{ij}(t) - x_j(t))^2 \\
&= \mathbb{E}\Big[\Pi_M\Big\{\hat{x}_{ij}(t-1) + \gamma\rho(t)(F(C_{ij} - \hat{x}_{ij}(t-1)) \\
&\quad -s_{ij}(t))\Big\} - \Pi_M(x_j(t))\Big]^2 \\
&\le \mathbb{E}\Big[\varepsilon_{ij}(t-1) + \gamma\rho(t)(F(C_{ij} - \hat{x}_{ij}(t-1)) - s_{ij}(t)) \\
&\quad +\rho(t)\sum_{l\in N_j}(x_j(t-1) - x_l(t-1)) \\
&\quad -\rho(t)\sum_{l\in N_j}\varepsilon_{jl}(t-1) + \rho(t)\sum_{l\in N_j}\eta_{jl}(t-1)\Big]^2,\ t \ge 1.
\end{aligned} \tag{23}
$$

Let

$$
\begin{aligned}
\varphi_{ij}(t) &= \varepsilon_{ij}(t) + \gamma\rho(t)(F(C_{ij} - \hat{x}_{ij}(t)) - s_{ij}(t+1)) \\
&\quad +\rho(t)\sum_{l\in N_j}(x_j(t) - x_l(t)) - \rho(t)\sum_{l\in N_j}\varepsilon_{jl}(t) \\
&\quad +\rho(t)\sum_{l\in N_j}\eta_{jl}(t),
\end{aligned}
$$

and $\varphi(t) = (\varphi_{1r_1}(t), \ldots, \varphi_{1r_{d_1}}(t), \varphi_{2r_{d_1+1}}(t), \ldots, \varphi_{2r_{d_1+d_2}}(t), \ldots, \varphi_{nr_{d_1+\ldots+d_{n-1}+1}}(t), \ldots, \varphi_{nr_{d_1+\ldots+d_n}}(t))^T$. Then,

$$
\begin{aligned}
\varphi(t) &= \varepsilon(t) + \gamma\rho(t)(\hat{F}(t) - S(t+1)) \\
&\quad +\rho(t)Q(Lx(t) - P(\varepsilon(t) - \eta(t))),
\end{aligned} \tag{24}
$$

where $\hat{F}(t) = \Big(F(\hat{C}_{1r_1}(t)), \ldots, F(\hat{C}_{1r_{d_1}}(t)), F(\hat{C}_{2r_{d_1+1}}(t)), \ldots, F(\hat{C}_{2r_{d_1+d_2}}(t)), \ldots, F(\hat{C}_{nr_{d_1+\ldots+d_{n-1}+1}}(t)), \ldots, F(\hat{C}_{nr_{d_1+\ldots+d_n}}(t))\Big)^T$, $\hat{C}_{ij} = C_{ij} - \hat{x}_{ij}(t)$, $S(t) = \Big(s_{1r_1}(t), \ldots, s_{1r_{d_1}}(t), s_{2r_{d_1+1}}(t), \ldots, s_{2r_{d_1+d_2}}(t), \ldots, s_{nr_{d_1+\ldots+d_{n-1}+1}}(t), \ldots, s_{nr_{d_1+\ldots+d_n}}(t)\Big)^T$, $\eta(t) = (\eta_{1r_1}(t), \ldots, \eta_{1r_{d_1}}(t), \eta_{2r_{d_1+1}}(t), \ldots, \eta_{2r_{d_1+d_2}}(t), \ldots, \eta_{nr_{d_1+\ldots+d_{n-1}+1}}(t), \ldots, \eta_{nr_{d_1+\ldots+d_n}}(t))^T$, and $Q = (q_{1r_1}, \ldots, q_{1r_{d_1}}, q_{2r_{d_1+1}}, \ldots, q_{nr_{d_1+\ldots+d_{n-1}+1}}, \ldots, q_{nr_{d_1+\ldots+d_n}})^T$ with $q_{ij} = (0, \ldots, \underbrace{1}_{j\ th\ position}, \ldots, 0)^T$.

Following (23) and (24), we can derive that

$$
\begin{aligned}
R(t) &= \mathbb{E}(\varepsilon^T(t)\varepsilon(t)) \\
&= \sum_{i=1}^{n}\sum_{j\in N_i}\mathbb{E}(\varepsilon_{ij}^2(t)) \\
&\le \mathbb{E}(\varphi^T(t-1)\varphi(t-1)),\ t \ge 1.
\end{aligned} \tag{25}
$$

Since $x_i(t)$ and $\hat{x}_{ij}(t)$ are bounded, $\varepsilon(t)$ is also bounded. Thus, from (24), we have

$$
\begin{aligned}
&\mathbb{E}(\varphi^T(t-1)\varphi(t-1)) \\
&\le \Phi(t) + 2\rho(t)\mathbb{E}(\varepsilon^T(t-1)QP\eta(t-1)) \\
&\quad +2\gamma\rho^2(t)\mathbb{E}((\hat{F}(t-1) - S(t))^TQP\eta(t-1)) \\
&\quad +2\rho^2(t)\mathbb{E}(x^T(t-1)LQ^TQP\eta(t-1)) \\
&\quad -2\rho^2(t)\mathbb{E}(\varepsilon^T(t-1)P^TQ^TQP\eta(t-1)) \\
&\quad +\rho^2(t)\mathbb{E}(\eta^T(t-1)P^TQ^TQP\eta(t-1)) \\
&\quad +c_1\rho^2(t),
\end{aligned} \tag{26}
$$

where $\Phi(t) = R(t-1) + 2\gamma\rho(t)\mathbb{E}\Big(\varepsilon^T(t-1)(\hat{F}(t-1) - S(t))\Big) + 2\rho(t)\mathbb{E}(\varepsilon^T(t-1)QLx(t-1)) - 2\rho(t)\mathbb{E}(\varepsilon^T(t-1)QP\varepsilon(t-1)) + c_0\rho^2(t)$, $c_0 > 0$, $c_1 > 0$ are constants, and $P = (p_{1r_1}, \ldots, p_{1r_{d_1}}, \ldots, p_{nr_{d_1+\ldots+d_{n-1}+1}}, \ldots, p_{nr_{d_1+\ldots+d_n}})$ with $p_{ij} = (0, \ldots, \underbrace{1}_{i\ th\ position}, \ldots, 0)^T$.

By Lemma 1 in [24], we can obtain that

$$
\begin{aligned}
\Phi(t) &\le \Big(1 - (2\gamma f_M - \tfrac{\lambda_n d_M}{\delta} - 2d_M)\rho(t)\Big)R(t-1) \\
&\quad +\delta\rho(t)V(t-1) + c_2\rho^2(t),
\end{aligned} \tag{27}
$$

where $c_2 > 0$.

Since $x_i(t)$ and $\hat{x}_{ij}(t)$ are bounded, it yields from (12) that $|\sum_{j\in N_i}\eta_{ij}(t)|$ is bounded. Therefore, for $t \ge 1$, it holds $P\eta(t) = (\sum_{j\in N_1}\eta_{1j}(t), \ldots, \sum_{j\in N_n}\eta_{nj}(t))^T$ is also bounded.

Then, by virtue of (26) and (27), we have

$$
\begin{aligned}
&\mathbb{E}(\varphi^T(t-1)\varphi(t-1)) \\
&\le \Big(1 - (2\gamma f_M - \tfrac{\lambda_n d_M}{\delta} - 2d_M)\rho(t)\Big)R(t-1) \\
&\quad +\delta\rho(t)V(t-1) + c_3\rho^2(t) \\
&\quad +2\rho(t)\mathbb{E}(\varepsilon^T(t-1)QP\eta(t-1)),
\end{aligned} \tag{28}
$$

where $c_3 > 0$ is a constant.

Now, we analyze the term $\mathbb{E}(\varepsilon^T(t-1)QP\eta(t-1))$. Obviously,

$$
\begin{aligned}
&\mathbb{E}(\varepsilon^T(t-1)QP\eta(t-1)) \\
&= \sum_{i=1}^{n}\sum_{j\in N_i}\mathbb{E}(\varepsilon_{ij}(t-1)q_{ij}^TP\eta(t-1)) \\
&\le \sum_{i=1}^{n}\sum_{j\in N_i}\mathbb{E}\Big[\tfrac{1}{2}\varepsilon_{ij}^2(t-1) + \tfrac{1}{2}(q_{ij}^TP\eta(t-1))^2\Big] \\
&= \sum_{i=1}^{n}\sum_{j\in N_i}\mathbb{E}\Big[\tfrac{1}{2}\varepsilon_{ij}^2(t-1) + \tfrac{1}{2}(\sum_{l\in N_j}\eta_{jl}(t-1))^2\Big].
\end{aligned} \tag{29}
$$

Based on the triggering condition (12), we can get

$$
\begin{aligned}
&|\sum_{j\in N_i}\eta_{ij}(t-1)|^2\\
\le\ & 3\alpha_M^2\Big[\sum_{j\in N_i}(x_i(t-1)-x_j(t-1))\\
&+\sum_{j\in N_i}\varepsilon_{ij}(t-1)\Big]^2+3M^2g_i^2(t-1)\\
&+\tfrac{3M^2}{\nu_i^2}\Upsilon_i^2(t-1)\\
\le\ & 6\alpha_M^2 d_i\Big[\sum_{j\in N_i}(x_i(t-1)-x_j(t-1))^2\\
&+\sum_{j\in N_i}\varepsilon_{ij}^2(t-1)\Big]+3M^2g_i^2(t-1)\\
&+\tfrac{3M^2}{\nu_i^2}\Upsilon_i^2(t-1).
\end{aligned}
\tag{30}
$$

Then, (29) in conjunction with (30) implies

$$
\begin{aligned}
&\mathbb{E}(\varepsilon^T(t-1)QP\eta(t-1))\\
\le\ & \tfrac{1}{2}\mathbb{E}(\varepsilon_{1r_1}^2(t-1)+\ldots+\varepsilon_{1r_{d_1}}^2(t-1)+\varepsilon_{2r_{d_1+1}}^2(t-1)\\
&+\ldots+\varepsilon_{2r_{d_1+d_2}}^2(t-1)+\ldots+\varepsilon_{nr_{d_1+\ldots+d_n}}^2(t-1))\\
&+3\alpha_M^2 d_M\mathbb{E}\Big[\sum_{i=1}^n\sum_{j\in N_i}\tfrac{d_i+d_j}{2}(x_i(t-1)-x_j(t-1))^2\\
&+d_1\sum_{j\in N_1}\varepsilon_{1j}^2(t-1)+\ldots+d_n\sum_{j\in N_n}\varepsilon_{nj}^2(t-1)\Big]\\
&+\tfrac{3}{2}g(t-1)+\tfrac{3d_MM^2}{2\nu_m^2}\tilde{\Upsilon}(t-1)\\
\le\ & (\tfrac{1}{2}+3\alpha_M^2d_M^2)R(t-1)+6\alpha_M^2d_M^2V(t-1)\\
&+\tfrac{3}{2}g(t-1)+\tfrac{3d_MM^2}{2\nu_m^2}\tilde{\Upsilon}(t-1).
\end{aligned}
\tag{31}
$$

By (25), (28) and (31), we can obtain

$$
\begin{aligned}
R(t)\ \le\ & \Big(1-(2\gamma f_M-\tfrac{\lambda_n d_M}{\delta}-2d_M\\
&-(1+6\alpha_M^2d_M^2))\rho(t)\Big)R(t-1)\\
&+(\delta+12\alpha_M^2d_M^2)\rho(t)V(t-1)+c_3\rho^2(t)\\
&+3\rho(t)g(t-1)+\tfrac{3M^2d_M\rho(t)}{\nu_m^2}\tilde{\Upsilon}(t-1).
\end{aligned}
\tag{32}
$$

Thus, the proof of Lemma 5.2 is finished by letting $c_3=\bar{c}$. ∎

8.3 Proof of Lemma 5.3

By the presentation of matrix P below (26), system (17) can be rewritten as

$$x(t+1)=(I-\rho(t)L)x(t)+\rho(t)P(\varepsilon(t)-\eta(t)),$$

where $\varepsilon(t)$ is in Lemma 5.2 and $\eta(t)$ is below (24).

Then, it holds that

$$
\begin{aligned}
V(t)\ =\ & \mathbb{E}\Big(\Psi(t)+\rho^2(t)\eta^T(t-1)P^TLP\eta(t-1)\\
&-2\rho^2(t)\varepsilon^T(t-1)P^TLP\eta(t-1)\\
&-2\rho(t)x^T(t-1)(I-\rho(t)L)LP\eta(t-1)\Big),
\end{aligned}
\tag{33}
$$

where $\Psi(t)=x^T(t-1)(I-\rho(t)L)L(I-\rho(t)L)x(t-1)+2\rho(t)x^T(t-1)(I-\rho(t)L)LP\varepsilon(t-1)+\rho^2(t)\varepsilon^T(t-1)P^TLP\varepsilon(t-1)$.

From Lemma 2 in [24], we know that

$$
\begin{aligned}
\mathbb{E}(\Psi(t))\ \le\ & (1-\tfrac{\lambda_2^2}{\lambda_n}\rho(t))V(t-1)\\
&+\tfrac{2\lambda_n^2d_M}{\lambda_2^2}\rho(t)R(t-1)+c_4\rho^2(t),\ t>T_1,
\end{aligned}
\tag{34}
$$

where $c_4>0$ is a constant.

Now, we analyze the term $\mathbb{E}(-2\rho(t)x^T(t-1)(I-\rho(t)L)LP\eta(t-1))$.

Following Theorem 5 (ii) in [30], we have

$$
\begin{aligned}
&\mathbb{E}(2x^T(t-1)(I-\rho(t)L)LP\eta(t-1))\\
\le\ & 2\sqrt{\mathbb{E}(x^T(t-1)(I-\rho(t)L)L(I-\rho(t)L)x(t-1))}\\
&\times\sqrt{\mathbb{E}(\eta^T(t-1)P^TLP\eta(t-1))}\\
\le\ & 2\sqrt{c_5V(t-1)\tfrac{\lambda_n}{c_5}\mathbb{E}(\eta^T(t-1)P^TP\eta(t-1))}\\
\le\ & c_5V(t-1)+\tfrac{\lambda_n}{c_5}\mathbb{E}(\eta^T(t-1)P^TP\eta(t-1)),
\end{aligned}
\tag{35}
$$

where $c_5>0$ is a constant.

By (30), we can get

$$
\begin{aligned}
&\mathbb{E}(\eta^T(t-1)P^TP\eta(t-1))\\
=\ & \mathbb{E}\Big[\sum_{i=1}^n(\sum_{j\in N_i}\eta_{ij}(t-1))^2\Big]\\
\le\ & 6\alpha_M^2d_M(2V(t-1)+R(t-1))+3g(t-1)\\
&+\tfrac{3M^2}{\nu_m^2}\tilde{\Upsilon}(t-1).
\end{aligned}
\tag{36}
$$

Based on the analysis in Lemma 5.2, we know that $\varepsilon(t-1)$ and $P\eta(t-1)$ are bounded for $t\ge 1$. Then, from (33)-(36), we can obtain

$$
\begin{aligned}
V(t)\ \le\ & \Big(1-(\tfrac{\lambda_2^2}{\lambda_n}-c_5-\tfrac{12\alpha_M^2\lambda_nd_M}{c_5})\rho(t)\Big)V(t-1)\\
&+\Big(\tfrac{2\lambda_n^2d_M}{\lambda_2^2}+\tfrac{6\alpha_M^2\lambda_nd_M}{c_5}\Big)\rho(t)R(t-1)+c_6\rho^2(t)\\
&+3\tfrac{\lambda_n}{c_5}\rho(t)g(t-1)\\
&+\tfrac{3M^2d_M\lambda_n\rho(t)}{c_5\nu_m^2}\tilde{\Upsilon}(t-1),\ t>T_1,
\end{aligned}
\tag{37}
$$

where $c_6 > 0$ is a constant.
Therefore, the proof of Lemma 5.3 is finished by letting $c_5 = \tilde{c}$ and $c_6 = \hat{c}$. ∎

8.4 Proof of Lemma 5.4

The proofs of case (i) and case (ii) under the condition $\hat{\mu} = 1$ are similar to those of Lemma 4 in [23], so they are omitted here. Next, we consider the case (ii) under the condition $0 < \hat{\mu} < 1$ and $\hat{\mu}(1 + \hat{\alpha}) > 1$.
From the case (i), we can obtain

$$\begin{aligned}&\sum_{i=1}^{k}\prod_{l=i+1}^{k}(1-\hat{\beta}\rho(l))\rho^{\hat{\alpha}+1}(i)\\&=O\big(\sum_{i=1}^{k}\frac{exp(\frac{\hat{\beta}}{1-\hat{\mu}}((i+\hat{b})^{1-\hat{\mu}}-(k+\hat{b})^{1-\hat{\mu}}))}{(i+\hat{b})^{\hat{\mu}(1+\hat{\alpha})}}\big).\end{aligned}$$

For $i = 1, \ldots, k$, there exists a constant $\hat{M} > 0$, such that $exp(\frac{\hat{\beta}}{1-\hat{\mu}}((i+\hat{b})^{1-\hat{\mu}} - (k+\hat{b})^{1-\hat{\mu}}) \leq \hat{M}\frac{1}{(k-i+1)^{\hat{\mu}(1+\hat{\alpha})}}$. Thus,

$$\begin{aligned}&O\big(\sum_{i=1}^{k}\frac{exp(\frac{\hat{\beta}}{1-\hat{\mu}}((i+\hat{b})^{1-\hat{\mu}}-(k+\hat{b})^{1-\hat{\mu}}))}{(i+\hat{b})^{\hat{\mu}(1+\hat{\alpha})}}\big)\\&=O\big(\sum_{i=1}^{k}\frac{1}{((i+\hat{b})(k-i+1))^{\hat{\mu}(1+\hat{\alpha})}}\big)\\&=O((k+\hat{b})^{-\hat{\mu}(1+\hat{\alpha})}).\end{aligned}$$

The proof is completed. ∎

8.5 Proof of Theorem 5.5

For $t > T_1$, by (19) and (20), we have

$$\begin{aligned}U(t) &\leq (I-\rho(t)H)U(t-1)+\rho^2(t)C+\rho(t)g(t-1)C_1\\&\quad+\rho(t)\tilde{\Upsilon}(t-1)C_2,\end{aligned}$$

which yields

$$\begin{aligned}&\|U(t)\|\\&\leq \|I-\rho(t)H\|\|U(t-1)\|+\|C\|\rho^2(t)\\&\quad+\|C_1\|\rho(t)g(t-1)+\|C_2\|\rho(t)\tilde{\Upsilon}(t-1),\ t>T_1.\end{aligned} \tag{38}$$

By choosing proper constants δ and $\tilde{c}$, we can derive $H^T = H$. Since $\rho(t)$ is non-increasing and satisfies $\lim_{t\to\infty}\rho(t) = 0$, there exists $T_2 > 0$ such that $\rho(t) < \frac{1}{\lambda_{\max}(H)}$ for $t > T_2$. Then, it holds

$$\|I-\rho(t)H\| \leq (1-\lambda_{\min}(H))\rho(t),\ t>T_2.$$

This in conjunction with (38) implies

$$\begin{aligned}&\|U(t)\|\\&\leq (1-\lambda_{\min}(H)\rho(t))\|U(t-1)\|+\|C\|\rho^2(t)\\&\quad+\|C_1\|\rho(t)g(t-1)+\|C_2\|\rho(t)\tilde{\Upsilon}(t-1),\ t>T_3,\end{aligned} \tag{39}$$

where $T_3 = \max\{T_1, T_2\}$.
When $t > T_3$, through an iterative procedure, we can get from (39) that

$$\begin{aligned}\|U(t)\| \leq& \prod_{i=1}^{t}(1-\lambda_{\min}(H)\rho(i))\|U(0)\|\\&+\|C\|\sum_{i=1}^{t}\prod_{l=i+1}^{t}(1-\lambda_{\min}(H)\rho(l))\rho^2(i)\\&+C_3\sum_{i=1}^{t}\prod_{l=i+1}^{t}(1-\lambda_{\min}(H)\rho(l))\rho(i)\big(g(i-1)\\&+\tilde{\Upsilon}(t-1)\big),\end{aligned}$$

where $C_3 = \max\{\|C_1\|, \|C_2\|\}$.
By noting $g_i(t) = O(t^{-\psi_2})$, the proof is completed from Lemma 4.6 and Lemma 5.4. ∎

8.6 Proof of Theorem 5.6

Let $\tilde{c} = \frac{\lambda_2^2}{2\lambda_n d_M}$ and $\delta = h_3 - 12\alpha_M^2 d_M^2$. Then, it holds that $H^T = H$. The minimum eigenvalue of matrix H is

$$\lambda_{\min}(H) = \frac{h_1+h_4-\sqrt{(h_1+h_4)^2-4(h_1h_4-h_2^2)}}{2}.$$

By noting $0 < \alpha_M < \frac{\lambda_2}{2\sqrt{6}\lambda_n d_M}\sqrt{\frac{\lambda_2^2}{\lambda_n}(1-\frac{1}{2d_M})-2\tilde{\psi}}$, we can get that $h_4 > 2\tilde{\psi}$. We next distinguish the following cases.
Case (i). Two sufficient conditions for $\lambda_{\min}(H) > 0$ are $h_4 > 0$ and $h_1h_4 > h_2^2$. And by solving these two inequalities, we can get

$$0<\alpha_M<\frac{\lambda_2^2}{2\sqrt{6}\lambda_n d_M}\sqrt{(1-\frac{1}{2d_M})\frac{1}{\lambda_n}},$$

and

$$\gamma > \frac{1}{2f_M}(\frac{h_2^2}{h_4}+\zeta_1).$$

Moreover, two sufficient conditions for $\lambda_{\min}(H) < \tilde{\psi}$ are $h_1 + h_4 > 2\tilde{\psi}$ and $h_1h_4 - h_2^2 < \tilde{\psi}(h_1 + h_4) - \tilde{\psi}^2$. By solving these two inequalities, we can get $\gamma < \zeta_2$.
To sum up, if $\frac{1}{2f_M}(\frac{h_2^2}{h_4} + \zeta_1) < \gamma < \zeta_2$, we can get $0 < \lambda_{\min}(H) < \tilde{\psi}$.
Case (ii). Two sufficient condition for $\lambda_{\min}(H) = \tilde{\psi}$ are $h_1+h_4 > 2\tilde{\psi}$ and $h_1h_4-h_2^2 = \tilde{\psi}(h_1+h_4)-\tilde{\psi}^2$. By solving them, we can get $\gamma = \zeta_2$, which yields $\lambda_{\min}(H) = \tilde{\psi}$.

Case (iii). Two sufficient condition for $\lambda_{\min}(H) > \tilde{\psi}$ are $h_1 + h_4 > 2\tilde{\psi}$ and $h_1 h_4 - h_2^2 > \tilde{\psi}(h_1 + h_4) - \tilde{\psi}^2$. By solving these two inequalities, we can get $\gamma > \zeta_2$, which implies $\lambda_{\min}(H) > \tilde{\psi}$.
By virtue of the above analysis, the proof is completed. ∎

8.7 *Proof of Theorem 5.7*

Let $v(t) = \varepsilon^T(t)\varepsilon(t)$ and $w(t) = x^T(t)Lx(t)$. Similar to (19) and (20), we have

$$\begin{aligned}
&\mathbb{E}(v(t)|\mathcal{D}_{t-1})\\
&\le (1 - h_1\rho(t))v(t-1) + h_2\rho(t)w(t-1) + \bar{c}\rho^2(t)\\
&\quad +3\rho(t)g(t-1) + \tfrac{3M^2 d_M \rho(t)}{\nu_m^2}\tilde{\Upsilon}(t-1),\ t \ge 1,
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
\mathbb{E}(w(t)|\mathcal{D}_{t-1}) \le\ & (1 - h_4\rho(t))w(t-1) + h_3\rho(t)v(t-1)\\
&+\hat{c}\rho^2(t) + 3\tfrac{\lambda_n}{\tilde{c}}\rho(t)g(t-1)\\
&+\tfrac{3M^2 d_M \lambda_n \rho(t)}{\tilde{c}\nu_m^2}\tilde{\Upsilon}(t-1),\ t > T_1,
\end{aligned} \tag{41}$$

where $\mathcal{D}_t$ is given in (2).
Let $z(t) = v(t) + w(t)$. By noting $h_1 \ge h_3$ and $h_4 \ge h_2$, and from (40)-(41), we can get

$$\begin{aligned}
\mathbb{E}(z(t)|\mathcal{D}_{t-1}) \le\ & z(t-1) + (\bar{c} + \hat{c})\rho^2(t)\\
&+3(1 + \tfrac{\lambda_n}{\tilde{c}})\rho(t)g(t-1)\\
&+\tfrac{3M^2 d_M}{\nu_m^2}(1 + \tfrac{\lambda_n}{\tilde{c}})\rho(t)\tilde{\Upsilon}(t-1),\ t > \hat{T},
\end{aligned} \tag{42}$$

where $\hat{T} > 0$ is a sufficiently large constant.
Since $\hat{\mu} + \tilde{\psi} > 1$, we can derive from Lemma 4.6 that

$$\sum_{k=1}^{\infty} \rho^2(k) < \infty, \tag{43}$$

$$\sum_{k=1}^{\infty} \rho(k)g(k-1) < \infty, \tag{44}$$

and

$$\sum_{k=1}^{\infty} \rho(k)\tilde{\Upsilon}(k-1) < \infty. \tag{45}$$

Then, from (42)-(45) and by Lemma 1.2.2 in [35], $z(t)$ converges a.s. to a finite limit. Moreover, by Theorem 5.5 and Theorem 5.6, we can get that $\lim_{t\to\infty} \mathbb{E}(z(t)) = 0$. Thus, following Lemma 2 in [33] on Page 67, it holds that $\lim_{t\to\infty} z(t) = 0$ a.s., which yields $\lim_{t\to\infty} \varepsilon_{ij}(t) = 0$ a.s. and $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ a.s. for any $i = 1, \dots, n$ and $j \in N_i$. ∎

8.8 *Proof of Theorem 5.8*

Similar to (40)-(42), we can derive

$$\begin{aligned}
z(t) \le\ & (1 - \tilde{h}\rho(t))z(t-1) + (\bar{c} + \hat{c})\rho^2(t)\\
&+3(1 + \tfrac{\lambda_n}{\tilde{c}})\rho(t)g(t-1)\\
&+\tfrac{3M^2 d_M}{\nu_m^2}(1 + \tfrac{\lambda_n}{\tilde{c}})\rho(t)\tilde{\Upsilon}(t-1)\\
&+2\gamma\rho(t)\varepsilon^T(t-1)\big(\tilde{F}(t) - S(t)\big),
\end{aligned} \tag{46}$$

where the elements of $\tilde{F}(t)$ is $F(C_{ij} - x_j(t))$ with the same order as $\varepsilon(t)$.
By (46), we have

$$\begin{aligned}
\tfrac{z(t)}{\rho(t)} - \tfrac{z(t-1)}{\rho(t-1)} \le\ & (\tfrac{1}{\rho(t)} - \tfrac{1}{\rho(t-1)} - \tilde{h})z(t-1) + (\bar{c} + \hat{c})\rho(t)\\
&+3(1 + \tfrac{\lambda_n}{\tilde{c}})g(t-1)\\
&+\tfrac{3M^2 d_M}{\nu_m^2}(1 + \tfrac{\lambda_n}{\tilde{c}})\tilde{\Upsilon}(t-1)\\
&+2\gamma\varepsilon^T(t-1)\big(\tilde{F}(t) - S(t)\big).
\end{aligned} \tag{47}$$

Through an iterative procedure, it yields from (47) that

$$\begin{aligned}
\tfrac{z(t)}{\rho(t)} \le\ & \tfrac{z(0)}{\rho(0)} + \sum_{k=1}^{t}(\tfrac{1}{\rho(k)} - \tfrac{1}{\rho(k-1)} - \tilde{h})z(k-1)\\
&+(\bar{c} + \hat{c})\sum_{k=1}^{t}\rho(k) + 3(1 + \tfrac{\lambda_n}{\tilde{c}})\sum_{k=1}^{t} g(k-1)\\
&+\tfrac{3M^2 d_M}{\nu_m^2}(1 + \tfrac{\lambda_n}{\tilde{c}})\sum_{k=1}^{t}\tilde{\Upsilon}(k-1)\\
&+2\gamma\sum_{k=1}^{t}\varepsilon^T(k-1)\big(\tilde{F}(k) - S(k)\big).
\end{aligned} \tag{48}$$

By noting $\hat{\mu} + \tilde{\psi} > 1$, we have $\hat{\mu} > \frac{1}{2}$ and $\hat{\mu} + 2\psi > 1$. Thus,

$$\rho(t)\sum_{k=1}^{t}\rho(k) = \begin{cases} O(\frac{\ln t}{t}), & \hat{\mu} = 1,\\ O(\frac{1}{t^{2\hat{\mu}-1}}), & \frac{1}{2} < \hat{\mu} < 1. \end{cases} \tag{49}$$

By noting $g(t) = O(\frac{1}{t^{2\psi_2}})$ and $\hat{\Upsilon}(t) = O(\frac{1}{t^{2\psi}})$ with $\psi \le \psi_2$, we can get

$$\begin{aligned}
&\rho(t)\sum_{k=1}^{t} g(k-1)\ \&\ \rho(t)\sum_{k=1}^{t}\tilde{\Upsilon}(k-1)\\
&= \begin{cases} O(\frac{\ln t}{t^{\hat{\mu}}}), & 2\psi = 1,\\ O(\frac{1}{t^{2\psi+\hat{\mu}-1}}), & 2\psi \ne 1. \end{cases}
\end{aligned} \tag{50}$$

Since $\mathbb{E}(F(C_{ij} - x_j(t)) - s_{ij}(t)|\mathcal{D}_{t-1}) = 0$, we know that $\{F(C_{ij} - x_j(t)) - s_{ij}(t), \mathcal{D}_t\}$ is a martingale difference sequence for any $i = 1, \dots, n$ and $j \in N_i$. Thus, by

Theorem 1.3.10 in [36], we have

$$\begin{aligned}&\sum_{k=1}^{t}\varepsilon_{ij}(k-1)(F(C_{ij}-x_j(k))-s_{ij}(k))\\&=O(\sqrt{\Delta(t)}log^{\epsilon}\sqrt{\Delta(t)})\end{aligned} \quad (51)$$

where $\Delta(t)=\sum_{k=1}^{t}\varepsilon_{ij}^2(k-1)$ and $\epsilon>\frac{1}{2}$ is arbitrary.
If $\Delta(t)$ converges to a finite number Δ_0, then by (51), it holds

$$\sum_{k=1}^{t}\varepsilon^T(k-1)\big(\tilde{F}(k)-S(k)\big)=O(1). \quad (52)$$

If $\lim\limits_{t\to\infty}\Delta(t)=\infty$, then by (51), it holds

$$\sum_{k=1}^{t}\varepsilon^T(k-1)\big(\tilde{F}(k)-S(k)\big)=h_\rho\sum_{k=1}^{t}z(k-1)o(1), \quad (53)$$

where $h_\rho=\sup\limits_{t}\{\frac{1}{\rho(t)}-\frac{1}{\rho(t-1)}-\tilde{h}\}\le 0$.
By (51)-(53), we can derive

$$\rho(t)\sum_{k=1}^{t}\varepsilon^T(k-1)\big(\tilde{F}(k)-S(k)\big)\begin{cases}=O(\frac{1}{t^{\bar{\mu}}}), & \Delta(t)\to\Delta_0,\\ \le 0, & \Delta(t)\to\infty.\end{cases} \quad (54)$$

Then, the proof is completed is from (49), (50) and (54).▮

8.9 Proof of Theorem 6.4

Based on Theorem 5.8, we have $\vartheta_{ij}(t)\le m_1r(t)$, a.s., where $m_1>0$ is a constant. Then, we have

$$\begin{aligned}|z_i(t)|&\le\textstyle\sum_{j\in N_i}|\varepsilon_{ij}(t)|+\sum_{j\in N_i}|x_i(t)-x_j(t)|\\&\le m_2r^{\frac{1}{2}}(t),\ a.s.,\end{aligned} \quad (55)$$

where $m_2>0$ is a constant.
By (9) and Lemma 4.1, we can get

$$\begin{aligned}&|z_i(t+1)-z_i(t)|\\&=|\sum_{j\in N_i}\big(\hat{x}_{ij}(t+1)-\hat{x}_{ij}(t)-x_i(t+1)+x_i(t)\big)|\\&\le\gamma\rho(t)\sum_{j\in N_i}|F(C_{ij}-\hat{x}_{ij}(t))-s_{ij}(t+1)|\\&\quad+\sum_{j\in N_i}|x_i(t+1)-x_i(t)|.\end{aligned} \quad (56)$$

By (17), (21) and (55), we can derive

$$|x_i(t+1)-x_i(t)|\le\rho(t)\big(m_3r^{\frac{1}{2}}(t)+\varsigma_i(t)M\big),\ a.s., \quad (57)$$

where $m_3>0$ is a constant.
Thus, from (56), (57), and by noting $|F(C_{ij}-\hat{x}_{ij}(t))-s_{ij}(t+1)|\le 2$, we can obtain

$$\begin{aligned}&|z_i(t+1)-z_i(t_k^i)|\\&\le|z_i(t)-z_i(t_k^i)|+\rho(t)\big(2\gamma d_i+m_3r^{\frac{1}{2}}(t)+\varsigma_i(t)M\big)\\&\le|z_i(t)-z_i(t_k^i)|+m_4\varpi(t),\ a.s.,\end{aligned} \quad (58)$$

where $m_4>0$ is a constant.
Let $\tau_k^i=t_{k+1}^i-t_k^i$. Then, through an iterative procedure, we have

$$\begin{aligned}&|z_i(t_{k+1}^i)-z_i(t_k^i)|\\&=|z_i(t_k^i+\tau_k^i)-z_i(t_k^i)|\\&\le|z_i(t_k^i+\tau_k^i-1)-z_i(t_k^i)|+m_4\varpi(t_k^i+\tau_k^i-1)\\&\ \vdots\\&\le m_4\textstyle\sum_{s=t_k^i}^{t_k^i+\tau_k^i-1}\varpi(s)\\&\le m_4\tau_k^i\varpi(t_k^i),\ a.s..\end{aligned} \quad (59)$$

By (21), it holds

$$|z_i(t_{k+1}^i)-z_i(t_k^i)|>\alpha_i(t_{k+1}^i)\phi_i(t_{k+1}^i)+\varsigma_i(t_{k+1}^i)M. \quad (60)$$

Thus, it yields from (59) and (60) that

$$m_4\tau_k^i\varpi(t_k^i)>\varsigma_i(t_{k+1}^i)M,$$

which yields

$$\tau_k^i>\tilde{M}\varpi^{-1}(t_k^i)\varsigma_i(t_{k+1}^i), \quad (61)$$

where $\tilde{M}>0$ is a constant.
By (61), and from the case (i) to the case (ii) of Assumption 6.2, there exists a positive integer s, such that for any $k>s$, the following inequality hold

$$\tau_k^i>\tilde{M}(t_k^i)^{\mu}. \quad (62)$$

Through an iterative procedure, we can get from (62) that

$$t_k^i>t_s^i+\tilde{M}\sum_{j=s}^{k-1}(t_j^i)^{\mu}. \quad (63)$$

By noting $t_j^i\ge j$, we have from (63) that

$$t_k^i>t_s^i+\tilde{M}\sum_{j=s}^{k-1}j^{\mu}. \quad (64)$$

By the O'Stolz rule in [37], we can obtain that

$$\sum_{j=s}^{k-1}j^{\mu}\ge(\frac{1}{\mu+1}-\varepsilon)k^{\mu+1},\ k\to\infty, \quad (65)$$

where $\varepsilon > 0$ is a sufficiently small constant.
Thus, by (64) and (65), it holds

$$t_k^i > t_s^i + \hat{M} k^{\mu+1}, \ k \to \infty, \tag{66}$$

where $\hat{M} > 0$ is a constant.
Let $t_k^i \leq t$. Then, (66) yields

$$k < \left(\frac{t - t_s^i}{\hat{M}}\right)^{\frac{1}{\mu+1}},$$

which implies

$$K_i(t) \leq \lfloor \left(\frac{t - t_s^i}{\hat{M}}\right)^{\frac{1}{\mu+1}} \rfloor, \tag{67}$$

where $\lfloor a \rfloor$ represents the maximal integer that is less than or equal to a.
Since $0 < \kappa < \frac{\mu}{\mu+1}$, (67) yields

$$\lim_{t\to\infty} \frac{K_i(t)}{t^{1-\kappa}} \leq \lim_{t\to\infty} \frac{1}{\hat{M}^{\frac{1}{\mu+1}}} t^{\frac{1}{\mu+1}+\kappa-1} = 0. \tag{68}$$

Then, from (68), we can get $\mathbb{P}\{\lim_{t\to\infty} \chi_i(t) t^{\kappa} = 0\} = 1$, which completes the proof. ∎

8.10 Proof of Theorem 6.8

Let $\hat{\beta}_{ij}(t) = \mathbb{P}(\beta_{ij}(t) = 1 | \mathcal{D}_{t-1})$. Then, by (8), we have

$$\begin{aligned} \hat{\beta}_{ij}(t) &= \mathbb{P}(s_{ij}(t) = 0) I_{\{C_{ij} - \hat{x}_{ij}(t-1) > \chi\}} \\ &\quad + \mathbb{P}(s_{ij}(t) = 1) I_{\{C_{ij} - \hat{x}_{ij}(t-1) \leq \chi\}} \\ &= (1 - F(C_{ij} - x_j(t))) I_{\{C_{ij} - \hat{x}_{ij}(t-1) > \chi\}} \\ &\quad + F(C_{ij} - x_j(t)) I_{\{C_{ij} - \hat{x}_{ij}(t-1) \leq \chi\}}. \end{aligned} \tag{69}$$

Thus, $\hat{\beta}_{ij}(t)$ is $\mathcal{D}_{t-1}$ measurable. Moreover, by noting $\mathbb{E}(\beta_{ij}(t)|\mathcal{D}_{t-1}) = \mathbb{P}(\beta_{ij}(t) = 1|\mathcal{D}_{t-1})$ and by (69), we can get that $\{\beta_{ij}(t) - \hat{\beta}_{ij}(t), \mathcal{D}_t, t \geq 1\}$ is a martingale difference sequence.
Let $f_{ij}(t) = \sum_{k=1}^{t} \big(\beta_{ij}(k) - \hat{\beta}_{ij}(k)\big)$. Then, for any $s \in [1, t)$,

$$\begin{aligned} \mathbb{E}(f_{ij}(t)|\mathcal{D}_s) &= \mathbb{E}(f_{ij}(s)|\mathcal{D}_s) \\ &\quad + \mathbb{E}\big(\sum_{k=s+1}^{t} (\beta_{ij}(k) - \hat{\beta}_{ij}(k))|\mathcal{D}_s\big). \end{aligned} \tag{70}$$

For any $l = s+1, \ldots, t$, it holds $\mathcal{D}_s \subset \mathcal{D}_{l-1}$ and

$$\begin{aligned} \mathbb{E}\big(\beta_{ij}(l) - \hat{\beta}_{ij}(l)|\mathcal{D}_s\big) &= \mathbb{E}\Big(\mathbb{E}\big(\beta_{ij}(l) - \hat{\beta}_{ij}(l)|\mathcal{D}_{l-1}\big)|\mathcal{D}_s\Big) \\ &= 0. \end{aligned} \tag{71}$$

Then, by (70)-(71), we can derive $\mathbb{E}(f_{ij}(t)|\mathcal{D}_s) = f_{ij}(s)$, $s \in [1, t)$, and thus $\{f_{ij}(t), t \geq 1\}$ is a martingale.
In addition, by noting $\mathbb{E} f_{ij}^2(t) < \infty$ and

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}(f_{ij}(k) - f_{ij}(k-1))^2}{k^2} < \infty,$$

it yields from Lemma 6.7 that

$$\lim_{t\to\infty} \frac{\sum_{k=1}^{t} (\beta_{ij}(k) - \hat{\beta}_{ij}(k))}{t} = 0, a.s. \tag{72}$$

Let $\tilde{\beta}_{ij}(t) = (1 - F(C_{ij} - x_j(t))) I_{\{C_{ij} - x_j(t) > \chi\}} + F(C_{ij} - x_j(t)) I_{\{C_{ij} - x_j(t) \leq \chi\}}$. Then, $\mathbb{E}\big(\tilde{\beta}_{ij}(t)\big) = F_{\min}(C_{ij} - x_j(t))$. Moreover, by Theorem 5.7, we have $\hat{\beta}_{ij}(t) \xrightarrow{a.s.} \tilde{\beta}_{ij}(t)$ as $t \to \infty$.
By (3), (72) and the law of large numbers [33], we can derive

$$\bar{\beta}_{ij} = \lim_{t\to\infty} \frac{\sum_{k=1}^{t} F_{\min}(C_{ij} - x_j(k))}{t} \leq \frac{1}{2} \ a.s.$$

The proof is completed. ∎

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