Stability in Inverse Problem of Determining Two Parameters for the Moore-Gibson-Thompson Equation with Memory Terms*

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Abstract In this paper, the authors consider the inverse problem for the Moore-Gibson-Thompson equation with a memory term and variable diffusivity, which introduce a sort of delay in the dynamics, producing nonlocal effects in time. The Hölder stability of simultaneously determining the spatially varying viscosity coefficient and the source term is obtained by means of the key pointwise Carleman estimate for the Moore-Gibson-Thompson equation. For the sake of generality in mathematical tools, the analysis of this paper is discussed within the framework of Riemannian geometry.

Keywords Carleman estimate, memory term, Moore-Gibson-Thompson equation, stability, Riemannian geometry.

1 Introduction

Inverse problem starts with concerning a mapping between objects of interest (which is called parameters), and acquired information about these objects (which is called data or measurements) (see [1], Subsection 1.1). Roughly speaking, inverse problem refers to the determination of parameters (or changes in medium properties including the shape and position of the medium) from indirectly related data (measurements). The study of inverse problems mainly lies in two aspects: theory and application.

The theoretical research on inverse problems provides an important guidance for the practical applications. It focuses on the uniqueness and stability of determining parameters. The

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uniqueness property tells us what kind of measurements can determine the parameters, and the stability property is useful in designing the reconstruction algorithm of parameters. Based on theoretical analysis, the application of inverse problem concerns the practical environment, involving in the reconstruction algorithm and product design. A typical example is the Calderón problem, which has been successfully applied to the electrical impedance tomography (EIT).

Recovery of parameters with respect to measurements for partial differential equations (PDEs) is a key field in inverse problems. The study of inverse problems for PDEs is essential in practical applications, such as, in the field of medical imaging, geophysical prospecting and non-destructive testing on materials, etc. In this paper, we consider the inverse problem of determining parameters for the Moore-Gibson-Thompson equation (MGT equation for short) with memory terms.

The MGT equation considered in this paper is a linearized type of the nonlinear acoustic waves describing the high-intensity ultrasound accounting for thermal flux and molecular relaxation times (see [2, 3] for more details on this model). The review paper [4] provided an overview of established PDEs models of nonlinear sound propagation, as well as of more recent developements, along with a very useful collection of references. The MGT equation arises in the context of a branch of physics and mathematics. It is widely used in medical and industrial applications, including medical imaging, thermotherapy, ultrasound cleaning and sonochemistry since it deals more specifically with sound waves of sufficiently large amplitudes, e.g., see the monographs[5–7]. Usually, for the information of the viscosity of the fluid, as well as the external source added to the system, it is hard or even impossible to directly detect them. Therefore, it is necessary to indirectly get the information of these physical parameters by placing some sensors. For this reason, it is important to study the inverse problems for the MGT equation. In other words, we need to find suitable measurements that can uniquely determine the physical parameters.

This third order in time equation (both linear and nonlinear) was previously developed by several authors from various aspects. See for example the works [8–11] for a variety of problems related to this equation. The memory term (appearing in the MGT equation) reflects the memory effects of materials due to viscoelasticity. Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. It usually appears in fluids with complex microstructure, such as polymers. Viscoelastic materials often encountered in biological science, materials sciences as well as in many industrial processes, e.g., in the chemical, food, and oil industries. Compared with Newtonian fluids, these phenomenon and mathematical models are more suitable for diverse viscoelastic materials, see [12] for more details.

The MGT equation differs from the classical second order (in time) wave equation in various topics, such as the analysis of well-posedness, dynamic behavior, controllability, and stability etc. More precisely, this equation displays, a variety of dynamical behaviors for its solution that depend on the physical parameters appearing in the equation. Indeed, the well-posedness of solutions fails, even in the simplest case when the diffusivity vanishes (e.g., see [13, Theorem 1.1]). In other words, the diffusivity does affect both well-posedness and stability. While for the second order (in time) equations, the presence of the structural damping is irrelevant for the well-poseness. The controllability properties of the MGT equations, compared to the classical wave equations, are much more complicated, see [9, 14]. Even for the one dimensional MGT equation, it is not exact and null controllable by a control supported on the boundary, see [15]. The studying of the linearized MGT equation with memory term has been investigated in [12, 16] concerning the energy decay of solutions.

Inverse problems of both linear and nonlinear partial (ordinary) differential equations (ODEs, PDEs) have attracted much attention with lots of literature on inverse elliptic equations, parabolic equations, Schrödinger equations, hyperbolic equations, and plate equations, etc. See an incomplete list [17–27] and the references therein for these topics. Among those, the original work [22] created the Carleman estimates method to derive the strong Lipschitz stability of recovering parameters, after which many papers have been published along this line, see [19, 28–32] and the references therein.

For the inverse MGT equation, few results are available in literature. The work [33] studied the inverse MGT equation of recovering the spatially varying viscosity. Both canonical recovery problems are investigated: (i) Uniqueness and (ii) stability, by using just one boundary measurement. Their approach relies on the Carleman estimates and dynamical decomposition of the MGT equation. Later, [28] devoted to the inverse problem of recovering a space varying coefficient of the MGT equation, from knowledge of the trace of the solution on some open subset of the boundary. They obtained a Lipschitz stability for this inverse problem, and designed a convergent algorithm for the reconstruction of the unknown coefficient.

This paper is devoted to studying the inverse problem of simultaneously determining two space varying parameters appearing in the MGT equation with a memory term. To prove the stability of recovering the parameters, we use the Bukhgeim-Klibanov method created in [22], which is based on the idea of applications of the Carleman estimates to prove the result for the inverse problem of PDEs. The main features and findings of this paper are summarized as follows: (i) We establish a pointwise Carleman estimate for the MGT equation within the frame work of Riemannian geometry (the Carleman estimate is valid whenever the MGT equation is considered on a Riemannian manifold). (ii) We consider the variable diffusivity, which affects the original metric from the view of geometry. (iii) We simultaneously recover two space varying parameters from two groups of measurements measured on a small subdomain near the boundary.

The rest of this paper is organized as follows: In Section 2, we give some preliminaries and the main theorem of this paper. In Section 3, we prove a pointwise Carleman estimate for the MGT equation, which is essential in dealing with the inverse problem. Section 4 focuses on proving the main theorem of this paper. Finally, some conclusions are given in Section 5.

2 Preliminaries and Main Results

2.1 Preliminaries and Statement of the Inverse Problem

Let Ω be a bounded domain of dimension $n \geq 2$ with smooth boundary $\partial\Omega = \Gamma$, and $Q =$

$\Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ for $T>0$. Denote by ∂_t^m ($m \in \mathbb{N}$) the m-th partial derivative with respect to t, and $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ the Laplacian on Ω. Consider the following MGT equation with a memory term:

$$\begin{cases} u_{ttt} + \alpha u_{tt} - b\Delta u_t - c^2\Delta u + \displaystyle\int_0^t k(t-l)\Delta u(x,l)dl = F(x,t), & (x,t) \in Q, \\ u = 0, & (x,t) \in \Sigma, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = \mu(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where the unknown $u = u(x,t)$ denotes the acoustic velocity potential, and the coefficients $\alpha(x), b(x), c(x)$ are all positive functions, referring to the viscosity parameter (friction damping), the diffusivity and speed of sound, respectively. $F(x,t) = q(x)R(x,t)$ is the source term, which comes from the external ultrasonic transmitting device, providing external power to the system. Let $\mathcal{P} = \partial_t^3 + \alpha\partial_t^2 - b\Delta\partial_t - c^2\Delta$, $L_b = \partial_t^2 - b\Delta$ and $\gamma = \alpha - \frac{c^2}{b}$. Then,

$$\mathcal{P}u = L_b u_t + \frac{c^2}{b} L_b u + \gamma u_{tt}.$$

For simplicity, we assume that $b(x), c(x)$ are smooth functions with positive lower bound throughout this paper. It is clear that, the variable coefficient $b(x)$ appearing in $b\Delta u_t$ conformally changes the Euclidean metric. Therefore, for the sake of generality in mathematical tools, the analysis of this paper is discussed within the framework of Riemannian geometry. More precisely, in the system (2.1), we consider the Laplace-Beltrami operator Δ_g on the Riemannian manifold (Ω, g), instead of the Laplacian Δ. Thus, it is necessary to introduce some notations related to this work.

Denote by $\mathcal{X}(\Omega)$ the set of all smooth vector fields on Ω. Let $\langle X, Y\rangle = g(X,Y)$ be the inner product for vector fields $X, Y \in \mathcal{X}(\Omega)$. Particularly, $|X|_g^2 = g(X,X) = |X|^2$. Let D be the Levi-Civita connection in the metric g. The gradient of a smooth function v is denoted by Dv, which is a vector field on Ω satisfying $\langle X, Dv\rangle = Xv$. For any $x \in \Omega$, denote by Ω_x the tangent space at x, and D^2v the usual Hessian tensor of v. It is known that

$$D^2v(X,Y) = \langle D_Y Dv, X\rangle = YXv - (D_Y X)v = D^2v(Y,X), \quad \forall X, Y \in \Omega_x.$$

Here and after, the subscript g is omitted if there are no confusions.

Denote by $L^p(\Omega)$ the set of measurable functions v on Ω satisfying

$$\int_\Omega |v|^p dg < +\infty, \quad p > 0,$$

where dg is the volume element in the metric g. For an integer $m \geq 0$, define

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega |D^s v|^2 dg < +\infty, \ s = 0, 1, \cdots, m \right\},$$

where $D^s v$ denotes the s-th covariant derivative of v (with the convention $D^0 v = v$). Particularly, we have $L^2(\Omega) = H^0(\Omega)$. Denote the function space $H^m(\mathcal{O}_T)$ by

$$H^m(\mathcal{O}_T) = \bigcap_{s=0}^m H^s(0,T; H^{m-s}(\mathcal{O})), \quad m = 0, 1, \cdots,$$

where $\mathcal{O}_T = \mathcal{O} \times (0, T)$, and $\mathcal{O}$ is an open subspace of Ω. In particular, we have $H^0(\mathcal{O}_T) = L^2(\mathcal{O}_T)$.

Statement of the inverse problem Let $u_{kj}(x,t) = u(x,t;\mu_j,\alpha_k,q_k)$ be the solution to the system (2.1) with respect to the inputs $u_{kjtt}(x,0) = \mu_j(x)$ and the unknowns $\alpha_k(x)$ and $q_k(x)$ for $k,j = 1,2$. For given $T > 0$, define

$$\Lambda_T^{\alpha_k,q_k}(\mu_j(x)) = u_{kj}|_{\mathcal{O}\times(0,T)},$$

where $\mathcal{O}$ is an arbitrary neighborhood of Γ inside Ω. That is, the measurements are taken on a small neighborhood of Γ. The inverse problem considered in this paper is: Deriving a stability result of determining $\alpha(x)$ and $q(x)$ by two groups of measurements. More precisely, we are going to prove

$$||\alpha_1 - \alpha_2||_{L^2(\Omega)} + ||q_1 - q_2||_{L^2(\Omega)} \le C\left(||u_{11} - u_{21}||_{X(\mathcal{O}_T)} + ||u_{12} - u_{22}||_{X(\mathcal{O}_T)}\right)^{\lambda},$$

where $X(\mathcal{O}_T)$ is a proper Sobolev space on the domain $\mathcal{O}_T = \mathcal{O} \times (0,T)$ and $\lambda \in (0,1)$ is a constant. By the fact that the map $(\alpha, q) \to u(x,t;\alpha,q)$ is nonlinear, it is clear that the mentioned inverse problem with respect to α, is nonlinear. We shall turn the above inverse problem for the original u-system (2.1) into an auxiliary linear inverse problem for the following w-system:

$$\begin{cases} w_{ttt} - \alpha_1 w_{tt} - b\Delta w_t - c^2\Delta w = -\overline{\alpha} u_{2tt} + \overline{q} R, & (x,t) \in Q, \\ w = 0, & (x,t) \in \Sigma, \\ w(x,0) = w_t(x,0) = w_{tt}(x,0) = 0, & x \in \Omega, \end{cases}$$

where $w = u_1 - u_2$, $\overline{\alpha} = \alpha_1 - \alpha_2$, and $\overline{q} = q_1 - q_2$.

2.2 Main Assumptions and Main Theorem

Let M_0 be a large positive constant. For functions $\alpha(x), q(x)$, $b(x)$ and the solution $u(x,t)$, define two admissible sets as follows.

$$\mathcal{U}_1 = \{(\alpha,q) \in L^\infty(\Omega) \times L^\infty(\Omega) : ||p||^2_{L^\infty(\Omega)} + ||\alpha||^2_{L^\infty(\Omega)} \le M_0,\ (\alpha,q)|_{\omega_0} = (\alpha_0,q_0)|_{\omega_0}\}, \quad (2.2)$$

$$\mathcal{U}_b = \{b \in C^2(\overline{\Omega}) : M_0^{-1} \le ||b||_{C^2(\overline{\Omega})} \le M_0,\ D\vartheta(\ln b) \le 2(2-m)\}, \quad m \in (0,1), \quad (2.3)$$

where $\alpha_0(x), q_0(x)$ are known functions in ω_0, and ω_0 is a small neighborhood of Γ inside Ω. The following are the main assumptions of this paper.

(A.1) There exists a strictly positive function $\vartheta : \overline{\Omega} \to \mathbb{R}$ of calss C^3 in the metric g such that

$$D^2\vartheta(X,X) \ge \rho|X|^2, \quad \forall\ X \in \Omega_x,\ x \in \Omega \quad (2.4)$$

and

$$\min_{x\in\overline{\Omega}} \vartheta(x) \ge d_0 > 0, \quad \max_{x\in\overline{\Omega}} \vartheta(x) \le d_1, \quad (2.5)$$

where ρ, d_0, d_1 are positive constants.

(A.2) $\vartheta(x)$ has no critical point on $\overline{\Omega}$, namely,

$$\inf_{x\in\Omega}|D\vartheta| = m_0 > 0. \tag{2.6}$$

(A.3) $R(x,t) \in W^{1,\infty}(Q)$ and the kernel $k(t)$ is a positive smooth function defined on $[0,+\infty)$.

(A.4) Functions $R, q, \alpha, u_0, u_1, \mu$ and k are selected such that the system (2.1) is well-posed and

$$u \in \mathcal{U}_2 = \Big\{ v \in W^{3,\infty}(Q) : \ ||v||^2_{W^{3,\infty}(Q)} \le M_0 \Big\}. \tag{2.7}$$

(A.5) $\mu_j \in C^\infty(\overline{\Omega})$ $(j=1,2)$, and $|(\mu_1(x)-\mu_2(x))R(x,0)| \ge r > 0$ for $x \in \overline{\Omega}$, where r is some positive constant.

Remark 2.1 By translation and re-scaling, we can achieve

$$D^2\vartheta(X,X) \ge 2|X|^2, \quad \inf_{x\in\Omega}\frac{|D\vartheta|^2}{\vartheta} > 4. \tag{2.8}$$

Assumption (A.1) is widely used in the Carleman estimates and inverse PDEs, which guarantees the interior information of solutions to the system arrives at boundary at a finite time, see for example [30, 34]. The existence of convex functions depends on the curvature of (Ω, g). Particularly, for the Euclidean case, we can take $\vartheta(x) = |x-x_0|^2$ with $x_0 \in \mathbb{R}^n\backslash\overline{\Omega}$. For general Riemannian manifolds, such ϑ exists locally. There are a number of non-trivial examples to give such ϑ, see [34, Chapter 2.3].

The regularity assumption of the solution u to the system (2.1) can be attained with sufficient smooth functions R, α, q and (u_0, u_1, μ_j) (see [35]). It is pointed out that, in [35], the assumption that $\int_0^\infty k(l)dl < 1$, $k'(s) \le 0$ and there exists a $\varrho > 0$ such that

$$(\alpha - \varrho)k'(l) - k''(l) \le 0,$$

for some $l > 0$, were additionally assumed. See also [16] for the well-posedness discussions of the system (2.1).

We are now in a position to introduce the main theorem of this paper.

Theorem 2.2 *Let ω be an arbitrary neighborhood of Γ inside Ω such that $\omega_0 \subset \omega$. Let $(\alpha_k, q_k) \in \mathcal{U}_1, b \in \mathcal{U}_b$, $\beta > 0$ sufficiently small, and $T > \sup_{x\in\Omega}(\beta^{-1}\vartheta(x))^{\frac{1}{2}}$. Suppose that $u_{kj}(x,t) \in \mathcal{U}_2$ are the solutions to the system* (2.1) *with respect to the parameters α_k, q_k and the initial data*

$$(u_{kj}(x,0), u_{kjt}(x,0), u_{kjtt}(x,0)) = (u_0(x), u_1(x), \mu_j(x)), \quad k,j = 1,2.$$

Then, under the assumptions (A.1)–(A.5)*, there exist positive constants $C = C(M_0, \Omega, T)$ and $\lambda \in (0,1)$ such that*

$$||\alpha_1 - \alpha_2||_{L^2(\Omega)} + ||q_1 - q_2||_{L^2(\Omega)} \le C\Bigg[\sum_{j=1}^{2}||u_{1j} - u_{2j}||_{H^3(\omega\times(0,T))}\Bigg]^\lambda. \tag{2.9}$$

The proof of the above theorem will be given in Section 4. We have the following corollary immediately.

Corollary 2.3 *Under the condition of Theorem* 2.1*, we have*

$$u_{1k} = u_{2k} \quad in\ \omega \times (0,T)\ for\ k=1,2 \quad imply\ \alpha_1 = \alpha_2\ and\ q_1 = q_2 \quad \text{a.e. } in\ \Omega.$$

Remark 2.4 We want to remark more on the condition that $b \in \mathcal{U}_b$. It is clear that the presence of $b(x)$ in $b\Delta_g u_t$ conformally changes the original metric g. Since the existence of strictly convex function mentioned above relies on the metric g, it is necessary to impose a certain condition on $b(x)$ (see the set $\mathcal{U}_b$). In the following, we give a more concrete explanation.

Let $x = (x_1, x_2, \cdots, x_n)$ be the natural local coordinate system of Ω, and $\partial_{x_i} (i = 1, 2, \cdots, n)$ be the natural frame vector fields at x. Then, we have

$$\Delta_g v = (\det G(x))^{-\frac{1}{2}} \sum_{i,j=1}^{n} \partial_{x_i}[(\det G(x))^{\frac{1}{2}} g^{ij} \partial_{x_j} v], \tag{2.10}$$

where $G(x)$ is a positive definite matrix given by

$$G(x) = (g(\partial_{x_i}, \partial_{x_j}))_{1\le i,j\le n}.$$

Let $g_1 = b^{-1}g$. Then, by (2.10), we find that

$$\Delta_{g_1} v = b\Delta_g v + \frac{2-n}{2} g(Db, Dv). \tag{2.11}$$

Let $\widetilde{D}$ be the Levi-Civita connection in the metric g_1. By the basic theorem of Riemannian geometry, i.e., the Levi-Civita connection is uniquely determined by the Riemannian metric, see [36, Theorem 3.6]. By [36, PP. 55, Identity (9)], it is known that the relation between $\widetilde{D}$ and D is given by

$$\widetilde{D}_X Y = D_X Y - \frac{1}{2} g(D\ln b, Y)X - \frac{1}{2} g(D\ln b, X)Y + \frac{1}{2} g(X,Y) D\ln b, \quad \forall X, Y \in \mathcal{X}(\Omega).$$

For function e^{ϑ}, it is not difficult to find

$$\begin{aligned}
\widetilde{D}^2 \mathrm{e}^{\vartheta}(X,X) &= g_1(\widetilde{D}_X \widetilde{D}\mathrm{e}^{\vartheta}, X) \\
&= b^{-1} g(\widetilde{D}_X \widetilde{D}\mathrm{e}^{\vartheta}, X) \\
&= b^{-1} g(D_X(bD\mathrm{e}^{\vartheta}), X) - \frac{1}{2}\mathrm{e}^{\vartheta}[D\vartheta(\ln b)|X|_g^2 + (X\ln b)(X\vartheta) - (X\vartheta)(X\ln b)] \\
&= \mathrm{e}^{\vartheta}\left[D^2\vartheta(X,X) - \frac{1}{2}D\vartheta(\ln b)|X|_g^2 + (X\vartheta)^2 + (X\ln b)(X\vartheta)\right] \\
&= \mathrm{e}^{\vartheta}\left\{D^2\vartheta(X,X) - \frac{1}{2}D\vartheta(\ln b)|X|_g^2 + \left[(X\vartheta) + \frac{1}{2}(X\ln b)\right]^2 - \frac{1}{4}(X\ln b)^2\right\} \\
&\ge b\mathrm{e}^{\vartheta}[2 - 1/2 D\vartheta(\ln b)]|X|_{g_1}^2 + \mathrm{e}^{\vartheta}\left\{\left[(X\vartheta) + \frac{1}{2}(X\ln b)\right]^2 - \frac{1}{4}(X\ln b)^2\right\},
\end{aligned}$$

where (2.8) was used in the last inequality. We can always achieve that

$$\widetilde{D}^2 \mathrm{e}^{\vartheta}(X,X) \geq b\mathrm{e}^{\vartheta}[2 - 1/2D\vartheta(\ln b)]|X|_{g_1}^2$$

by re-scaling ϑ (i.e., replace θ by $l\vartheta$, where $l > 0$ is large enough), if necessary. Therefore, we conclude that, if $b \in \mathcal{U}_b$, then e^{ϑ} is a strictly convex function in the metric g_1, satisfying the assumptions (A.1) and (A.2) (if necessary, we can re-scale e^{ϑ} by $l\mathrm{e}^{\vartheta}$ for some $l > 0$). For the case where $\vartheta = |x - x_0|^2$, the similar assumption of $b \in \mathcal{U}_b$ is widely used in inverse problems, e.g., see [19, Condition (1.7)], [37, Condition (2.2)].

3 Carleman Estimates for the MGT Equation

In this section, we will prove a Carleman estimate for

$$\mathcal{A}u := \mathcal{P}u + \int_0^t k(t-l)\Delta u(x,l)dl. \tag{3.1}$$

To do so, we shall firstly establish a pointwise Carleman estimate for $\mathcal{P}u$.

Let

$$\varphi(x,t) = \vartheta(x) - \beta t^2, \quad (x,t) \in Q, \tag{3.2}$$

where $\vartheta(x)$ is the strictly convex function given by the assumption (A.1), and $\beta \in (0,1)$ is a small constant. For arbitrary $\varepsilon \geq 0$ and $0 < \delta < d_0$, denote by

$$\begin{aligned} &\Omega(\varepsilon) = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}, \quad \Omega(0) = \Omega, \\ &Q(\varepsilon,\delta) = \{(x,t) \in \Omega(\varepsilon) \times (0, +\infty) : \varphi(x,t) > \delta\}, \quad Q(0,0) = Q, \end{aligned}$$

where d_0 is given by (2.5) and $\delta > 0$ is a small constant.

3.1 A Pointwise Carleman Estimate for $\mathcal{P}u$

The following is the key pointwise Carleman estimate for $\mathcal{P}u$.

Theorem 3.1 *Let $u \in H^3(\overline{Q(0,\delta)})$. Under the assumptions* (A.1) *and* (A.2)*, there exist positive constants $C = C(M_0,T)$, β_0 and s_0 such that*

$$\begin{aligned} \mathrm{e}^{2s\varphi}(\mathcal{P}u)^2 \geq\ & C[s(u_{tt}^2 + |Du_t|^2 + |Du|^2) + s^3(u^2 + u_t^2)]\mathrm{e}^{2s\varphi} \\ & + \mathrm{div}X + \partial_t Y, \quad (x,t) \in Q(0,\delta) \end{aligned} \tag{3.3}$$

holds for all $0 < \beta \leq \beta_0$ and $s \geq s_0$, where X is vector field on Ω, and Y is a scalar function with respect to u, φ, respectively. Moreover, X, Y satisfy

$$|X| \leq C[s(u_{tt}^2 + |Du_t|^2 + |Du|^2) + s^3(u^2 + u_t^2)]\mathrm{e}^{2s\varphi}, \quad (x,t) \in Q(0,\delta), \tag{3.4}$$

$$|Y| \leq C[|\Delta u|^2 + s(u_{tt}^2 + |Du_t|^2 + |Du|^2) + s^3(u^2 + u_t^2)]\mathrm{e}^{2s\varphi}, \quad (x,t) \in Q(0,\delta), \tag{3.5}$$

and $Y(x,0) = 0$ if $u(x,0) = u_t(x,0) = 0$.

To prove the above theorem, inspired by [38], we firstly establish a pointwise Carleman estimate for the hyperbolic operator $L_0 = \partial_t^2 - \Delta$. The Carleman-type estimate was firstly introduced by Carleman[39] in 1939 to study the uniqueness of elliptic equations in two dimensions. It has become an important tool in studying the uniqueness, control, and inverse problems for PDEs (see [22, 30, 40] and the references therein). The pointwise Carleman estimates for hyperbolic equations have simple weight functions, and can be effectively used to deal with the inverse hyperbolic PDEs with memory terms.

Lemma 3.2 *Let $u \in H^2(\overline{Q(0,\delta)})$ and the assumptions* (A.1) *and* (A.2) *hold. Then, there exist positive constants $C = C(M_0, T)$, β_* and s_* such that*

$$\begin{aligned} e^{2s\varphi}(L_0 v)^2 \geq & -\frac{c^2}{b}(v_t^2 e^{2s\varphi})_t + Cs(v_t^2 + |Dv|^2 + s^2 v^2)e^{2s\varphi} \\ & + \mathrm{div} U_\varphi(v) + \partial_t V_\varphi(v), \quad (x,t) \in Q(0,\delta) \end{aligned} \tag{3.6}$$

holds for all $0 < \beta \leq \beta_*$ and $s \geq s_*$. Moreover, U_φ, V_φ satisfy

$$|(U_\varphi, V_\varphi)| \leq Cs(v_t^2 + |Dv|^2 + s^2 v^2)e^{2s\varphi} \tag{3.7}$$

and V_φ is given by (A.13).

Proof See the Appendix. ∎

As we have discussed in Remark 1.2, with the condition $b \in \mathcal{U}_b$, e^{ϑ} is a strictly convex function satisfying the assumptions (A.1) and (A.2). Based on Lemma 3.2, and noticing that the lower order term $\langle Db, Dv \rangle$ can be absorbed, we can obtain a similar Carleman estimate for

$$L_b v = v_{tt} - b\Delta v = v_{tt} - b\Delta_g v = v_{tt} - \Delta_{g_1} v + \frac{2-n}{2}\langle Db, Dv \rangle.$$

Corollary 3.3 *Let $u \in H^2(\overline{Q(0,\delta)})$ and $b \in \mathcal{U}_b$. Under the assumptions* (A.1) *and* (A.2), *there exist positive constants $C = C(M_0, T)$, $\widetilde{\beta}$ and $\widetilde{s}$ such that*

$$\begin{aligned} e^{2s\varphi}(L_b v)^2 \geq & -\frac{c^2}{b}(v_t^2 e^{2s\varphi})_t + Cs(v_t^2 + |Dv|^2 + s^2 v^2)e^{2s\varphi} \\ & + \mathrm{div} U_\varphi(v) + \partial_t V_\varphi(v), \quad (x,t) \in Q(0,\delta) \end{aligned} \tag{3.8}$$

holds for all $0 < \beta \leq \widetilde{\beta}$ and $s \geq \widetilde{s}$. Moreover, (3.7) *holds for U_φ, V_φ.*

Remark 3.4 Clearly, without the operation of introducing the new metric $g_1 = b^{-1}g$, using the same arguments in Lemma 3.2, we can directly prove Corollary 3.3 for L_b, and we can also find a similar condition that $b \in \mathcal{U}_b$. By introducing the conformal metric g_1 discussed in Remark 2.2, the reason of the condition $b \in \mathcal{U}_b$ is clarified in a direct way. Compared to the pointwise Carleman estimate established in, e.g., [40] for wave equations on Euclidean spaces, the estimate (3.8) includes an extra term $-\frac{c^2}{b}(v_t^2 e^{2s\varphi})_t$, which is crucial in obtaining $Y_\varphi(x, 0) = 0$.

Proof of Theorem 3.1 By Corollary 3.3, with $v = u$ and $v = u_t$, we have

$$C_1[s(u_t^2+|Du|^2)+s^3u^2]\mathrm{e}^{2s\varphi}-\frac{c^2}{b}(u_t^2\mathrm{e}^{2s\varphi})_t+\mathrm{div}U_\varphi(u)+\partial_tV_\varphi(u)\le|L_bu|^2\mathrm{e}^{2s\varphi},$$
$$C_2[s(u_{tt}^2+|Du_t|^2)+s^3u_t^2]\mathrm{e}^{2s\varphi}-\frac{c^2}{b}(u_{tt}^2\mathrm{e}^{2s\varphi})_t+\mathrm{div}U_\varphi(u_t)+\partial_tV_\varphi(u_t)\le|L_bu_t|^2\mathrm{e}^{2s\varphi}.$$

In the above first inequality, taking $C=\min\{C_1,C_2\}$, we have

$$\begin{aligned}&C[s(u_{tt}^2+|Du_t|^2+|Du|^2)+s^3(u^2+u_t^2)]\mathrm{e}^{2s\varphi}-\frac{c^2}{b}[(u_t^2+u_{tt}^2)\mathrm{e}^{2s\varphi}]_t\\&+\mathrm{div}(U_\varphi(u)+U_\varphi(u_t))+\partial_tV_\varphi(u)+\partial_tV_\varphi(u_t)\\\le&(|L_bu|^2+|L_bu_t|^2)\mathrm{e}^{2s\varphi}.\end{aligned}$$

Observe that

$$\begin{aligned}|\mathcal{P}u-\gamma u_{tt}|^2\mathrm{e}^{2s\varphi}&=\left|\frac{c^2}{b}L_bu+L_bu_t\right|^2\mathrm{e}^{2s\varphi}\\&=\left(\frac{c^4}{b^2}|L_bu|^2+|L_bu_t|^2\right)\mathrm{e}^{2s\varphi}+\frac{c^2}{b}(|L_bu|^2\mathrm{e}^{2s\varphi})_t-\frac{2sc^2}{b}\varphi_t|L_bu|^2\mathrm{e}^{2s\varphi}\\&\ge(C_0|L_bu|^2+|L_bu_t|^2)\mathrm{e}^{2s\varphi}+\frac{c^2}{b}(u_{tt}^2\mathrm{e}^{2s\varphi})_t+c^2[(b|\varDelta u|^2-2u_{tt}\varDelta u)\mathrm{e}^{2s\varphi}]_t,\end{aligned}$$

where $C_0=\min_{x\in\overline{\Omega}}\frac{c^4}{b^2}>0$, and $\varphi_t=-2\beta t\le0$ was used in the last step. Taking s sufficiently large (there exists a positive constant s_0, such that $s\ge s_0$), it follows that

$$\begin{aligned}&C[s(u_{tt}^2+|Du_t|^2+|Du|^2)+s^3(u^2+u_t^2)]\mathrm{e}^{2s\varphi}+c^2[(b|\varDelta u|^2-2u_{tt}\varDelta u-b^{-1}u_t^2)\mathrm{e}^{2s\varphi}]_t\\&+\mathrm{div}(U_\varphi(u)+U_\varphi(u_t))+\partial_tV_\varphi(u)+\partial_tV_\varphi(u_t)\\\le&|\mathcal{P}u|^2\mathrm{e}^{2s\varphi}.\end{aligned}\tag{3.9}$$

Let $X=C_0U_\varphi(u)+U_\varphi(u_t)$ and

$$Y=C_0V_\varphi(u)+V_\varphi(u_t)+c^2(b|\varDelta u|^2-2u_{tt}\varDelta u-C_0b^{-1}u_t^2)\mathrm{e}^{2s\varphi}.\tag{3.10}$$

In view of (A.13) with v replaced by u, and by $\psi=\varDelta\vartheta-\beta-1$, we obtain

$$Y(x,0)=0,$$

provided that $u(x,0)=u_t(x,0)=0$. Moreover, (3.4) and (3.5) follow from (3.7) and (3.10). Therefore, the proof of Theorem 3.1 is completed. ∎

3.2 Carleman Estimate for the System (2.1)

Recalling the positive constants ε,δ given in the beginning of Section 2, for $t\ge0$, we set $\Omega_t(\varepsilon,\delta)=Q(\varepsilon,\delta)\cap(\Omega\times\{t\})$ and for $x\in\Omega$,

$$t_\delta(x)=\begin{cases}\sqrt{\dfrac{\vartheta(x)-\delta}{\beta}}, & \delta<\vartheta(x),\\ 0, & \text{else.}\end{cases}$$

For given $T > \sup_{x\in\Omega}(\beta^{-1}\vartheta(x))^{-\frac{1}{2}}$, we have $(0, t_\delta(x)) \subset (0, T)$. Accordingly, the boundary of $Q(\varepsilon, \delta)$ is given by

$$\partial Q(\varepsilon, \delta) = \Sigma_1 \cup \Sigma_2 \cup (\Omega(\varepsilon) \times \{0\}),$$

where $\Sigma_1 = \{(x, t) \in \Omega(\varepsilon) \times (0, T) : \varphi(x, t) = \delta\}$, and

$$\Sigma_2 = \partial\Omega(\varepsilon) \times (0, t_\delta(x)).$$

Denote by $dQ = dgdt$ the volume element of $\Omega \times (0, T)$. Let ω be a neighborhood of Γ inside Ω and $\omega_T = \omega \times (0, T)$. With the above preparations at hand, we prove

Theorem 3.5 *Let $T > \sup_{x\in\Omega}(\beta^{-1}\vartheta(x))^{-\frac{1}{2}}$ and $0 < \delta < \frac{d_0}{2}$. Suppose that $u \in H^3(Q)$ solves (2.1) with $u(x, 0) = u_t(x, 0) = 0$. Then, under the assumptions (A.1) and (A.2), for any $\varepsilon > 0$ with $\Omega(\varepsilon)\backslash\Omega(2\varepsilon) \subset \omega$, there exist $\beta_- > 0$, $s_- > 0$ and $C = C(T, \Omega, M_0) > 0$ such that*

$$\begin{aligned}
&\int_{Q(\varepsilon,\delta)} [s(|u_{tt}|^2 + |Du_t|^2 + |Du|^2) + s^3(u_t^2 + u^2)]\mathrm{e}^{2s\varphi} dQ \\
\leq &C \int_{Q(\varepsilon,\delta)} \left| \int_0^t k(t-l)\Delta u(x, l)dl \right|^2 \mathrm{e}^{2s\varphi} dQ \\
&+ \int_{Q(\varepsilon,\delta)} |\mathcal{A}u|^2 \mathrm{e}^{2s\varphi} dQ + C\mathrm{e}^{Cs}||u||^2_{H^2(\omega_T)} + Cs^3\mathrm{e}^{4s\delta}M_0
\end{aligned} \tag{3.11}$$

holds for all $0 < \beta \leq \beta_-$ and $s \geq s_-$.

Proof Let $\chi \in C_0^3(Q)$ be a cut-off function satisfying

$$\chi(x, t) = \begin{cases} 1, & (x, t) \in Q(2\varepsilon, 2\delta), \\ 0, & (x, t) \in Q\backslash Q(\varepsilon, \delta). \end{cases} \tag{3.12}$$

For $y(x, t) = \chi(x, t)u(x, t)$, we have

$$\mathcal{P}y = \chi\mathcal{P}u + 3\chi_t u_{tt} - \chi_t\Delta u + (3\chi_{tt} + 2\alpha\chi_t - \Delta\chi)u_t - 2D\chi(u + u_t) - 2D\chi_t(u) + (\mathcal{P}\chi)u.$$

Integrating the inequality (3.3) over $Q(\varepsilon, \delta)$ yields

$$\begin{aligned}
&\int_{Q(\varepsilon,\delta)} [s(|(\chi u)_{tt}|^2 + |D(\chi u)_t|^2 + |D(\chi u)|^2) + s^3(|(\chi u)_t|^2 + |\chi u|^2)]\mathrm{e}^{2s\varphi} dQ \\
\leq &C \int_{Q(\varepsilon,\delta)} |\mathcal{P}u|^2 \mathrm{e}^{2s\varphi} dQ + C \int_{Q(\varepsilon,\delta)} (|u_{tt}|^2 + |\Delta u|^2 + |Du_t|^2 + |D_{x,t}u|^2 + u^2) \\
&\times (|D\chi_t|^2 + |D_{x,t}\chi|^2 + |\chi_{ttt}|^2 + |\chi_{tt}|^2 + |\Delta\chi|^2 + |\Delta\chi_t|^2)\mathrm{e}^{2s\varphi} dQ \\
&+ \int_{Q(\varepsilon,\delta)} \mathrm{div}X(\chi u)dQ + \int_{Q(\varepsilon,\delta)} \partial_t Y(\chi u)dQ.
\end{aligned} \tag{3.13}$$

Notice that $u(x, 0) = u_t(x, 0) = 0$ and $\chi(x, t) = 0$ for $(x, t) \in Q\backslash Q(\varepsilon, \delta)$. Then, we have

$$\int_{Q(\varepsilon,\delta)} Y_t(\chi u)dQ = 0.$$

By a similar argument with [41, Corollary 2], and observing that $X(\chi u)=0$ on $\Sigma_1\cup\Sigma_2$, we have

$$\int_{Q(\varepsilon,\delta)}\operatorname{div}X(\chi u)dgdt=\int_{\partial Q(\varepsilon,\delta)\backslash(\Omega(\varepsilon)\times\{0\})}\langle X(\chi u),\widetilde{\nu}\rangle\,d\Gamma dt,$$

where $\widetilde{\nu}$ is the unit normal vector field of $\partial\Omega_t(\varepsilon,\delta)$ and $d\Gamma$ is the surface element. Moreover, by the fact that

$$\operatorname{supp}(\Delta\chi_t,\Delta\chi,D_{x,t}\chi,\chi_{ttt},\chi_{tt},D\chi_t)\subset Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta),$$

the second term on the right-hand side of (3.13) satisfies

$$\begin{aligned}&\int_{Q(\varepsilon,\delta)}(|u_{tt}|^2+|\Delta u|^2+|Du_t|^2+|D_{x,t}u|^2+u^2)\\&\times(|D\chi_t|^2+|D_{x,t}\chi|^2+|\chi_{ttt}|^2+|\chi_{tt}|^2+|\Delta\chi|^2+|\Delta\chi_t|^2)\mathrm{e}^{2s\varphi}dQ\\ \le&C\int_{Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta)}(|u_{tt}|^2+|\Delta u|^2+|Du_t|^2+|D_{x,t}u|^2+u^2)\mathrm{e}^{2s\varphi}dQ,\end{aligned}\tag{3.14}$$

where $D_{x,t}=(Du,u_t)$ and $|D_{x,t}u|^2=|Du|^2+u_t^2$. Since $Q(\varepsilon,\delta)=(Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta))\cup Q(2\varepsilon,2\delta)$ and $\chi(x,t)=1$ in $Q(2\varepsilon,2\delta)$, via (3.13)–(3.14), we have

$$\begin{aligned}&\int_{Q(\varepsilon,\delta)}[s(|u_{tt}|^2+|Du_t|^2+|Du|^2)+s^3(u_t^2+u^2)]\mathrm{e}^{2s\varphi}dQ\\ \le&C\int_{Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta)}[s(|u_{tt}|^2+|\Delta u|^2+|Du_t|^2)+s^3(|D_{x,t}u|^2+u^2)]\mathrm{e}^{2s\varphi}dQ\\&+C\int_{Q(\varepsilon,\delta)}|\mathcal{P}u|^2\mathrm{e}^{2s\varphi}dQ.\end{aligned}\tag{3.15}$$

We proceed to deal with the following terms:

$$\int_{Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta)}[s(|u_{tt}|^2+|\Delta u|^2+|Du_t|^2)+s^3(|D_{x,t}u|^2+u^2)]\mathrm{e}^{2s\varphi}dQ.$$

Notice that

$$Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta)\subset(Q(\varepsilon,\delta)\backslash Q(2\varepsilon,\delta))\cup(Q(\varepsilon,\delta)\backslash Q(\varepsilon,2\delta))$$

and $\delta<\varphi(x,t)\le 2\delta$ for $(x,t)\in Q(\varepsilon,\delta)\backslash Q(\varepsilon,2\delta)$, and $Q(\varepsilon,\delta)\backslash Q(2\varepsilon,\delta)\subset\omega_T$. Then, we obtain

$$\begin{aligned}&\int_{Q(\varepsilon,\delta)\backslash Q(2\varepsilon,2\delta)}[s(|u_{tt}|^2+|\Delta u|^2+|Du_t|^2)+s^3(|D_{x,t}u|^2+u^2)]\mathrm{e}^{2s\varphi}dQ\\ \le&C\mathrm{e}^{Cs}||u||^2_{H^2(\omega_T)}+Cs^3\mathrm{e}^{4s\delta}M_0,\end{aligned}\tag{3.16}$$

where M_0 was introduced in (2.7). Inserting (3.16) into (3.15) yields (3.11). Therefore, the proof of Theorem 3.5 is finished. ▮

4 Proof of Theorem 2.2

Let $u_{kj}=u_{kj}(x,t;\mu_j(x))$ solve the system (2.1) with respect to the unknowns $\alpha_k(x)$ (or $\gamma_k(x)$), $q_k(x)$ and the initial data $\partial_t^2u_{kj}(x,0)=\mu_j(x)$ for $k,j=1,2$. Let

$$w_j(x,t) = u_{1j}(x,t) - u_{2j}(x,t), \quad \widehat{q}(x) = q_1(x) - q_2(x), \quad \widehat{\gamma}(x) = \gamma_2(x) - \gamma_1(x).$$

Then, $\widehat{w} = (w_1, w_2)^{\mathrm{T}}$ solves

$$\begin{cases} L_b\widehat{w}_t + \dfrac{c^2}{b}L_b\widehat{w} + \gamma_1\widehat{w}_{tt} + \displaystyle\int_0^t k(t-l)\varDelta\widehat{w}(x,l)dl = A(\widehat{\gamma},\widehat{q})^{\mathrm{T}}, & (x,t)\in Q, \\ \widehat{w} = 0, & (x,t)\in \varSigma, \\ \widehat{w}(x,0) = \widehat{w}_t(x,0) = \widehat{w}_{tt}(x,0) = 0, & x\in\varOmega, \end{cases} \tag{4.1}$$

where the superscript T denotes the transpose, and

$$A = A(x,t) = \begin{pmatrix} u_{21tt}(x,t) & R(x,t) \\ u_{22tt}(x,t) & R(x,t) \end{pmatrix}.$$

Let $\overline{w} = \widehat{w}_t$. Notice that

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^t k(t-l)\varDelta\widehat{w}(x,l)dl &= k(0)\varDelta\widehat{w}(x,t) + \int_0^t k'(t-l)\varDelta\widehat{w}(x,l)dl \\ &= \int_0^t k(t-l)\varDelta\overline{w}(x,l)dl. \end{aligned}$$

Then,

$$\begin{cases} L_b\overline{w}_t + \dfrac{c^2}{b}L_b\overline{w} + \gamma_1\overline{w}_{tt} + \displaystyle\int_0^t k(t-l)\varDelta\overline{w}(x,l)dl = A_t(\widehat{\gamma},\widehat{q})^{\mathrm{T}}, & (x,t)\in Q, \\ \overline{w} = 0, & (x,t)\in \varSigma, \\ \overline{w}(x,0) = \overline{w}_t(x,0) = 0, \overline{w}_{tt}(x,0) = A(x,0)(\widehat{\gamma},\widehat{q})^{\mathrm{T}}, & x\in\varOmega, \end{cases} \tag{4.2}$$

where

$$A(x,0) = \begin{pmatrix} \mu_1(x) & R(x,0) \\ \mu_2(x) & R(x,0) \end{pmatrix}.$$

Applying Theorem 3.5 to the system (4.2), via the assumption that $||A_t||_{L^\infty(Q)} \le M_0$ (see the assumption (A.4)), we have

$$\begin{aligned} &\int_{Q(\varepsilon,\delta)} [s(|\overline{w}_{tt}|^2 + |D\overline{w}_t|^2 + |D\overline{w}|^2) + s^3(|\overline{w}|^2 + |\overline{w}_t|^2)]\mathrm{e}^{2s\varphi}dQ \\ &\le C\int_Q (|\widehat{\gamma}|^2 + |\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ + C\mathrm{e}^{Cs}||\overline{w}||^2_{H^2(\omega_T)} + Cs^3\mathrm{e}^{4\delta s}M_0 \\ &\quad + \int_{Q(\varepsilon,\delta)} \left|\int_0^t k(t-l)\varDelta\overline{w}(x,l)dl\right|^2 \mathrm{e}^{2s\varphi}dQ. \end{aligned} \tag{4.3}$$

By [40, Lemma 3.1.1] (or [41, Lemma 1]), there exists a positive constant C such that

$$\int_{Q(\varepsilon,\delta)} \left|\int_0^t v(x,l)dl\right|^2 \mathrm{e}^{2s\varphi}dQ \le \frac{C}{s}\int_{Q(\varepsilon,\delta)} |v(x,t)|^2\mathrm{e}^{2s\varphi}dQ$$

holds for all $v(x,t) \in L^2(Q(\varepsilon,\delta))$. Using the first equation in (4.1) and noticing that

$$\varDelta\widehat{w}(x,t) = \int_0^t \varDelta\widehat{w}_t(x,l)dl,$$

we have

$$\begin{aligned}
&\int_{Q(\varepsilon,\delta)}\left|\int_0^t k(t-l)\Delta\overline{w}(x,l)dl\right|^2 \mathrm{e}^{2s\varphi}dQ\\
&\le C\int_{Q(\varepsilon,\delta)}|\Delta\widehat{w}_t|^2\mathrm{e}^{2s\varphi}dQ\\
&\le C\int_{Q(\varepsilon,\delta)}|\Delta\widehat{w}|^2\mathrm{e}^{2s\varphi}dQ + C\int_{Q(\varepsilon,\delta)}\left|\int_0^t k(t-l)\Delta\widehat{w}(x,l)dl\right|^2\mathrm{e}^{2s\varphi}dQ\\
&\quad +C\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|\overline{w}_t|^2+|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ\\
&\le C\int_{Q(\varepsilon,\delta)}\left|\int_0^t \Delta\widehat{w}_t(x,l)dl\right|^2\mathrm{e}^{2s\varphi}dQ + C\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|\overline{w}_t|^2+|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ\\
&\le Cs^{-1}\int_{Q(\varepsilon,\delta)}|\Delta\widehat{w}_t|^2\mathrm{e}^{2s\varphi}dQ + C\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|\overline{w}_t|^2+|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ.
\end{aligned}$$

Thus,

$$\int_{Q(\varepsilon,\delta)}|\Delta\widehat{w}_t|^2\mathrm{e}^{2s\varphi}dQ \le C\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|\overline{w}_t|^2+|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ. \tag{4.4}$$

Combining (4.3) with (4.4), for sufficiently large s, we have

$$\begin{aligned}
&\int_{Q(\varepsilon,\delta)}[s(|\overline{w}_{tt}|^2+|D\overline{w}_t|^2+|D\overline{w}|^2)+s^3(|\overline{w}|^2+|\overline{w}_t|^2)]\mathrm{e}^{2s\varphi}dQ\\
&\le C\int_Q(|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi}dQ + C\mathrm{e}^{Cs}||\overline{w}||^2_{H^2(\omega_T)} + Cs^3\mathrm{e}^{4\delta s}M_0.
\end{aligned} \tag{4.5}$$

By the assumption (A.5), we know that

$$|\det A(x,0)| = |R(x,0)(\mu_1(x)-\mu_2(x))| \ge r > 0, \quad x\in\overline{\widehat{\Omega}} = \Omega\backslash\omega.$$

Therefore, there exists a positive constant C such that

$$|\widehat{\gamma}|^2+|\widehat{q}|^2 \le C|\overline{w}_{tt}(x,0)|^2 = C\sum_{j=1}^2(|\overline{w}_{1tt}(x,0)|^2+|\overline{w}_{2tt}(x,0)|^2).$$

Recalling the cut-off function χ given by (3.12), we have

$$\begin{aligned}
\int_{\Omega(\varepsilon)}(|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi(x,0)}dg &\le C\int_{\Omega(\varepsilon)}\chi^2(x,0)|\overline{w}_{tt}(x,0)|^2\mathrm{e}^{2s\varphi(x,0)}dg\\
&= -\int_{\Omega(\varepsilon)}\frac{\partial}{\partial t}\int_0^{t_{\delta(x)}}\chi^2(x,t)|\overline{w}_{tt}(x,t)|^2\mathrm{e}^{2s\varphi(x,t)}dQ\\
&= -\int_{Q(\varepsilon,\delta)}[2\chi\chi_t|\overline{w}_{tt}|^2+\chi^2\overline{w}_{tt}^{\mathrm{T}}\overline{w}_{ttt}+2s\varphi_t\chi^2|\overline{w}_{tt}|^2]\mathrm{e}^{2s\varphi}dQ.
\end{aligned} \tag{4.6}$$

We proceed to deal with the following integral:

$$\int_{Q(\varepsilon,\delta)}\chi^2\overline{w}_{tt}^{\mathrm{T}}\overline{w}_{ttt}\mathrm{e}^{2s\varphi}dQ.$$

Indeed, by the first equation in (4.2), we have

$$\begin{aligned}
&\chi^2\overline{w}_{tt}^{\mathrm{T}}\overline{w}_{ttt}\mathrm{e}^{2s\varphi}\\
&=\chi^2\overline{w}_{tt}^{\mathrm{T}}\Big[-\alpha\overline{w}_{tt}+c^2\Delta\overline{w}+b\Delta\overline{w}_t-\int_0^t k(t-l)\Delta\overline{w}(x,l)dl+A_t(\widehat{\gamma},\widehat{q})^{\mathrm{T}}\Big]\mathrm{e}^{2s\varphi}\\
&\le C\chi^2\left[|\overline{w}_{tt}|^2+|\Delta\overline{w}|^2+|\widehat{\gamma}|^2+|\widehat{q}|^2+\left|\int_0^t k(t-l)\Delta\overline{w}(x,l)dl\right|^2\right]\mathrm{e}^{2s\varphi}+\chi^2\overline{w}_{tt}^{\mathrm{T}}\Delta\overline{w}_t\mathrm{e}^{2s\varphi}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\chi^2\overline{w}_{tt}^{\mathrm{T}}\Delta\overline{w}_t\mathrm{e}^{2s\varphi}&=\mathrm{div}(\chi^2\mathrm{e}^{2s\varphi}\overline{w}_{tt}^{\mathrm{T}}D\overline{w}_t)-\frac{1}{2}(\chi^2|D\overline{w}_t|^2\mathrm{e}^{2s\varphi})_t\\
&\quad+(\chi\chi_t+s\varphi\chi^2)|D\overline{w}_t|^2\mathrm{e}^{2s\varphi}-2\overline{w}_{tt}^{\mathrm{T}}[\chi\left\langle D\chi,D\overline{w}_t\right\rangle+s\chi^2\left\langle D\varphi,D\overline{w}_t\right\rangle]\mathrm{e}^{2s\varphi}\\
&\le\mathrm{div}(\chi^2\mathrm{e}^{2s\varphi}\overline{w}_{tt}^{\mathrm{T}}D\overline{w}_t)-\frac{1}{2}(\chi^2|D\overline{w}_t|^2\mathrm{e}^{2s\varphi})_t+Cs(|\overline{w}_{tt}|^2+|D\overline{w}_t|^2)\mathrm{e}^{2s\varphi},
\end{aligned}$$

where

$$\left\langle D(\chi\text{ or }\varphi),D\overline{w}_t\right\rangle=\left\langle D(\chi\text{ or }\varphi),Dw_{1tt}\right\rangle+\left\langle D(\chi\text{ or }\varphi),Dw_{2tt}\right\rangle.$$

Notice that $\frac{1}{2}(\chi^2|D\overline{w}_t|^2\mathrm{e}^{2s\varphi})|_0^{t_\delta(x)}=0$, $D\overline{w}_t(x,0)=0$ on $\Omega(\varepsilon)$ and $\chi^2\mathrm{e}^{2s\varphi}\overline{w}_{tt}^{\mathrm{T}}=0$ on $\Sigma_1\cup\Sigma_2$. Then, we have

$$\begin{aligned}
&\int_{Q(\varepsilon,\delta)}\chi^2\overline{w}_{tt}^{\mathrm{T}}\overline{w}_{ttt}\mathrm{e}^{2s\varphi}dQ\\
\le&Cs\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|D\overline{w}_t|^2)\mathrm{e}^{2s\varphi}dQ+C\int_{Q(\varepsilon,\delta)}(|\widehat{\gamma}|^2+|\widehat{q}|^2+|\Delta\overline{w}|^2)\mathrm{e}^{2s\varphi}dQ\\
&+Cs^{-1}\int_{Q(\varepsilon,\delta)}|\Delta\widehat{w}_t|^2\mathrm{e}^{2s\varphi}dQ\\
\le&C\left[s\int_{Q(\varepsilon,\delta)}(|\overline{w}_{tt}|^2+|D\overline{w}_t|^2)+\int_{Q(\varepsilon,\delta)}(|\widehat{\gamma}|^2+|\widehat{q}|^2+|\Delta\widehat{w}_t|^2)\right]\mathrm{e}^{2s\varphi}dQ.
\end{aligned}\tag{4.7}$$

With (4.5), (4.6) and (4.7), we derive that

$$\int_{\Omega(\varepsilon)}(|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi(x,0)}dg\le C\int_Q(|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi(x,t)}dQ+C\mathrm{e}^{Cs}||\overline{w}||^2_{H^2(\omega_T)}+Cs^3\mathrm{e}^{4\delta s}M_0.$$

Notice that $\widehat{\gamma}=\widehat{q}=0$ in $\Omega\backslash\Omega(\varepsilon)$, and

$$\begin{aligned}
\int_Q(|\widehat{\gamma}|^2+|\widehat{q}|^2)\mathrm{e}^{2s\varphi(x,t)}dQ&=\int_\Omega\mathrm{e}^{2s\varphi(x,0)}(|\widehat{\gamma}|^2+|\widehat{q}|^2)\int_0^T\mathrm{e}^{-2s\beta t^2}dt\\
&\le\frac{C}{\sqrt{s}}\int_\Omega(|\widehat{\gamma}|^2+|\widehat{q}|^2)dg.
\end{aligned}$$

Then, we have

$$\mathrm{e}^{2sd_0}\int_\Omega(|\widehat{\gamma}|^2+|\widehat{q}|^2)dg\le C\mathrm{e}^{Cs}||\overline{w}||^2_{H^2(\omega_T)}+Cs^3\mathrm{e}^{4\delta s}M_0,$$

which implies that, there exists a sufficient large constant $\overline{s}>0$ such that

$$\mathrm{e}^{2sd_0}(||\widehat{\gamma}||^2_{L^2(\Omega)}+||\widehat{q}||^2_{L^2(\Omega)})\le C\mathrm{e}^{Cs}||\overline{w}||^2_{H^2(\omega_T)}+C\mathrm{e}^{4\delta s}M_0$$

holds for all $s \geq \overline{s}$. We choose $\overline{\varepsilon} > 0$ small enough such that $\overline{d}_0 = d_0 - \overline{\varepsilon} > 2\delta$, and $\overline{C} = C - 2d_0 + \overline{\varepsilon}$. Since $\varphi(x, 0) \geq d_0$, we have

$$||\widehat{\gamma}||^2_{L^2(\Omega)} + ||\widehat{q}||^2_{L^2(\Omega)} \leq C\mathrm{e}^{-2(\overline{d}_0 - 2\delta)s} M_0 + C\mathrm{e}^{\overline{C}s} ||\overline{w}||^2_{H^2(\omega_T)}, \quad s \geq \overline{s}. \tag{4.8}$$

Setting $s = s + \overline{s}$ and replacing C by $C\mathrm{e}^{\overline{C}_0 \overline{s}}$, we have (4.8) holds for all $s \geq 0$. Therefore, by minimizing the right-hand side of (4.8) with respect to $s \geq 0$, we can finally deduce that there exists a positive constant $C = C(\Omega, T, M_0, s)$ such that

$$||\widehat{\gamma}||^2_{L^2(\Omega)} + ||\widehat{q}||^2_{L^2(\Omega)} \leq C ||\overline{w}||^{2\lambda}_{H^2(\omega_T)},$$

where $\lambda \in (0, 1)$ is a constant. Thus, we complete the proof of Theorem 2.2.

5 Conclusion

This paper considers the inverse problem of simultaneously determining two space varying parameters appearing in the MGT equation with memory terms. By establishing a key Carleman estimate of the MGT equation, the Hölder stability of determining two parameters from the knowledge of solutions to a small sub-domain near the boundary is obtained. It is natural and interesting to consider the stability in inverse problems of recovering the parameters from the boundary measurements (e.g., see [21, 41] for wave equations). Also, the recovery of time-dependent coefficients for the MGT equation (even for the nonlinear Jordan-Moore-Gibson-Thompson equation) is worthy to be considered.

Conflict of Interest

The authors declare no conflict of interest.

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Appendix Proof of Lemma 3.2

Let $z = \mathrm{e}^{s\varphi} v$. Then,

$$\begin{aligned} \mathrm{e}^{s\varphi} L_0 v &= \mathrm{e}^{s\varphi} L_0(\mathrm{e}^{-s\varphi} z) \\ &= z_{tt} - \Delta z + s\left(s\varphi_t^2 + \Delta\varphi - \varphi_{tt} - s|D\varphi|^2 - \psi\right) z - 2s\varphi_t z_t + s\psi z + 2sDz(\varphi), \end{aligned}$$

where $\psi(x) \in C^2(\overline{\Omega})$ that will be given later. Let

$$z_1 = z_{tt} - \Delta z + s\left(s\varphi_t^2 + \Delta\varphi - \varphi_{tt} - s|D\varphi|^2 - \psi\right) z$$

and

$$z_2 = -2s\varphi_t z_t + 2sDz(\varphi), \quad z_3 = s\psi z.$$

Then, we have

$$e^{2s\varphi}(L_0 v)^2 \geq 2(z_1 z_2 + z_1 z_3 + z_2 z_3) + z_1^2. \tag{A.1}$$

We next estimate each term on the right-hand side of (A.1) by three steps.

Step 1 We firstly concentrate on estimating the term $z_1 z_2$. Taking

$$a(x,t) = s(s\varphi_t^2 + \Delta\varphi - \varphi_{tt} - s|D\varphi|^2 - \psi),$$

we have

$$\begin{aligned} z_1 z_2 =& (z_{tt} - \Delta z + az)(-2s\varphi_t z_t + 2sDz(\varphi)) \\ =& -s(\varphi_t z_t^2)_t + s\varphi_{tt} z_t^2 + 2sDz(\varphi)z_{tt} - 2sDz(\varphi)\Delta z + 2s\varphi_t z_t \Delta z - s(\varphi_t a z^2)_t \\ &+ s(\varphi_t a)_t z^2 + 2saDz(\varphi)z. \end{aligned}$$

It is easy to see

$$\begin{aligned} 2sz_{tt}Dz(\varphi) &= 2s(Dz(\varphi)z_t)_t - 2s(Dz(\varphi))_t z_t \\ &= 2s(Dz(\varphi)z_t)_t - 2sDz(\varphi_t)z_t - s\mathrm{div}(z_t^2 D\varphi) + s(\Delta\varphi)z_t^2. \end{aligned}$$

By the fact that

$$\begin{aligned} D\left\langle D\varphi, Dz\right\rangle(z) &= Dz\left\langle D\varphi, Dz\right\rangle \\ &= \left\langle D_{Dz}D\varphi, Dz\right\rangle + \left\langle D_{Dz}Dz, D\varphi\right\rangle \\ &= D^2\varphi(Dz, Dz) + D^2 z(D\varphi, Dz) \\ &= D^2\varphi(Dz, Dz) + D^2 z(Dz, D\varphi) \\ &= D^2\varphi(Dz, Dz) + \frac{1}{2}\left\langle D\varphi, D|Dz|^2\right\rangle, \end{aligned}$$

we can obtain

$$\begin{aligned} 2sDz(\varphi)\Delta z &= 2s\mathrm{div}(Dz(\varphi)Dz) - 2sD\left\langle D\varphi, Dz\right\rangle(z) \\ &= s[\mathrm{div}(2Dz(\varphi)Dz - |Dz|^2 D\varphi) + |Dz|^2\Delta\varphi - D^2\varphi(Dz, Dz)]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} 2s\varphi_t z_t \Delta z + 2saDz(\varphi)z =& s\mathrm{div}(az^2 D\varphi) - s\left(Da(\varphi) + a\Delta\varphi\right)z^2 + s\varphi_{tt}|Dz|^2 \\ &+ 2s\mathrm{div}(\varphi_t z_t Dz) - 2sDz(\varphi_t)z_t - s(\varphi_t|Dz|^2)_t. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} z_1 z_2 =& s(\varphi_{tt} + \Delta\varphi)z_t^2 + s(\varphi_{tt} - \Delta\varphi)|Dz|^2 + s\left[(\varphi_t a)_t - (Da(\varphi) + a\Delta\varphi)\right]z^2 \\ &- 4sDz(\varphi_t)z_t + 2sD^2\varphi(Dz, Dz) + \mathrm{div}U_1 + V_{1t}, \end{aligned} \tag{A.2}$$

where

$$U_1 = 2s(\varphi_t z_t - Dz(\varphi))Dz + s(az^2 - z_t^2 - |Dz|^2)D\varphi$$

and

$$V_1 = s(2Dz(\varphi)z_t - \varphi_t z_t^2 - \varphi_t a z^2 - \varphi_t |Dz|^2).$$

Step 2 We estimate the terms $z_1 z_3$ and $z_2 z_3$. Directly calculating yields

$$\begin{aligned} z_1 z_3 =& (z_{tt} - \Delta z + az)s\psi z \\ =& s\left(\frac{1}{2}\psi_{tt} + a\psi\right)z^2 - s\psi z_t^2 + s\psi|Dz|^2 + sDz(\psi)z + \mathrm{div}U_2 + V_{2t}, \end{aligned} \tag{A.3}$$

where

$$U_2 = -s\psi z Dz, \quad V_2 = s\psi z_t z - \frac{s}{2}\psi_t z^2.$$

Moreover, we have

$$z_2 z_3 = s^2\left[(\varphi_t\psi)_t - D\varphi(\psi) - \psi\Delta\varphi\right]z^2 + \mathrm{div}U_3 + V_{3t}, \tag{A.4}$$

where

$$U_3 = s^2\psi z^2 D\varphi, \qquad V_3 = -s^2\varphi_t\psi z^2.$$

Let

$$\xi(x,t) = s\varphi_t^2 + \Delta\varphi - \varphi_{tt} - s|D\vartheta|^2 - \psi = 4s\beta^2 t^2 + 2\beta + \Delta\vartheta - s|D\vartheta|^2 - \psi.$$

Notice that

$$\begin{aligned} &\frac{c^4}{b^2}z_t^2 + z_1^2 \geq -\frac{2c^2}{b}z_t z_1 \\ &= -\frac{c^2}{b}(z_t^2 + s\xi z^2 + |Dz|^2)_t + 2\mathrm{div}\left(\frac{c^2}{b}z_t Dz\right) - 2\left\langle D\frac{c^2}{b}, z_t Dz\right\rangle + \frac{c^2 s}{b}\xi_t z^2. \end{aligned} \tag{A.5}$$

Then, it follows from (A.2)–(A.5) that

$$\begin{aligned} \mathrm{e}^{2s\varphi}(L_0 v)^2 \geq& 2(z_1 z_2 + z_1 z_3 + z_2 z_3) \\ \geq& -(z_t^2)_t + \mathrm{div}U + V_t - 8sDz(\varphi_t)z_t + 2sDz(\varphi)z \\ &+ \left(\Psi_1 - \frac{c^4}{b^2}\right)z_t^2 + \Psi_2|Dz|^2 + \Psi_3 z^2 + 4sD^2\varphi(Dz, Dz), \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} U_\varphi(z) =& 2(U_1 + U_2 + U_3) + \frac{2c^2}{b}z_t Dz \\ =& 4s\left[\varphi_t z_t - Dz(\varphi) + \frac{c^2}{2bs}z_t + \frac{1}{2}\psi z\right]Dz + 2s\left[(a + s\psi)z^2 - z_t^2 - |Dz|^2\right]D\varphi, \end{aligned} \tag{A.7}$$

$$\begin{aligned} \widetilde{V}_\varphi(z) =& 2(V_1 + V_2 + V_3) - \frac{c^2}{b}(s\xi z^2 - |Dz|^2) \\ =& 4sDz(\varphi)z_t + 2s\psi z_t z - 2s\varphi_t z_t^2 - \left(2s\varphi_t + \frac{c^2}{b}\right)|Dz|^2 \\ &- s\left(2\varphi_t a + \psi_t + 2s\varphi_t\psi + \frac{c^2}{b}\xi\right)z^2, \end{aligned} \tag{A.8}$$

and

$$\begin{cases} \Psi_1(x,t) = 2s(\varphi_{tt} + \Delta\varphi) - 2s\psi, \\ \Psi_2(x,t) = 2s(\varphi_{tt} - \Delta\varphi) + 2s\psi, \\ \Psi_3(x,t) = 2s\left[(\varphi_t a)_t - D\varphi(a) - a\Delta\varphi\right] + s\psi_{tt} \\ \qquad\qquad +2sa\psi + 2s^2(\varphi_t\psi)_t - 2s^2 D\varphi(\psi) - 2s^2\psi\Delta\varphi + \dfrac{c^2 s}{b}\xi_t. \end{cases} \tag{A.9}$$

Step 3 We proceed to estimate the terms on the right-hand side of (A.6). By means of (A.9), we have

$$\Psi_1(x,t) = 2s(\Delta\vartheta - 2\beta - \psi), \quad \Psi_2(x,t) = 2s(\psi - \Delta\vartheta - 2\beta).$$

Directly calculating yields

$$\begin{aligned} a(x,t) &= s^2\left(4\beta^2t^2 - |D\vartheta|^2\right) + s(\Delta\vartheta + 2\beta - \psi) = s^2\left(4\beta^2t^2 - |D\vartheta|^2\right) + O(s), \\ -2sD\varphi(a) &= 2s^3\left\langle D|D\vartheta|^2, D\vartheta\right\rangle + O(s^2) = 4s^3D^2\vartheta(D\vartheta, D\vartheta) + O(s^2), \\ 2s\left[(\varphi_t a)_t + a\psi - a\Delta\varphi\right] &= -48\beta^3s^3t^2 + 4\beta s^3|D\vartheta|^2 + 8\beta^2s^3t^2\psi - 2s^3\psi|D\vartheta|^2 \\ &\quad - 8\beta^2s^3t^2\Delta\vartheta + 2s^3|D\vartheta|^2\Delta\vartheta + O(s^2) \end{aligned} \tag{A.10}$$

and

$$s\psi_{tt} + 2s^2(\varphi_t\psi)_t - 2s^2D\varphi(\psi) - 2s^2\psi\Delta\varphi = O(s^2). \tag{A.11}$$

Thus, by (A.9), (A.10) and (A.11), taking $\psi(x) = \Delta\vartheta(x) - \beta - 1$, we obtain

$$\Psi_3(x,t) = 2s^3(3\beta+1)|D\vartheta|^2 + 4s^3D^2\vartheta(D\vartheta, D\vartheta) - 8\beta^2s^3(7\beta+1)t^2 + O(s^2). \tag{A.12}$$

Notice that

$$2sDz(\psi)z \le \varepsilon_0|Dz|^2 + \frac{s^2}{\varepsilon_0}|D\psi|^2z^2, \quad \varepsilon_0 > 0.$$

Then, we conclude that, for sufficiently large s and sufficiently small ε_0,

$$\begin{aligned} \mathrm{e}^{2s\varphi}(L_0v)^2 \ge& -\frac{c^2}{b}(z_t^2)_t + \left[2s(1-\beta) - \frac{c^4}{b^2}\right]z_t^2 - 2s\left(3\beta + 1 + \frac{\varepsilon_0}{2s}\right)|Dz|^2 \\ &+ 4sD^2\vartheta(Dz, Dz) + \left(\Psi_3 - \frac{s^2}{\varepsilon_0}|D\psi|^2\right)v^2\mathrm{e}^{2s\varphi} + \mathrm{div}U_\varphi + \partial_t\widetilde{V}_\varphi. \end{aligned}$$

Recalling that

$$z_t = (s\varphi_t v + v_t)\mathrm{e}^{s\varphi}, \quad Dz = (svD\varphi + Dv)\mathrm{e}^{s\varphi},$$

we have

$$\begin{aligned} &-\frac{c^2}{b}(z_t^2)_t = -\frac{c^2}{b}\left[\mathrm{e}^{2s\varphi}(v_t^2 - 4s\beta tvv_t + 4s^2\beta^2t^2v^2)\right]_t, \\ &2z_t^2 \ge \mathrm{e}^{2s\varphi}(v_t^2 - 2s^2\varphi_t^2v^2), \quad 2|Dz|^2 \ge \mathrm{e}^{2s\varphi}(|Dv|^2 - 2s^2|D\vartheta|^2v^2). \end{aligned}$$

At this moment,

$$\begin{aligned}V_\varphi(v) &= 4sDz(\varphi)z_t + 2s\psi z_t z + 4s\beta t z_t^2 + \left(4s\beta t - \frac{c^2}{b}\right)|Dz|^2 \\ &\quad + s\left[4\beta t(a+s\psi) - \psi_t - \frac{c^2}{b}\xi\right]z^2 + \frac{4c^2}{b}s\beta t(vv_t - \beta t v^2)\mathrm{e}^{2s\varphi} \\ &= \mathrm{e}^{2s\varphi}\Big\{4s\langle svD\vartheta + Dv, D\vartheta\rangle(-2s\beta tv + v_t) + 4s\beta t(v_t - 2s\beta tv) \\ &\quad + \left(4s\beta t - \frac{c^2}{b}\right)|svD\vartheta + Dv|^2 + s\left[4\beta t(a+s\psi) - \frac{c^2}{b}\xi\right]v^2 \\ &\quad + \frac{4c^2}{b}s\beta t(vv_t - \beta t v^2)\Big\}. \end{aligned} \tag{A.13}$$

Moreover, it follows from (A.7), (A.8) and (A.13) that

$$|(U_\varphi, V_\varphi)| \le Cs(v_t^2 + |Dv|^2 + s^2v^2)\mathrm{e}^{2s\varphi}.$$

Recalling that $D^2\vartheta(X,X) \ge 2|X|^2$ for all $X \in \mathcal{M}_x$, $x \in \Omega$, we have

$$\begin{aligned}\mathrm{e}^{2s\varphi}(L_0v)^2 &\ge \sigma s v_t^2 \mathrm{e}^{2s\varphi} + (\Psi_3 - 8s^3\beta^2t^2 - 6s^3\sigma|D\vartheta|^2 + O(s^2))v^2\mathrm{e}^{2s\varphi} \\ &\quad + s\left(3\sigma - \frac{\varepsilon_0}{2s}\right)|Dv|^2\mathrm{e}^{2s\varphi} + \mathrm{div}U_\varphi + \partial_t V_\varphi,\end{aligned}$$

where $\sigma = 1-\beta > 0$. We are left to deal with the coefficient

$$\Psi_3 - 8s^3\beta^2t^2 - 6s^3\sigma|D\vartheta|^2.$$

It is not difficult to check that

$$\begin{aligned}\Psi_3(x,t) - 8s^3\beta^2t^2 - 6s^3\sigma|D\vartheta|^2 &\ge 4s^3[(3\beta+1)|D\vartheta|^2 - 2\beta(7\beta+2)\beta t^2] + O(s^2) \\ &\ge 8s^3[2(3\beta+1)\vartheta - \beta(7\beta+2)\beta t^2] + O(s^2) \\ &\ge 8s^3(7\beta+2)(\vartheta - \beta t^2) + O(s^2) \\ &\ge 8s^3\delta(7\beta+2) + O(s^2),\end{aligned}$$

for $(x,t) \in Q(0,\delta)$ and $0 < \beta < \frac{2+3\sqrt{2}}{7}$. Thus, the proof of Lemma 3.2 is finished. ∎