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Adaptive stabilization of unstable and nonminimum-phase stochastic systems *

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Abstract: The system under consideration is the discrete time stochastic system $A(z)y_n = zB(z)u_n + w_n$ driven by a martingale difference sequence $\{w_n\}$, where $A(z)$ and $B(z)$ are polynomials in backward shift operator z with unknown coefficients and both $A(z)$ and $B(z)$ may be unstable. With the purpose of demonstrating theoretical possibility rather than designing a practically applicable control law, this paper constructs an adaptive control that stabilizes the system and simultaneously guarantees strong consistency of the least squares estimates for unknown coefficients.

Keywords: Unstable system; nonminimum-phase stochastic system; adaptive stabilization; consistency.

1. Introduction

Consider the single-input single-output stochastic system

$$A(z)y_n = zB(z)u_n + w_n, \quad \forall n \geqslant 0, \qquad y_n = u_n = 0, \quad \forall n < 0, \tag{1.1}$$

where y_n, u_n and w_n are the system output, input and unknown disturbance, respectively, and $A(z)$ and $B(z)$ are polynomials in backward shift operator z:

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \quad p \geqslant 0, \ a_p \neq 0, \tag{1.2}$$

$$B(z) = b_1 + \cdots + b_q z^{q-1}, \quad q \geqslant 1, \ b_q \neq 0. \tag{1.3}$$

Set

$$\theta = \begin{bmatrix} -a_1 & \cdots & -a_p & b_1 & \cdots & b_q \end{bmatrix}^{\mathrm{T}}, \tag{1.4}$$

which is the unknown parameter of the system.

In adaptive control, to stabilize a nonminimum-phase and unstable system is a problem important in practice and appealing in theory. In the case where $w_n \equiv 0$ ($\forall n \geqslant 0$) in (1.1), this problem is considered in [16,7,12,1]. In the case where $w_n \not\equiv 0$, this problem is the research topic of many papers, where, besides the standard coprimeness assumption on $A(z)$ and $zB(z)$, some additional conditions are usually required. For example, in [15, 8,9,14] a lower bound of controllability (and observability) degree of the

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systems is assumed to be known; in [6] it is required that a positive constant $\delta > 0$ is available so that $|\det[B_n \;\; A_n B_n \;\; \cdots \;\; A_n^{s-1} B_n]| > \delta > 0$, where

$$A_n = \begin{bmatrix} a_{1n} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ a_{sn} & & & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} b_{1n} \\ \vdots \\ b_{sn} \end{bmatrix}, \quad s = \max(p, q)$$

and $\theta_n = [a_{1n} \;\; \cdots \;\; a_{pn} \;\; b_{1n} \;\; \cdots \;\; b_{qn}]^{\mathrm{T}}$ is the estimate at time n for unknown parameter θ, $a_{in} = 0$, $b_{jn} = 0$ for $i > p$, $j > q$, while in [11,2–5] stability is imposed either on $A(z)$ or on $B(z)$ when an adaptive control problem is solved, while controlling the system in the sense of adaptive LQ or adaptive tracking.

It is worth mentioning that Giri, M'Saad, Dugart and Dion [10] have introduced a robust adaptive stabilization method for time-varying and ill-modelled systems with minimal priori knowledge. They assume that the system noise $\{w_n\}$ consists of unmodelled dynamics and uniformly bounded external disturbances.

In this paper, as in [10], except coprimeness we assume no additional condition on $A(z)$ and $B(z)$. For the case where $\limsup_{n\to\infty} n^{-1}\sum_{i=0}^{n} w_i^2 < \infty$, we give an adaptive control that stabilizes the system in the long run average sense, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(u_i^2 + y_i^2\right) < \infty \quad \text{a.s.} \tag{1.5}$$

and simultaneously leads to strong consistency of the least squares estimate for θ. Unlike [10], we do not assume that the external disturbance $\{w_n\}$ is uniformly bounded. For example, it may be a Gaussian white noise process. The key techniques used in the paper are 'explosive excitation' [13], 'diminishing excitation' [2] and 'random truncations' [3,4]. The purpose of this work is to show the ability of adaptive control by exposing the minimum condition under which the system can adaptively be stabilized. To design a practically applicable control law is beyond the task we aim at.

2. Definition of adaptive control

In this paper, for a polynomial $X(z) = \sum_{i=0}^{\mu} x_i z^i$, the norm is defined as follows:

$$\| X(z) \| = \left(\sum_{i=0}^{\mu} |x_i|^2 \right)^{1/2}.$$

For estimating the unknown parameter θ we use the LS algorithm by which the estimate θ_n is recursively defined as follows:

$$\theta_{n+1} = \theta_n + \mu_n P_n \phi_n \left(y_{n+1}^{\mathrm{T}} - \phi_n^{\mathrm{T}} \theta_n \right), \tag{2.1}$$

$$P_{n+1} = P_n - \mu_n P_n \phi_n \phi_n^{\mathrm{T}} P_n, \; \mu_n = \left(1 + \phi_n^{\mathrm{T}} P_n \phi_n\right)^{-1}, \tag{2.2}$$

$$\phi_n^{\mathrm{T}} = [y_n \;\; \cdots \;\; y_{n-p+1} \;\; u_n \;\; \cdots \;\; u_{n-q+1}] \tag{2.3}$$

with $P_0 = I$ and arbitrary initial value

$$\theta_0^{\mathrm{T}} = \left[-a_{10} \;\; \cdots \;\; -a_{p0} \;\; b_{10} \;\; \cdots \;\; b_{q0}\right].$$

For any $n \geqslant 0$ write θ_n in the component form

$$\theta_n^{\mathrm{T}} = \left[-a_{1n} \;\; \cdots \;\; -a_{pn} \;\; b_{1n} \;\; \cdots \;\; b_{qn}\right]. \tag{2.4}$$

If $A(z)$ and $zB(z)$ are coprime, then there exist two polynomials

$$G(z)=1+\sum_{j=1}^{q-1} g_j z^j, \qquad H(z)=\sum_{j=0}^{p-1} h_j z^j \tag{2.5}$$

such that

$$A(z)G(z)-zB(z)H(z)=1. \tag{2.6}$$

Set

$$M_1^{\mathrm{T}}=\begin{pmatrix} 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p \end{pmatrix} \quad (p+q \text{ by } q), \tag{2.7}$$

$$M_2^{\mathrm{T}}=\begin{pmatrix} 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q \end{pmatrix} \quad (p+q \text{ by } p), \tag{2.8}$$

$$M=[M_1 \quad M_2], \tag{2.9}$$

$$\psi^{\mathrm{T}}=\begin{bmatrix}1 & g_1 & \cdots & g_{q-1} & h_0 & \cdots & h_{p-1}\end{bmatrix}, \tag{2.10}$$

and

$$e^{\mathrm{T}}=[1 \quad 0 \quad \cdots \quad 0]_{1\times(p+q)}. \tag{2.11}$$

Replacing a_i, b_j, g_k, h_s by their estimates a_{in}, b_{jn}, g_{kn} and h_{sn} respectively in (1.2), (1.3), (2.5), (2.7)–(2.10), $i=1,\ldots,p$, $j=1,\ldots,q$, $k=1,\ldots,q-1$, $s=0,\ldots,p-1$, we correspondingly denote $A(z)$, $B(z)$, $G(z)$, $H(z)$, M_1, M_2, M and ψ by $A_n(z)$, $B_n(z)$, $G_n(z)$, $H_n(z)$, M_{1n}, M_{2n}, M_n and ψ_n respectively; for example, $A_n(z)=1+\sum_{i=1}^{p} a_{in} z^i$, $G_n(z)=1+\sum_{j=1}^{q-1} g_{jn} z^j$ and $G_n(z)u_n=u_n+\sum_{j=1}^{q-1} g_{jn} u_{n-j}$.

We state the following simple fact as a lemma.

Lemma 1. *If $A(z)$ and $zB(z)$ are coprime and $\theta_n \to \theta$ a.s. as $n\to\infty$, then there is an integer-valued n_1, possibly depending on sample path such that for any $n \geqslant n_1$ the equation*

$$A_n(z)G_n(z)-zB_n(z)H_n(z)=1 \tag{2.12}$$

has a unique solution $(G_n(z), H_n(z))$ satisfying

$$\deg(G_n(z)) \leqslant q-1, \qquad \deg(H_n(z)) \leqslant p-1 \tag{2.13}$$

and

$$\| G_n(z) \|^2 + \| H_n(z) \|^2 \leqslant 1 + \| G(z) \|^2 + \| H(z) \|^2. \tag{2.14}$$

Proof. It is easy to see that (2.6) has a unique solution $(G(z), H(z))$ if and only if the equation $M\psi=e$ has a unique solution or $\det(M)\neq 0$. Therefore, under the conditions of the lemma, $\det(M_n)\neq 0$ for sufficiently large n, and (2.10) has a unique solution with (2.13) satisfied.

Noticing that $\lim_{n\to\infty}\psi_n=\lim_{n\to\infty}M_n^{-1}e=M^{-1}e=\psi$ we conclude that for any sample path (with possible exception of set of probability 0) there exists an integer n_1 such that for any $n\geqslant n_1$, (2.14) is fulfilled. □

From (2.6) it is clear that

$$\begin{aligned} y_n &= A(z)G(z)y_n - zB(z)H(z)y_n \\ &= G(z)[A(z)y_n - zB(z)u_n] + zB(z)[G(z)u_n - H(z)y_n] \\ &= G(z)w_n + zB(z)[G(z)u_n - H(z)y_n] \end{aligned} \tag{2.15}$$

and

$$u_n = H(z)w_n + A(z)[G(z)u_n - H(z)y_n]. \tag{2.16}$$

Therefore, if θ is known and u_n is defined from

$$G(z)u_n - H(z)y_n = 0, \tag{2.17}$$

then by (2.15) and (2.16) we get $y_n = G(z)w_n$ and $u_n = H(z)w_n$. In this case, the system will be stabilized by controller (2.17) provided that $\limsup_{n\to\infty} n^{-1}\sum_{i=1}^n w_i^2 < \infty$.

The 'certainty equivalence principle' suggests us to define adaptive control from

$$G_n(z)u_n - H_n(z)y_n = 0. \tag{2.18}$$

However, in the present case the closeness of θ_n to θ is not guaranteed. Consequently, it is not clear if (2.12) is solvable or not. Even if $G_n(z)$ and $H_n(z)$ can be defined from (2.12) we still do not know whether or not they are close to $G(z)$ and $H(z)$ respectively. So it is important that θ_n somehow approximates θ. If this is the case, then adaptive control defined by (2.18) may hopefully stabilize the system, and a stabilized system will in turn lead to a better estimate for θ if the diminishing excitation technique is applied. For first step of approximating θ we apply an explosive excitation input, by which we mean such an input that yields $\lambda_{\min}^{(n)}/n \to \infty$ a.s. as $n \to \infty$, where $\lambda_{\min}^{(n)}$ denotes the minimum eigenvalue of $P_{n+1}^{-1} := I + \sum_{i=0}^n \phi_i\phi_i^{\mathrm{T}}$. To explain why an explosive excitation input will lead to an appropriate estimate for θ we need the following lemma.

Lemma 2. *Let $\{w_n\}$ in (1.1) be any disturbance (deterministic or stochastic) satisfying the following condition:*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n w_i^2 < \infty. \tag{2.19}$$

Then the accuracy of the LS *estimate θ_n for θ is expressed by*

$$\|\theta_n - \theta\|^2 = \mathrm{O}\left(n/\lambda_{\min}^{(n)}\right)$$

where $\lambda_{\min}^{(n)}$ denotes the minimum eigenvalue of $P_{n+1}^{-1} := I + \sum_{i=0}^n \phi_i\phi_i^{\mathrm{T}}$.

Proof. We will complete the proof with a similar argument used in Lemma 1 of [2]. Denote the estimation error by $\tilde{\theta}_n$, i.e. $\tilde{\theta}_n = \theta - \theta_n$. Then from (1.1), (2.1) and (2.2) it follows that

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n - P_n\phi_n\left(\tilde{\theta}_{n+1}^{\mathrm{T}}\phi_n + w_{n+1}\right), \tag{2.20}$$

and $P_{k+1}^{-1} = P_k^{-1} + \phi_k\phi_k^{\mathrm{T}}$. Thus, we have

$$\tilde{\theta}_{k+1}^{\mathrm{T}}P_{k+1}^{-1}\tilde{\theta}_{k+1} = \left[\tilde{\theta}_{k+1}^{\mathrm{T}}\phi_k\right]^2 + \tilde{\theta}_k^{\mathrm{T}}P_k^{-1}\tilde{\theta}_k - 2\left(\tilde{\theta}_{k+1}^{\mathrm{T}}\phi_k + w_{k+1}\right)\tilde{\theta}_k^{\mathrm{T}}\phi_k + \phi_k^{\mathrm{T}}P_k\phi_k\left[\tilde{\theta}_{k+1}^{\mathrm{T}}\phi_k + w_{k+1}\right]^2. \tag{2.21}$$

By (2.20) we get

$$\tilde{\theta}_k = \tilde{\theta}_{k+1} + P_k\phi_k\left(\tilde{\theta}_{k+1}^{\mathrm{T}}\phi_k + w_{k+1}\right).$$

Substituting this into the third term on the right hand side of (2.21) leads to

$$\begin{aligned}\tilde{\theta}_{k+1}^{\mathrm{T}} P_{k+1}^{-1} \tilde{\theta}_{k+1} &= \left[\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k\right]^2 + \tilde{\theta}_k^{\mathrm{T}} P_k^{-1} \tilde{\theta}_k + \phi_k^{\mathrm{T}} P_k \phi_k \left[\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k + w_{k+1}\right]^2 \\ &\quad - 2\left(\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k + w_{k+1}\right)\left[\tilde{\theta}_{k+1}^{\mathrm{T}} + \left(\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k + w_{k+1}\right) \phi_k^{\mathrm{T}} P_k\right] \phi_k \\ &= \tilde{\theta}_k^{\mathrm{T}} P_k^{-1} \tilde{\theta}_k - \left[\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k\right]^2 - 2 w_{k+1} \tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k - \phi_k^{\mathrm{T}} P_k \phi_k \left[\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k + w_{k+1}\right]^2 \\ &\leqslant \tilde{\theta}_k^{\mathrm{T}} P_k^{-1} \tilde{\theta}_k - \left[\tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k\right]^2 - 2 w_{k+1} \tilde{\theta}_{k+1}^{\mathrm{T}} \phi_k .\end{aligned}$$

Summing up both sides and using (2.19) an noticing $2|ab| = 2(a/\sqrt{2})(\sqrt{2}b) \leqslant \frac{1}{2}a^2 + 2b^2$ we have

$$\begin{aligned}\tilde{\theta}_{n+1}^{\mathrm{T}} P_{n+1}^{-1} \tilde{\theta}_{n+1} &\leqslant \mathrm{O}(1) - \sum_{i=0}^{n} \left\|\tilde{\theta}_{i+1}^{\mathrm{T}} \phi_i\right\|^2 - 2 \sum_{i=0}^{n} w_{i+1} \tilde{\theta}_{i+1}^{\mathrm{T}} \phi_i \\ &\leqslant \mathrm{O}(1) - \tfrac{1}{2} \sum_{i=0}^{n} \left\|\tilde{\theta}_{i+1}^{\mathrm{T}} \phi_i\right\|^2 + 2 \sum_{i=0}^{n} w_{i+1}^2 = \mathrm{O}(n),\end{aligned}$$

which implies the desired result. □

Lemma 2 tells us that under an explosive excitation input the LS estimate is consistent. However, the stabilization purpose (1.5) does not allow us to apply such an input for a period longer than finite. Thus we need to define stopping times σ_i at which we turn off the explosive excitation input and switch on the control defined by the certainty equivalence principle until τ_i at which the accuracy of the LS estimate θ_n becomes unsatisfactory and we have to apply the explosive excitation input again. After defining stopping times

$$0 =: \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots$$

it is most important to show that there is some integer i such that $\sigma_i < \infty$ and $\tau_i = \infty$, because otherwise the requirement (1.5) will never be met.

Let $\{\varepsilon_n\}$ be a real sequence with the following properties:

$$0 < \varepsilon_n < 1, \qquad \varepsilon_n \to 0, \qquad \varepsilon_n n^{\alpha} \to \infty \quad \text{for some } \alpha \in \left(0, \tfrac{1}{4}\right). \tag{2.22}$$

(For example, $\varepsilon_n = [\log(10+n)]^{-1}$, $\forall n \geqslant 0$.) We now define stopping times σ_i and τ_i for any $i \geqslant 1$:

$$\begin{aligned}\sigma_i = \min\Bigg\{ n > \tau_{i-1}: &\sum_{j=0}^{n-1} \phi_j \phi_j^{\mathrm{T}} - (\log s_n)^3 \varepsilon_n^{-4} I > 0;\ \det M_n \neq 0; \\ &\|G_n(z)\|^2 + \|H_n(z)\|^2 \leqslant \frac{1}{\max(p, q)\varepsilon_n}; \\ &\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_n\right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j\right)^2 \right| \leqslant 2\varepsilon_n^2 \Bigg\},\end{aligned} \tag{2.23}$$

$$\tau_i = \min\left\{ n > \sigma_i: \left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i}\right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j\right)^2 \right| > \varepsilon_{\sigma_i}^2 + \varepsilon_n^2 \right\}, \tag{2.24}$$

where s_n is given by $s_0 = 1$,

$$s_n = n \max\left\{1, \frac{1}{k}\sum_{j=0}^{k-1}\left(y_j^2 + u_j^2\right), k = 1,\dots,n\right\}, \quad \forall n \geqslant 1. \tag{2.25}$$

Let $\{v_n\}$ be independent of $\{w_n\}$ and be mutually independent with the property

$$Ev_n = 0, \quad Ev_n^2 = 1, \ |v_n| \leqslant \sigma, \quad \forall n \geqslant 0, \ v_n = 0, \quad \forall n < 0 \tag{2.26}$$

with $\sigma > 0$ being a constant.

The explosive excitation signal $\{u'_n\}$ is defined from

$$D(z)u'_n = v_n, \quad u'_0 = 0, \tag{2.27}$$

where $D(z) = 1 + \sum_{i=1}^{p+q} d_i z^i$ is an unstable polynomial of degree $p+q$, i.e.

$$D(z) \neq 0, \quad |z| \geqslant 1 \quad \text{and} \quad d_{p+q} \neq 0. \tag{2.28}$$

We now show that $\{u'_n\}$ really is explosive.

Lemma 3. *If $A(z)$ and $zB(z)$ are coprime, $\lim_{n\to\infty} n^{-1} \log \sum_{i=0}^{n} w_i^2 = 0$ and $u_n = u'_n$, then*

$$\liminf_{n\to\infty} \frac{\lambda_{\min}^{(n)}}{a^n} := c > 0 \quad a.s. \tag{2.29}$$

for some constant $a > 1$.

Proof. Replacing $-b_i$ by b_i, $i = 1,\dots,q$, in M_2 given by (2.8) we denote it by M'_2. Set $\Phi_n = A(z)\phi_n$ and $D = \begin{bmatrix} M_2'^{\mathrm{T}} \\ M_1^{\mathrm{T}} \end{bmatrix}$. From (1.1) it is easy to see that

$$\Phi_n = DU_n + W_n,$$

where $U_n = [u_n \ \cdots \ u_{n-(p+q)+1}]^{\mathrm{T}}$ and $W_n = [w_n \ \cdots \ w_{n-p+1} \ 0 \ \cdots \ 0]^{\mathrm{T}}$ (q zeros).

By coprimeness of $A(z)$ and $zB(z)$ we know that $\det(D) \neq 0$.

From (2.26)–(2.28) and Theorem 2 of [13] (see (A.2) in the Appendix) it follows that

$$\liminf_{n\to\infty} \mu^n \lambda_{\min}\left(\sum_{i=0}^{n} U_i U_i^{\mathrm{T}}\right) \geqslant c > 0 \quad \text{a.s.} \tag{2.30}$$

where c possibly depends on sample path, $\mu := \max\{|z_i|: D(z_i) = 0, i = 1,\dots,p+q\} < 1$, and $\lambda_{\min}(X)$ denotes the minimum eigenvalue of a matrix X.

It is not difficult to see that for any $\eta \in \mathbb{R}^{p+q}$ with $\|\eta\| = 1$,

$$\sum_{i=0}^{n} \|\eta^{\mathrm{T}}\Phi_i\|^2 \geqslant \tfrac{1}{2}\sum_{i=0}^{n} \|\eta^{\mathrm{T}} DU_i\|^2 - \sum_{i=0}^{n} \|\eta^{\mathrm{T}} W_i\|^2,$$

which implies that

$$\begin{aligned}\lambda_{\min}\left(\sum_{i=0}^{n} \Phi_i\Phi_i^{\mathrm{T}}\right) &\geqslant \tfrac{1}{2}\lambda_{\min}\left(\sum_{i=0}^{n} DU_iU_i^{\mathrm{T}}D^{\mathrm{T}}\right) - \lambda_{\max}\left(\sum_{i=0}^{n} W_iW_i^{\mathrm{T}}\right) \\ &\geqslant \tfrac{1}{2}\lambda_{\min}(DD^{\mathrm{T}})\lambda_{\min}\left(\sum_{i=0}^{n} U_iU_i^{\mathrm{T}}\right) - p\sum_{i=0}^{n} \|w_i\|^2. \end{aligned} \tag{2.31}$$

On the other hand, noticing that $\phi_i = 0$ for any $i < 0$ we have

$$\lambda_{\min}\left(\sum_{i=0}^{n} \Phi_i \Phi_i^{\mathrm{T}}\right) = \inf_{\|x\|=1} \sum_{i=0}^{n} \left(x^{\mathrm{T}} \Phi_i\right)^2 = \inf_{\|x\|=1} \sum_{i=0}^{n} \left(x^{\mathrm{T}} \phi_i + \sum_{j=1}^{p} a_j x^{\mathrm{T}} \phi_{i-j}\right)^2$$
$$\leqslant (p+1)\left(1 + \sum_{j=1}^{p} a_j^2\right) \inf_{\|x\|=1} \sum_{i=0}^{n} \left(x^{\mathrm{T}} \phi_i\right)^2 \leqslant (p+1)\left(1 + \sum_{j=1}^{p} a_j^2\right) \lambda_{\min}\left(\sum_{i=0}^{n} \phi_i \phi_i^{\mathrm{T}}\right). \tag{2.32}$$

Combining (2.30)–(2.32), we find that $\lambda_{\min}(\Sigma_{i=0}^{n} \phi_i \phi_i^{\mathrm{T}})$ diverges exponentially fast and (2.29) is true. □

Finally, we define adaptive control u_n at time n is given by

$$u_n = \begin{cases} u_n' & \text{if } n \in [\tau_i, \sigma_{i+1}) \text{ for some } i \geqslant 0, \\ H_{\sigma_i}(z) y_n - \left(G_{\sigma_i}(z) - 1\right) u_n + \dfrac{v_n}{(n+1)^{\varepsilon}} & \text{if } n \in [\sigma_i, \tau_i) \text{ for some } i \geqslant 1, \end{cases} \tag{2.33}$$

where $\varepsilon \in [0, 1/4(p+q))$.

We note that in the interval $[\sigma_i, \tau_i)$ we add a diminishing signal $v_n/(n+1)^{\varepsilon}$ to the control defined by the certainty equivalence principle. The purpose of this is to make the LS estimate for θ strongly consistent without damaging asymptotic behavior of the system. In contrast to [8,9,15] the upper bound of $\| G(z) \|^2 + \| H(z) \|^2$ is not assumed to be available in this paper. So in (2.23) we use a sequence $\{1/\max(p, q)\varepsilon_n\}$ diverging to infinity to dominate the estimated values of $\{\| G_n(z) \|^2 + \| H_n(z) \|^2\}$.

3. Main results

For convenience we formulate some known results on LS estimates as Theorem 1. For its proof we refer to [2,13].

Theorem 1. *If $A(z)$ and $zB(z)$ are coprime, $\{w_n, \mathscr{F}_n\}$ is a martingale difference sequence with*

$$\sup_{n \geqslant 0} E\left[w_{n+1}^2 \mid \mathscr{F}_n\right] < \infty \quad \text{and} \quad \sum_{i=0}^{n} w_i^2 = \mathrm{O}(n), \tag{3.1}$$

and u_n is $\mathscr{F}_n$-measurable, then for any $\gamma > 1$,

$$\sum_{j=0}^{n-1} \xi_j^2 = \mathrm{O}\left(\log r_n (\log \log r_n)^{\gamma}\right), \qquad \| \theta_n - \theta \|^2 = \mathrm{O}\left(\frac{\log r_n (\log \log r_n)^{\gamma}}{\lambda_{\min}^{(n)}}\right), \tag{3.2}$$

where

$$\xi_n = y_n - \phi_{n-1}^{\mathrm{T}} \theta_n - w_n, \qquad r_n = 1 + \sum_{j=0}^{n-1} \| \phi_j \|^2, \tag{3.3}$$

and $\{\mathscr{F}_n\}$ is a family of nondecreasing σ-algebras.

Further, if, in addition, $u_n = u_n^s + v_n/(n+1)^{\varepsilon}$, $\varepsilon \in [0, 1/4(p+q))$, where v_n is given by (2.26) and u_n^s is measurable with respect to $\mathscr{F}_{n-1}' := \sigma\{w_i, 0 \leqslant i \leqslant n, v_j, 0 \leqslant j \leqslant n-1\}$ with $\Sigma_{i=1}^{n-1}(u_i^s)^2 = \mathrm{O}(n)$, and if the system output satisfies $\Sigma_{i=1}^{n-1}(y_i)^2 = \mathrm{O}(n)$, then

$$\| \theta_n - \theta \|^2 = \mathrm{O}\left(\frac{\log n (\log \log n)^{\gamma}}{n^{1-2\varepsilon(p+q)}}\right), \quad \forall \gamma > 1. \tag{3.4}$$

Theorem 2. *If $A(z)$ and $zB(z)$ are coprime, $\{w_n, \mathcal{F}_n\}$ is a martingale difference sequence with (3.1) satisfied, then the adaptive control (2.33) stabilizes the closed-loop system (1.1) and the LS estimate θ_n given by (2.1)–(2.3) is strongly consistent, precisely,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \left(y_j^2 + u_j^2\right) < \infty \quad a.s., \tag{3.5}$$

$$\|\theta_n - \theta\|^2 = \mathrm{O}\left(\frac{(\log n)(\log\log n)^{\gamma}}{n^{1-2\varepsilon(p+q)}}\right) \quad \text{for any } \gamma > 1. \tag{3.6}$$

Proof. The first step is to show that there exists an integer i such that $\sigma_i < \infty$ and $\tau_i = \infty$.

By (2.25) it is clear that s_n/n is nondecreasing,

$$s_n = \mathrm{O}(nr_n) \quad \text{and} \quad r_n = \mathrm{O}(s_n). \tag{3.7}$$

We now show that it is impossible that $\tau_i < \infty$ and $\sigma_{i+1} = \infty$ for some $i \geqslant 0$. If it were not true, i.e. if there were an $i \geqslant 0$ such that $\tau_i < \infty$ and $\sigma_{i+1} = \infty$, then $u_n = u_n'$, $\forall n \geqslant \tau_i$. Noticing that $\{v_n\}$ is bounded, it is easy to see that there exists $b > 1$ such that

$$r_n = \mathrm{O}(b^n), \tag{3.8}$$

which together with (3.7) leads to $\log s_n = \mathrm{O}(n)$. Thus, from (2.22) and (2.29) it follows that for some integer $N_0 \geqslant 0$ and all $n \geqslant N_0$, the first inequality in the definition (2.23) is true.

By Lemmas 2 and 3,

$$\|\theta_n - \theta\|^2 = \mathrm{O}(n/a^n) \quad \text{a.s.} \tag{3.9}$$

From this and Lemma 1 we see that for some integer $N_1 \geqslant N_0 \geqslant 0$ and all $n \geqslant N_1$, the second and third inequalities in the definition (2.23) hold.

By (1.1), (3.1)–(3.3), (3.7)–(3.9) and Schwarz inequality we have

$$\begin{aligned}
&\left|\frac{1}{s_n}\sum_{j=0}^{n-1}\left(y_j - \phi_{j-1}^{\mathrm{T}}\theta_n\right)^2 - \frac{1}{s_n}\sum_{j=0}^{n-1}\left(y_j - \phi_{j-1}^{\mathrm{T}}\theta_j\right)^2\right| \\
&\leqslant \frac{2}{s_n}\sum_{j=0}^{n-1}\left|w_j\left(\phi_{j-1}^{\mathrm{T}}\theta - \phi_{j-1}^{\mathrm{T}}\theta_n\right)\right| + \frac{1}{s_n}\sum_{j=0}^{n-1}\left(\phi_{j-1}^{\mathrm{T}}\theta - \phi_{j-1}^{\mathrm{T}}\theta_n\right)^2 \\
&\quad + \frac{2}{s_n}\sum_{j=0}^{n-1}|w_j\xi_j| + \frac{1}{s_n}\sum_{j=0}^{n-1}\xi_j^2 \quad \text{(by (1.1) and (3.3))} \\
&\leqslant \frac{2}{s_n}\left(\sum_{j=0}^{n-1} w_j^2\right)^{1/2} r_n^{1/2}\|\theta_n - \theta\| + \frac{r_n}{s_n}\|\theta_n - \theta\|^2 + \frac{1}{s_n}\sum_{j=0}^{n-1}\xi_j^2 \\
&\quad + \frac{2}{s_n}\left(\sum_{j=0}^{n-1} w_j^2\right)^{1/2}\left(\sum_{j=0}^{n-1}\xi_j^2\right)^{1/2} \quad \text{(by Schwarz inequality)} \\
&= \mathrm{O}\left(\sqrt{\frac{n\log^{\gamma} n}{a^n}}\right) \quad \forall \gamma > 1 \quad \text{(by (3.1), (3.2), (3.7)–(3.9))}.
\end{aligned} \tag{3.10}$$

From this and (2.22) it follows that there exists an integer $N_2 \geqslant N_1 \geqslant 0$ such that for any $n \geqslant N_2$,

$$\left|\frac{1}{s_n}\sum_{j=0}^{n-1}\left(y_j - \phi_{j-1}^{\mathrm{T}}\theta_n\right)^2 - \frac{1}{s_n}\sum_{j=0}^{n-1}\left(y_j - \phi_{j-1}^{\mathrm{T}}\theta_j\right)^2\right| \leqslant 2\varepsilon_n^2, \tag{3.11}$$

i.e. the last inequality in the definition (2.23) is true.

Therefore, we have $\sigma_{i+1} \leqslant N_2$. This contradicts $\sigma_{i+1} = \infty$.
We now prove that $\tau_i = \infty$ for some i. By (3.2) it follows that

$$\| \theta_{\sigma_i} - \theta \|^2 = \mathrm{O}\left((\log s_{\sigma_i})^2 / \lambda_{\min}^{(\sigma_i - 1)} \right),$$

which incorporating the definition of σ_i implies that

$$\| \theta_{\sigma_i} - \theta \|^2 = \mathrm{o}\left(\varepsilon_{\sigma_i}^4 \right). \tag{3.12}$$

Similar to (3.10), by (3.2) we obtain that

$$\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i} \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j \right)^2 \right| = \mathrm{O}\left(\| \theta_{\sigma_i} - \theta \| + \frac{\log s_n}{\sqrt{s_n}} \right). \tag{3.13}$$

By (3.12), (3.13) and $s_n \geqslant n$ we know that there exists and i_0 such that for any $i \geqslant i_0$ and any $n \geqslant \sigma_i$,

$$\left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i} \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j \right)^2 \right| \leqslant \varepsilon_{\sigma_i}^2 + \varepsilon_n^2.$$

Thus $\tau_{i_0} = \infty$. For simplicity of notations we write i_0 as i, i.e. $\sigma_i < \infty$, $\tau_i = \infty$ a.s.
By (2.12) we have

$$y_n = G_{\sigma_i}(z) \left[A_{\sigma_i}(z) y_n - z B_{\sigma_i}(z) u_n \right] + z B_{\sigma_i}(z) \left[G_{\sigma_i}(z) u_n - H_{\sigma_i}(z) y_n \right], \tag{3.14}$$

$$u_n = H_{\sigma_i}(z) \left[A_{\sigma_i}(z) y_n - z B_{\sigma_i}(z) u_n \right] + A_{\sigma_i}(z) \left[G_{\sigma_i}(z) u_n - H_{\sigma_i}(z) y_n \right]. \tag{3.15}$$

Hence, from (3.14), (3.15) and (2.33) we get, for any $n \geqslant n_0 := \sigma_i + \max(p, q)$,

$$y_n = G_{\sigma_i}(z) \left[A_{\sigma_i}(z) y_n - z B_{\sigma_i}(z) u_n \right] + z B_{\sigma_i}(z) \frac{v_n}{(n+1)^{\varepsilon}}, \tag{3.16}$$

$$u_n = H_{\sigma_i}(z) \left[A_{\sigma_i}(z) y_n - z B_{\sigma_i}(z) u_n \right] + A_{\sigma_i}(z) \frac{v_n}{(n+1)^{\varepsilon}}. \tag{3.17}$$

By noticing $|v_j| \leqslant \sigma$ and the elementary inequality $(\alpha + \beta)^2 \leqslant (1+\nu)\alpha^2 + ((1+\nu)/\nu)\beta^2$, $\forall \nu > 0$, $\forall \alpha$, $\forall \beta$, from (3.16) and (3.17) it follows that for any $n \geqslant n_0$,

$$\begin{aligned}
\frac{1}{n} \sum_{j=0}^{n-1} \left(y_j^2 + u_j^2 \right) &= \frac{1}{n} \sum_{j=n_0}^{n-1} \left(y_j^2 + u_j^2 \right) + \frac{1}{n} \sum_{j=0}^{n_0 - 1} \left(y_j^2 + u_j^2 \right) \\
&\leqslant \frac{1+\nu}{n} \max(p, q) \left(\| G_{\sigma_i}(z) \|^2 + \| H_{\sigma_i}(z) \|^2 \right) \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i} \right)^2 \\
&\quad + \frac{1+\nu}{\nu n} [1 + \max(p, q)] \left(\| A_{\sigma_i}(z) \|^2 + \| B_{\sigma_i}(z) \|^2 \right) \sum_{j=1}^{n-1} \frac{v_j^2}{(j+1)^{2\varepsilon}} \\
&\quad + \frac{1}{n} \sum_{j=0}^{n_0 - 1} \left(y_j^2 + u_j^2 \right) \quad \text{(by (3.16) and (3.17))}
\end{aligned}$$

$$= \frac{1+\nu}{\varepsilon_{\sigma_i}} \frac{1}{n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i} \right)^2 + \mathrm{O}(1) \quad \left(\text{by (2.23) and } |v_j| \leqslant \sigma \right)$$

$$\leqslant \frac{1+\nu}{\varepsilon_{\sigma_i}} \frac{s_n}{n} \left| \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_{\sigma_i} \right)^2 - \frac{1}{s_n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j \right)^2 \right|$$

$$+ \frac{1+\nu}{\varepsilon_{\sigma_i}} \frac{1}{n} \sum_{j=0}^{n-1} \left(y_j - \phi_{j-1}^{\mathrm{T}} \theta_j \right)^2 + \mathrm{O}(1)$$

$$\leqslant (1+\nu) \left(\varepsilon_{\sigma_i} + \frac{\varepsilon_n^2}{\varepsilon_{\sigma_i}} \right) \frac{s_n}{n} + \mathrm{O}\left(\frac{(\log s_n)^2}{n} \right) + \mathrm{O}(1), \tag{3.18}$$

where for the last inequality we have used (2.24) and $\tau_i = \infty$.

Since $\varepsilon_{\sigma_i} < 1$ and $\varepsilon_n \to 0$, we can take $\nu > 0$ small enough and n_1 sufficiently large so that $(1+\nu)(\varepsilon_{\sigma_i} + \varepsilon_n^2/\varepsilon_{\sigma_i}) \leqslant \beta < 1$ for all $n \geqslant n_1$. Therefore, for any $n \geqslant n_1$,

$$\frac{1}{n} \sum_{j=0}^{n-1} \left(y_j^2 + u_j^2 \right) \leqslant \beta \frac{s_n}{n} + c_1, \tag{3.19}$$

where $c_1 < \infty$ is independent of n.

Noticing that s_n/n is nondecreasing from (3.19) we get for any $n \geqslant n_1$ and any $l \in [n_1, n]$,

$$\frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2 \right) \leqslant \beta \frac{s_l}{l} + c_1 \leqslant \beta \frac{s_n}{n} + c_1,$$

which together with (2.25) yields

$$\frac{s_n}{n} \leqslant \max\left\{ 1;\ \frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2 \right),\ l = 1, \ldots, n_1 - 1;\ \beta \frac{s_n}{n} + c_1 \right\},$$

or

$$\frac{s_n}{n} \leqslant \beta \frac{s_n}{n} + c_2 \tag{3.20}$$

where

$$c_2 = 1 + c_1 + \max\left\{ \frac{1}{l} \sum_{j=0}^{l-1} \left(y_j^2 + u_j^2 \right),\ l = 1, \ldots, n_l - 1 \right\}.$$

Consequently, $s_n/n \leqslant (1-\beta)^{-1} c_2$, i.e., (3.5) follows, while (3.6) implied by Theorem 1. □

4. Conclusion

For the stochastic system, which possibly is of nonminimum-phase and open-loop unstable, we basically use the adaptive control constructed by the certainty equivalence principle and excited by a diminishing signal, but with explosive inputs applied at some random time intervals of finite length. As results, both adaptive stabilization and consistent parameter estimate are achieved.

Appendix

Consider the autoregressive AR(p) model

$$\bar{y}_n = \beta_1 \bar{y}_{n-1} + \cdots + \beta_p \bar{y}_{n-p} + \bar{\varepsilon}_n, \tag{A.1}$$

where $\bar{y}_n$ is the observation and $\bar{\varepsilon}_n$ is the random disturbance at stage n, and $\beta_1, \ldots, \beta_p$ are the parameters of the model.

Let

$$f(z) = z^p - \beta_1 z^{p-1} - \cdots - \beta_p,$$

$$Y_n = \left[\bar{y}_n, \ldots, \bar{y}_{n-p+1}\right]^{\mathrm{T}}, \qquad L_n = \left[\bar{\varepsilon}_n, 0, \ldots, 0\right]^{\mathrm{T}} \quad (p-1 \text{ zeros})$$

$$F = \begin{pmatrix} \beta_1, \ldots, \beta_{p-1} & \beta_p \\ I_{p-1} & 0 \end{pmatrix} \quad \text{and} \quad X_n = \left[Y_p, Y_{p+1}, \ldots, Y_{n-1}\right]^{\mathrm{T}}.$$

Then Lai and Wei [13] obtain the following results.

Theorem A. *Suppose that in the* AR(p) *model* (A.1), $\{\bar{\varepsilon}_n\}$ *is a martingale difference sequence with respect to an increasing sequence of* σ*-fields* $\{\mathscr{Y}_n\}$ *such that*

$$\liminf_{n\to\infty} E\left(\bar{\varepsilon}_n^2 \mid \mathscr{Y}_{n-1}\right) > 0 \quad a.s.$$

Assume that the roots z_j *of the characteristic polynomial* $f(z)$ *lie outside the unit circle, i.e.,* $|z_j| > 1$ *for* $j = 1, \ldots, p$. *Let* $\overline{m} = \min_{1\le j\le p} |z_j|$ *and* $\overline{M} = \max_{1\le j\le p} |z_j|$. *Then we have*

(i) $F^{-n} Y_{n+p}$ *converges a.s. to* $Z := Y_p + \sum_{i=1}^{\infty} F^{-i} D_{i+p}$. *Moreover,* $x^{\mathrm{T}} Z$ *has a continuous distribution for all* $x \in \mathbb{R}^p - \{0\}$.

(ii) $F^{-n} X_n^{\mathrm{T}} X_n (F^{-n})^{\mathrm{T}}$ *converges a.s. to* $\Sigma := \sum_{i=p+1}^{\infty} F^{-i}(ZZ^{\mathrm{T}})(F^{-i})^{\mathrm{T}}$. *Moreover,* Σ *is positive definite with probability* 1. *Consequently,*

$$\lim_{n\to\infty} n^{-1} \log \lambda_{\min}\left(X_n^{\mathrm{T}} X_n\right) = 2 \log \overline{m} \quad a.s., \tag{A.2}$$

$$\lim_{n\to\infty} n^{-1} \log \lambda_{\max}\left(X_n^{\mathrm{T}} X_n\right) = 2 \log \overline{M} \quad a.s. \tag{A.3}$$

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