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Adaptive stabilization of unstable and nonminimum-phase stochastic systems *

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Abstract: The system under consideration is the discrete time stochastic system $A(z)y_n = zB(z)u_n + w_n$ driven by a martingale difference sequence $\{w_n\}$, where $A(z)$ and $B(z)$ are polynomials in backward shift operator z with unknown coefficients and both $A(z)$ and $B(z)$ may be unstable. With the purpose of demonstrating theoretical possibility rather than designing a practically applicable control law, this paper constructs an adaptive control that stabilizes the system and simultaneously guarantees strong consistency of the least squares estimates for unknown coefficients.

Keywords: Unstable system; nonminimum-phase stochastic system; adaptive stabilization; consistency.

1. Introduction

Consider the single-input single-output stochastic system

$$A(z)y_n = zB(z)u_n + w_n, \quad \forall n \geqslant 0, \qquad y_n = u_n = 0, \quad \forall n < 0, \tag{1.1}$$

where y_n, u_n and w_n are the system output, input and unknown disturbance, respectively, and $A(z)$ and $B(z)$ are polynomials in backward shift operator z:

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \quad p \geqslant 0, \ a_p \neq 0, \tag{1.2}$$

$$B(z) = b_1 + \cdots + b_q z^{q-1}, \quad q \geqslant 1, \ b_q \neq 0. \tag{1.3}$$

Set

$$\theta = \begin{bmatrix} -a_1 & \cdots & -a_p & b_1 & \cdots & b_q \end{bmatrix}^{\mathrm{T}}, \tag{1.4}$$

which is the unknown parameter of the system.

In adaptive control, to stabilize a nonminimum-phase and unstable system is a problem important in practice and appealing in theory. In the case where $w_n \equiv 0$ ($\forall n \geqslant 0$) in (1.1), this problem is considered in [16,7,12,1]. In the case where $w_n \not\equiv 0$, this problem is the research topic of many papers, where, besides the standard coprimeness assumption on $A(z)$ and $zB(z)$, some additional conditions are usually required. For example, in [15, 8,9,14] a lower bound of controllability (and observability) degree of the

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systems is assumed to be known; in [6] it is required that a positive constant $\delta > 0$ is available so that $|\det[B_n \ A_nB_n \ \cdots \ A_n^{s-1}B_n]| > \delta > 0$, where

$$A_n = \begin{bmatrix} a_{1n} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ a_{sn} & & & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} b_{1n} \\ \vdots \\ b_{sn} \end{bmatrix}, \quad s = \max(p, q)$$

and $\theta_n = [a_{1n} \ \cdots \ a_{pn} \ b_{1n} \ \cdots \ b_{qn}]^{\mathrm{T}}$ is the estimate at time n for unknown parameter θ, $a_{in} = 0$, $b_{jn} = 0$ for $i > p$, $j > q$, while in [11,2–5] stability is imposed either on $A(z)$ or on $B(z)$ when an adaptive control problem is solved, while controlling the system in the sense of adaptive LQ or adaptive tracking.

It is worth mentioning that Giri, M'Saad, Dugart and Dion [10] have introduced a robust adaptive stabilization method for time-varying and ill-modelled systems with minimal priori knowledge. They assume that the system noise $\{w_n\}$ consists of unmodelled dynamics and uniformly bounded external disturbances.

In this paper, as in [10], except coprimeness we assume no additional condition on $A(z)$ and $B(z)$. For the case where $\limsup_{n\to\infty} n^{-1}\sum_{i=0}^{n} w_i^2 < \infty$, we give an adaptive control that stabilizes the system in the long run average sense, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(u_i^2 + y_i^2\right) < \infty \quad \text{a.s.} \tag{1.5}$$

and simultaneously leads to strong consistency of the least squares estimate for θ. Unlike [10], we do not assume that the external disturbance $\{w_n\}$ is uniformly bounded. For example, it may be a Gaussian white noise process. The key techniques used in the paper are 'explosive excitation' [13], 'diminishing excitation' [2] and 'random truncations' [3,4]. The purpose of this work is to show the ability of adaptive control by exposing the minimum condition under which the system can adaptively be stabilized. To design a practically applicable control law is beyond the task we aim at.

2. Definition of adaptive control

In this paper, for a polynomial $X(z) = \sum_{i=0}^{\mu} x_i z^i$, the norm is defined as follows:

$$\| X(z) \| = \left(\sum_{i=0}^{\mu} | x_i |^2 \right)^{1/2}.$$

For estimating the unknown parameter θ we use the LS algorithm by which the estimate θ_n is recursively defined as follows:

$$\theta_{n+1} = \theta_n + \mu_n P_n \phi_n \left(y_{n+1}^{\mathrm{T}} - \phi_n^{\mathrm{T}} \theta_n \right), \tag{2.1}$$

$$P_{n+1} = P_n - \mu_n P_n \phi_n \phi_n^{\mathrm{T}} P_n, \ \mu_n = \left(1 + \phi_n^{\mathrm{T}} P_n \phi_n\right)^{-1}, \tag{2.2}$$

$$\phi_n^{\mathrm{T}} = [y_n \ \cdots \ y_{n-p+1} \ u_n \ \cdots \ u_{n-q+1}] \tag{2.3}$$

with $P_0 = I$ and arbitrary initial value

$$\theta_0^{\mathrm{T}} = \left[-a_{10} \ \cdots \ -a_{p0} \ b_{10} \ \cdots \ b_{q0}\right].$$

For any $n \geqslant 0$ write θ_n in the component form

$$\theta_n^{\mathrm{T}} = \left[-a_{1n} \ \cdots \ -a_{pn} \ b_{1n} \ \cdots \ b_{qn}\right]. \tag{2.4}$$

If $A(z)$ and $zB(z)$ are coprime, then there exist two polynomials

$$G(z)=1+\sum_{j=1}^{q-1} g_j z^j, \qquad H(z)=\sum_{j=0}^{p-1} h_j z^j \tag{2.5}$$

such that

$$A(z)G(z)-zB(z)H(z)=1. \tag{2.6}$$

Set

$$M_1^{\mathrm{T}}=\begin{pmatrix} 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p \end{pmatrix} \quad (p+q \text{ by } q), \tag{2.7}$$

$$M_2^{\mathrm{T}}=\begin{pmatrix} 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q \end{pmatrix} \quad (p+q \text{ by } p), \tag{2.8}$$

$$M=[M_1 \quad M_2], \tag{2.9}$$

$$\psi^{\mathrm{T}}=[1 \quad g_1 \quad \cdots \quad g_{q-1} \quad h_0 \quad \cdots \quad h_{p-1}], \tag{2.10}$$

and

$$e^{\mathrm{T}}=[1 \quad 0 \quad \cdots \quad 0]_{1\times(p+q)}. \tag{2.11}$$

Replacing a_i, b_j, g_k, h_s by their estimates a_{in}, b_{jn}, g_{kn} and h_{sn} respectively in (1.2), (1.3), (2.5), (2.7)–(2.10), $i=1,\dots,p$, $j=1,\dots,q$, $k=1,\dots,q-1$, $s=0,\dots,p-1$, we correspondingly denote $A(z)$, $B(z)$, $G(z)$, $H(z)$, M_1, M_2, M and ψ by $A_n(z)$, $B_n(z)$, $G_n(z)$, $H_n(z)$, M_{1n}, M_{2n}, M_n and ψ_n respectively; for example, $A_n(z)=1+\sum_{i=1}^{p} a_{in}z^i$, $G_n(z)=1+\sum_{j=1}^{q-1} g_{jn}z^j$ and $G_n(z)u_n=u_n+\sum_{j=1}^{q-1} g_{jn}u_{n-j}$.

We state the following simple fact as a lemma.

Lemma 1. *If $A(z)$ and $zB(z)$ are coprime and $\theta_n \to \theta$ a.s. as $n\to\infty$, then there is an integer-valued n_1, possibly depending on sample path such that for any $n \geqslant n_1$ the equation*

$$A_n(z)G_n(z)-zB_n(z)H_n(z)=1 \tag{2.12}$$

has a unique solution $(G_n(z), H_n(z))$ satisfying

$$\deg(G_n(z)) \leqslant q-1, \qquad \deg(H_n(z)) \leqslant p-1 \tag{2.13}$$

and

$$\|G_n(z)\|^2+\|H_n(z)\|^2 \leqslant 1+\|G(z)\|^2+\|H(z)\|^2. \tag{2.14}$$

Proof. It is easy to see that (2.6) has a unique solution $(G(z), H(z))$ if and only if the equation $M\psi=e$ has a unique solution or $\det(M)\neq 0$. Therefore, under the conditions of the lemma, $\det(M_n)\neq 0$ for sufficiently large n, and (2.10) has a unique solution with (2.13) satisfied.

Noticing that $\lim_{n\to\infty}\psi_n=\lim_{n\to\infty}M_n^{-1}e=M^{-1}e=\psi$ we conclude that for any sample path (with possible exception of set of probability 0) there exists an integer n_1 such that for any $n\geqslant n_1$, (2.14) is fulfilled. □

From (2.6) it is clear that

$$\begin{aligned} y_n &= A(z)G(z)y_n - zB(z)H(z)y_n \\ &= G(z)[A(z)y_n - zB(z)u_n] + zB(z)[G(z)u_n - H(z)y_n] \\ &= G(z)w_n + zB(z)[G(z)u_n - H(z)y_n] \end{aligned} \tag{2.15}$$

and

$$u_n = H(z)w_n + A(z)[G(z)u_n - H(z)y_n]. \tag{2.16}$$

Therefore, if θ is known and u_n is defined from

$$G(z)u_n - H(z)y_n = 0, \tag{2.17}$$

then by (2.15) and (2.16) we get $y_n = G(z)w_n$ and $u_n = H(z)w_n$. In this case, the system will be stabilized by controller (2.17) provided that $\limsup_{n\to\infty} n^{-1}\sum_{i=1}^n w_i^2 < \infty$.

The 'certainty equivalence principle' suggests us to define adaptive control from

$$G_n(z)u_n - H_n(z)y_n = 0. \tag{2.18}$$

However, in the present case the closeness of θ_n to θ is not guaranteed. Consequently, it is not clear if (2.12) is solvable or not. Even if $G_n(z)$ and $H_n(z)$ can be defined from (2.12) we still do not know whether or not they are close to $G(z)$ and $H(z)$ respectively. So it is important that θ_n somehow approximates θ. If this is the case, then adaptive control defined by (2.18) may hopefully stabilize the system, and a stabilized system will in turn lead to a better estimate for θ if the diminishing excitation technique is applied. For first step of approximating θ we apply an explosive excitation input, by which we mean such an input that yields $\lambda_{\min}^{(n)}/n \to \infty$ a.s. as $n \to \infty$, where $\lambda_{\min}^{(n)}$ denotes the minimum eigenvalue of $P_{n+1}^{-1} := I + \sum_{i=0}^n \phi_i\phi_i^{\mathrm{T}}$. To explain why an explosive excitation input will lead to an appropriate estimate for θ we need the following lemma.

Lemma 2. *Let $\{w_n\}$ in (1.1) be any disturbance (deterministic or stochastic) satisfying the following condition*:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n w_i^2 < \infty. \tag{2.19}$$

Then the accuracy of the LS *estimate θ_n for θ is expressed by*

$$\|\theta_n - \theta\|^2 = \mathrm{O}(n/\lambda_{\min}^{(n)})$$

where $\lambda_{\min}^{(n)}$ denotes the minimum eigenvalue of $P_{n+1}^{-1} := I + \sum_{i=0}^n \phi_i\phi_i^{\mathrm{T}}$.

Proof. We will complete the proof with a similar argument used in Lemma 1 of [2]. Denote the estimation error by $\tilde\theta_n$, i.e. $\tilde\theta_n = \theta - \theta_n$. Then from (1.1), (2.1) and (2.2) it follows that

$$\tilde\theta_{n+1} = \tilde\theta_n - P_n\phi_n(\tilde\theta_{n+1}^{\mathrm{T}}\phi_n + w_{n+1}), \tag{2.20}$$

and $P_{k+1}^{-1} = P_k^{-1} + \phi_k\phi_k^{\mathrm{T}}$. Thus, we have

$$\tilde\theta_{k+1}^{\mathrm{T}}P_{k+1}^{-1}\tilde\theta_{k+1} = [\tilde\theta_{k+1}^{\mathrm{T}}\phi_k]^2 + \tilde\theta_k^{\mathrm{T}}P_k^{-1}\tilde\theta_k - 2(\tilde\theta_{k+1}^{\mathrm{T}}\phi_k + w_{k+1})\tilde\theta_k^{\mathrm{T}}\phi_k + \phi_k^{\mathrm{T}}P_k\phi_k[\tilde\theta_{k+1}^{\mathrm{T}}\phi_k + w_{k+1}]^2. \tag{2.21}$$

By (2.20) we get

$$\tilde\theta_k = \tilde\theta_{k+1} + P_k\phi_k(\tilde\theta_{k+1}^{\mathrm{T}}\phi_k + w_{k+1}).$$