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Linear quadratic mean field social control with common noise: A directly decoupling method☆

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ABSTRACT

This paper is concerned with mean field linear quadratic social control with common noise, where the weight matrices of individual costs are indefinite We first obtain a set of forward–backward stochastic differential equations (FBSDEs) from variational analysis, and then construct a centralized feedback representation by decoupling the FBSDEs. By using solutions of two Riccati equations, we design a set of decentralized control laws, which is further shown to be asymptotically social optimal. The necessary and sufficient conditions are given for uniform stabilization of the systems by exploiting the relation between population state average and aggregate effect. An explicit expression of the optimal social cost is given in terms of two Riccati equations. Besides, the decentralized optimal solution is provided for mean field social control problem with finite agents.

1. Introduction

1.1. Background and motivation

The subject of mean field (MF) games and control has attracted increasing attention from communities of system control, mathematics and economics (Bensoussan, Frehse, & Yam, 2013; Caines, Huang, & Malhame, 2017; Carmona & Delarue, 2018; Gomes & Saude, 2014). A MF model involves a large number of interactive players (agents), where a key feature is while the influence of each agent is negligible, the impact of the overall population is significant to each one. The main idea of MF approximations is to replace the average interaction of an agent with all other agents by aggregation effect. By consistent MF approximations, the dimensionality difficulty is overcome. By now, the linear quadratic (LQ) framework has been commonly adopted in MF studies because of its analytical tractability and close connection to practical applications. In this aspect, some relevant works include (Bensoussan, Sung, Yam, & Yung, 2016; Elliott, Li, & Ni, 2013; Huang, Caines, & Malhamé, 2007; Li & Zhang, 2008; Moon & Basar, 2017; Wang & Zhang, 2012b). Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point analysis and designed ϵ-Nash equilibria for large-population LQ games with discounted costs (Huang et al., 2007). The NCE approach was then applied to the more general cases with ergodic costs (Li & Zhang, 2008) and with Markov jump parameters (Wang & Zhang, 2012b), respectively. The works (Bensoussan et al., 2016; Carmona & Delarue, 2013) employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of equilibrium strategies over a finite horizon. For other aspects of MF games, readers are referred to Carmona and Delarue (2013), Huang, Malhamé, and Caines (2006) and Lasry and Lions (2007) for nonlinear MF games, Weintraub, Benkard, and Roy (2008) for oblivious equilibrium in dynamic games, Huang (2010) and Wang and Zhang (2012a), for MF games with major players, Huang and Huang (2017) and Moon and Basar (2017) for robust MF games.

Mean field games with common noise are a kind of large-population games with correlated players who share a common random noise (Carmona, Delarue, & Lacker, 2016). The common noise may be interpreted as a passive version of the major player (Huang & Wang, 2015). Such modeling can accommodate considerable situations since common noise may represent some external factors with influence on all players. This is well-framed in reality, particularly in finance and economics; for instance, the physical environment for all particles or a financial policy for all market participants (Carmona, Fouque, & Sun, 2015; Guéant,

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Lasry, & Lions, 2011). Due to the impact of common noise, all the players are not independent of each other, but correlative or strong coupling. Consequently, MF games with common noise are within a more general setting, and the related analysis becomes more technical. The work (Carmona et al., 2016) studied strong and weak solutions to MF games with common noise from the fixed point analysis in random measure flows. The well-posedness problem was further considered in Ahuja (2016). The work (Graber, 2016) investigated a general LQ-MF type control and connected it to mean field games with common noise. The MF limit of large-population symmetric games was derived with and without common noise, respectively (Lacker, 2016).

Besides noncooperative games, the social optima in MF models with weak coupling have drawn more research interests. By social optimization, all players in a large-population system cooperate to optimize the common social cost—the sum of individual costs. Accordingly, we formulate a type of team decision problem (Radner, 1962). Different from Nash games, all players in a team problem share the same cost and cooperate to reach a social optimum although they may have different information sets (Ho, 1980). The work (Huang, Caines, & Malhamé, 2012) studied social optima in MF-LQ control, and provided an asymptotic team-optimal solution. Subsequently, Wang and Zhang (2017) investigated a MF social optimal problem where the Markov jump parameter appears as a common source of randomness. The work (Huang & Nguyen, 2016) designed socially optimal strategies by analyzing forward backward stochastic differential equations (FBSDEs). In Salhab, Ny, and Malhame (2018), authors investigated dynamic cooperative collective choice by finding a social optimum. Furthermore, some recent studies focused on stochastic dynamic teams and their MF limit, such as Sanjari and Yuksel (2019).

1.2. Novelty and contributions

This paper investigates decentralized control of MF-LQ social control with common noise, where the weight matrices in individual costs are indefinite. The presence of common noise leads to the strong correlations of all agents, which makes the corresponding analysis much more complicated (Carmona et al., 2016). Most previous works on MF social control applied the fixed-point method and person-by-person optimality (see e.g. Huang et al., 2012; Salhab et al., 2018; Wang & Zhang, 2017). However, due to the appearance of common noise, the corresponding fixed-point analysis is very complicated (Huang, Wang, & Yong, 2020). In this paper, we obtain decentralized control of the problem by decoupling directly high-dimensional FBSDEs with MF approximations. This procedure shares a similar philosophy with the direct method (Huang & Zhou, 2020; Lasry & Lions, 2007; Wang, Zhang, & Zhang, 2020). In recent years, some progress has been made for the study of the optimal LQ control by tackling FBSDEs. Readers are referred to Carmona and Delarue (2013), Wang et al. (2020), Yong (2013), Zhang (2021), Zhang, Qi, and Fu (2019) etc. for details.

For the finite-horizon problem, we first obtain a set of high-dimensional FBSDEs by tackling the large-scale social control problem, and give a centralized feedback-type control laws (only depending on the state of a representative agent and population state average) by decoupling the FBSDEs. By the MF heuristics, we design a set of decentralized control laws, which is further shown to be asymptotically optimal. An explicit form of the asymptotic social optimal cost is given in terms of solutions to two Riccati equations. Besides, we apply the result to obtain decentralized solution to a class of MF social control problem with finite agents. For the infinite-horizon case, we first design a set of decentralized control laws with help of two Riccati equations, and then show the consistency of MF approximations by the Lyapunov function method. By exploiting the relationship between population state average and aggregate effect, two equivalent conditions are given for uniform stabilization of all the subsystems.

The main contributions of the paper are listed as follows.

- For the finite-horizon problem, we first obtain the existence conditions of centralized optimal control by variational analysis, and then design a feedback decentralized control by decoupling FBSDEs and applying MF approximations.
- With the help of condition expectation, we give the *decentralized optimal* solution to a class of MF social control problem with finite agents.
- For the infinite-horizon problem, feedback decentralized control laws are designed with help of the solutions to Riccati equations. Two equivalent conditions are given for uniform stabilization of all the subsystems by making full use of the relation between population state average and aggregate effect.
- Under some basic assumptions we obtain asymptotic optimality of decentralized control. No fixed-point equations or additional assumptions are needed. Besides, an explicit expression of the optimal social cost is given by virtue of two Riccati equations.

1.3. Organization and notation

The organization of the paper is as follows. In Section 2, we first design the asymptotical optimal control for finite-horizon MF social control problems, and then give the optimal decentralized control of MF social problems with finite agents. In Section 3, we design decentralized controls for the infinite-horizon case and further give necessary and sufficient conditions for uniform stabilization of the systems. In Section 4, a numerical example is given to verify the results. Section 5 concludes the paper.

The following notation will be used throughout this paper. We use $|\cdot|$ to denote the norm of a Euclidean space, or the Frobenius norm for matrices. For a symmetric matrix Q and a vector z, $|z|_Q^2 = z^TQz$. $Q^\dagger$ is the Moore–Penrose pseudoinverse[1] of the matrix Q; $\mathcal{R}(Q)$ denotes the range of a matrix (or an operator) Q. For a vector or matrix M, M^T denotes its transpose, and $M > 0$ ($M \geq 0$) means that M is positive definite (nonnegative definite). Let $\mathbb{S}^n$ denote the space of all symmetric $n \times n$ matrices. Let $L^\infty(0,T;\mathbb{R}^{n\times k})$ (resp., $C([0,T];\mathbb{R}^{n\times k})$) be the space of all $\mathbb{R}^{n\times k}$-valued bounded (resp. continuous bounded) functions; let $L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ be the space of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-adapted processes $x(\cdot)$ satisfying $\mathbb{E}\int_0^T|x(t)|^2dt < \infty$. For convenience of presentation, we use $c, c_0, c_1, \ldots$ to denote generic positive constants, which may vary from place to place.

2. Finite horizon mean field LQ social control

Consider a large population system with N agents. The ith agent $\mathcal{A}_i$ evolves by the following stochastic differential equation (SDE):

$$\begin{aligned}dx_i(t) = {}&[A(t)x_i(t) + G(t)x^{(N)}(t) + B(t)u_i(t) + f(t)]dt\\&+ [C(t)x_i(t) + D(t)u_i(t) + \sigma(t)]dW_i(t)\\&+ [C_0(t)x_i(t) + D_0(t)u_i(t) + \sigma_0(t)]dW_0(t),\end{aligned} \tag{1}$$

where $1 \leq i \leq N$, $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and input of the agent $\mathcal{A}_i$. $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$ is the population state

[1] $Q^\dagger$ is a unique matrix satisfying $QQ^\dagger Q = Q^\dagger$, $Q^\dagger QQ^\dagger = Q$, $(Q^\dagger Q)^T = Q^\dagger Q$, and $(QQ^\dagger)^T = QQ^\dagger$.

average. $\{W_i(t), 0 \le i \le N\}$ are a sequence of independent 1-dimensional Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$. A, B, G, C, D, C_0 and D_0 are deterministic matrix-valued functions with compatible dimensions, where $\mathcal{F}_t^0 = \sigma(W_0(s), 0 \le s \le t)$. f, σ and σ_0 are $\mathcal{F}_t^0$-adapted vector-valued processes in $\mathbb{R}^n$, reflecting the impact on each agent by the environment. The cost functional of $\mathcal{A}_i$ is given by

$$J_i^{\rm F}(u) = \mathbb{E}\int_0^T \left\{ \left|x_i(t) - \Gamma(t)x^{(N)}(t) - \eta(t)\right|^2_{Q(t)} + |u_i(t)|^2_{R(t)} \right\} dt + \mathbb{E}|x_i(T) - \Gamma_0 x^{(N)}(T) - \eta_0|^2_H. \quad (2)$$

where Q, R and Γ are bounded deterministic matrix-valued functions with compatible dimensions. Denote $u = \{u_1, \dots, u_i, \dots, u_N\}$. Let $\mathcal{F}_t^i = \sigma(x_i(0), W_i(s), 0 \le s \le t)$, and $\mathcal{G}_t^i = \sigma(x_i(s), W_i(s), W_0(s), 0 \le s \le t)$, $i = 1, \dots, N$. Denote $\mathcal{F}_t = \sigma\{\bigcup_{i=0}^N \mathcal{F}_t^i\}$. Define the centralized control set as

$$\mathcal{U}_c = \left\{ (u_1, \dots, u_N) \middle| \, u_i(t) \text{ is adapted to } \mathcal{F}_t, \ \mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty \right\},$$

and the decentralized control set as

$$\mathcal{U}_d = \left\{ (u_1, \dots, u_N) \middle| u_i(t) \text{ is adapted to } \mathcal{G}_t^i, \ \mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty \right\}.$$

In this section, we mainly study the two problems:

(P1). Seek a set of centralized control laws to optimize the social cost $J_{\rm soc}^{\rm F}$ for the system (1)–(2), i.e., $\inf_{u\in\mathcal{U}_c} J_{\rm soc}^{\rm F}(u)$, where $J_{\rm soc}^{\rm F}(u) = \sum_{i=1}^N J_i^{\rm F}(u)$.

(P1′). Seek a set of decentralized controls to optimize social cost $J_{\rm soc}^{\rm F}$ for the system (1)–(2), i.e., $\inf_{u\in\mathcal{U}_d} J_{\rm soc}^{\rm F}(u)$.

Assume

(A1) The coefficients satisfy the following conditions:

(i) $A, G, C, C_0, \Gamma \in L^\infty(0, T; \mathbb{R}^{n\times n})$, and $B, D, D_0 \in L^\infty(0, T; \mathbb{R}^{n\times r})$;

(ii) $Q \in L^\infty(0, T; \mathbb{S}^n)$, and $R \in L^\infty(0, T; \mathbb{S}^r)$;

(iii) $f, \sigma, \sigma_0, \eta \in L^2_{\mathcal{F}^0}(0, T; \mathbb{R}^n)$;

(iv) $\Gamma_0 \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{S}^n$ are bounded; $\eta_0 \in \mathbb{R}^n$ is $\mathcal{F}_T^0$-adapted, and $\mathbb{E}(|\eta_0|^2) < \infty$.

(A2) $x_i(0), i = 1, \dots, N$ are mutually independent and have the same mathematical expectation. $x_i(0) = x_{i0}$, $\mathbb{E}[x_i(0)] = \bar{x}_0$, $i = 1, \dots, N$. There exists a constant c_0 (independent of N) such that $\max_{1\le i\le N} \mathbb{E}|x_i(0)|^2 < c_0$.

We now provide a practical example.

Example 2.1 (*Systemic Risk Model*)**.** Consider a model of inter-bank borrowing and lending where the log-monetary reserves x_i, $i = 1, \dots, N$ evolve by

$$dx_i(t) = -A(t)[x_i(t) - x^{(N)}(t)]dt + u_i(t)dt + \sigma(W_0)(\sqrt{1-\rho^2}dW_i(t) + \rho dW_0(t)),$$

where W_0 is a common Brownian motion, and $W_i, i = 1, \dots, N$ are idiosyncratic Brownian motions, independent of W_0. $A(t)$ is the rate of mean-reversion in interactions from inter-bank borrowing and lending; $\sigma(W_0)$ is stochastic volatility of bank reserves. $\rho \in [-1, 1]$ is the correlation coefficient between idiosyncratic noise and common noise. Each bank controls its rate of lending or borrowing to minimize $J_{\rm soc}^{\rm F}(u) = \sum_{i=1}^N J_i^{\rm F}(u)$ where

$$J_i^{\rm F}(u) = \mathbb{E}\int_0^T \left[Q(t)[x_i(t) - x^{(N)}(t)]^2 + u_i^2(t) \right] dt.$$

Compared with the original model in Carmona et al. (2015), social optimization is considered here with applications in government regulation, community welfare, etc.

From now on, time variable t may be suppressed if no confusion occurs. To simplify the statement, denote by

$$\begin{cases} Q_\Gamma \triangleq \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma, & H_{\Gamma_0} \triangleq \Gamma_0^T H + H\Gamma_0 - \Gamma_0^T H\Gamma_0, \\ \bar{\eta} \triangleq Q\eta - \Gamma^T Q\eta, & \bar{\eta}_0 \triangleq H\eta_0 - \Gamma_0^T H\eta_0. \end{cases}$$

2.1. Design of control laws

Definition 2.1 (*Zălinescu, 1983*)**.** Problem (P1) is convex, if for any $0 < \lambda < 1$ and $u, v \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{nr})$,

$$J_{\rm soc}^{\rm F}(\lambda u + (1-\lambda)v) \le \lambda J_{\rm soc}^{\rm F}(u) + (1-\lambda)J_{\rm soc}^{\rm F}(v).$$

Particularly, (P1) is uniformly convex, if we have

$$J_{\rm soc}^{\rm F}(\lambda u + (1-\lambda)v) \le \lambda J_{\rm soc}^{\rm F}(u) + (1-\lambda)J_{\rm soc}^{\rm F}(v) - \lambda(1-\lambda)\mathbb{E}\int_0^T |u - v|^2 dt.$$

We first obtain the necessary and sufficient conditions for the existence of centralized optimal control of (P1).

Theorem 2.1. *Assume (A1)–(A2) hold. Then (P1) has an optimal control $\check{u} \in \mathcal{U}_c$ if and only if (P1) is convex in u and the following equation system admits a set of solutions $(\check{x}_i, p_i, \{\beta_i^j\}_{j=0}^N$, $i = 1, \dots, N)$:*

$$\begin{cases} d\check{x}_i = \left(A\check{x}_i + B\check{u}_i + G\check{x}^{(N)} + f \right) dt + (C\check{x}_i + D\check{u}_i + \sigma)dW_i \\ \qquad + (C_0\check{x}_i + D_0\check{u}_i + \sigma_0)dW_0, \quad \check{x}_i(0) = x_{i0}, \\ dp_i = -\left(A^T p_i + G^T p^{(N)} + C_0^T\beta_i^0 + C^T\beta_i^i \right) dt \\ \qquad - \left(Q\check{x}_i - Q_\Gamma\check{x}^{(N)} - \bar{\eta} \right) dt + \sum_{j=0}^N \beta_i^j dW_j, \\ p_i(T) = H\check{x}_i(T) - H_{\Gamma_0}\check{x}^{(N)}(T) - \bar{\eta}_0, \quad i = 1, \dots, N, \end{cases} \quad (3)$$

where $p^{(N)} = \frac{1}{N}\sum_{i=1}^N p_i$ and the optimal control $\check{u}_i$ satisfies the stationary condition

$$R\check{u}_i + B^T p_i + D^T\beta_i^i + D_0^T\beta_i^0 = 0. \quad (4)$$

Particularly, if Problem (P1) is uniformly convex, then (P1) admits an optimal control necessarily.

Proof. See Appendix A. □

Eqs. (3)–(4) are a fully coupled FBSDE. It is nonstandard due to MF terms appearing in both forward and backward equations, thus its solvability becomes quite technical. Related works on solvability and control problems of FBSDEs may be referred to Yong and Zhou (1999), Peng and Wu (1999) and Wang, Wu, and Xiong (2013).

Now we make the following assumption:

(A3) Problem (P1) is uniformly convex.[2]

By (A3), (P1) admits a unique optimal control. The next step is to obtain a proper form for deriving the decentralized feedback representation of optimal control.

2 By Sun, Li, and Yong (2016, Theorem 4.5), a necessary and sufficient condition to guarantee (A3) is that the corresponding Riccati equation admits a strongly regular solution.

Denote $\check{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{x}_i$, $\check{x}_{i0}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{x}_{i0}$ and $\check{u}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{u}_i$. It follows from (3) that

$$\begin{aligned} d\check{x}^{(N)} =& \left[(A+G)\check{x}^{(N)} + B\check{u}^{(N)} + f \right] dt \\ &+\frac{1}{N}\sum_{i=1}^N (C\check{x}_i + D\check{u}_i + \sigma)dW_i \\ &+(C_0\check{x}^{(N)} + D_0\check{u}^{(N)} + \sigma_0)dW_0,\ \check{x}^{(N)}(0) = \check{x}_{i0}^{(N)}. \end{aligned} \tag{5}$$

We now use the idea inspired by Yong and Zhou (1999) and Zhang et al. (2019) to decouple the FBSDE (3). Let $p_i = P_N\check{x}_i + K_N\check{x}^{(N)} + \varphi_N$, where $P_N, K_N \in \mathbb{R}^{n\times n}$, and φ_N satisfies

$$d\varphi_N = \check{\varphi}_N dt + \gamma_N dW_0,\ \varphi_N(T) = -\bar{\eta}_0.$$

Here $\check{\varphi}_N$ and γ_N are to be determined. Then by (3), (5) and Itô's formula,

$$\begin{aligned} dp_i =& P_N \Big[\big(A\check{x}_i + B\check{u}_i + G\check{x}^{(N)} + f \big) dt + (C\check{x}_i + D\check{u}_i + \sigma)dW_i \\ &+ (C_0\check{x}_i + D_0\check{u}_i + \sigma_0)dW_0 \Big] + \dot{P}_N\check{x}_i dt + \dot{K}_N\check{x}^{(N)}dt \\ &+ K_N \Big\{ \Big[(A+G)\check{x}^{(N)} + B\check{u}^{(N)} + f \Big] dt \\ &+ \frac{1}{N}\sum_{j=1}^N (C\check{x}_j + D\check{u}_j + \sigma)dW_j \\ &+ (C_0\check{x}^{(N)} + D_0\check{u}^{(N)} + \sigma_0)dW_0 \Big\} + \check{\varphi}_N dt + \gamma_N dW_0 \end{aligned} \tag{6}$$

Comparing this with (3), it follows that

$$\begin{aligned} \beta_i^0 =& P_N(C_0\check{x}_i + D_0\check{u}_i + \sigma_0) \\ &+ K_N(C_0\check{x}^{(N)} + D_0\check{u}^{(N)} + \sigma_0) + \gamma_N, \\ \beta_i^i =& (P_N + \frac{1}{N}K_N)(C\check{x}_i + D\check{u}_i + \sigma), \\ \beta_i^j =& \frac{1}{N}K_N(C\check{x}_j + D\check{u}_j + \sigma),\ 1 \le j \ne i \le N. \end{aligned}$$

From (4), we have for any $i = 1, \ldots, N$,

$$\Upsilon_N\check{u}_i + \Psi_N\check{x}_i + \Theta_N\check{x}^{(N)} + D_0^TK_ND_0\check{u}^{(N)} + \psi_N = 0,$$

where

$$\begin{aligned} \Upsilon_N =& R + D^T \Big(P_N + \frac{K_N}{N} \Big) D + D_0^TP_ND_0, \\ \Psi_N =& B^TP_N + D^T \Big(P_N + \frac{1}{N}K_N \Big) C + D_0^TP_NC_0, \\ \Theta_N =& B^TK_N + D_0^TK_NC_0, \\ \psi_N =& D^T(P_N + \frac{K_N}{N})\sigma + D_0^T(P_N + K_N)\sigma_0 \\ &+ B^T\varphi_N + D_0^T\gamma_N. \end{aligned} \tag{7}$$

This further implies

$$\check{u}^{(N)} = -(\Upsilon_N + D_0^TK_ND_0)^\dagger[(\Psi_N + \Theta_N)\check{x}^{(N)} + \psi_N].$$

Thus, we obtain that the optimal control is given by

$$\begin{aligned} \check{u}_i =& -\Upsilon_N^\dagger\Psi_N(\check{x}_i - \check{x}^{(N)}) - (\Upsilon_N + D_0^TK_ND_0)^\dagger \\ &\times [(\Psi_N + \Theta_N)\check{x}^{(N)} + \psi_N]. \end{aligned} \tag{8}$$

This together with (6) gives

$$\begin{aligned} &\dot{P}_N + A^TP_N + P_NA + C^T \Big(P_N + \frac{K_N}{N} \Big) C + Q \\ &+ C_0^TP_NC_0 - \Psi_N^T\Upsilon_N^\dagger\Psi_N = 0,\ P_N(T) = H, \end{aligned} \tag{9}$$

$$\begin{aligned} &\dot{K}_N + (A+G)^TK_N + K_N(A+G) + C_0^TK_NC_0 + G^TP_N \\ &+ P_NG - (\Psi_N + \Theta_N)^T(\Upsilon_N + D_0^TK_ND_0)^\dagger(\Psi_N + \Theta_N) \\ &+ \Psi_N^T\Upsilon_N^\dagger\Psi_N - Q_\Gamma = 0,\ K_N(T) = -H_{\Gamma_0}, \end{aligned} \tag{10}$$

$$\begin{aligned} d\varphi_N =& -\Big\{ [A + G - B(\Upsilon_N + D_0^TK_ND)^\dagger(\Psi_N + \Theta_N)]^T\varphi_N \\ &+ [C - D(\Upsilon_N + D_0^TK_ND_0)^\dagger(\Psi_N + \Theta_N)]^T \\ &\times (P_N + \frac{1}{N}K_N)\sigma + [C_0 - D_0(\Upsilon_N + D_0^TK_ND_0)^\dagger \\ &\times (\Psi_N + \Theta_N)]^T[(P_N + K_N)\sigma_0 + \gamma_N] \\ &+ (P_N + K_N)f - \bar{\eta} \Big\} dt + \gamma_N dW_0,\ \varphi_N(T) = -\bar{\eta}_0. \end{aligned} \tag{11}$$

Note that (11) is a linear backward SDE. If (9)–(10) admit solutions, then (11) has a solution. From the above discussion and Theorem 2.1, we have the following result.

Proposition 2.1. *Assume that (A1)–(A3) hold, and (9)–(10) respectively admits a solution such that*

$$\mathcal{R}(\Psi_N) \subseteq \mathcal{R}(\Upsilon_N), \quad \psi \in \mathcal{R}(\Upsilon_N + D_0^TK_ND_0),$$
$$\mathcal{R}(\Psi_N + \Theta_N) \subseteq \mathcal{R}(\Upsilon_N + D_0^TK_ND_0).$$

Then Problem (P1) has an optimal control given by (8).

Denote

$$\begin{cases} \Upsilon = R + D^TPD + D_0^TPD_0,\ \bar{\Upsilon} = \Upsilon + D_0^TKD_0, \\ \Psi = B^TP + D^TPC + D_0^TPC_0, \\ \Theta = B^TK + D_0^TKC_0, \\ \psi = D^TP\sigma + D_0^T(P+K)\sigma_0 + B^T\varphi + D_0^T\gamma, \end{cases} \tag{12}$$

where P, K and (φ, γ) satisfy

$$\begin{aligned} &\dot{P} + A^TP + PA + C^TPC + C_0^TPC_0 + \\ &+ Q - \Psi^T\Upsilon^\dagger\Psi = 0,\ P(T) = H, \end{aligned} \tag{13}$$

$$\begin{aligned} &\dot{K} - \Big[(A+G)^TK + K(A+G) + C_0^TKC_0 \\ &- (\Psi + \Theta)^T\bar{\Upsilon}^\dagger(\Psi + \Theta) + \Psi^T\Upsilon^\dagger\Psi \\ &+ G^TP + PG - Q_\Gamma = 0,\ K(T) = -H_{\Gamma_0}, \end{aligned} \tag{14}$$

$$\begin{aligned} d\varphi =& -\Big\{ [A + G - B(\Upsilon + D_0^TKD_0)^\dagger(\Psi + \Theta)]^T\varphi \\ &+ [C - D\bar{\Upsilon}^\dagger(\Psi + \Theta)]^TP\sigma + (P+K)f \\ &+ [C_0 - D_0\bar{\Upsilon}^\dagger(\Psi + \Theta)]^T[(P+K)\sigma_0 + \gamma] - \bar{\eta} \Big\} dt \\ &+ \gamma dW_0,\ \varphi(T) = -\bar{\eta}_0. \end{aligned} \tag{15}$$

For further analysis, we assume

(A4) (13)–(15) have solutions such that $\Upsilon, \bar{\Upsilon} \ge 0$, and

$$\mathcal{R}(\Psi(t)) \subseteq \mathcal{R}(\Upsilon(t)), \quad \mathcal{R}(\Psi(t) + \Theta(t)) \subseteq \mathcal{R}(\bar{\Upsilon}(t)),$$
$$\psi(t) \in \mathcal{R}(\bar{\Upsilon}(t)), \quad t \in [0, T].$$

Remark 2.1. If (A4) holds, then by the continuous dependence of the solution on the parameter (see e.g. Khalil (2002, Theorem 3.5) or Huang and Zhou (2020, Theorem 4)), we obtain for a sufficiently large N, (9)–(11) admit solutions, respectively.

Let $N \to \infty$. By the law of large numbers, we may approximate $x^{(N)}$ in (5) with $\bar{x}$, which satisfies

$$\begin{aligned} d\bar{x} =& \Big\{ [A + G - B\bar{\Upsilon}^\dagger(\Psi + \Theta)]\bar{x} - B\bar{\Upsilon}^\dagger\psi + f \Big\} dt \\ &+ \Big\{ [C_0 - D_0\bar{\Upsilon}^\dagger(\Psi + \Theta)]\bar{x} - D_0\bar{\Upsilon}^\dagger\psi + \sigma_0 \Big\} dW_0, \end{aligned} \tag{16}$$

with $\bar{x}(0) = \bar{x}_0$. By Proposition 2.1, the decentralized control law for agent $\mathcal{A}_i$, $i = 1, \ldots, N$ may be taken as

$$\begin{aligned} \hat{u}_i(t) =& -\Upsilon^\dagger(t)\Psi(t)[\hat{x}_i(t) - \bar{x}(t)] \\ &- \bar{\Upsilon}^\dagger(t) \Big[(\Psi(t) + \Theta(t))\bar{x}(t) + \psi(t) \Big], \end{aligned} \tag{17}$$

where Υ, $\bar{\Upsilon}$, Ψ, Θ, ψ and $\bar{x}$ are determined by (12)–(16), and $\hat{x}_i$ satisfies

$$\begin{aligned} d\hat{x}_i = &\big\{\bar{A}\hat{x}_i + G\hat{x}^{(N)} + B[\Upsilon^\dagger\Psi - \bar{\Upsilon}^\dagger(\Psi+\Theta)]\bar{x} - B\bar{\Upsilon}^\dagger\psi \\ &+ f\big\}dt + \big\{\bar{C}\hat{x}_i + D[\Upsilon^\dagger\Psi - \bar{\Upsilon}^\dagger(\Psi+\Theta)]\bar{x} \\ &- D\bar{\Upsilon}^\dagger\psi + \sigma\big\}dW_i + \big\{\bar{C}_0\hat{x}_i + D_0[\Upsilon^\dagger\Psi \\ &- \bar{\Upsilon}^\dagger(\Psi+\Theta)]\bar{x} - D_0\bar{\Upsilon}^\dagger\psi + \sigma_0\big\}dW_0, \end{aligned} \tag{18}$$

with

$$\bar{A} \triangleq A - B\Upsilon^\dagger\Psi,\ \bar{C} \triangleq C - D\Upsilon^\dagger\Psi,\ \bar{C}_0 \triangleq C_0 - D_0\Upsilon^\dagger\Psi. \tag{19}$$

Remark 2.2. Previous works (e.g., Bensoussan et al., 2016; Carmona & Delarue, 2013; Graber, 2016; Huang et al., 2012) considered LQ-MF models by the fixed-point approach. To achieve asymptotic optimality, an additional condition like the solvability of fixed point equations is needed, which is not easy to be verified. Furthermore, for MF social control with common noise, the tackling process is more complicated. In Huang et al. (2020) and Qiu, Huang, and Xie (2021), authors first constructed a new auxiliary system by two-step duality and then derived a consistency condition system, which is a MF FBSDE with embedding representation. Here, we get rid of fixed-point conditions and MF FBSDEs thoroughly (Note that φ and $\bar{x}$ are fully decoupled).

We are in a position to give asymptotic social optimality of the decentralized control.

Theorem 2.2. *Assume that (A1)–(A4) hold. For Problem (P1′), the set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (17) has asymptotic social optimality, i.e.,*

$$\Big|\frac{1}{N}J^{\mathrm{F}}_{\mathrm{soc}}(\hat{u}) - \frac{1}{N}\inf_{u\in\mathcal{U}_c} J^{\mathrm{F}}_{\mathrm{soc}}(u)\Big| = O(\frac{1}{\sqrt{N}}).$$

Proof. See Appendix B. □

2.2. Asymptotically social optimal cost

We now give an explicit expression of the asymptotic average social cost in terms of two Riccati equations.

Theorem 2.3. *Assume that (A1)–(A4) hold and $\{x_{i0}\}$ have the same variance. Then for (P1′), the asymptotic average social optimum is given by*

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}J^{\mathrm{F}}_{\mathrm{soc}}(\hat{u}) = &\mathbb{E}\Big[(x_{i0}-\bar{x}_0)^T P(0)(x_{i0}-\bar{x}_0) \\ &+ \bar{x}_0^T\Pi(0)\bar{x}_0 + 2\varphi^T(0)\bar{x}_0\Big] + q_T, \end{aligned}$$

where P, Π are given by (13) and (14) ($\Pi = P + K$), and

$$\begin{aligned} q_T \triangleq \mathbb{E}\int_0^T &\Big[|\sigma|_P^2 + |\sigma_0|_\Pi^2 + 2\varphi^T f + 2\gamma^T\sigma_0 + |\eta|_Q^2 \\ &- |B^T\varphi + D^T P\sigma + D_0^T\Pi\sigma_0 + D_0^T\gamma|^2_{\bar{\Upsilon}^{-1}}\Big]dt + \mathbb{E}[|\eta_0|_H^2]. \end{aligned}$$

Proof. See Appendix C. □

Remark 2.3. The MF-type control problem (see e.g., Carmona & Delarue, 2018; Graber, 2016) is a closely related problem to the MF social control. The setups of both problems are different. The MF social control is multi-agent optimization, and there are a large number of agents with their own states and controls. In contrast, the mean field type control is single-agent optimization. However, the optimal solutions of both problems coincide for the infinite population case (see details in Appendix C).

2.3. Mean field social optimal control with finite agents

We now consider the case $G = 0$. Denote $\mathcal{H}_t^i = \sigma(x_i(0), W_i(s), W_0(s), s \le t)$ and

$$\begin{aligned} \mathcal{U}_d' = \Big\{&(u_1,\ldots,u_N)\big|u_i(t) \text{ is adapted to } \mathcal{H}_t^i, \\ &\mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty\Big\}. \end{aligned}$$

In analogy to Theorem 2.1, we obtain the result for the existence of decentralized optimal control of (P1′).

Proposition 2.2. *Assume that (A1)–(A2) hold and (P1′) is convex in u. Then $\{u_i^*, i = 1,\ldots,N\}$ is a set of optimal control laws of (P1′) with respect to $\mathcal{U}_d'$ if and only if the following equation system admits a set of solutions $(x_i^*, p_i^*, \beta_i^{j,*}, i = 1,\ldots,N, j = 0, 1,\ldots,N)$:*

$$\begin{cases} dx_i^* = \big(Ax_i^* + Bu_i^* + f\big)dt + (Cx_i^* + Du_i^* + \sigma)dW_i \\ \qquad + (C_0x_i^* + D_0u_i^* + \sigma_0)dW_0,\ x_i^*(0) = x_{i0}, \\ dp_i^* = -\big[A^Tp_i^* + C_0^T\beta_i^{0,*} + C^T\beta_i^{i,*}\big]dt \\ \qquad - \big(Qx_i^* - Q_\Gamma x_*^{(N)} - \bar{\eta}\big)dt + \sum_{j=0}^N \beta_i^{j,*}dW_j, \\ p_i(T) = Hx_i^*(T) - H_{\Gamma_0}x_*^{(N)}(T) - \bar{\eta}_0,\ i = 1,\ldots,N, \end{cases} \tag{20}$$

where $x_*^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i^*$ and the optimal control u_i^* satisfies the stationarity condition

$$Ru_i^* + \mathbb{E}[B^Tp_i^* + D^T\beta_i^{i,*} + D_0^T\beta_i^{0,*}|\mathcal{H}_t^i] = 0. \tag{21}$$

Proof. See Appendix D. □

Remark 2.4. The main difference between Theorems 2.1 and 2.2 lies in different stationarity conditions. Intuitively, (21) is obtained by applying conditional expectation into the zero variational condition.

Note that W_i, W_j ($j \neq i$) and W_0 are mutually independent, and $x_j^*(t) \in \mathcal{H}_t^j$. We have

$$\mathbb{E}[x_j^*(t)|\mathcal{H}_t^i] = \mathbb{E}[x_j^*(t)|\mathcal{F}_t^0] = \mathbb{E}_{\mathcal{F}^0}[x_i^*(t)],\ j \neq i. \tag{22}$$

It follows from (20) that

$$\begin{cases} d\mathbb{E}_{\mathcal{F}^0}[x_i^*] = \big(A\mathbb{E}_{\mathcal{F}^0}[x_i^*] + B\mathbb{E}_{\mathcal{F}^0}[u_i^*] + f\big)dt \\ \qquad + (C_0\mathbb{E}_{\mathcal{F}^0}[x_i^*] + D_0\mathbb{E}_{\mathcal{F}^0}[u_i^*] + \sigma_0)dW_0, \\ d\mathbb{E}_{\mathcal{H}^i}[p_i^*] = -\big(A^T\mathbb{E}_{\mathcal{H}^i}[p_i^*] + C_0^T\mathbb{E}_{\mathcal{H}^i}[\beta_i^{0,*}]\big)dt \\ \qquad - \big[C^T\mathbb{E}_{\mathcal{H}^i}[\beta_i^{i,*}] + (Q - \frac{1}{N}Q_\Gamma)x_i^* - \bar{\eta} \\ \qquad - \frac{N-1}{N}Q_\Gamma\mathbb{E}_{\mathcal{F}^0}[x_i^*]\big]dt + \beta_i^{0,*}dW_0 + \beta_i^{i,*}dW_i, \\ x_i^*(0) = x_{i0},\ \mathbb{E}_{\mathcal{H}^i}[p_i(T)] = \big(H - \frac{1}{N}H_{\Gamma_0}\big)x_i^*(T) \\ \qquad - \frac{N-1}{N}H_{\Gamma_0}\mathbb{E}_{\mathcal{F}^0}[x_i^*(T)] - \bar{\eta}_0,\ i = 1,\ldots,N. \end{cases} \tag{23}$$

Define

$$\begin{aligned} u_i^* = &-\tilde{\Upsilon}_N^\dagger\tilde{\Psi}_N(x_i^* - \mathbb{E}_{\mathcal{F}^0}[x_i^*]) - (\tilde{\Upsilon}_N + D_0^TK_N^*D_0)^\dagger \\ &\times [(\tilde{\Psi}_N + \tilde{\Theta}_N)\mathbb{E}_{\mathcal{F}^0}[x_i^*] + \psi_N^*], \end{aligned} \tag{24}$$

where $\tilde{\Upsilon}_N$, $\tilde{\Psi}_N$, $\tilde{\Theta}_N$ are given by

$$\begin{aligned} \tilde{\Upsilon}_N &= R + D^TP_N^*D + D_0^TP_N^*D_0, \\ \tilde{\Psi}_N &= B^TP_N^* + D^TP_N^*C + D_0^TP_N^*C_0, \\ \tilde{\Theta}_N &= B^TK_N^* + D_0^TK_N^*C_0, \\ \psi_N^* &= D^TP_N^*\sigma + D_0^T(P_N^* + K_N^*)\sigma_0 + B^T\varphi_N^* + D_0^T\gamma_N^*, \end{aligned}$$

and P_N^*, K_N^* and φ_N^* satisfy

$$\dot{P}_N^* + A^T P_N^* + P_N^* A + C^T P_N^* C + C_0^T P_N^* C_0 + Q - \frac{1}{N} Q_\Gamma - \tilde{\Psi}_N^T \Upsilon_N^{-1} \tilde{\Psi}_N = 0, \; P_N^*(T) = H - \frac{1}{N} H_{\Gamma_0}, \tag{25}$$

$$\begin{aligned}&\dot{K}_N^* + A^T K_N^* + K_N^* A + C_0^T K_N^* C_0 + \tilde{\Psi}_N^T \tilde{\Upsilon}_N^{-1} \tilde{\Psi}_N \\ &- (\tilde{\Psi}_N + \tilde{\Theta}_N)^T (\tilde{\Upsilon}_N^{-1} + D_0^T K_N^* D_0)(\tilde{\Psi}_N + \tilde{\Theta}_N) \\ &- \frac{N-1}{N} Q_\Gamma = 0, \; K_N^*(T) = -\frac{N-1}{N} H_{\Gamma_0},\end{aligned} \tag{26}$$

$$\begin{aligned} d\varphi_N^* = &- \Big\{ [A - B(\tilde{\Upsilon}_N + D_0^T K_N^* D)^{-1}(\tilde{\Psi}_N + \tilde{\Theta}_N)]^T \varphi_N^* \\ &+ [C - D(\tilde{\Upsilon}_N + D_0^T K_N^* D)^{-1}(\tilde{\Psi}_N + \tilde{\Theta}_N)]^T P_N^* \sigma \\ &+ [C_0 - D_0(\tilde{\Upsilon}_N + D_0^T K_N^* D)^{-1}(\tilde{\Psi}_N + \tilde{\Theta}_N)]^T \\ &\times [(P_N^* + K_N^*)\sigma_0 + \gamma_N^*] + (P_N^* + K_N^*) f - \bar{\eta} \Big\} dt \\ &+ \gamma_N^* dW_0, \; \varphi_N^*(T) = -\bar{\eta}_0,\end{aligned} \tag{27}$$

Theorem 2.4. *Assume that (A1)–(A2) hold and (P1′) is convex in u. Eqs. (9)–(10) admit a set of solutions. Then $\{u_1^*, \ldots, u_N^*\}$ given by (24) is social optimal with respect to the decentralized control set $\mathcal{U}_d'$, i.e., for any $u \in \mathcal{U}_d'$, the following holds: $J_{soc}^F(u^*) \le J_{soc}^F(u)$.*

Proof. See Appendix D. □

Remark 2.5. In general, there are two methods to tackle Problem (P1′). The first one is to find asymptotic optimal solutions; another one is to seek the optimal solution for (fixed) N agents. Most works on MF control focused on the former one. Indeed, the latter one can be obtained in some cases. Theorem 2.4 provides a social optimal solution for (P1′) with N agents.

3. Infinite horizon mean field LQ control

In this section, we consider the infinite-horizon MF social control problem. For simplicity, suppose $A(t)$, $B(t)$, $C(t)$, $D(t)$, $C_0(t)$, $D_0(t)$, $\Gamma(t)$, $Q(t)$ and $R(t)$ are constant matrices. Assume $Q \ge 0$ and $R > 0$; $f, \sigma, \sigma_0, \eta \in L^2_{\mathcal{F}^0}(0, \infty; \mathbb{R}^n)$. The cost of $\mathcal{A}_i$ is given by

$$J_i(u) = \mathbb{E} \int_0^\infty \Big\{ \big| x_i(t) - \Gamma x^{(N)}(t) - \eta(t) \big|_Q^2 + |u_i(t)|_R^2 \Big\} dt. \tag{28}$$

In what follows, we study the following problem.

(P2). Seek a set of decentralized control laws to optimize the social cost J_{soc} for the system (1), (28) where $J_{soc}(u) = \sum_{i=1}^N J_i(u)$.

We first introduce some definitions. Consider the system

$$dy = (Ay + Bu)dt + (Cy + Du)dW(t) + (C_0 y + D_0 u)dW_0(t), \; y(0) = y, \tag{29}$$

$$z = Fy, \tag{30}$$

where $y(t) \in \mathbb{R}^n$, and $W(t)$, $W_0(t)$ are 1-dimensional Brownian motions.

Definition 3.1. The system (29) with $u = 0$ (or simply $[A, C, C_0]$) is said to be mean-square stable, if for any initial y, there exists $c > 0$ such that $\mathbb{E}\left[\int_0^\infty |y(t)|^2 dt \right] \le c$.

Definition 3.2. The system (29) (or $[A, C, C_0; B, D, D_0]$) is said to be (mean-square) stabilizable, if there exists a control law $u^*(t) = Ky(t)$ such that for any initial y, the closed-loop system is mean-square stable,

$$dy(t) = (A + BK)y(t)dt + (C + DK)y(t)dW(t) + (C_0 + D_0 K)y(t)dW_0(t).$$

Definition 3.3 (Zhang, Zhang, & Chen, 2008)**.** The system (29)–(30) (or simply $[A, C, C_0; F]$) is said to be exactly detectable, if there exists a $T_0 \ge 0$ such that for any $T > T_0$, $z(t) = 0$, $u(t) = 0$, *a.s.*, $0 \le t \le T$ implies $\lim_{t\to\infty} \mathbb{E}[y^T(t)y(t)] = 0$.

We now introduce the following basic assumptions:

(A5) The system $[A, C, C_0; B, D, D_0]$ is stabilizable, and $[A + G, C_0; B, D_0]$ is stabilizable.

(A6) $Q \ge 0$, $R > 0$, $[A, C, C_0; \sqrt{Q}]$ is exactly detectable, and $[A + G, C_0; \sqrt{Q}(I - \Gamma)]$ is exactly detectable.

Similar to Theorem 2.1, we have the following result.

Theorem 3.1. *Assume (A2), (A5)–(A6) hold. Then for a sufficiently large N, (P2) has an optimal control $\check{u} \in \mathcal{U}_c$ such that $\check{x}_i, i \ge 1$ in $L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^{Nn})$ if and only if the following equation system admits a set of solutions $(\check{x}_i, p_i, \{\beta_i^j\}_{j=0}^N, i \ge 1)$ in $L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^{N(N+3)n})$:*

$$\begin{cases} d\check{x}_i = \big(A\check{x}_i + B\check{u}_i + G\check{x}^{(N)} + f \big) dt + (C\check{x}_i + D\check{u}_i + \sigma)dW_i \\ \qquad + (C_0\check{x}_i + D_0\check{u}_i + \sigma_0)dW_0, \quad \check{x}_i(0) = x_{i0}, \\ dp_i = -\big(A^T p_i + G^T p^{(N)} + C_0^T \beta_i^0 + C^T \beta_i^i \big) dt \\ \qquad - \big(Q\check{x}_i - Q_\Gamma \check{x}^{(N)} - \bar{\eta} \big) dt + \sum_{j=0}^N \beta_i^j dW_j, \end{cases}$$

where the optimal control $\check{u}_i$ $(i = 1, \ldots, N)$ satisfies

$$R\check{u}_i + B^T p_i + D^T \beta_i^i + D_0^T \beta_i^0 = 0.$$

Proof. *(Sufficiency)*. The proof is similar to that of Theorem 2.1. *(Necessity)*. Since $\check{x}_i \in L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^n)$, we have $\mathbb{E}\int_0^\infty |\check{x}^{(N)}|^2 dt \le \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^\infty |\check{x}_i|^2 dt < \infty$. Based on a similar derivation as in Section 2.1, we only need to prove $p_i \in L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^n)$. Let $p_i = P_N\check{x}_i + K_N\check{x}^{(N)} + \varphi_N$. Similarly, we have that P_N, K_N, φ_N satisfy

$$A^T P_N + P_N A + C^T \Big(P_N + \frac{K_N}{N} \Big) C + Q + C_0^T P_N C_0 - \Psi_N^T \Upsilon_N^{-1} \Psi_N = 0, \tag{31}$$

$$\begin{aligned}&(A+G)^T K_N + K_N(A+G) + C_0^T K_N C_0 + G^T P_N \\ &+ P_N G - (\Psi_N + \Theta_N)^T (\Upsilon_N + D_0^T K_N D_0)^{-1} (\Psi_N + \Theta_N) \\ &+ \Psi_N^T \Upsilon_N^{-1} \Psi_N - Q_\Gamma = 0,\end{aligned} \tag{32}$$

$$\begin{aligned} d\varphi_N = &-\Big\{ [A + G - B(\Upsilon_N + D_0^T K_N D)^{-1}(\Psi_N + \Theta_N)]^T \varphi_N \\ &+ [C - D(\Upsilon_N + D_0^T K_N D_0)^{-1}(\Psi_N + \Theta_N)]^T \\ &\times (P_N + \frac{1}{N} K_N)\sigma + [C_0 - D_0(\Upsilon_N + D_0^T K_N D_0)^{-1} \\ &\times (\Psi_N + \Theta_N)]^T [(P_N + K_N)\sigma_0 + \gamma_N] \\ &+ (P_N + K_N) f - \bar{\eta} \Big\} dt + \gamma_N dW_0.\end{aligned} \tag{33}$$

Let $\Pi_N = P_N + K_N$. Then Π_N satisfies

$$\begin{aligned}&(A+G)^T \Pi_N + \Pi_N (A+G) + Q - Q_\Gamma \\ &+ C_0^T \big(\Pi_N + \frac{K_N}{N}\big) C_0 + C^T P_N C \\ &- (\Psi_N + \Theta_N)^T \bar{\Upsilon}_N^{-1} (\Psi_N + \Theta_N) = 0.\end{aligned} \tag{34}$$

From (A5)–(A6) and the continuous dependence of the solution on the parameter (see e.g. Freiling, Jank, Lee, and Abou-Kandil (1996)), for a sufficiently large N, (31) and (34) admit solutions such that $[A + G - B\bar{\Upsilon}_N^{-1}(\Psi_N + \Theta_N), C_0 - D_0\bar{\Upsilon}_N^{-1}(\Psi_N + \Theta_N)]$ is stable. By Sun and Yong (2018, Lemma 2.5), (33) admits a unique solution $\varphi_N \in L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^n)$. Thus, we obtain $p_i \in L^2_{\mathcal{F}}(0, \infty; \mathbb{R}^n)$. □

Based on the above discussion, we may construct the following decentralized control laws for Problem (P2):

$$\hat{u}_i = -\Upsilon^{-1}\Psi(x_i - \bar{x}) - \bar{\Upsilon}^{-1}[(\Psi + \Theta)\bar{x} + \psi], \quad t \geq 0, \quad i = 1, \ldots, N, \tag{35}$$

where

$$\Upsilon = R + D^T PD + D_0^T PD_0,$$
$$\bar{\Upsilon} = R + D^T PD + D_0^T \Pi D_0,$$
$$\Psi = B^T P + D^T PC + D_0^T PC_0,$$
$$\Theta = B^T(\Pi - P) + D_0^T(\Pi - P)C_0,$$
$$\psi = D^T P\sigma + D_0^T \Pi \sigma_0 + B^T \varphi + D_0^T \gamma.$$

In the above, P and Π satisfy

$$A^T P + PA + C^T PC + C_0^T PC_0 + Q - \Psi^T \Upsilon^{-1}\Psi = 0, \tag{36}$$

$$(A+G)^T \Pi + \Pi(A+G) - (\Psi+\Theta)^T \bar{\Upsilon}^{-1}(\Psi+\Theta) + C_0^T \Pi C_0 + C^T PC + Q - Q_\Gamma = 0, \tag{37}$$

and $\varphi, \gamma, \bar{x} \in L^2_{\mathcal{F}}([0,\infty), \mathbb{R}^n)$ are determined by

$$d\varphi = -\Big\{ [A + G - B\bar{\Upsilon}^{-1}(\Psi+\Theta)]^T \varphi + \Pi f + [C_0 - D_0\bar{\Upsilon}^{-1}(\Psi+\Theta)]^T(\Pi\sigma_0 + \gamma) + [C - D\bar{\Upsilon}^{-1}(\Psi+\Theta)]^T P\sigma - \bar{\eta} \Big\} dt + \gamma dW_0, \tag{38}$$

$$d\bar{x} = \Big\{[A + G - B\bar{\Upsilon}^{-1}(\Psi+\Theta)]\bar{x} - B\bar{\Upsilon}^{-1}\psi + f\Big\}dt + \Big\{[C_0 - D_0\bar{\Upsilon}^{-1}(\Psi+\Theta)]\bar{x} - D_0\bar{\Upsilon}^{-1}\psi + \sigma_0\Big\}dW_0,$$
$$\bar{x}(0) = \bar{x}_0. \tag{39}$$

Here, (38) is a backward SDE and (φ, γ) is its solution.

Lemma 3.1. *Assume (A2), (A5)–(A6) hold. Then we have*

(i) Eq. (36) admits a unique solution $P \geq 0$ such that $[\bar{A}, \bar{C}, \bar{C}_0]$ is mean-square stable.

(ii) Eq. (37) admits a unique solution $\Pi \geq 0$ such that $[A + G - B\bar{\Upsilon}^{-1}(\Psi+\Theta), C_0 - D_0\bar{\Upsilon}^{-1}(\Psi+\Theta)]$ is mean square stable;

(iii) (38)–(39) have a solution $\varphi, \bar{x} \in L^2_{\mathcal{F}^0}(0,\infty;\mathbb{R}^n)$.

Proof. From (A5)–(A6), we obtain that (36) admits a unique solution $P \geq 0$ such that $[\bar{A}, \bar{C}, \bar{C}_0]$ is mean-square stable (i.e., P is a stabilizing solution; see e.g., Rami, Chen, Moore, & Zhou, 2001; Zhang et al., 2008). Note that $[A + G, C_0; (C^T PC + Q - Q_\Gamma)^{1/2}]$ is exactly detectable. It follows that (37) admits a unique stabilizing solution $\Pi \geq 0$. By Sun and Yong (2018, Lemma 2.5), (38) admits a unique solution $\varphi \in L^2_{\mathcal{F}^0}(0,\infty;\mathbb{R}^n)$. It is straightforward that $\bar{x} \in L^2_{\mathcal{F}^0}(0,\infty;\mathbb{R}^n)$. □

For further analysis, we introduce the following assumption. We will show that this assumption is also necessary for the uniform stability of closed-loop systems.

(A7) $[\bar{A} + G, \bar{C}_0]$ is mean-square stable, where $\bar{A}$ and $\bar{C}_0$ are given by (19).

We now show that the closed-loop system under decentralized control (17) is uniformly mean-square stable.

Theorem 3.2. *Assume that (A2), (A5)–(A7) hold. Then there exist an integer N_0 and a constant c such that $N > N_0$ the following hold:*

$$\max_{1\leq i\leq N} \mathbb{E}\int_0^\infty \left(|\hat{x}_i(t)|^2 + |\hat{u}_i(t)|^2\right) dt < c, \tag{40}$$

$$\mathbb{E}\int_0^\infty |\hat{x}^{(N)}(t) - \bar{x}(t)|^2 dt = O(\frac{1}{N}). \tag{41}$$

Proof. See Appendix E. □

Remark 3.1. Due to the appearance of common noise, the mean-square approximation error between population state average $\hat{x}^{(N)}$ and aggregate effect $\bar{x}$ relies on the states of all agents while the second moment of the state $\hat{x}_i$ conversely depends on the approximation error $\hat{x}^{(N)} - \bar{x}$. Thus, we need to analyze jointly $\hat{x}^{(N)} - \bar{x}$ and $\hat{x}_i, i = 1, \ldots, N$. By tackling the relevant integral inequalities, we obtain the uniform stabilization of the systems and the consistency of MF approximation.

We now give two equivalent conditions for uniform stabilization of all the subsystems.

Theorem 3.3. *Let (A6) hold. Assume that (36)–(37) have symmetric solutions. Then for (P2) the following statements are equivalent:*

(i) For any initial condition $(\hat{x}_1(0), \ldots, \hat{x}_N(0))$ satisfying (A2), the following holds,

$$\sum_{i=1}^N \mathbb{E}\int_0^\infty \left(|\hat{x}_i(t)|^2 + |\hat{u}_i(t)|^2\right) dt < \infty; \tag{42}$$

(ii) (36) and (37) admit unique solutions such that $P \geq 0, \Pi \geq 0$, and $[\bar{A} + G, \bar{C}_0]$ is stable;

(iii) (A5) and (A7) hold.

Proof. See Appendix E. □

We now state the asymptotic optimality of the decentralized control.

Theorem 3.4. *Assume (A2), (A5)–(A7) hold. For Problem (P2), the set of decentralized strategies $\{\hat{u}_1, \ldots, \hat{u}_N\}$ given by (35) has asymptotic social optimality, i.e.,*

$$\left|\frac{1}{N}J_{soc}(\hat{u}) - \frac{1}{N}\inf_{u\in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{nr})} J_{soc}(u)\right| = O(\frac{1}{\sqrt{N}}).$$

Proof. By a similar argument to the proof of Theorem 2.2 with Theorem 3.2, the conclusion follows. □

4. Numerical example

In this section, two examples are given to illustrate the effectiveness of the proposed decentralized control.

Example 4.1. Consider a scalar system with 30 agents in Problem (P2). For the system (1) and (28), take $A = -1, B = C = D = C_0 = D_0 = G = R = \Gamma = 1, Q = 0.5, f(t) = e^{-2t}, \sigma(t) = \sigma_0(t) = 0$ and $\eta(t) = \frac{1}{1+t^2}$. The initial states of 30 agents are taken independently from a normal distribution $N(10, 1)$. Note that $B = D = D_0 \neq 0$. Both systems $[A, C, C_0; B, D, D_0]$ and $[A + G, C_0; B, D_0]$ are mean-square stabilizable necessarily. Besides, it can be verified that $[\bar{A} + G, \bar{C}_0] = [-0.5598, 0.4402]$ is mean-square stable. We obtain that (A2), (A5)–(A7) hold. Under the control law (35), the state trajectories of agents are shown in Fig. 1. It appears that after the transient phase, all the states of agents reach the agreement and converge to 0 gradually.

The trajectories of $\bar{x}$ and $\hat{x}^{(N)}$ in (P2) are shown in Fig. 2. It can be seen that $\bar{x}$ and $\hat{x}^{(N)}$ coincide well, which illustrate the consistency of MF approximations.

Example 4.2. We now consider the model in Example 4.1 where $A = 1, G = -0.8, R = -0.1$, and the other parameters are the same as above. Then we have $P = 0.5$ and $\Pi = -0.0648$. However, it can be verified that $[\bar{A}, \bar{C}, \bar{C}_0]$, $[A + G - B\bar{\Upsilon}^{-1}(\Psi + \Theta), C_0 - D_0\bar{\Upsilon}^{-1}(\Psi+\Theta)]$ and $[\bar{A} + G, \bar{C}_0]$ are mean-square stable. From Fig. 3, it can seen that curves of $\bar{x}$ and $\hat{x}^{(N)}$ still coincide well.

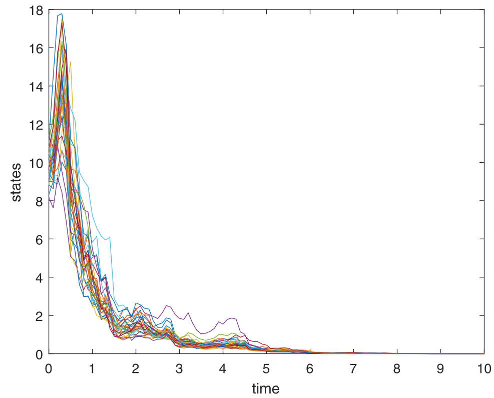

Fig. 1. Curves of 30 agents.

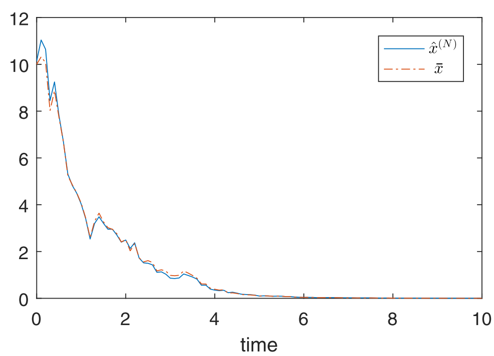

Fig. 2. Curves of $\bar{x}$ and $\hat{x}^{(N)}$.

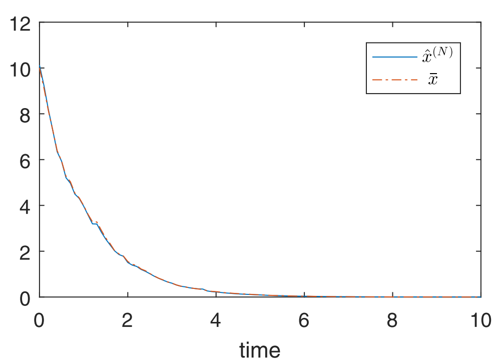

Fig. 3. Curves of $\bar{x}$ and $\hat{x}^{(N)}$.

5. Concluding remarks

In this paper, we have considered uniform stabilization and social optimality for MF-LQ systems with common noise. By tackling coupled high-dimensional FBSDEs, we design the decentralized control laws for finite-horizon and infinite-horizon problems, respectively, and then show their asymptotical social optimality. The necessary and sufficient conditions are given for uniform stabilization of all the subsystems.

An interesting generalization is to consider MF Stackelberg games with common noise. In this situation, the resulting FBSDEs are more complicated. How to obtain nonconservative conditions to ensure the solvability of FBSDEs is worthy of further study. Besides, it also deserves investigating further for applications in economics, smart grids and etc. (Guéant et al., 2011; Jiang, Jia, & Guan, 2022).

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Appendix A. Proof of Theorem 2.1

Proof. Suppose that $\check{u}_i$ is a candidate of the optimal control of Problem (P1). Denote by $\check{x}_i$ the state of agent i under the control $\check{u}_i$. For any $v_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^r)$ and $\lambda \in \mathbb{R}(\lambda \neq 0)$, let $\acute{u}_i = \check{u}_i + \lambda v_i$. Denote by $\acute{x}_i$ the solution of the following perturbed state equation

$$
\begin{aligned}
d\acute{x}_i = &\Big[A\acute{x}_i + B(\check{u}_i + \lambda v_i) + \frac{G}{N}\sum_{i=1}^{N}\acute{x}_i + f \Big] dt + [C\acute{x}_i \\
&+ D(\check{u}_i + \lambda v_i) + \sigma] dW_i + [C_0\acute{x}_i + D_0(\check{u}_i + \lambda v_i) \\
&+ \sigma_0] dW_0, \ \ \acute{x}_i(0) = x_{i0}, \ \ i = 1, 2, \ldots, N.
\end{aligned}
$$

Let $y_i = (\acute{x}_i - \check{x}_i)/\lambda$. It can be verified that y_i satisfies

$$
\begin{aligned}
dy_i = &[Ay_i + Gy^{(N)} + Bv_i]dt + (Cy_i + Dv_i)dW_i \\
&+ (C_0 y_i + D_0 v_i)dW_0, \quad y_i(0) = 0.
\end{aligned}
$$

Let $\{p_i, \beta_i^j, i = 1, \ldots, N, j = 0, 1, \ldots, N\}$ be a set of solutions to (3). Then by Itô's formula,

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}[\langle p_i(T), y_i(T)\rangle] \\
= &\sum_{i=1}^{N} \mathbb{E}\int_0^T \Big[\langle -(Q\check{x}_i - Q_\Gamma \check{x}^{(N)} - \bar{\eta}), y_i\rangle \\
&+ \langle B^T p_i + D^T\beta_i^i + D_0^T\beta_i^i, v_i\rangle \Big] dt.
\end{aligned} \tag{A.1}
$$

From (2), we have

$$
J_{\mathrm{soc}}^{\mathrm{F}}(\acute{u}) - J_{\mathrm{soc}}^{\mathrm{F}}(\check{u}) = 2\lambda \mathcal{I}_1 + \lambda^2 \mathcal{I}_2 \tag{A.2}
$$

where $\check{u} = (\check{u}_1, \ldots, \check{u}_N)$, and

$$
\begin{aligned}
\mathcal{I}_1 \triangleq &\sum_{i=1}^{N} \mathbb{E}\Big\{ \int_0^T \Big[\Big\langle Q\Big(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)\Big), y_i - \Gamma y^{(N)}\Big\rangle \\
&+ \langle R\check{u}_i, v_i\rangle \Big] dt + \Big\langle H\Big(\check{x}_i(T) - (\Gamma_0\check{x}^{(N)}(T) + \eta_0)\Big), \\
&y_i(T) - \Gamma_0 y^{(N)}(T)\Big\rangle \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_2 \triangleq &\sum_{i=1}^{N} \mathbb{E}\Big\{ \int_0^T \Big(|y_i - \Gamma y^{(N)}|_Q^2 + |v_i|_R^2 \Big) dt \\
&+ |y_i(T) - \Gamma_0 y^{(N)}(T)|_H^2 \Big\}.
\end{aligned}
$$

A direct calculation with (A.1) gives

$$
\mathcal{I}_1 = \sum_{i=1}^{N} \mathbb{E}\int_0^T \Big\langle R\check{u}_i + B^T p_i + D^T\beta_i^i + D_0^T\beta_i^0, v_i\Big\rangle dt.
$$

By Lim and Zhou (1999) and Huang and Huang (2017), $\mathcal{I}_2 \geq 0$ equals that (P1) is convex. From a similar argument in Wang et al. (2020), $\check{u} = (\check{u}_1, \dots, \check{u}_N)$ is an optimal control of (P1) if and only if (P1) is convex and FBSDE (3) admits a solution $(\check{x}_i, p_i, \beta_i^j, i, j = 1, \dots, N)$.

If Problem (P1) is uniformly convex, by Yong and Zhou (1999) and Sun et al. (2016), (P1) admits a unique optimal control. □

Appendix B. Proof of Theorem 2.2

To prove Theorem 2.2, we need some lemmas. Consider the linear SDEs

$$\begin{aligned} dz_i(t) = &[A(t)z_i(t) + G(t)z^{(N)}(t) + b(t)]dt \\ &+ [C(t)z_i(t) + \sigma(t)]dW_i(t) + [C_0(t)z_i(t) \\ &+ \sigma_0(t)]dW_0(t),\ z_i(0) = z_{i0},\ 1 \leq i \leq N. \end{aligned} \tag{B.1}$$

Lemma B.1. *Suppose that A, G, C and C_0 are bounded, and $b, \sigma, \sigma_0 \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$. Then (B.1) admits a unique solution $z_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$. Particularly, there exists a constant c, such that*

$$\begin{aligned} \sup_{0\leq t\leq T} \sum_{i=1}^N \mathbb{E}\left[\ |z_i(t)|^2\ \right] \leq Nc \Bigg[&\max_{1\leq i\leq N} \mathbb{E}|z_{i0}|^2 \\ &+ \mathbb{E}\int_0^T \left[\ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2\ \right] ds \Bigg]. \end{aligned}$$

Proof. By a similar argument in Yong and Zhou (1999, Theorem 6.3), we obtain

$$\begin{aligned} \mathbb{E}\left[\ |z_i(t)|^2\ \right] \leq &c_0 \mathbb{E}\Bigg[|z_{i0}|^2 + \int_0^t |z^{(N)}(s)|^2 ds \\ &+ \int_0^t \left[\ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2\ \right] ds \Bigg] \\ \leq &\frac{c}{N}\int_0^t \sum_{i=1}^N \mathbb{E}|z_i(s)|^2 ds + c_0 h(t), \end{aligned}$$

where

$$h(t) \triangleq \max_{1\leq i\leq N} \mathbb{E}|z_{i0}|^2 + \mathbb{E}\int_0^T \left[\ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2\ \right] ds.$$

This further implies

$$\sum_{i=1}^N \mathbb{E}|z_i(t)|^2 \leq c_0 \int_0^t \sum_{i=1}^N \mathbb{E}|z_i(s)|^2 ds + Nc_0 h(t).$$

By Gronwall's inequality, we have $\sum_{i=1}^N \mathbb{E}|z_i(t)|^2 \leq Nc_0 e^{c_1 t} h(t)$. Let $c = c_0 e^{c_1 T}$. The lemma follows. □

Lemma B.2. *Let (A1)–(A4) hold. Under the control (17), the following holds:*

$$\sup_{0\leq t\leq T} \sum_{i=1}^N \mathbb{E}|\hat{x}_i(t)|^2 \leq Nc.$$

Proof. Since $\psi, f, \sigma_0 \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$, by (16) and Lemma B.1, we have $\sup_{0\leq t\leq T} \mathbb{E}|\bar{x}(t)|^2 \leq c$. From Lemma B.1 we obtain $\sup_{0\leq t\leq T} \sum_{i=1}^N \mathbb{E}|\hat{x}_i|^2 \leq c$. □

Lemma B.3. *Let (A1)–(A4) hold. Under the control (17), the following holds:*

$$\max_{0\leq t\leq T} \mathbb{E}|\hat{x}^{(N)}(t) - \bar{x}(t)|^2 = O(1/N).$$

Proof. Denote

$$\begin{aligned} \bar{f} &\triangleq B[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi + \Theta)]\bar{x} - B\bar{\Upsilon}^{-1}\psi + f, \\ \bar{\sigma} &\triangleq D[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi + \Theta)]\bar{x} - D\bar{\Upsilon}^{-1}\psi + \sigma, \\ \bar{\sigma}_0 &\triangleq D_0[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi + \Theta)]\bar{x} - D_0\bar{\Upsilon}^{-1}\psi + \sigma_0. \end{aligned}$$

Let $\zeta(t) = \hat{x}^{(N)}(t) - \bar{x}(t)$. It follows by (18) that

$$\begin{aligned} d\hat{x}^{(N)}(t) = &\left[\ (\bar{A}(t) + G(t))\hat{x}^{(N)}(t) + \bar{f}(t)\ \right] dt \\ &+ \frac{1}{N}\sum_{i=1}^N [\bar{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)]dW_i(t) \\ &+ [\bar{C}_0(t)\hat{x}^{(N)}(t) + \bar{\sigma}_0(t)]dW_0(t). \end{aligned}$$

From this and (16), we have

$$\begin{aligned} d\zeta(t) = &[\bar{A}(t) + G(t)]\zeta(t)dt \\ &+ \frac{1}{N}\sum_{i=1}^N [\bar{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)]dW_i(t) \\ &+ \bar{C}_0\zeta(t)dW_0(t),\ \zeta(0) = \hat{x}^{(N)}(0) - \bar{x}_0. \end{aligned}$$

By (A2) and Lemma B.1, one can obtain

$$\begin{aligned} &\sup_{0\leq t\leq T} \mathbb{E}\left|\hat{x}^{(N)}(t) - \bar{x}(t)\right|^2 = \sup_{0\leq t\leq T} \mathbb{E}|\zeta(t)|^2 \\ \leq &c\mathbb{E}\left\{ \mathbb{E}\left|\hat{x}^{(N)}(0) - \bar{x}_0\right|^2 \right. \\ &\left. + \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \int_0^T [\bar{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)]dW_i\right|^2 \right\} \\ \leq &\frac{c}{N}\left\{ \max_{1\leq i\leq N} \mathbb{E}|\hat{x}_{i0}|^2 + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T |\bar{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)|^2 dt \right\} \\ \leq &\frac{c}{N}\left\{ \max_{1\leq i\leq N} \mathbb{E}|\hat{x}_{i0}|^2 + \max_{1\leq i\leq N} \mathbb{E}\int_0^T |\bar{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)|^2 dt \right\}. \end{aligned}$$

By Lemma B.2, the proof is completed. □

Proof of Theorem 2.2. We first prove that for $u \in \mathcal{U}_c, J^{\mathrm{F}}_{\mathrm{soc}}(u) < \infty$ implies $\mathbb{E}\int_0^T (|x_i|^2 + |u_i|^2)dt < \infty$, for all $i = 1, \dots, N$. By (A3) and Wang et al. (2020, Proposition 3.1), we have

$$\delta_0 \sum_{i=1}^N \mathbb{E}\int_0^T |u_i(t)|^2 dt - c \leq J^{\mathrm{F}}_{\mathrm{soc}}(u) < \infty,$$

which implies $\sum_{i=1}^N \mathbb{E}\int_0^T |u_i(t)|^2 dt < c_1$. By Lemma B.1 and (1), we have

$$\sum_{i=1}^N \mathbb{E}\int_0^T |x_i(t)|^2 dt \leq Nc. \tag{B.2}$$

Denote $\tilde{x}_i = x_i - \hat{x}_i$, $\tilde{u}_i = u_i - \hat{u}_i$ and $\tilde{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \tilde{x}_i$. Then

$$\mathbb{E}\int_0^T \left(\ |\tilde{x}_i|^2 + |\tilde{u}_i|^2\ \right) dt < \infty. \tag{B.3}$$

It follows from (1) and (18) that

$$\begin{aligned} d\tilde{x}_i = &(A\tilde{x}_i + G\tilde{x}^{(N)} + B\tilde{u}_i)dt + (C\tilde{x}_i + D\tilde{u}_i)dW_i \\ &+ (C_0\tilde{x}_i + D_0\tilde{u}_i)dW_0,\ \tilde{x}_i(0) = 0. \end{aligned} \tag{B.4}$$

We have $J^{\mathrm{F}}_{\mathrm{soc}}(u) = \sum_{i=1}^N (J^{\mathrm{F}}_i(\hat{u}) + \tilde{J}^{\mathrm{F}}_i(\tilde{u}) + I_i)$, where

$$\begin{aligned} \tilde{J}^{\mathrm{F}}_i(\tilde{u}) \triangleq &\mathbb{E}\int_0^T \left[\ |\tilde{x}_i - \Gamma\tilde{x}^{(N)}|^2_Q + |\tilde{u}_i|^2_R\ \right] dt \\ &+ \mathbb{E}|\tilde{x}_i(T) - \Gamma_0\tilde{x}^{(N)}(T)|^2_H, \end{aligned}$$

and

$$I_i \triangleq 2\mathbb{E}\int_0^T \Big[\big(\hat{x}_i - \Gamma\hat{x}^{(N)} - \eta\big)^T Q \,\big(\tilde{x}_i - \Gamma\tilde{x}^{(N)}\big) + \hat{u}_i^T R\tilde{u}_i \Big] dt + 2\mathbb{E}\Big[\big(\hat{x}_i(T) - (\Gamma_0\hat{x}^{(N)}(T) + \eta_0)\big)^T \times H(\tilde{x}_i(T) - \Gamma_0\tilde{x}^{(N)}(T))\Big].$$

By (A3) together with Lim and Zhou (1999, Lemma 1), $\sum_{i=1}^N \tilde{J}_i^F(\tilde{u}) \ge 0$. We only need to prove $\frac{1}{N}\sum_{i=1}^N I_i = O(\frac{1}{\sqrt{N}})$. Note that

$$\begin{aligned}\sum_{i=1}^N I_i =& \sum_{i=1}^N 2\mathbb{E}\int_0^T \Big[\tilde{x}_i^T\,\big(Q\hat{x}_i - Q_\Gamma\bar{x} - \bar{\eta}\big) + \sum_{i=1}^N \hat{u}_i^T R\tilde{u}_i \Big] dt \\ &- \sum_{i=1}^N 2\Big(\mathbb{E}\int_0^T (\hat{x}^{(N)} - \bar{x})^T Q_\Gamma \tilde{x}_i dt \\ &+ \mathbb{E}\big[\tilde{x}_i^T(T) H_{\Gamma_0}(\hat{x}^{(N)}(T) - \bar{x}(T))\big]\Big) \\ &+ 2\sum_{i=1}^N \mathbb{E}\big[\tilde{x}_i^T(T)\,\big(H\hat{x}_i(T) - H_{\Gamma_0}\bar{x}(T) - \bar{\eta}_0\big)\big].\end{aligned} \tag{B.5}$$

Let $\hat{p}_i = P\hat{x}_i + K\bar{x} + \varphi$. By (9)–(11) and Itô's formula, we obtain

$$\begin{aligned}d\hat{p}_i =& -[A^T\hat{p}_i + G^T\hat{p}^{(N)} + C_0^T\hat{\beta}_i^0 + C^T\hat{\beta}_i^i + Q\hat{x}_i - Q_\Gamma\bar{x} \\ &- \bar{\eta}]dt + (G^TP + PG)(\hat{x}^{(N)} - \bar{x})dt + \hat{\beta}_i^0 dW_0 \\ &+ \hat{\beta}_i^i dW_i,\ \ \hat{p}_i(T) = H\hat{x}_i(T) - H_{\Gamma_0}\bar{x}(T) - \bar{\eta}_0,\end{aligned} \tag{B.6}$$

where $\hat{\beta}_i^i = P(C\hat{x}_i + D\hat{u}_i + \sigma)$, $\bar{u} = -\bar{\Upsilon}^{-1}(\Psi + \Theta)\bar{x}$, and

$$\begin{aligned}\hat{\beta}_i^0 =& P(C_0\hat{x}_i + D_0\hat{u}_i + \sigma_0) + K(C_0\bar{x} + D_0\bar{u} + \sigma_0) \\ &+ \Lambda\hat{x}_i + L\bar{x} + \gamma.\end{aligned}$$

Note that by (17), $R\hat{u}_i = -(B\hat{p}_i + D_0\hat{\beta}_i^0 + D\hat{\beta}_i^i)$. From (B.4) and (B.6), we obtain

$$\begin{aligned}&\sum_{i=1}^N \mathbb{E}\big[\tilde{x}_i^T(T)(H\hat{x}_i(T) - H_{\Gamma_0}\bar{x}(T) - \bar{\eta}_0)\big] \\ =& \mathbb{E}\int_0^T \sum_{i=1}^N \Big\{ -\tilde{x}_i^T\big[Q\hat{x}_i - Q_\Gamma\bar{x} - \bar{\eta}\big] - \hat{u}_i^T R\tilde{u}_i \Big\} dt \\ &+ N\mathbb{E}\int_0^T (\hat{x}^{(N)} - \bar{x})^T(G^TP + PG)\tilde{x}^{(N)} dt.\end{aligned}$$

From this and (B.5), we obtain

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N I_i =& 2\mathbb{E}\int_0^T (\hat{x}^{(N)} - \bar{x})^T(G^TP + PG - Q_\Gamma)\tilde{x}^{(N)} dt \\ &+ 2\mathbb{E}\big[(\hat{x}^{(N)}(T) - \bar{x}(T))^T H_{\Gamma_0}\tilde{x}^{(N)}(T)\big].\end{aligned}$$

By Lemma B.3, (B.2) and (B.3), we obtain

$$\begin{aligned}\Big|\frac{1}{N}\sum_{i=1}^N I_i\Big|^2 \le& c\mathbb{E}\int_0^T |\hat{x}^{(N)} - \bar{x}|^2 dt \cdot \mathbb{E}\int_0^T |\tilde{x}^{(N)}|^2 dt \\ &+ c\mathbb{E}|\hat{x}^{(N)}(T) - \bar{x}(T)|^2 \cdot \mathbb{E}|\tilde{x}^{(N)}(T)|^2,\end{aligned}$$

which implies $|\frac{1}{N}\sum_{i=1}^N I_i| = O(1/\sqrt{N})$. □

Appendix C. Proof of Theorem 2.3

To prove Theorem 2.3, we need two lemmas.

Denote $\mathbb{E}_{\mathcal{F}^0}(\cdot) \triangleq \mathbb{E}(\cdot|\mathcal{F}_t^0)$. Consider the (conditional) MF type system

$$\begin{aligned}dz_i =& (Az_i + Bu_i + G\mathbb{E}_{\mathcal{F}^0}(z_i) + f)dt + (Cz_i + Du_i + \sigma)dW_i \\ &+ (C_0z_i + D_0u_i + \sigma_0)dW_0,\ \ z_i(0) = x_{i0},\end{aligned} \tag{C.1}$$

with the cost functional

$$\begin{aligned}\mathcal{J}_i(u_i) =& \mathbb{E}\int_0^T (|z_i - \Gamma\mathbb{E}_{\mathcal{F}^0}(z_i) - \eta|_Q^2 + |u_i|_R^2)dt \\ &+ \mathbb{E}|z_i(T) - \Gamma_0\mathbb{E}_{\mathcal{F}^0}(z_i(T)) - \eta_0|_H^2.\end{aligned} \tag{C.2}$$

The admissible control set is given by

$$\mathcal{U}_i = \Big\{ u_i \Big|\ u_i(t) \text{ is adapted to } \sigma(z_i(0), W_i(0), W_0(s), 0 \le s \le t), \mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty \Big\}.$$

Lemma C.1. *Assume (A1)–(A4) hold. For the system (C.1)–(C.2), the optimal control is given by*

$$\hat{u}_i = -\Upsilon^{-1}\Psi\big[z_i - \mathbb{E}_{\mathcal{F}^0}(z_i)\big] - \bar{\Upsilon}^{-1}\big[(\Psi + \Theta)\mathbb{E}_{\mathcal{F}^0}(z_i) + \psi\big], \tag{C.3}$$

and the optimal cost is

$$\inf_{u_i\in\mathcal{U}_i} \mathcal{J}_i(u_i) = \mathbb{E}\big[(x_{i0} - \bar{x}_0)^T P(x_{i0} - \bar{x}_0) + \bar{x}_0^T \Pi\bar{x}_0 + 2\varphi^T(0)\bar{x}_0\big] + q_T.$$

Proof. The proof is similar to Theorem 2.6 of Graber (2016). □

After applying the control (C.3) into (C.1), we have

$$d\hat{z}_i \tag{C.4}$$

$$\begin{aligned}=& \big\{\bar{A}\hat{z}_i + G\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) + B[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi + \Theta)]\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) \\ &- B\bar{\Upsilon}^{-1}\psi + f\big\}dt + \big\{\bar{C}\hat{z}_i + D[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi \\ &+ \Theta)]\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) - D\bar{\Upsilon}^{-1}\psi + \sigma\big\}dW_i + \big\{\bar{C}_0\hat{z}_i \\ &+ D_0[\Upsilon^{-1}\Psi - \bar{\Upsilon}^{-1}(\Psi + \Theta)]\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) \\ &- D_0\bar{\Upsilon}^{-1}\psi + \sigma_0\big\}dW_0.\end{aligned} \tag{C.5}$$

Lemma C.2. *Assume that (A1)–(A4) hold. Then*

$$\mathbb{E}\int_0^T |\hat{x}_i - \hat{z}_i|^2 dt = O\Big(\frac{1}{N}\Big).$$

Proof. From (C.4),

$$\begin{aligned}d\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) =& \big\{\big[A + G - B\bar{\Upsilon}^{-1}(\Psi + \Theta)\big]\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) - B\bar{\Upsilon}^{-1}\psi \\ &+ f\big\}dt + \big\{[C_0 - D_0\bar{\Upsilon}^{-1}(\Psi + \Theta)]\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) \\ &- D_0\bar{\Upsilon}^{-1}\psi + \sigma_0\big\}dW_0.\end{aligned}$$

By comparing (C.4) with (16), we can verify that $\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i) = \bar{x}_0$. From (18),

$$\begin{aligned}d(\hat{x}_i - \hat{z}_i) =& \bar{A}(\hat{x}_i - \hat{z}_i)dt + G(\hat{x}^{(N)} - \mathbb{E}_{\mathcal{F}^0}(\hat{z}_i))dt \\ &+ \bar{C}(\hat{x}_i - \hat{z}_i)dW_i + \bar{C}_0(\hat{x}_i - \hat{z}_i)dW_0.\end{aligned}$$

This gives

$$\hat{x}_i(t) - \hat{z}_i(t) = \int_0^t \Phi_i(t-\tau)G(\tau)[\hat{x}^{(N)}(\tau) - \mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)(\tau)]d\tau,$$

where Φ_i satisfies

$$\begin{aligned}d\Phi_i(t) =& \bar{A}(t)\Phi_i(t)dt + \bar{C}(t)\Phi_i(t)dW_i \\ &+ \bar{C}_0(t)\Phi_i(t)dW_0,\ \ \Phi_i(0) = I.\end{aligned}$$

By Schwarz's inequality and Lemma B.3,

$$\begin{aligned}&\max_{0\le t\le T} \mathbb{E}|\hat{x}_i(t) - \hat{z}_i(t)|^2 dt \\ =& \max_{0\le t\le T} \mathbb{E}\Big|\int_0^t \Phi_i(t-\tau)G[\hat{x}^{(N)}(\tau) - \mathbb{E}_{\mathcal{F}^0}(\hat{z}_i(\tau))]d\tau\Big|^2 dt\end{aligned}$$

$$\leq \mathbb{E}\int_0^T |\Phi_i(T-\tau)|^2 d\tau \mathbb{E}\int_0^T \left|G(\hat{x}^{(N)}(\tau)-\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)(\tau))\right|^2 d\tau$$
$$\leq c\mathbb{E}\int_0^T \left|\hat{x}^{(N)}(\tau)-\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)(\tau)\right|^2 d\tau = O(\frac{1}{N}). \quad \square$$

Proof of Theorem 2.3. Note that $\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)=\bar{x}$. We have

$$\begin{aligned}\frac{1}{N}J_{\rm soc}^{\rm F}(\hat{u}) &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \Big[\left|\hat{x}_i-(\Gamma\hat{x}^{(N)}+\eta)\right|_Q^2 \\ &+\left|\Upsilon^{-1}\Psi(\hat{x}_i-\bar{x})+\bar{\Upsilon}^{-1}\left[(\Psi+\Theta)\bar{x}+\psi\right]\right|_R^2\Big]dt \\ &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T \Big[\big|\hat{z}_i-[\Gamma\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)+\eta]+\hat{x}_i-\hat{z}_i \\ &+\Gamma\hat{x}^{(N)}-\Gamma\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)\big|_Q^2+\big|\Upsilon^{-1}\Psi\left[\hat{x}_i-\hat{z}_i+\hat{z}_i\right. \\ &\left.-\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)\right]+\bar{\Upsilon}^{-1}\left[(\Psi+\Theta)\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i)+\psi\right]\big|_R^2\Big]dt.\end{aligned}$$

Note that $\mathcal{J}_{\rm soc}(\hat{u})\triangleq\sum_{i=1}^N\mathcal{J}_i(\hat{u})=N\mathcal{J}_i(\hat{u})$. By Schwarz's inequality, and Lemma C.2,

$$\begin{aligned}&\left|\frac{1}{N}J_{\rm soc}^{\rm F}(\hat{u})-\frac{1}{N}\mathcal{J}_{\rm soc}(\hat{u})\right| = \frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^T \left[|\hat{x}_i-\hat{z}_i|_Q^2\right. \\ &\left.+|\Gamma(\hat{x}^{(N)}-\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i))|_Q^2\right]dt+\frac{c_2}{N}\sum_{i=1}^N\Big(\mathbb{E}\int_0^T|\hat{x}_i-\hat{z}_i|_Q^2dt\Big)^{1/2} \\ &+\frac{c_2}{N}\sum_{i=1}^N\Big(\int_0^T|\Gamma(\hat{x}^{(N)}-\mathbb{E}_{\mathcal{F}^0}(\hat{z}_i))|_Q^2dt\Big)^{1/2}=O(1/\sqrt{N}).\end{aligned}$$

Letting $N\to\infty$, by Lemma C.1, the theorem follows. $\square$

Appendix D. Proofs of Proposition 2.2 and Theorem 2.4

Proof of Proposition 2.2. Denote by x_i^* the corresponding state under the control $u_i^*, i=1,\ldots,N$. Let $\delta u_i=u_i-u_i^*$, $\delta x_i=x_i-x_i^*$ and $\delta x^{(N)}=\frac{1}{N}\sum_{i=1}^N\delta x_i$, Assume $u_i^*, u_i\in\mathcal{U}_d'$. Denote by $\delta J_{\rm soc}^{\rm F}$ the variation of $J_{\rm soc}^{\rm F}$ with δu_i. Then we have

$$\begin{aligned}&\delta J_{\rm soc}^{\rm F} \\ &= \sum_{i=1}^N\mathbb{E}\Big\{\int_0^T\Big[\big\langle Q\big(x_i^*-(\Gamma x_*^{(N)}+\eta)\big),\delta x_i-\Gamma\delta x^{(N)}\big\rangle \\ &+\langle Ru_i^*,\delta u_i\rangle\Big]dt+\big\langle H\big(x_i^*(T)-(\Gamma_0x_*^{(N)}(T)+\eta_0)\big), \\ &\delta x_i(T)-\Gamma_0\delta x^{(N)}(T)\big\rangle\Big\}.\end{aligned}\tag{D.1}$$

Let $\{p_i^*,\beta_i^{j,*}, i=1,\ldots,N, j=0,1,\ldots,N\}$ be a solution to the second equation in (20). By Itô's formula,

$$\begin{aligned}&\sum_{i=1}^N\mathbb{E}[\langle p_i(T)^*,\delta x_i(T)\rangle] \\ &=\sum_{i=1}^N\mathbb{E}\int_0^T\Big[\big\langle-\big[Qx_i^*-Q_\Gamma x_*^{(N)}-\bar{\eta}+G^Tp_*^{(N)}\big],\delta x_i\big\rangle \\ &+\langle B^Tp_i^*+D^T\beta_i^{i,*}+D_0^T\beta_i^{i,*},\delta u_i\rangle\Big]dt.\end{aligned}\tag{D.2}$$

From (D.1) and (D.2), we obtain

$$\delta J_{\rm soc}^{\rm F}=\sum_{i=1}^N\mathbb{E}\int_0^T\Big\langle Ru_i^*+B^Tp_i^*+D^T\beta_i^{i,*}+D_0^T\beta_0^{i,*},\delta u_i\Big\rangle dt.$$

Note that $u_i^*,\delta u_i\in\mathcal{H}_t^i$. By the smoothing property of conditional mathematical expectation,

$$\begin{aligned}\delta J_{\rm soc}^{\rm F}&=\mathbb{E}\int_0^T\Big\langle Ru_i^*+B^Tp_i^*+D^T\beta_i^{i,*}+D_0^T\beta_0^{i,*},\delta u_i\Big\rangle dt \\ &=\mathbb{E}\int_0^T\Big\langle Ru_i^*+\mathbb{E}[B^Tp_i^*+D^T\beta_i^{i,*}+D_0^T\beta_i^{0,*}|\mathcal{H}_t^i],\delta u_i\Big\rangle dt.\end{aligned}\tag{D.3}$$

Since (P1′) is convex, by standard variation principle $\{u_i^*, i=1,\ldots,N\}$ is an optimal control of (P1′) if and only if $\delta J_{\rm soc}^{\rm F}=0$. From (D.3), $\delta J_{\rm soc}^{\rm F}=0$ is equivalent to $u_i^*=-R^{-1}\mathbb{E}[B^Tp_i^*+D^T\beta_i^{i,*}+D_0^T\beta_i^{0,*}|\mathcal{H}_t^i]$. $\square$

Proof of Theorem 2.4. Let $\mathbb{E}_{\mathcal{H}^i}[p_i^*]=P_N^*x_i^*+K_N^*\mathbb{E}_{\mathcal{F}^0}[x_i^*]+\varphi_N^*$, where $P_N^*,K_N^*\in\mathbb{R}^{n\times n}$ and φ_N^* satisfies $d\varphi_N^*=\check{\varphi}_N^*dt+\gamma_N^*dW_0$, $\varphi_N^*(T)=-\bar{\eta}_0$. By Itô's formula, we obtain

$$\begin{aligned}&d\mathbb{E}_{\mathcal{H}^i}[p_i^*]=\dot{P}_N^*x_i^*dt+P_N^*\big[(Ax_i^*+B\check{u}_i^*+f)dt \\ &+(Cx_i^*+Du_i^*+\sigma)dW_i+(C_0x_i^*+D_0u_i^*+\sigma_0)dW_0\big] \\ &+\dot{K}_N^*\mathbb{E}[x_i^*]\big]dt+K_N^*\big[\big(A\mathbb{E}_{\mathcal{F}^0}[x_i^*]+B\mathbb{E}_{\mathcal{F}^0}[u_i^*]+f\big)dt \\ &+\big(C_0\mathbb{E}_{\mathcal{F}^0}[x_i^*]+D_0\mathbb{E}_{\mathcal{F}^0}[u_i^*]+\sigma_0\big)dW_0\big] \\ &+\check{\varphi}_N^*dt+\gamma_N^*dW_0.\end{aligned}\tag{D.4}$$

By comparing this with (23), it follows that

$$\begin{aligned}\beta_i^{0,*}&=P_N^*(C_0x_i^*+D_0u_i^*+\sigma_0)+\gamma_N^* \\ &\quad+K_N^*\big(C_0\mathbb{E}_{\mathcal{F}^0}[x_i^*]+D_0\mathbb{E}_{\mathcal{F}^0}[u_i^*]+\sigma_0\big), \\ \beta_i^{i,*}&=P_N^*(Cx_i^*+Du_i^*+\sigma).\end{aligned}$$

From (21) and (22), we obtain

$$(\tilde{\Upsilon}_N+D_0^TK_N^*D_0)\mathbb{E}_{\mathcal{F}^0}[u_i^*]+(\tilde{\Psi}_N+\tilde{\Phi}_N)\mathbb{E}_{\mathcal{F}^0}[x_i^*]+\psi_N^*=0.$$

From (D.4), we obtain that $u_i^*, i=1,\ldots,N$ satisfy (24), and P_N^*, K_N^* and φ_N^* satisfy (25)–(27), respectively. $\square$

Appendix E. Proofs of Theorems 3.2 and 3.3

Proof of Theorem 3.2. After (35) is applied, we have

$$\begin{aligned}d\hat{x}_i=&\big(\bar{A}\hat{x}_i+G\hat{x}^{(N)}+\bar{f}\big)dt+(\bar{C}\hat{x}_i+\bar{\sigma})dW_i \\ &+(\bar{C}_0\hat{x}_i+\bar{\sigma}_0)dW_0,\end{aligned}\tag{E.1}$$

where $\bar{f}=B[\Upsilon^{-1}\Psi-\bar{\Upsilon}^{-1}(\Psi+\Theta)]\bar{x}-B\bar{\Upsilon}^{-1}\psi+f$, $\bar{\sigma}=D[\Upsilon^{-1}\Psi-\bar{\Upsilon}^{-1}(\Psi+\Theta)]\bar{x}-D\bar{\Upsilon}^{-1}\psi+\sigma$, and $\bar{\sigma}_0=D_0[\Upsilon^{-1}\Psi-\bar{\Upsilon}^{-1}(\Psi+\Theta)]\bar{x}-D_0\bar{\Upsilon}^{-1}\psi+\sigma_0$. From (18) and (39), we have

$$\begin{aligned}d\zeta(t)=&(\bar{A}+G)\zeta(t)dt+\frac{1}{N}\sum_{i=1}^N(\bar{C}\hat{x}_i(t)+\bar{\sigma})dW_i(t) \\ &+\bar{C}_0\zeta(t)dW_0(t),\ \ \zeta(0)=x^{(N)}(0)-\bar{x}_0,\end{aligned}\tag{E.2}$$

where $\zeta=\hat{x}^{(N)}-\bar{x}$. Let $\bar{P}$ satisfy

$$\bar{P}(\bar{A}+G)+(\bar{A}+G)^T\bar{P}+\bar{C}_0^T\bar{P}\bar{C}_0=-I.$$

From (A5)–(A6), there exists a constant c such that $P>0$. By Itô's formula, we have

$$\begin{aligned}&\mathbb{E}[\zeta^T(T)\bar{P}\zeta(T)-\zeta^T(0)\bar{P}\zeta(0)] \\ =&\mathbb{E}\Big[\int_0^T\zeta^T(t)[\bar{P}(\bar{A}+G)+(\bar{A}+G)^T\bar{P}+\bar{C}_0^T\bar{P}\bar{C}_0]\zeta(t)dt \\ &+\frac{1}{N^2}\sum_{i=1}^N\int_0^T(\bar{C}\hat{x}_i+\bar{\sigma})^T\bar{P}(\bar{C}\hat{x}_i+\bar{\sigma})dt\Big] \\ \leq&-\mathbb{E}\int_0^T\zeta^T(t)\zeta(t)dt+\frac{1}{N}\Big[c_1\max_{1\leq i\leq N}\mathbb{E}\int_0^T|\hat{x}_i(t)|_{\bar{P}}^2dt+c_1\Big]\end{aligned}$$

which gives

$$\mathbb{E}\int_0^T \zeta^T(t)\zeta(t)dt \le \frac{1}{N}\left[c_2 \max_{1\le i\le N}\mathbb{E}\int_0^T |\hat{x}_i(t)|^2 dt + c_2 \right]. \tag{E.3}$$

Let P satisfy

$$P\bar{A} + \bar{A}^T P + \bar{C}^T P\bar{C} + \bar{C}_0^T P\bar{C}_0 = -2I.$$

From (A5)–(A6), we have $P > 0$. By Itô's formula, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\hat{x}_i^T(T)P\hat{x}_i(T) - \hat{x}_i^T(0)P\hat{x}_i(0) \right]\\
\le &\mathbb{E}\left[\int_0^T \left[\hat{x}_i^T(P\bar{A} + \bar{A}^T P + \bar{C}^T P\bar{C} + \bar{C}_0^T P\bar{C}_0)\hat{x}_i \right.\right.\\
&+ \hat{x}^{(N)}(PG + G^T P)\hat{x}^{(N)} + \bar{\sigma}^T P\bar{\sigma} + \bar{\sigma}_0^T P\bar{\sigma}_0\\
&\left.\left. + 2\hat{x}_i^T(P\bar{f} + \bar{C}^T P\bar{\sigma} + \bar{C}_0^T P\bar{\sigma}_0) \right] dt \right]\\
\le &-\mathbb{E}\left[\int_0^T (\hat{x}_i^T\hat{x}_i)dt \right] + \alpha_T, \ \ a.s.,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_T = \mathbb{E}\Bigg[\int_0^T &\left[\hat{x}^{(N)}(PG + G^T P)\hat{x}^{(N)} + \bar{\sigma}^T P\bar{\sigma} \right.\\
&\left. + |P\bar{f} + \bar{C}^T P\bar{\sigma} + \bar{C}_0^T P\bar{\sigma}_0|^2 + \bar{\sigma}_0^T P\bar{\sigma}_0 \right] dt \Bigg].
\end{aligned}$$

This implies $\mathbb{E}\left[\int_0^T |\hat{x}_i|^2 dt \right] \le \mathbb{E}[x_{i0}^T P x_{i0}] + \alpha_T$, which with (E.3) further gives

$$\begin{aligned}
&\mathbb{E}\int_0^T |\hat{x}_i(t)|^2 dt \le c_3\mathbb{E}\int_0^T |\hat{x}^{(N)}(t)|^2 dt + c_3\\
&\le 2c_3\mathbb{E}\int_0^T \left(|\bar{x}(t)|^2 + |\zeta(t)|^2 \right) dt + c_3\\
&\le 2c_3\left[\mathbb{E}\int_0^T |\bar{x}(t)|^2 dt + \frac{c_2}{N}\max_{1\le i\le N}\mathbb{E}\int_0^T |\hat{x}_i(t)|^2 dt \right] + c_4.
\end{aligned}$$

Thus, there exists an integer N_0 such that $N > N_0$, the following holds:

$$\max_{1\le i\le N}\mathbb{E}\int_0^T |\hat{x}_i(t)|^2 dt \le 2c_5\mathbb{E}\int_0^T |\bar{x}(t)|^2 dt + c_6.$$

Note $\bar{x} \in L^2_{\mathcal{F}}([0,\infty), \mathbb{R}^n)$. It follows that

$$\max_{1\le i\le N}\mathbb{E}\int_0^\infty |\hat{x}_i(t)|^2 dt \le c.$$

This together with (E.3) gives (41). □

Proof of Theorem 3.3. (iii)⇒(i) has been given in Theorem 3.2. We now show (i)⇒(ii). By (E.1),

$$\begin{aligned}
\frac{d\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i)}{dt} = &\left[\bar{A}\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i) + G\mathbb{E}_{\mathcal{F}^0}(\hat{x}^{(N)}) + \bar{f} \right] dt\\
&+ \left[\bar{C}_0\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i) + \bar{\sigma}_0 \right] dW_0, \ \ \mathbb{E}_{\mathcal{F}^0}(\hat{x}_i(0)) = \bar{x}_0.
\end{aligned} \tag{E.4}$$

It follows from (A2) that

$$\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i) = \mathbb{E}_{\mathcal{F}^0}(\hat{x}_j) = \mathbb{E}_{\mathcal{F}^0}(\hat{x}^{(N)}), \ \ j \ne i.$$

By comparing (E.4) with (39), we obtain $\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i) = \bar{x}$. Note that

$$\begin{aligned}
\mathbb{E}[|\bar{x}|^2] &= \mathbb{E}\left[\mathbb{E}_{\mathcal{F}^0}(\hat{x}_i)^2 \right] = \mathbb{E}\left\{ \left| \mathbb{E}[\hat{x}_i|\mathcal{F}_t^0] \right|^2 \right\}\\
&\le \mathbb{E}\left\{ \mathbb{E}[|\hat{x}_i|^2|\mathcal{F}_t^0] \right\} = \mathbb{E}[|\hat{x}_i|^2].
\end{aligned}$$

It follows from (42) that

$$\mathbb{E}\int_0^\infty |\bar{x}(t)|^2 dt < \infty. \tag{E.5}$$

By (39), we have

$$\begin{aligned}
\bar{x}(t) = \Phi_0(t)\Bigg[&\bar{x}_0 + \int_0^t \Phi_0^{-1}(\tau)h(\tau)d\tau\\
&+ \int_0^t \Phi_0^{-1}(\tau)(\sigma_0(\tau) - D_0\bar{\Upsilon}^{-1}\psi(\tau))dW_0(\tau) \Bigg],
\end{aligned}$$

where $h = f - B\bar{\Upsilon}^{-1}\psi - C_0(\sigma_0 - D_0\bar{\Upsilon}^{-1}\psi)$, and Φ_0 satisfies

$$\begin{aligned}
d\Phi_0 = &(A + G - B\bar{\Upsilon}^{-1}(\Psi + \Theta))\Phi_0 dt\\
&+ (C_0 - D_0\bar{\Upsilon}^{-1}(\Psi + \Theta))\Phi_0 dW_0, \ \ \Phi_0(0) = I.
\end{aligned}$$

By the arbitrariness of $\bar{x}_0$ and (E.5), we obtain that $[A + G, B; C_0, D_0]$ is stabilizable. Note that $\mathbb{E}[x^{(N)}]^2 \le \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\hat{x}_i^2]$. Then from (42) we have

$$\mathbb{E}\int_0^\infty \left| \hat{x}^{(N)}(t) \right|^2 dt < \infty. \tag{E.6}$$

This leads to $\mathbb{E}\int_0^\infty |G\hat{x}^{(N)} + \bar{f}|^2 dt < \infty$. By (E.1), we obtain

$$\begin{aligned}
&\mathbb{E}|\hat{x}_i(t)|^2\\
= &\mathbb{E}\Bigg| \Phi_i(t)\Bigg(x_{i0} + \int_0^t \Phi_i^{-1}(\tau)[Gx^{(N)} + \bar{f} - \bar{C}\bar{\sigma} - \bar{C}_0\bar{\sigma}_0]dt\\
&+ \int_0^t \Phi_i^{-1}(\tau)\bar{\sigma}dW_i(\tau) + \int_0^t \Phi_i^{-1}(\tau)\bar{\sigma}_0 dW_0(\tau) \Bigg) \Bigg|^2,
\end{aligned}$$

where Φ_i satisfies

$$d\Phi_i = \bar{A}\Phi_i dt + \bar{C}\Phi_i dW_i + \bar{C}_0\Phi_i dW_0, \ \ \Phi_i(0) = I.$$

By (42) and the arbitrariness of x_{i0}, we obtain that $\mathbb{E}\int_0^\infty |\Phi_i(t)|^2 dt < \infty$, i.e., $[A, C, C_0; B, D, D_0]$ is stabilizable. On the other hand, from (E.5) and (E.6),

$$\mathbb{E}\int_0^\infty \left| \zeta(t) \right|^2 dt = \mathbb{E}\int_0^\infty \left| \hat{x}^{(N)}(t) - \bar{x}(t) \right|^2 dt < \infty.$$

Noting that $\zeta(t)$ satisfies (E.2), we obtain that (A7) holds.

(ii)⇒(iii). By a similar argument in Zhang et al. (2019, 2008) and Wang et al. (2020), we can show $[A, C, C_0; B, D, D_0]$ is stabilizable. Since $\Pi \ge 0$, there exists an orthogonal U such that $U^T\Pi U = \begin{bmatrix} 0 & 0 \\ 0 & \Pi_2 \end{bmatrix}$, where $\Pi_2 > 0$. From (37),

$$\begin{aligned}
&(U^T\bar{\mathbb{A}}U)^T U^T\Pi U + U^T\Pi U(U^T\bar{\mathbb{A}}U)\\
&+ (U^T\bar{\mathbb{C}}U)^T U^T\Pi U(U^T\bar{\mathbb{C}}_0 U) + U^T\Xi U = 0,
\end{aligned} \tag{E.7}$$

where $\bar{\mathbb{A}} = A + G - B\Lambda$, $\bar{\mathbb{C}}_0 = C_0 - D_0\Lambda$ with $\Lambda \triangleq \bar{\Upsilon}^{-1}(\Psi + \Phi)$, and

$$\begin{aligned}
\Xi = &C^T PC + Q - Q_\Gamma + \Lambda^T(R + D^T PD)\Lambda\\
&- \Lambda^T D^T PC - (D^T PC)^T\Lambda.
\end{aligned}$$

Let $\hat{\Upsilon} = R + D^T PD$. Then, Ξ can be deformed into

$$\begin{aligned}
\Xi = &(\Lambda - \hat{\Upsilon}^{-1}D^T PC)^T\hat{\Upsilon}(\Lambda - \hat{\Upsilon}^{-1}D^T PC)\\
&- C^T PD\hat{\Upsilon}^{-1}D^T PC + C^T PC + Q - Q_\Gamma.
\end{aligned} \tag{E.8}$$

Note that

$$\begin{bmatrix} P & PD \\ D^T P & R + D^T PD \end{bmatrix} = \begin{bmatrix} I & 0 \\ D^T & I \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I & D \\ 0 & I \end{bmatrix}.$$

Thus, we have $\begin{bmatrix} P & PD \\ D^T P & R + D^T PD \end{bmatrix} \ge 0$. By Schur's lemma (Rami et al., 2001), $P - PD\hat{\Upsilon}^{-1}D^T P \ge 0$. This gives $C^T(P - PD\hat{\Upsilon}^{-1}D^T P)C \ge 0$. From (E.8), $\Xi \ge 0$. By Zhang et al. (2008, Theorem 3.1) and

(A6), we obtain that $(\bar{\mathbb{A}}, \bar{\mathbb{C}}_0; \Xi^{1/2})$ is exactly detectable. Denote

$$U^T\bar{\mathbb{A}}U = \begin{bmatrix} \bar{\mathbb{A}}_{11} & \bar{\mathbb{A}}_{12} \\ \bar{\mathbb{A}}_{21} & \bar{\mathbb{A}}_{22} \end{bmatrix}, \; U^T\bar{\mathbb{C}}_0U = \begin{bmatrix} \bar{\mathbb{C}}^0_{11} & \bar{\mathbb{C}}^0_{12} \\ \bar{\mathbb{C}}^0_{21} & \bar{\mathbb{C}}^0_{22} \end{bmatrix},$$
$$U^T\Xi U = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix}.$$

By pre- and post-multiplying by ξ^T and ξ where $\xi = [\xi_1^T, 0]^T$, it follows that $0 = \xi^T U^T \Xi U \xi$. From the arbitrariness of ξ_1, we obtain $\Xi_{11} = 0$. Since Ξ is positive semi-definite, $\Xi_{12} = \Xi_{21} = 0$, and $\Xi_{22} \geq 0$. By comparing each block matrix of both sides of (E.7), we obtain $\bar{\mathbb{A}}_{21} = \bar{\mathbb{C}}^0_{21} = 0$. It follows from (E.7) that

$$\Pi_2\bar{\mathbb{A}}_{22} + \bar{\mathbb{A}}_{22}^T\Pi_2 + (\bar{\mathbb{C}}^0_{22})^T\Pi_2\bar{\mathbb{C}}^0_{22} + \Xi_{22} = 0. \tag{E.9}$$

Let $\nu = [\nu_1^T, \nu_2^T]^T = U^T\bar{y}^*$, where $\bar{y}^*$ satisfies $\dot{\bar{y}}^* = \bar{\mathbb{A}}\bar{y}^*$. Then, we have

$$d\nu_1 = (\bar{\mathbb{A}}_{11}\nu_1 + \bar{\mathbb{A}}_{12}\nu_2)dt + (\bar{\mathbb{C}}^0_{11}\nu_1 + \bar{\mathbb{C}}^0_{12}\nu_2)dW_0,$$
$$d\nu_2 = \bar{\mathbb{A}}_{22}\nu_2 dt + \bar{\mathbb{C}}^0_{22}\nu_2 dW_0.$$

Note that $U^T\Xi U = \begin{bmatrix} 0 & 0 \\ 0 & \Xi_{22} \end{bmatrix}$. Then, $\Xi^{1/2}\bar{y} = \Xi^{1/2}U\nu = 0$, which together with the detectability of $[\bar{\mathbb{A}}, \bar{\mathbb{C}}_0; \Xi^{1/2}]$ implies $\nu_1 \to 0$ and $[\bar{\mathbb{A}}_{11}, \bar{\mathbb{C}}^0_{11}]$ is stable. We now prove $[\bar{\mathbb{A}}_{22}, \bar{\mathbb{C}}^0_{22}]$ is stable. Denote $S(t) = \nu_2^T\Pi_2\nu_2$. By (E.9), $S(T) - S(0) = -\int_0^T \nu_2(t)^T\Xi_{22}\nu_2(t)dt \leq 0$, which implies $\lim_{t\to\infty} S(t)$ exists. By a similar argument in Zhang et al. (2019, 2008) and Wang et al. (2020), we obtain that $\nu_2 \to 0$. This with the fact that $[\bar{\mathbb{A}}_{11}, \bar{\mathbb{C}}^0_{11}]$ is stable gives that ν is stable, which leads to the stabilizability of $[A + G, C_0; B, D_0]$. □

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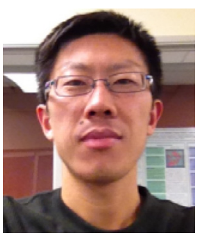

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