ON IDENTIFIABILITY FOR MULTIDIMENSIONAL ARMAX MODEL*†

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Abstract

This paper gives a definition of identifiability for multidimensional linear input-output systems and presents a necessary and sufficient condition for its satisfaction. For a class of identifiable systems it is also shown that the unknown coefficients of the system can consistently be estimated by a recursive algorithm.

1. Introduction

The basic idea of identifiability is the possibility of determining a system or its parameters from the input-output data. Several different definitions of identifiability are given in the survey paper [1] for one-dimensional systems. However, from the following example we shall see that the situation for multidimensional systems is quite different from the one-dimensional case.

Let the linear input-output system be described by

$$A(z)y_t = B(z)u_t, \tag{1.1}$$

where $A(z)$ and $B(z)$ are polynomials in the shift-back operator z.

If both y_t and u_t are one-dimensional, then coprimeness of $A(z)$ and $B(z)$ is necessary and sufficient for uniquely determining parameters of $A(z)$ and $B(z)$ from the data. But in the multidimensional case the left-coprimeness of $A(z)$ and $B(z)$ does not guarantee the uniqueness of representation (1.1).

Example 1.1. Let

$$A(z) = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + Iz^2, \quad B(z) = Iz + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^2.$$

They are left-coprime, since

$$A(z)M(z) + B(z)N(z) = I,$$

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where

$$M(z) = I - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z, \qquad N(z) = -Iz + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^2.$$

If we multiply $A(z)$ and $B(z)$ from the left by $I - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z$, then the system turns to

$$A'(z) y_t = B'(z) u_t, \tag{1.2}$$

where

$$A'(z) = I + Iz^2 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^3, \qquad B'(z) = Iz.$$

It is easy to see that $A'(z)$ and $B'(z)$ also are left-coprime. Thus the input-output data cannot uniquely define parameters of the system.

In this paper, we give a definition of identifiability for multidimensional linear systems and present a necessary and sufficient condition for identifiability. When this condition is satisfied, strongly consistent estimates for the unknown coefficients are derived.

2. Identifiability and Identification Methods

We now consider the system described by an ARMAX model:

$$\begin{aligned} A(z) y_t &= B(z) u_t + C(z) w_t, \qquad t > 0; \\ y_t &= w_t = 0, \quad u_t = 0, \qquad t \le 0, \end{aligned} \tag{2.1}$$

where y_t, u_t and w_t are m-output, n-input and m-driven noise, respectively; $A(z)$, $B(z)$ and $C(z)$ are given by the following equations:

$$A(z) \triangleq I + A_1 z + \cdots + A_p z^p, \qquad p \ge 0, \tag{2.2}$$

$$B(z) \triangleq B_1 z + \cdots + B_q z^q, \qquad q \ge 1, \tag{2.3}$$

$$C(z) \triangleq I + C_1 z + \cdots + C_r z^r, \qquad r \ge 0. \tag{2.4}$$

The driven noise $\{w_t, \mathcal{F}_t\}$ is assumed to be a martingale difference sequence with respect to a non-decreasing family of σ-algebras. It is also assumed that

$$\sup_{t \ge 0} E[\, \|w_{t+1}\|^2 \mathcal{F}_t \,] < \infty \qquad \text{a.s.,} \tag{2.5}$$

$$\liminf_{t \to \infty} \frac{1}{t^{1-\varepsilon^*}} \lambda_{\min} \left(\sum_{i=0}^{t} w_i w_i^\tau \right) > 0 \qquad \text{a.s.,} \tag{2.6}$$

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t} \|w_i\|^2 < \infty \qquad \text{a.s.,} \tag{2.7}$$

where

$$\varepsilon^* = \frac{1}{2\mu + 3}, \qquad \mu = (m+1)p + r + q \tag{2.8}$$

and $\lambda_{\min}(x)$ denotes the minimum eigenvalue of the matrix X.

In recent years there has been made some progress for consistently estimating the unknown coefficient θ

$$\theta^\tau = [-A_1 \cdots -A_p \; B_1 \cdots B_q \; C_1 \cdots C_r] \tag{2.9}$$

under various conditions. For example, in Theorem 4 of [2] it is assumed that $A(z)$ is stable, A_p is of row-full-rank and $C^{-1}(z) - \frac{1}{2}I$ is strictly positive real; in [3] it is required that both $C^{-1}(z) - \frac{1}{2}I$ and $C(z) - \frac{\bar{a}}{2}I$ are positive real for some $\bar{a} > 0$, $z^{-1}B(z)$ is stable, A_p is of row-full-rank and $A(z)$, $B(z)$ and $C(z)$ have no common left factor. Obviously, all these conditions are sufficient for identifying θ. Our purpose is to clarify what is the minimum requirement for this.

Definition 2.1. A system described by (2.1) is said to be identifiable if there are no polynomials $A'(z) = I + A'_1 z + \cdots + A'_{p'} z^{p'}$, $B'(z) = B'_1 z + \cdots + B'_{q'} z^{q'}$ and $C'(z) = I + C'_1 z + \cdots + C'_{r'} z^{r'}$ with $p' \le p$, $q' \le q$ and $r' \le r$, respectively, so that $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$ unless $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$.

Theorem 2.1. The system (2.1) is identifiable if and only if $A(z)$, $B(z)$ and $C(z)$ have no common left factor and rank $[A_p \; B_q \; C_r] = m$.

Proof. We first prove the necessity. Assume the system is identifiable. If the converse were true, then there would exist a non-unimodular polynomial matrix $D(z)$ and polynomials $A'(z)$, $B'(z)$ and $C'(z)$ with orders less than or equal to those of $A(z)$, $B(z)$ and $C(z)$, respectively, so that $[A(z) \; B(z) \; C(z)] = D(z)[A'(z) \; B'(z) \; C'(z)]$ which implies $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$. Thus by Definition 2.1 we have $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$, and hence $D(z) = I$. The obtained contradiction implies that $A(z)$, $B(z)$ and $C(z)$ have no common left factor.

Further, if rank $[A_p \; B_q \; C_r] \ne m$ then rank $[A_p \; B_q \; C_r]$ must be less than m, because $[A_p \; B_q \; C_r]$ has m rows. Hence, there is a non-zero square matrix D of dimension m so that $DA_p = 0$, $DB_q = 0$ and $DC_r = 0$. Thus, we have

$$\begin{aligned}
A'(z) &\triangleq (I + Dz)A(z) = I + (A_1 + D)z + \cdots + (DA_{p-1} + A_p)z^p, \\
B'(z) &\triangleq (I + Dz)B(z) = B_1 z + \cdots + (DB_{q-1} + B_q)z^q, \\
C'(z) &\triangleq (I + Dz)C(z) = I + (C_1 + D)z + \cdots + (DC_{r-1} + A_r)z^r
\end{aligned}$$

and

$$(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z) \quad \text{and} \quad (A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z),$$

which combining with Definition 2.1 implies that $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$. In particular, $A_1 + D = A_1$, i.e. $D = 0$. The contradiction shows rank $[A_p \; B_q \; C_r] = m$.

We now show the sufficiency. If $A(z)$, $B(z)$ and $C(z)$ in (2.1) have no common left factor and rank $[A_p \; B_q \; C_r] = m$, then we can show that the system is identifiable. In fact, if the system were not identifiable, then, by Definition 2.1, there would exist three polynomial matrices $A'(z) = I + A'_1 z + \cdots + A'_{p'} z^{p'}$, $B'(z) = B'_1 z + \cdots + B'_{q'} z^{q'}$ and $C'(z) = I + C'_1 z + \cdots + C'_{r'} z^{r'}$ with $p' \le p$, $q' \le q$ and $r' \le r$ such that $[A(z) \; B(z) \; C(z)] \ne [A'(z) \; B'(z) \; C'(z)]$, but $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$. Let $D(z) = A'(z)(A(z))^{-1}$. Then we have

$$[A'(z) \; B'(z) \; C'(z)] = D(z)[A(z) \; B(z) \; C(z)].$$

Since $A(z)$, $B(z)$ and $C(z)$ have no left common factor, there are three polynomial matrices $M_1(z)$, $M_2(z)$ and $M_3(z)$ such that

$$A(z)M_1(z)+B(z)M_2(z)+C(z)M_3(z)=I,$$

and hence $A'(z)M_1(z)+B'(z)M_2(z)+C'(z)M_3(z)=D(z)$, which implies that $D(z)=A'(z)(A(z))^{-1}$ is a polynomial matrix. Furthermore, since both $A'(z)$ and $A(z)$ have identity as their leading coefficient matrices, the leading coefficient matrix of $D(z)$ must be identity.

Set $D(z)=I+D_1z+\cdots+D_dz^d$ and $a=\max(p,q,r)$, and assume $A_i=0$ for $i>p$, $B_j=0$ for $j>q$ and $C_k=0$ for $k>r$. Then we have

$$\begin{aligned}&[A'(z)\quad B'(z)\quad C'(z)]\\ =&[I\quad 0\quad I]+[A_1+D_1\quad B_1\quad C_1+D_1]z\\ &+[A_2+D_2+D_1A_1\quad B_2+D_1B_1\quad C_2+D_2+D_1C_1]z^2\\ &+\cdots+[D_dA_a\quad D_dB_a\quad D_dC_a]z^{a+d}.\end{aligned}\tag{2.10}$$

In the case $d\geq 1$ we must have $D_dA_p=0$, since $\deg A'(z)=p'\leq p$. Similarly, we have $D_dB_q=0$, $D_dC_r=0$, and hence $D_d[A_p\ \ B_q\ \ C_r]=0$, which together with the fact that rank$[A_p\ \ B_q\ \ C_r]=m$, implies that $D_d=0$. Suppose that $D_h=0$ for $h=k+1,\cdots,d$. If $k\geq 1$, then from (2.10) and $D_h=0$ $(h=k+1,\cdots,d)$ it follows that $D_k[A_p\ \ B_q\ \ C_r]=0$, which obviously implies that $D_k=0$. Therefore, we have $D(z)=I$, and hence $[A(z)\ \ B(z)\ \ C(z)]=[A'(z)\ \ B'(z)\ \ C'(z)]$ which contradicts $[A(z)\ \ B(z)\ \ C(z)]\neq[A'(z)\ \ B'(z)\ \ C'(z)]$.

The proof is completed. ∎

Theorem 2.2. If $A(z)$ is stable, $C^{-1}(z)-\frac{1}{2}I$ is strictly positive real and the system (2.1) is identifiable, then a strongly consistent estimate θ_t for θ can be given on the basis of input-output data of the system.

Proof. Let $\{v_t\}$ be a sequence of n-dimensional mutually independent random vectors with continuous distributions and satisfying

$$Ev_t=0,\quad Ev_tv_t^\tau=\frac{1}{t^\varepsilon}I,\quad \|v_t\|^2\leq\frac{\sigma^2}{t^\varepsilon},\quad t\geq 1;\qquad v_t=0,\quad t\leq 0,\tag{2.11}$$

$$\varepsilon\in\left[0,\frac{1}{2\mu+3}\right),\qquad \mu=(m+1)p+q+r,$$

where σ^2 is a fixed positive constant.

Take $u_t=v_t$ and estimate θ by θ_t:

$$\begin{aligned}\theta_{t+1}=&\theta_t+a_tP_t\varphi_t(y_{t+1}^\tau-\varphi_t^\tau\theta_t),\\ P_{t+1}=&P_t-a_tP_t\varphi_t\varphi_t^\tau P_t,\qquad a_t=(1+\varphi_t^\tau P_t\varphi_t)^{-1},\\ \varphi_t^\tau=&[\,y_t^\tau\cdots y_{t-p+1}^\tau\quad u_t^\tau\cdots u_{t-q+1}^\tau\quad y_t^\tau-\varphi_{t-1}^\tau\theta_t\cdots y_{t-r+1}^\tau-\varphi_{t-r}^\tau\theta_{t-r+1}\,]\end{aligned}$$

with $P_0=I$ and with θ_0 arbitrary.

Set

$$\begin{aligned}\varphi_t^0=&[\,y_t^\tau\cdots y_{t-p+1}^\tau\quad u_t^\tau\ldots u_{t-q+1}^\tau\quad w_t^\tau\cdots w_{t-r+1}^\tau\,]^\tau,\\ r_t^0=&mp+nq+mr+\sum_{i=0}^{t-1}\|\varphi_i^0\|^2\end{aligned}$$