



A quantized output feedback MRAC scheme for discrete-time linear systems[☆]

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ABSTRACT

This paper presents a quantized output feedback model reference adaptive control (MRAC) scheme for a class of single-input and single-output discrete-time linear time-invariant systems with unknown parameters. Our method, firstly, integrates the well-known MRAC and quantized control techniques to construct a quantized output feedback adaptive control law with parameter update laws. Then, some vital technical lemmas are developed, fundamentally applicable to finite and infinite quantized output feedback MRAC. Moreover, we prove that in the case of infinite quantization, appropriately choosing the output quantizer's sensitivity affords the proposed adaptive control law to ensure closed-loop stability and achieve bounded or asymptotic output tracking. The significant advantage of the developed adaptive control scheme is combining the benefits of the classic MRAC and quantized control. Compared with current adaptive tracking control schemes, the developed scheme not only reduces the feedback information requirement, but also has full capability to achieve closed-loop stability and output tracking. The effectiveness of the proposed MRAC scheme is verified through several simulations.

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1. Introduction

Real control systems suffer from parametric, structural, and environmental uncertainties due to payload variation or system aging, component failures, and external disturbances. Moreover, due to the sensors' accuracy and measurement limitations and the computers' computational capabilities, controlling practical systems often suffers from measurement errors, saturation, and computational burden. Considering these real-world demands, the quantized feedback adaptive control technique provides a feasible solution to adaptively control uncertain systems involving quantized measurements.

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To date, controlling uncertain systems with quantized measurements has gained much research interest, with stability analysis, stabilizing control, tracking control, and their applications being extensively studied. For example, Liu et al. (2012) employed the cyclic-small-gain theorem and proposed a robust approach to solve the tracking problem of uncertain systems with finite output quantization. In De Persis and Mazenc (2010), the authors proposed a Lyapunov–Krasovskii function-based robust approach to analyze the stability of uncertain quantized time-delay nonlinear systems. Considering that adaptive control is a powerful tool to handle system uncertainties, some researchers developed various state feedback adaptive control methods to solve the stabilizing and tracking problems of uncertain control systems with quantized measurements (Liu, Li et al., 2021; Moustakis et al., 2018; Su & Chesi, 2018; Yu & Lin, 2016; Zhou et al., 2018). Since the system states may be difficult to obtain, some researchers addressed adaptive control problems for uncertain systems by using quantized output feedback (Sun et al., 2021; Wang et al., 2017; Yu & Lin, 2021; Zouari et al., 2017). The quantized control methods are exploited in several applications, such as attitude stabilization of the flexible spacecraft in Liu, Liu et al. (2021), vision-based landing control of the airplane in Sharon et al. (2010), and distributed networked control systems in Ge et al. (2017). The literature related to quantized control systems commonly proposes methods designed on the state-space

formulations. The well-known backstepping technique proposed in Krstic et al. (1995) is the main tool to design adaptive control laws for nonlinear systems with quantized measurements.

Although adaptive quantized feedback control theory and applications have progressed significantly, the quantized output feedback model reference adaptive control (MRAC) of linear time-invariant (LTI) systems covering continuous-time and discrete-time have not been studied yet. Recently, based on the classical reference control theory (Tao, 2003) and the finite quantized stabilizing control theory (Brockett & Liberzon, 2000), we developed a finite quantized output feedback model reference control scheme for a general class of minimum-phase discrete-time LTI systems in Zhang et al. (2022). However, in Zhang et al. (2022), we did not consider parameter uncertainties. Therefore a fundamental problem still remains: for a general class of minimum-phase discrete-time LTI systems with unknown parameters, whether a quantized output feedback version of the well-known MRAC law is effective or not. Essentially different from that in Zhang et al. (2022), the adaptive control for systems with unknown parameters involves the following new technical issues which are not addressed in Zhang et al. (2022):

- how to construct an adaptive control law and a parameterized tracking error model with parameter estimates and quantized error;
- how to develop adaptive parameter update laws based on a further derived error model using a new estimation signal depending on quantized error;
- how to perform closed-loop stability and output tracking analysis in the presence of the unbounded tracking error, quantized error and estimation error; and
- how to achieve global output tracking performance for systems with unknown initial conditions.

These concerns are all addressed in this work. In summary, the main contributions are as follows:

- (1) We solve the MRAC problem for a general class of minimum-phase discrete-time LTI systems using quantized output feedback. By combining the classic MRAC and quantized feedback control, the proposed scheme reduces the feedback information requirement and achieves closed-loop stability and output tracking. To the best of our knowledge, such a method has not been reported yet.
- (2) We derive some fundamental results applicable to finite and infinite quantized output feedback MRAC. We mainly address the infinite quantized output feedback MRAC with finite quantization-based results as support. For the case of infinite quantization, all parameters and signals in the developed MRAC law are analytically specified, and essential relation between the quantized and the exact output feedback MRAC is derived.
- (3) Compared with the current literature, the proposed MRAC scheme has several distinctive characteristics. First, we analytically construct an MRAC law independent of the system's initial conditions by solely employing the reference input, quantized output, and estimated parameters. Second, the proposed MRAC scheme depends on the standard design conditions, which are practically the same as the classic MRAC scheme (Tao, 2003). Third, the proposed MRAC scheme utilizes quantized output feedback without requiring the system states to be observable.

The remainder of this paper is as follows. Section 1 provides a notation description, while Section 2 introduces the controlled plant and the research problems, and reviews some fundamentals related to the output feedback model reference control, and Section 3, which is the main part of this paper, gives the design

details of the proposed quantized output feedback MRAC scheme. Section 4 presents the simulation results and, finally, Section 5 concludes this work.

Notation: In this work we use $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}^+$ to denote the sets of complex numbers, real numbers, integers, and positive integers, respectively. Furthermore, z and z^{-1} denote the forward and backward shift operators, i.e., $zx(t) = x(t+1)$ and $z^{-1}x(t) = x(t-1)$, where $t \in \{0, 1, 2, 3, \dots\}$, $x(t) \triangleq x(tT)$ for a sampling period $T > 0$, and $x(t)$ denotes any signal of any finite dimension. We also use the notation L^∞ and $[\cdot]: L^\infty$ for a signal space defined as $L^\infty = \{X(t) : \|X(\cdot)\|_\infty < \infty\}$ with $\|X(\cdot)\|_\infty \triangleq \sup_{t \geq 0} |X(t)|$ and $[\cdot]$ is defined as $[X(t)] \triangleq \max\{k \in \mathbb{Z} : k < X(t)\}$, where $X(t) \in \mathbb{R}$ denotes any signal on \mathbb{R} . Moreover, $y(t) = G(z)u(t)$ refers to the output $y(t)$ of a discrete-time LTI system represented by a transfer function $G(z)$ with an input $u(t)$. This notation is simple to combine time and z -domain signal operations, is helpful for control design and analysis purposes, avoiding causality contradiction problems and complex convolution expressions for control system presentation. A similar notation is exploited in Chen and Zhang (1990), Goodwin and Sin (2014) and Tao (2003).

2. Problem statement

This section presents the system model and the control problems addressed, and reviews some fundamentals of output feedback model reference control (MRC).

2.1. System model and control problems

System model. We consider the following single-input and single-output (SISO) LTI system:

$$A(z)y(t) = k_p B(z)u(t), \quad t \geq 0, \quad (1)$$

where $k_p \neq 0$ is the constant high-frequency gain, and $A(z)$ and $B(z)$ are monic polynomials of degrees n and m , respectively with constant coefficients, i.e.,

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

$$B(z) = z^m + b_{m-1}z^{m-1} + \dots + b_1z + b_0.$$

Without loss of generality, we assume that $A(z)$ is unstable, i.e., system (1) has at least one unstable pole. Note that $n - m$ is the input-output delay, named the system relative degree (Tao, 2003). Let $a_i, i = 0, 1, \dots, n-1$, and $b_j, j = 0, 1, \dots, m-1$, are all unknown. The values of $y(t)$ cannot be accurately measured and can only be finite quantized, denoted as $q(y(t), \Delta(t))$, where $q: \mathbb{R} \times \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is the output quantizer.

Dynamic quantizer. In this paper, we use the following type of the output quantizer:

$$q(y(t), \Delta(t)) = \begin{cases} M, & \text{if } y(t) > (M + \frac{1}{2})\Delta(t), \\ \left[\frac{y(t)}{\Delta(t)} + \frac{1}{2} \right], & \text{if } -(M + \frac{1}{2})\Delta(t) < y(t) \leq (M + \frac{1}{2})\Delta(t), \\ -M, & \text{if } y(t) \leq -(M + \frac{1}{2})\Delta(t). \end{cases} \quad (2)$$

where $\Delta(t)$ depends on t and is the sensitivity of q such that $\Delta(t) > 0$. In this paper, we will claim that $q(y(t), \Delta(t))$ saturates if $|y(t)| \geq (M + \frac{1}{2})\Delta(t)$; and $q(y(t), \Delta(t))$ does not saturate if $|y(t)| < (M + \frac{1}{2})\Delta(t)$.

This paper considers the finite and infinite quantized output cases simultaneously. For the finite quantized case, M is a positive integer, while for the infinite quantized case, $M = \infty$.

As stated in Brockett and Liberzon (2000), the quantizer (2) has some certain physical meanings and potential applications,

e.g., vision-based control. The authors in Brockett and Liberzon (2000) proposed (2) to address the stabilization problem of system (1) with known parameters. In this paper we employ (2) to address the MRAC problem for system (1) with unknowns a_i and b_j . For a complete clarification on the quantizer (2), the reader is referred to Brockett and Liberzon (2000).

Control objective. The control objective of this paper is to develop a quantized output feedback MRAC law $u(t)$ for system (1) with unknowns a_i and b_j , ensuring that the closed-loop signals are bounded, and $y(t) - y^*(t)$ converges to a specific small residual set asymptotically, where $y^*(t)$ is any given reference output signal such that $y^*(t) \in L^\infty$.

Remark 1. It is well known that the exact output feedback MRAC scheme can achieve asymptotic tracking. However, for the quantized output feedback case, the measurement error always exists and cannot be ignored. Moreover, it is unknown and may be unbounded. This leads that the quantized output feedback MRAC scheme is difficult to still achieve asymptotic tracking. In this paper, we derive a result (Corollary 1) which clarifies some specified conditions when the quantized feedback case can degrade into the exact feedback case.

Assumption. To meet the control objective, the following assumptions are required.

- (A1): The polynomial $B(z)$ is stable.
- (A2): The degree n of $A(z)$ is known.
- (A3): The relative degree $n^* = n - m$ is known.
- (A4): The sign of the high-frequency gain k_p is known, and $|k_p| \leq k_p^0$ for some known $k_p^0 > 0$.
- (A5): An upper bound on the magnitudes of $\lambda_i(A(z))$ is known with $\lambda_i(A(z))$ being the zeros of $A(z)$ on the complex z -plane.

Assumption (A1) is a consequence of zero-pole cancellations in MRAC of LTI systems. The proposed control law will cancel the zeros of the system (1) and replaces them with those of the reference model. For stability, such cancellations should occur inside the unit circle of the complex z -plane, which implies that $B(z)$ should be stable. Assumption (A2) is used for determining the parameter estimate vector's dimension and can be relaxed as: an upper bound on n is known. The reader is referred to Tao (2003) for further details. Assumption (A3) is used for choosing the reference output signal. The reference output model is

$$y^*(t) = W_m(z)r(t), \quad W_m(z) = \frac{1}{P_m(z)}, \quad (3)$$

where $P_m(z)$ is a stable polynomial of degree n^* and $r(t) \in \mathbb{R}$ is an external reference input signal such that $r(t) \in L^\infty$. For discrete-time MRAC, $P_m(z)$ is generally chosen as z^{n^*} (Ioannou & Sun, 1996; Tao, 2003), so do we in this paper. Assumption (A4) means that the control direction of system (1) is known. The literature commonly exploits the Nussbaum and other multiple-model techniques to relax the control gain sign condition (see some representative results, e.g., Chen et al. (2019), Ge and Wang (2003) and Zhao et al. (2021)). Nevertheless, this paper does not address the sign issue that requires further study. Assumption (A5) is needed for the finite quantization case and means that an upper bound on the maximum growth rate of the open-loop output is known and is used for the control law design and global convergence analysis. If it is known that the quantizer $q(\gamma(0), \Delta(0))$ does not saturate in prior, then Assumption (A5) is not needed. In the following analysis, this issue will be explained whenever necessary.

It should be noted that the zeros and poles of system (1) are allowed to be non-coprime, i.e., the state-space form of system (1)

is allowed to be unobservable. In the literature, the observable condition is commonly employed when using output feedback. However, in this work, the observable condition is not required.

This paper indicates that under the above standard assumptions, a quantized output feedback version of the standard MRAC scheme in Tao (2003) is still effective, ensuring closed-loop stability and output tracking.

2.2. Fundamentals of output feedback MRC

Now, we review some fundamentals of the output feedback MRC of discrete-time LTI systems in Tao (2003) that will be utilized in the quantized output feedback MRAC design.

Matching equation. Before introducing the MRC law, we first present the following lemma which specifies a fundamental equation for the control law design.

Lemma 2 (Tao, 2003). There exist constant vectors $\theta_1^* \in \mathbb{R}^{n-1}$ and $\theta_2^* \in \mathbb{R}^n$ such that

$$\theta_1^{*T} \omega_1(z)A(z) + k_p \theta_2^{*T} \omega_2(z)B(z) = A(z) - B(z)z^{n^*}, \quad (4)$$

where

$$\omega_1(z) = [z^{-n+1}, \dots, z^{-1}]^T, \quad \omega_2(z) = [z^{-n+1}, \dots, z^{-1}, 1]^T.$$

The proof of this lemma is presented in Tao (2003), while (4) is the well-known matching equation for output feedback MRC of LTI systems (Tao, 2003).

Output feedback MRC law. Utilizing θ_1^* and θ_2^* from (4), the MRC law is designed as

$$u(t) = \theta_1^{*T} \phi_1(t) + \theta_2^{*T} \phi_2(t) + \frac{1}{k_p} r(t), \quad t \geq 0, \quad (5)$$

with

$$\phi_1(t) = \omega_1(z)u(t), \quad \phi_2(t) = \omega_2(z)y(t). \quad (6)$$

The following lemma specifies the capability of law (5).

Lemma 3 (Tao, 2003). Unique θ_1^* and θ_2^* exist that meet (4) and ensure the MRC law (5) leads to exact output tracking

$$y(t + n^*) - y^*(t + n^*) = 0, \quad \forall t \geq 0,$$

for system (1).

The proof of Lemma 3 is presented in Tao (2003). The parameters θ_1^* and θ_2^* of Lemma 3 are named *matching parameters*, as these parameters afford the control law (6) to exactly match the closed-loop system to the reference model (3).

Remark 4. If $A(z)$ and $B(z)$ are not coprime, then θ_1^* and θ_2^* satisfying (4) are not unique. However, Lemma 3 indicates that regardless of $A(z)$ and $B(z)$ being coprime or not, the θ_1^* and θ_2^* parameters in (5) are unique. This property is proven in the proof of Theorem 6.1 in Tao (2003), and also can be concluded from the proof of Lemma 5 presented in this paper. Moreover, if the MRC law (5) uses an arbitrary pair of θ_1^* and θ_2^* satisfying (4), this leads to a tracking error of $B(z)e(t + n - m) = 0$, which cannot imply exact tracking if not all zeros of $B(z)$ are at the origin. \square

Lemmas 2–3 are the fundamentals of MRC of LTI systems and are the foundation of the quantized output feedback MRAC scheme.

3. Quantized output feedback MRAC design

This section develops a quantized output feedback MRAC scheme for system (1), where the parameters a_i , b_j and k_p are unknown.

3.1. Quantized output feedback MRAC structure

In this sub-section, we construct a basic structure of the quantized output feedback MRAC law. Then, we specify the corresponding parameter update laws, and finally, we provide a closed-loop tracking error equation.

Basic structure of adaptive control law. Motivated by (5), we design the quantized output feedback MRAC law as

$$u(t) = \begin{cases} 0, & \text{if } t \in [0, t_0), \\ \theta_1^T(t)\phi_1(t) + \theta_2^T(t)\phi_q(t) \\ + \theta_3(t)r(t), & \text{if } t \in [t_0, \infty), \end{cases} \quad (7)$$

where t_0 is the initial operation time of the adaptive control law (t_0 will be specified in Lemma 6), $\theta_1(t)$ and $\theta_2(t)$ are estimates of θ_1^* and θ_2^* in Lemma 3, respectively, $\theta_3(t)$ is an estimate of $\frac{1}{k_p}$, and

$$\phi_q(t) = \omega_2(z)(\Delta(t)q(y(t), \Delta(t))) \quad (8)$$

with $\Delta(t)$ to be designed later. Note that $\theta_i(t)$, $i = 1, 2, 3$, need to be updated by some parameter update laws. With (8), we introduce some notation that will be used next:

$$\phi(t) = [\phi_1^T(t), \phi_q^T(t), r(t)]^T \in \mathbb{R}^{2n}, \quad (9)$$

$$\theta(t) = [\theta_1^T(t), \theta_2^T(t), \theta_3(t)]^T \in \mathbb{R}^{2n}, \quad (10)$$

$$\tilde{\theta}(t) = \theta(t) - \theta^*, \quad \theta^* = [\theta_1^{*T}, \theta_2^{*T}, \frac{1}{k_p}]^T \in \mathbb{R}^{2n}. \quad (11)$$

Parameter update laws. We define a quantized output based tracking error and a quantized output based estimation error as

$$e_q(t) = \Delta(t)q(y(t), \Delta(t)) - y^*(t) \in \mathbb{R}, \quad (12)$$

$$\epsilon_q(t) = e_q(t) + \rho(t)\xi(t) \in \mathbb{R}, \quad (13)$$

respectively, where $\rho(t)$ is the estimate of k_p , and

$$\xi(t) = \theta^T(t)\phi(t - n^*) - \theta^T(t - n^*)\phi(t - n^*) \in \mathbb{R}. \quad (14)$$

Note that signals $e_q(t)$, $\epsilon_q(t)$ and $\xi(t)$ are all available at the current time instant. Thus, we employ these signals and design the update laws for $\theta(t)$ and $\rho(t)$ as

$$\theta(t+1) = \theta(t) - \frac{\text{sign}[k_p]\Gamma\epsilon_q(t)\phi(t - n^*)}{m^2(t)} + f_\theta(t), \quad (15)$$

$$\rho(t+1) = \rho(t) - \frac{\gamma\epsilon_q(t)\xi(t)}{m^2(t)} + f_\rho(t), \quad (16)$$

where Γ and γ are adaptive gains such that $\gamma \in (0, 2)$ and $\Gamma = \text{diag}\{\gamma_1, \dots, \gamma_{2n}\}$, $0 < \gamma_i < 2/k_p^0$, with $i = 1, \dots, 2n$, and k_p^0 defined in Assumption (A4),

$$m(t) = \sqrt{1 + \phi^T(t - n^*)\phi(t - n^*) + \xi^2(t)}, \quad (17)$$

and $f_\theta(t) = [f_{\theta_1}(t), \dots, f_{\theta_{2n}}(t)]^T$ and $f_\rho(t)$ are modification terms. This paper exploits the parameter projection technique to design $f_\theta(t)$ and $f_\rho(t)$, for which the knowledge of a convex set containing θ^* and k_p is needed. Thus, we make the following assumption:

(A6) The lower bound θ_i^a and upper bound θ_i^b of the i th component θ_i^* of θ^* , $i = 1, 2, \dots, 2n$, and those of k_p , denoted as ρ^a and ρ^b , are known.

This assumption is standard for a robust adaptive control design (Tao, 2003, Ioannou and Sun (1996)). Given Assumption (A6), we design $f_\theta(t)$ and $f_\rho(t)$ as

$$f_{\theta_i}(t) = \begin{cases} 0, & \text{if } \theta_i(t) \in [\theta_i^a, \theta_i^b], \\ \theta_i^b - \theta_i(t) - p_{\theta_i}(t), & \text{if } \theta_i(t) + p_i(t) > \theta_i^b, \\ \theta_i^a - \theta_i(t) - p_{\theta_i}(t), & \text{if } \theta_i(t) + p_i(t) < \theta_i^a, \end{cases} \quad (18)$$

$$f_\rho(t) = \begin{cases} 0, & \text{if } \rho(t) \in [\rho^a, \rho^b], \\ \rho^b - \rho(t) - p_\rho(t), & \text{if } \rho(t) + p_\rho(t) > \rho^b, \\ \rho^a - \rho(t) - p_\rho(t), & \text{if } \rho(t) + p_\rho(t) < \rho^a, \end{cases} \quad (19)$$

where $p_{\theta_i}(t)$ is the i th component of $p_\theta(t)$ with $p_\theta(t) = -\frac{\text{sign}[k_p]\Gamma\epsilon_q(t)\phi(t - n^*)}{m^2(t)}$ and $p_\rho(t) = -\frac{\gamma\epsilon_q(t)\xi(t)}{m^2(t)}$.

Quantized output-based tracking error equation. Let the quantized error and the tracking error be

$$s(y(t), \Delta(t)) = \Delta(t)q(y(t), \Delta(t)) - y(t), \quad (20)$$

$$e(t) = y(t) - y^*(t), \quad (21)$$

respectively. Since $y(t)$ is not available, the two error signals $s(y(t), \Delta(t))$ and $e(t)$ are not available.

The following lemma specifies a critical relationship between the tracking, parameter estimation, and quantized errors, which is crucial for designing $\Delta(t)$ and performing stability analysis.

Lemma 5. *The quantized output feedback MRAC law (7), applied to system (1), provides the tracking error equation as*

$$e(t + n^*) = k_p \tilde{\theta}^T(t)\phi(t) + d_0(t), \quad t \geq t_0, \quad (22)$$

where $\phi(t)$ and $\tilde{\theta}(t)$ are defined in (9)–(11), respectively, and

$$d_0(t) = k_p \tilde{\theta}_2^{*T} \omega_2(z)s(y(t), \Delta(t)).$$

The proof of Lemma 5 is given in the Appendix. This equation implies $e(t + n^*) = 0$ if $\theta(t) = \theta^*$ and $s(y(t), \Delta(t)) = 0$. However, since the parameter update law (15) does not rely on persistent excitation condition or exact feedback, the convergence of $\theta(t)$ cannot be guaranteed. Moreover, the boundedness of $s(y(t), \Delta(t))$ in (20) also cannot be guaranteed for the finite quantized feedback case.

3.2. Technical lemmas

This sub-section derives some technical lemmas that are crucial for the stability and output tracking analysis. The reading flow is optimized by presenting some long proofs in the Appendix. We define

$$\tilde{\phi}(t) = [\phi_1^T(t), \phi_q^T(t)]^T, \quad (23)$$

$$\lambda = \lambda_0 + \text{an upper bound of } \max\{\text{magnitudes of } \lambda_i(A(z))\},$$

where $\lambda_0 > 0$ is a constant. Given Assumption (A5), we see that λ is available for the adaptive control design.

The following lemmas will sequentially specify a key time instant of the quantizer being non-saturated, an estimation error $\epsilon_q(t)$ expression, the parameter update laws capabilities, two key characteristics of $\tilde{\phi}(t)$, and a property of the quantized error $s(y(t), \Delta(t))$.

Lemma 6. *Given Assumption (A5), if $u(t) = 0$ and $\Delta(t) = c_0 \lambda^{kt}$ with $c_0 > 0$ and $k \geq 1$ being any two constants, then there always exists a well-defined number t_0 as*

$$t_0 \triangleq \min \{t \geq 1 : |q(y(t), \Delta(t))| \leq M - 1\}.$$

Proof. If $u(t) = 0$, $y(t)$ grows at most exponentially and $|y(t)| \leq k_0(\lambda - \lambda_0)^t$ with k_0 being some constant. If $\Delta(t) = c_0 \lambda^{kt}$ with $c_0 > 0$ and $k \geq 1$, $\Delta(t)$ grows faster than $|y(t)|$. Thus, no matter whether $q(y(0), \Delta(0))$ saturates or not, there exists some finite time instant t_s such that $q(y(t), \Delta(t))$ will never saturate for all $t \geq t_s$. Then, the lemma's result follows the definition of $q(y(t), \Delta(t))$ in (2). \square

Remark 7. Since it is unknown whether $q(y(0), \Delta(0))$ is saturated or not, we introduce t_0 . Lemma 6 shows that regardless $q(y(0), \Delta(0))$ is saturated or not, t_0 exists and $q(y(t_0), \Delta(t_0))$ is not saturated. If $M = \infty$ or given that $q(y(0), \Delta(0))$ is not saturated in prior, then t_0 will be zero and Assumption (A5) is no longer required. \square

Lemma 8. The estimation error $\epsilon_q(t)$ in (13) satisfies

$$\epsilon_q(t) = k_p \tilde{\rho}^T(t) \phi(t - n^*) + \tilde{\rho}(t) \xi(t) + d_1(t), \tag{24}$$

where $\tilde{\rho}(t) = \rho(t) - k_p$ and

$$d_1(t) = (1 + k_p \theta_2^{*T} \omega_2(z) z^{-n^*}) s(y(t), \Delta(t)). \tag{25}$$

Proof. From (12)–(14), we have

$$\begin{aligned} \epsilon_q(t) &= \Delta(t) q(y(t), \Delta(t)) - y^*(t) + \tilde{\rho}(t) \xi(t) + k_p \xi(t) \\ &= s(y(t), \Delta(t)) + e(t) + \tilde{\rho}(t) \xi(t) + k_p \xi(t). \end{aligned}$$

Then, based on (14) and (22), we have

$$\begin{aligned} \epsilon_q(t) &= k_p (\theta^T(t - n^*) - \theta^*) \phi(t - n^*) + \tilde{\rho}(t) \xi(t) \\ &\quad + k_p (\theta^T(t) \phi(t - n^*) - \theta^T(t - n^*) \phi(t - n^*)) \\ &\quad + d_0(t - n^*) + s(y(t), \Delta(t)) \end{aligned} \tag{26}$$

which derives the lemma's result. \square

Lemma 9. The parameter update laws (15)–(16) ensure that $\theta(t) \in L^\infty$, $\rho(t) \in L^\infty$, $\frac{\epsilon_q(t)}{m(t)} \in L^\infty$, and

$$\sum_{t=t_1}^{t_2} \frac{\epsilon_q^2(t)}{m^2(t)} \leq \alpha_1 + \beta_1 \sum_{t=t_1}^{t_2} \frac{d_1^2(t)}{m^2(t)}, \tag{27}$$

$$\sum_{t=t_1}^{t_2} \|\theta(t + t_0) - \theta(t)\|_2^2 \leq \alpha_2 + \beta_2 \sum_{t=t_1}^{t_2} \frac{d_1^2(t)}{m^2(t)}, \tag{28}$$

for all $t_2 > t_1 \geq 0$, any finite $t_0 > 0$, and some constants $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$, with $d_1(t)$ in (25).

The proof of Lemma 9 is given in the Appendix.

Remark 10. For the finite quantized feedback case, the modification terms $f_\theta(t)$ and $f_\rho(t)$ in (15)–(16) are necessary to ensure the boundedness of $\theta(t)$ and $\rho(t)$. This is because the finite quantized output feedback leads to the unboundedness of the quantized error, and the unbounded quantized error may lead to the unboundedness of parameter estimates. However, for the infinite quantized output feedback case, the quantized error is bounded, and the modification terms can be eliminated from (15)–(16). In other words, for the case of $M = \infty$, Assumption (A6) is no longer required. \square

Lemma 11. The signal $\bar{\phi}(t)$ satisfies

$$\bar{\phi}(t + 1) = A^* \bar{\phi}(t) + b^*(y(t + n^*) - d_0(t)) + d_2(t) \tag{29}$$

for A^* being stable, where

$$d_2(t) = [0, \dots, 0, z^{-n+1} A(z) s(y(t), \Delta(t))]^T \in \mathbb{R}^{2n-1},$$

$$A^* = \begin{bmatrix} E_{n-2} & 0_{(n-2) \times n} \\ \theta_1^{*T} & \theta_2^{*T} \\ 0_{(n-1) \times (n-1)} & E_{n-1} \\ A_1^{*T} & A_2^{*T} \end{bmatrix} \in \mathbb{R}^{(2n-1) \times (2n-1)},$$

$$b^* = \begin{bmatrix} b_1^* \\ b_2^* \end{bmatrix} \in \mathbb{R}^{2n-1}, \quad b_1^* = \begin{bmatrix} 0, \dots, 0, \frac{1}{k_p} \end{bmatrix}^T \in \mathbb{R}^{n-1},$$

$$E_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{i \times (i+1)}, \quad i = n - 1, n,$$

$$\{A_1^{*T} \in \mathbb{R}^{n-1}, A_2^{*T} \in \mathbb{R}^n, b_2^* \in \mathbb{R}^n\}$$

$$= \begin{cases} \{[k_p z^{*T}, k_p, 0, \dots, 0], -p^{*T}, [0, \dots, 0]\} & \text{if } n^* > 1, \\ \{k_p z^{*T} + k_p \theta_1^{*T}, -p^{*T} + k_p \theta_2^{*T}, [0, \dots, 0, 1]\} & \text{if } n^* = 1, \end{cases}$$

with $p^* = [a_0, \dots, a_{n-1}]^T \in \mathbb{R}^n$ and $z^* = [b_0, \dots, b_{m-1}]^T \in \mathbb{R}^m$.

The proof of Lemma 11 is provided in the Appendix. Since A^* is stable, there exists a nonsingular matrix $G^* \in \mathbb{C}^{(2n-1) \times (2n-1)}$ such that $\|G^* A^* G^{*-1}\|_2 < 1$, where $\|\cdot\|_2$ denotes the induced Euclidean matrix norm, i.e., for a given vector norm $\|\cdot\|$ on \mathbb{C}^{2n-1} and any matrix $Q \in \mathbb{C}^{(2n-1) \times (2n-1)}$, $\|Q\|_2 = \sup_{\|x\|=1} \|Qx\|$. We introduce G^* because the vector norm $\|\cdot\|$ on \mathbb{R}^{2n-1} used next is defined as $\|x\| = \|G^* x\|_2$ for any $x \in \mathbb{R}^{2n-1}$.

Lemma 12. The signal $\bar{\phi}(t)$ satisfies

$$\|\bar{\phi}(t + 1)\| \leq (c_0 + c_1 \omega(t)) \|\bar{\phi}(t)\| + \|d_4(t)\| \tag{30}$$

for some constants $c_0 \in (0, 1)$, $c_1 > 0$, and

$$\omega(t) = \left| \frac{\epsilon_q(t + n^*)}{m(t + n^*)} \right| + \|\theta(t + n^*) - \theta(t)\|_2, \tag{31}$$

$$\begin{aligned} d_4(t) &= (-b^*(1 + k_p \theta_2^{*T} \omega_2(z)) + [0, \dots, 0, z^{-n+1} A(z)]^T) \\ &\quad \cdot s(y(t), \Delta(t)) + b^* y^*(t + n^*). \end{aligned}$$

Proof. With (13), (14), and (21), it follows from (29) that

$$\begin{aligned} \bar{\phi}(t + 1) &= A^* \bar{\phi}(t) + b^*(y^*(t + n^*) - d_0(t) + \epsilon_q(t + n^*) \\ &\quad - \rho(t + n^*)(\theta(t + n^*) - \theta(t))^T \phi(t)) + d_3(t), \end{aligned} \tag{32}$$

where $d_3(t) = (-b^* + [0, \dots, 0, z^{-n+1} A(z)]^T) s(y(t), \Delta(t))$.

Then, with the definitions of $\phi(t)$, $\bar{\phi}(t)$, and $\omega(t)$ in (9), (23) and (31) and in addition to the boundedness of $y^*(t)$, we derive from (31) and (32) that (30) holds. \square

Lemma 13. If $q(y(t), \Delta(t))$ is not saturated, then

$$|s(y(t), \Delta(t))| \leq \frac{1}{2} \Delta(t). \tag{33}$$

Proof. If $q(y(t), \Delta(t))$ is not saturated, it derives from (2) that $q(y(t), \Delta(t)) = \left[\frac{y(t)}{\Delta(t)} + \frac{1}{2} \right]$. Then, from the definition of $[\cdot]$ in the last paragraph of Section 1, we have $\frac{y(t)}{\Delta(t)} - \frac{1}{2} \leq q(y(t), \Delta(t)) \leq \frac{y(t)}{\Delta(t)} + \frac{1}{2}$ which follows (33). \square

All the lemmas of this paper apply to finite and infinite quantized output feedback cases. Next, we will use these lemmas to analyze the closed-loop performance.

3.3. Stability and output tracking analysis

This sub-section analyzes the system's performance and discusses the finite and infinite quantized cases.

Finite quantized output feedback case. Based on the technical lemmas presented above, we provide the following:

Theorem 14. Given Assumptions (A1)–(A6), if $\Delta(t)$ is chosen such that $q(y(t), \Delta(t))$ is not saturated for all $t \geq t_0$ and $\Delta(t)$ grows at most exponentially with $\Delta(t) = c_0 \lambda^{kt}$, $t < t_0$, then the adaptive

control law (7) with parameter update laws (15)–(16), applied to system (1) with unknown a_i, b_j and k_p , ensures that $\phi(t) \in L^\infty$ and

$$\sum_{t=t_1}^{t_2} (y(t) - y^*(t))^2 \leq \mu_1 + \mu_2 \sum_{t=t_1}^{t_2} \Delta^2(t) \quad (34)$$

for all $t_2 > t_1 \geq t_0 + n + n^*$ and some constants $\mu_i > 0, i = 1, 2$.

The proof of Theorem 14 is given in the Appendix. Theorem 14 is fundamental for the quantized output feedback MRAC of uncertain LTI systems and reveals a relationship between the classic MRAC and the finite quantized output feedback control. The closed-loop stability is not clarified in Theorem 14, and may be solved based on the well-known saturation techniques of LTI systems under some additional conditions for saturation designs. This problem is rather complicated requiring further study.

Next, based on Theorem 14, we derive some more specific results for the infinite quantization case.

Infinite quantized output feedback case. For infinite quantization, i.e., $M = \infty$, the choice of $\Delta(t)$ in Theorem 14 motivates us to design $\Delta(t)$ as $\sigma(t)$ such that $\sigma(t) \in (0, 1)$ is a designed signal that may be arbitrary. As mentioned in Remarks 7 and 10, Assumptions (A5) and (A6) are no longer required for the infinite quantized output feedback case.

Theorem 15. Given the Assumptions (A1)–(A4), if $M = \infty$ and $\Delta(t)$ is chosen as $\sigma(t)$, then the adaptive control law (7) with parameter update laws (15)–(16), applied to system (1) with unknown a_i, b_j and k_p , ensures all closed-loop signals are bounded and

$$\sum_{t=t_1}^{t_2} (y(t) - y^*(t))^2 \leq \mu_3 + \mu_4 \sum_{t=t_1}^{t_2} \Delta^2(t) \quad (35)$$

for all $t_2 > t_1 \geq n + n^*$ and some constants $\mu_i > 0, i = 3, 4$.

Proof. If $M = \infty$, then (2) becomes $q(y(t), \Delta(t)) = \left[\frac{y(t)}{\Delta(t)} + \frac{1}{2} \right]$, $\forall y \in \mathbb{R}$. In this case, regardless of the sensitivity $\Delta(t)$ changes, $q(y(t), \Delta(t))$ is always not saturated. In other words, the inequality $|s(y(t), \Delta(t))| \leq \frac{1}{2} \Delta(t)$ always holds. Especially, for the particular choice of $\Delta(t)$, the derivations in the proof of Theorem 14 are applicable to perform the proof of Theorem 15. Moreover, the boundedness of $y(t)$ can be derived from (34). From (7), we derive that $u(t)$ is bounded and thus, all closed-loop signals are bounded. The conclusion of Theorem 15 results from the above analysis. \square

Theorem 15 specifies an analytical solution to the MRAC problem for system (1) utilizing a quantized output feedback. Specifically, all signals and parameters in the adaptive control law (7) and the update laws (15)–(16) are specified.

Considering that signal $\sigma(t)$ can be designed arbitrarily, based on Theorem 15 we obtain the following:

Corollary 1. Given Assumptions (A1)–(A4), if $M = \infty$ and $\Delta(t)$ is chosen as $\sigma(t)$ with $\sum_{t=t_3}^{\infty} \sigma(t) < \infty$ for some $t_3 \geq 0$, then the adaptive control law (7) with parameter update laws (15)–(16), applied to system (1) with unknown a_i, b_j and k_p , ensures all closed-loop signals are bounded and asymptotic output tracking:

$$\lim_{t \rightarrow \infty} (y(t) - y^*(t)) = 0.$$

Proof. Based on the property $\sum_{t=t_3}^{\infty} \sigma(t) < \infty$ for a finite $t_3 \geq 0$, it is straightforward to obtain $\sum_{t=t_1}^{\infty} (y(t) - y^*(t))^2 < \infty$. Thus, we derive that $\lim_{t \rightarrow \infty} (y(t) - y^*(t)) = 0$. The closed-loop stability analysis can be performed based on the proof of Theorem 15. \square

Corollary 1 reveals an essential relationship between the infinite quantized output feedback MRAC and the exact output feedback MRAC. In fact, the condition $\sum_{t=t_3}^{\infty} \sigma(t) < \infty$ implies that $\lim_{t \rightarrow \infty} \sigma(t) = 0$ and $\lim_{t \rightarrow \infty} \Delta(t) = 0$. Then, it follows from Lemma 13 that $\sum_{t=t_3}^{\infty} \sigma(t) < \infty$ implies the convergence of the quantized error $s(y(t), \Delta(t))$. One can verify that, with the quantized error $s(y(t), \Delta(t))$ being zero, Corollary 1 degrades into the classic MRAC result of system (1) with an exact output feedback.

Although it may not be realistic to let sensitivity $\Delta(t)$ decay to zero, Corollary 1 still indicates that the tracking error $e(t)$ for system (1) with quantized output feedback can converge to an arbitrarily small residual set of the origin by appropriately adjusting sensitivity $\Delta(t)$.

So far, we have given a positive answer to the question: a quantized output feedback version of the classic MRAC scheme is still effective or not? Specifically, several adaptive control methods under different design conditions have been formulated.

4. Simulation study

This section provides a representative example to illustrate the design procedure and validity of the theoretical results.

Simulation model. Consider the following system

$$A_s(z)y(t) = k_{ps}B_s(z)u(t), \quad (36)$$

where $k_{ps} = 1$, and

$$A_s(z) = (z + 1)(z - 2) \left(z + \frac{1}{2} \right), \quad (37)$$

$$B_s(z) = z \left(z + \frac{1}{2} \right). \quad (38)$$

It follows from (37) and (38) that $A_s(z)$ is unstable and $B_s(z)$ is stable. Moreover, $A_s(z)$ and $B_s(z)$ have a common factor $z + \frac{1}{2}$ which corresponds to the uncontrollable or unobservable mode of the state-space form of (36). In a word, the controlled plant is minimum-phase, and its state-space form is allowed to be unobservable or uncontrollable.

Specification of θ_1^*, θ_2^* and $y^*(t)$. Based on the system parameters in (37) and (38) and following the procedure of deriving θ_1^* and θ_2^* for standard output feedback MRC law of general LTI systems in Tao (2003), we calculate θ_1^* and θ_2^* as

$$\theta_1^* = \left[0, -\frac{1}{2} \right]^T, \quad \theta_2^* = \left[-1, -\frac{5}{2}, -\frac{1}{2} \right]^T. \quad (39)$$

Moreover, $\phi_1(t)$ and $\phi_2(t)$ are specified as

$$\phi_1(t) = \omega_1(z)u(t), \quad \omega_1(z) = [z^{-2}, z^{-1}]^T, \quad (40)$$

$$\phi_2(t) = \omega_2(z)y(t), \quad \omega_2(z) = [z^{-2}, z^{-1}, 1]^T. \quad (41)$$

The reference output signal is chosen as

$$y^*(t) = r(t - 1) = \frac{9}{2} \sin(t) - \frac{3}{2} \cos(0.5t).$$

One can verify that the exact output tracking can be achieved by applying the standard MRC law (5) with θ_1^* and θ_2^* in (39) and $\phi_1(t)$ and $\phi_2(t)$ in (40)–(41) to the simulation model (36). The parameters θ_1^* and θ_2^* , in addition to k_{ps} , are employed to describe the simulation model and the adaptive control law to be designed will only use their estimates.

Quantized output feedback MRAC law. The quantizer $q(y(t), \Delta(t))$ used in the simulation is in the form of (2) with $M = \infty$. In this case, $t_0 = 0$, and the signal $\sigma(t)$ is chosen as $\sigma(t) = \frac{1}{100} + \frac{1}{2+t}$.

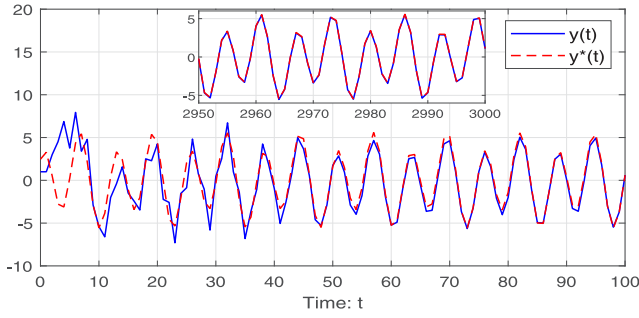


Fig. 1. Trajectories of $y(t)$ and $y^*(t)$.

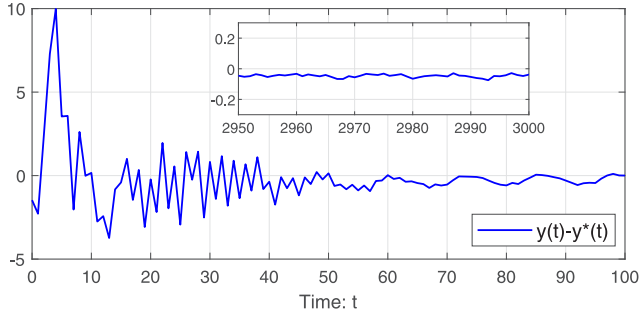


Fig. 2. Trajectory of the tracking error $y(t) - y^*(t)$.

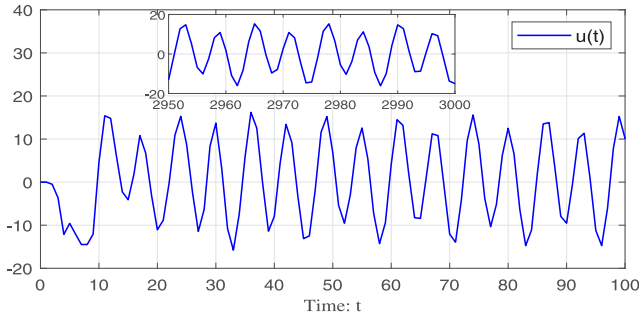


Fig. 3. Trajectory of the adaptive control law (42).

Based on (7), we define $\theta_i(t)$ as the estimates of θ_i^* with $i = 1, 2, 3$ and $\theta_3^* = 1/k_{ps} = 1$. They are in the form of

$$\theta_1(t) = [\theta_{11}(t), \theta_{12}(t)]^T \in \mathbb{R}^2,$$

$$\theta_2(t) = [\theta_{21}(t), \theta_{22}(t), \theta_{23}(t)]^T \in \mathbb{R}^3,$$

$$\theta_3(t) \in \mathbb{R}.$$

Thus, the adaptive control law for system (36) is designed as

$$\begin{aligned} u(t) = & \theta_{11}(t)u(t-2) + \theta_{12}(t)u(t-1) + \theta_3(t)y^*(t+1) \\ & + \theta_{23}(t)\Delta(t)q(y(t), \Delta(t)) \\ & + \theta_{22}(t)\Delta(t-1)q(y(t-1), \Delta(t-1)) \\ & + \theta_{21}(t)\Delta(t-2)q(y(t-2), \Delta(t-2)), \end{aligned} \quad (42)$$

for all $t \geq 0$. Note that $u(-2), u(-1), y(-2), y(-1)$ are all set to zero. The trajectory of the adaptive control law is shown in Fig. 3.

Parameter update laws. Let $\rho(t)$ be the estimate of k_{ps} . Define

$$\begin{aligned} \theta(t) = & [\theta_{11}(t), \theta_{12}(t), \theta_{21}(t), \theta_{22}(t), \theta_{23}(t), \theta_3(t)]^T \in \mathbb{R}^6, \\ \phi(t) = & [u(t-2), u(t-1), \Delta(t-2)q(y(t-2), \Delta(t-2)), \\ & \Delta(t-1)q(y(t-1), \Delta(t-1)), \Delta(t)q(y(t), \Delta(t)), \\ & y^*(t+1)]^T \in \mathbb{R}^6, \end{aligned}$$

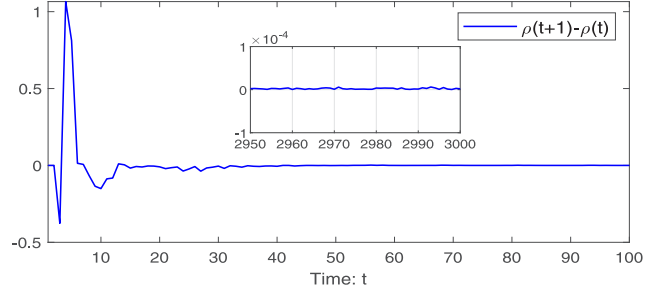
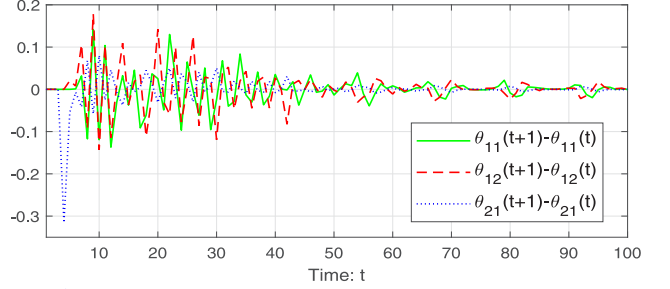
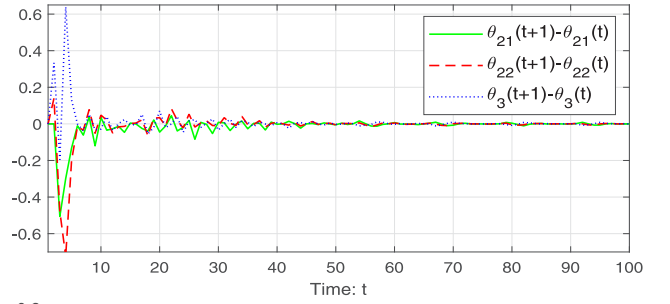


Fig. 4. Trajectories of $\theta(t+1) - \theta(t)$ and $\rho(t+1) - \rho(t)$.

$$e_q(t) = \Delta(t)q(y(t), \Delta(t)) - y^*(t) \in \mathbb{R},$$

$$\epsilon_q(t) = e_q(t) + \rho(t)\xi(t) \in \mathbb{R},$$

$$\xi(t) = \theta^T(t)\phi(t-1) - \theta^T(t-1)\phi(t-1) \in \mathbb{R},$$

$$m(t) = \sqrt{1 + \phi^T(t-1)\phi(t-1) + \xi^2(t)} \in \mathbb{R}.$$

To update $\theta_i(t)$ in (42) and the introduced signal $\rho(t)$, we design the parameter update laws as

$$\theta(t+1) = \theta(t) - \frac{\Gamma \epsilon_q(t)\phi(t-n^*)}{m^2(t)}, \quad (43)$$

$$\rho(t+1) = \rho(t) - \frac{\gamma \epsilon_q(t)\xi(t)}{m^2(t)}, \quad (44)$$

where $\Gamma = 0.7I$ and $\gamma = 1.9$.

Simulation figures. By setting the initial values of $\theta(t)$ and $\rho(t)$ as zero, we apply the quantized output feedback MRAC law (42) with the update laws (43) and (44) to the simulation model (36), and derive the following system response.

Fig. 1 presents the response of $y(t)$ of system (36) vs. the reference output $y^*(t)$. Fig. 2 gives the trajectory of the tracking error $e(t)$. From the two figures, we observe that the output tracking error $y(t) - y^*(t)$ may not converge to zero asymptotically, due to the existence of the quantized error. However, the figures highlight that the tracking error converges to a small residual set of the origin, indicating that the convergence of $y(t) - y^*(t)$ shown in Figs. 1 and 2 is consistent with the theoretical result (35) in Theorem 15.

To verify some key properties of the parameter estimates shown in Lemma 8, we present Figs. 4 and 5. Specifically, Fig. 4 shows the response of the parameter adaption of $\theta(t)$ and $\rho(t)$,

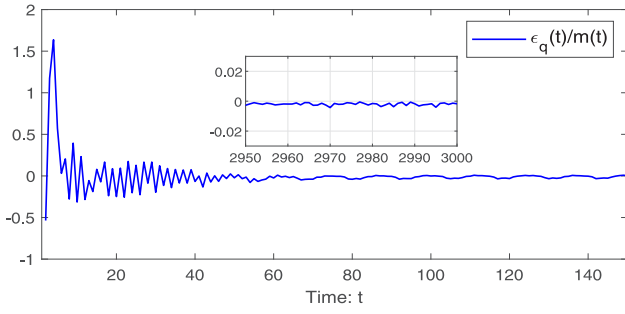


Fig. 5. Trajectory of $\epsilon_q(t)/m(t)$.

and Fig. 5 gives the response of the auxiliary signal $\epsilon_q(t)/m(t)$. Similar to the output tracking case, from Figs. 4 and 5, we see that the signals $\theta(t+1) - \theta(t)$ and $\rho(t+1) - \rho(t)$ may not converge to zero asymptotically. This is because the quantized error is always non-zero, even for the case when t goes to infinity. However, the signal trajectories shown in the two figures verify the theoretical results (27) and (28) in Lemma 9.

In summary, the simulation study verifies the validity of the proposed quantized output feedback MRAC scheme. Particularly, recalling that $A_s(z)$ and $B_s(z)$ in the model (36) are not coprime, the simulation also verifies the non-dependence of the proposed method on the coprime condition of zero and pole polynomials.

5. Concluding remarks

This paper provides a positive answer to the fundamental issue: whether a quantized output feedback version of the classic MRAC scheme still works or not for a general class of minimum-phase discrete-time LTI systems with unknown parameters. Moreover, we establish a basic quantized output feedback MRAC framework where the finite and infinite quantized output feedback cases are both addressed. Especially for the infinite quantized case, an analytical MRAC solution by using quantized output feedback is developed to ensure closed-loop stability and bounded output tracking.

Future work would be appealing to address the following questions: (i) how to integrate the saturation control technique into the proposed adaptive control scheme to solve the closed-loop stability problem that is still not solved in finite quantized output feedback case? (ii) whether a quantized output feedback version of the classical pole placement based adaptive control scheme still works for a general class of discrete-time LTI systems with unstable zeros and poles?

Appendix. Proofs of lemmas and theorems

Proof of Lemma 5. From (4), we have

$$(\theta_1^{*T} \omega_1(z) - 1)A(z) = (-k_p \theta_2^{*T} \omega_2(z) - z^{n^*})B(z). \quad (A.1)$$

We first consider the case when $A(z)$ and $B(z)$ are coprime. It follows from (A.1) that, if z_i is a zero of $B(z)$, it must be a zero of $\theta_1^{*T} \omega_1(z) - 1$, otherwise (A.1) does not hold for $z = z_i$ with $B(z)$ and $A(z)$ coprime. Thus, we conclude that there exists a polynomial $F(z) = -z^m + f_{n^*-2}z^{m-1} + \dots + f_0z^{-n+1}$ with $f_i, i = 0, \dots, n^* - 2$, being constants such that

$$F(z)B(z) = \theta_1^{*T} \omega_1(z) - 1. \quad (A.2)$$

Together with (A.1), we obtain

$$k_p \theta_2^{*T} \omega_2(z) + F(z)A(z) = -z^{n^*}. \quad (A.3)$$

Then, operating both sides of (A.3) on $y(t)$ yields $k_p \theta_2^{*T} \omega_2(z)y(t) + F(z)A(z)y(t) = -y(t + n^*)$. In addition to (1) and (6), it leads to

$$k_p \theta_2^{*T} \phi_2(t) + k_p F(z)B(z)u(t) = -y(t + n^*). \quad (A.4)$$

Substituting (A.2) to (A.4) derives

$$k_p \theta_2^{*T} \phi_2(t) + k_p (\theta_1^{*T} \omega_1(z) - 1)u(t) = -y(t + n^*).$$

Using (6) again, we get

$$k_p \theta_2^{*T} \phi_2(t) + k_p \theta_1^{*T} \phi_1(t) - k_p u(t) = -y(t + n^*). \quad (A.5)$$

Substituting the quantized output feedback law (7) to (A.5), we obtain

$$\begin{aligned} & k_p \theta_2^{*T} \phi_2(t) + k_p \theta_1^{*T} \phi_1(t) - k_p \theta_2^T(t) \phi_q(t) \\ & - k_p \theta_1^T(t) \phi_1(t) - k_p \theta_3(t) r(t) = -y(t + n^*). \end{aligned} \quad (A.6)$$

Note that $\theta_3^* = \frac{1}{k_p}$ and $r(t) = y^*(t + n^*)$. Thus, adding $r(t) - k_p \theta_2(t) \phi_2(t)$ to both sides of (A.6), we derive (22).

Considering the case when $A(z)$ and $B(z)$ are not coprime, we rewrite $B(z)$ as $B(z) = B_1(z)B_2(z)$ such that $B_1(z)$ has degree n_1 , and $B_2(z)$ and $A(z)$ are coprime. Then, there exist unique $\bar{\theta}_1^* \in \mathbb{R}^{n-1-n_1}$ and $\theta_2^* \in \mathbb{R}^n$ such that

$$\begin{aligned} & z^{n-1-n_1} \bar{\theta}_1^{*T} \bar{\omega}_1(z)A(z) + k_p \theta_2^{*T} z^{n-1} \omega_2(z)B_2(z) \\ & = z^{n-1-n_1} A(z) - z^{n-1} B_2(z) z^{n^*} \end{aligned} \quad (A.7)$$

with $\bar{\omega}_1(z) = [z^{-n+n_1+1}, \dots, z^{-1}]^T$. With some manipulations, (A.7) becomes

$$\begin{aligned} & z^{-n_1} \bar{\theta}_1^{*T} \bar{\omega}_1(z)A(z) + k_p \theta_2^{*T} \omega_2(z)B_2(z) \\ & = z^{-n_1} A(z) - B_2(z) z^{n^*}. \end{aligned} \quad (A.8)$$

Similar to (A.2), there exists some polynomial of the form $\bar{F}(z) = -z^{-m+n_1} + \bar{f}_{n^*+n_1-2}z^{-m+n_1-1} + \dots + \bar{f}_0z^{-n_1+1}$ such that $\bar{F}(z)B_2(z) = \bar{\theta}_1^{*T} \bar{\omega}_1(z) - 1$. Let $F(z) = z^{-n_1} \bar{F}(z)$. Then, in addition to (A.8), we also obtain (A.3), based on which the lemma's result follows. Note that, for the non-coprime case, the parameter θ_1^* is uniquely determined from the equation $\theta_1^{*T} \omega_1(z) - 1 = (\theta_1^{*T} \omega_1(z) - 1)B_1(z)z^{-n_1} = F(z)B(z)$. \square

Proof of Lemma 9. With the modification terms $f_{\theta_i}(t)$ and $f_{\rho}(t)$ in (18) and (19), one can verify that the parameter estimates $\theta(t) \in L^\infty$ and $\rho(t) \in L^\infty$.

From the definitions of $\phi(t)$ in (9), $\epsilon_q(t)$ in (13), and $m(t)$ in (17), one can also verify that $\frac{\epsilon_q(t)}{m(t)} \in L^\infty$.

Now, we show that the inequalities (27) and (28) hold. We introduce a positive definite function $V(\tilde{\theta}, \tilde{\rho}) = |k_p| \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \gamma^{-1} \tilde{\rho}^2$. Then, with (24), we obtain

$$\begin{aligned} & V(\tilde{\theta}(t+1), \tilde{\rho}(t+1)) - V(\tilde{\theta}(t), \tilde{\rho}(t)) \\ & = (\tilde{\theta}(t) - \frac{\text{sign}[k_p] \Gamma \epsilon_q(t) \phi(t - n^*)}{m^2(t)} + f_{\theta}(t))^T \Gamma^{-1} (\tilde{\theta}(t) - \frac{\text{sign}[k_p] \Gamma \epsilon_q(t) \phi(t - n^*)}{m^2(t)} + f_{\theta}(t)) \\ & \quad - \tilde{\theta}^T(t) \Gamma^{-1} \tilde{\theta}(t) \\ & \quad + \gamma^{-1} \left[\left(\rho(t) - \frac{\gamma \epsilon_q(t) \xi(t)}{m^2(t)} + f_{\rho}(t) \right)^2 - \rho^2(t) \right] \\ & = - \left(2 - \frac{|k_p| \phi^T(t - n^*) \Gamma \phi(t - n^*) + \gamma \xi^2(t)}{m^2(t)} \right) \frac{\epsilon_q^2(t)}{m^2(t)} \\ & \quad + 2|k_p| f_{\theta}^T(t) \Gamma^{-1} (\tilde{\theta}(t) + p_{\theta}(t) + f_{\theta}(t)) \\ & \quad + 2\gamma^{-1} f_{\rho}(t) (\tilde{\rho}(t) + p_{\rho}(t) + f_{\rho}(t)) - \gamma^{-1} f_{\rho}^2(t) \\ & \quad - |k_p| f_{\theta}^T(t) \Gamma^{-1} f_{\theta}(t) + \frac{2\epsilon_q(t) d_1(t)}{m^2(t)}. \end{aligned} \quad (A.9)$$

Let $\bar{\gamma} = \max\{|k_p^0|\gamma_1, \dots, |k_p^0|\gamma_{2n}, \gamma\}$. Then, based on the definition of Γ below (16), we obtain that $\bar{\gamma} < 2$. Together with the definition of $m(t)$ in (17), we have $\frac{|k_p|\phi^T(t-n^*)\Gamma\phi(t-n^*)+\gamma\xi^2(t)}{m^2(t)} \leq \bar{\gamma} < 2$.

With the definition of Γ below (16), we have

$$2|k_p|f_\theta^T(t)\Gamma^{-1}(\tilde{\theta}(t) + p_\theta(t) + f_\theta(t)) = \sum_{i=1}^{2n} 2\gamma_i^{-1}|k_p|f_{\theta_i}(t)(\tilde{\theta}_i(t) + p_{\theta_i}(t) + f_{\theta_i}(t))$$

which follows from (18) that

$$2|k_p|f_\theta^T(t)\Gamma^{-1}(\tilde{\theta}(t) + p_\theta(t) + f_\theta(t)) \leq 0.$$

Similarly, with (19), one can also verify that

$$2\gamma^{-1}f_\rho(t)(\tilde{\rho}(t) + p_\rho(t) + f_\rho(t)) \leq 0. \tag{A.10}$$

Thus, combining (A.9)–(A.10) yields that

$$V(\tilde{\theta}(t+1), \tilde{\rho}(t+1)) - V(\tilde{\theta}(t), \tilde{\rho}(t)) \leq -\frac{(2-\bar{\gamma})}{2} \frac{\epsilon_q^2(t)}{m^2(t)} + \frac{4}{2-\bar{\gamma}} \frac{d_1^2(t)}{m^2(t)}$$

which follows the boundedness of $\tilde{\theta}(t)$ and $\tilde{\rho}(t)$ that (27) holds. With (15), it follows that (28) holds. \square

Proof of Lemma 11. Recalling Eq. (A.5), we have

$$\theta_1^{*T}\phi_1(t) + \theta_2^{*T}\phi_q(t) = u(t) - \frac{1}{k_p}(y(t+n^*) - d_0(t)). \tag{A.11}$$

Combining (1), (23), and (A.11), we derive

$$\bar{\phi}(t+1) = A^*\bar{\phi}(t) + b^*(y(t+n^*) - d_0(t)) + d_2(t). \tag{A.12}$$

Next, we demonstrate that A^* is stable. Note that when using exact output feedback, i.e., $s(y(t), \Delta(t)) = 0$, the dynamic system (A.12) with respect to $\bar{\phi}(t)$ becomes

$$\bar{\phi}(t+1) = A^*\bar{\phi}(t) + b^*y(t+n^*) \tag{A.13}$$

which reveals that A^* is stable. Let the first variable of $\bar{\phi}(t)$ be the output of system (A.13). Thus, for system (A.13) with $y(t+n^*)$ as the virtual input and $z^{-n+1}u(t)$ as the virtual output, we derive an input–output description for system (A.13):

$$z^{-n+1}u(t) = c^*(zI_{2n-1} - A^*)^{-1}b^*y(t+n^*), \tag{A.14}$$

where $c^* = [1, 0, \dots, 0] \in \mathbb{R}^{1 \times (2n-1)}$. With $A(z)y(t) = k_p B(z)u(t)$, it derives from (A.14) that

$$A(z)z^{-n+1}u(t) = k_p c^*(zI_{2n-1} - A^*)^{-1}b^*z^{n^*}B(z)u(t).$$

This equation implies that

$$\det\{zI_{2n-1} - A^*\} = z^{n+n^*-1}B(z)$$

which follows from Assumption (A1) that the eigenvalues of A^* are all inside the unit circle of the z -complex plane, i.e., A^* is stable. \square

Proof of Theorem 14. Based on Lemma 6, we observe that $q(y(t), \Delta(t))$ is not saturated at $t = t_0$. Now, given that $q(y(t), \Delta(t))$ is not saturated for all $t \geq t_0$, we provide the following analysis. It follows from Lemma 13 that

$$|s(y(t), \Delta(t))| \leq \frac{1}{2}\Delta(t).$$

Then, (27), (28), and (30) can be rewritten as

$$\sum_{t=t_1}^{t_2} \frac{\epsilon_q^2(t)}{m^2(t)} \leq \alpha_1 + \beta_1 \sum_{t=t_1}^{t_2} \frac{\Delta^2(t)}{m^2(t)}, \tag{A.15}$$

$$\sum_{t=t_1}^{t_2} \|\theta(t+t_0) - \theta(t)\|_2^2 \leq \alpha_2 + \beta_2 \sum_{t=t_1}^{t_2} \frac{\Delta^2(t)}{m^2(t)}, \tag{A.16}$$

$$\|\bar{\phi}(t+1)\| \leq (c_0 + c_1\omega(t))\|\bar{\phi}(t)\| + c_2\Delta(t) + |b^*y^*(t+n^*)|, \tag{A.17}$$

for some positive constant c_2 independent of $\Delta(t)$. To reduce notation, we still use β_1 and β_2 in (A.15) and (A.16), respectively.

With (A.15) and (A.16), we derive from (31) that

$$\sum_{t=t_1}^{t_1+t_2} \omega^2(t) \leq c_4 + c_3 \sum_{t=t_1-n^*}^{t_1+t_2} \frac{\Delta^2(t)}{1 + \|\bar{\phi}(t)\|^2} \tag{A.18}$$

for some positive constants c_3 and c_4 independent of $\Delta(t)$. Combining (14), (26) and (32), we obtain

$$\bar{\phi}(t+1) = A^*\bar{\phi}(t) + b^*y^*(t+n^*) + b^*k_p(\theta^T(t) - \theta^*)\phi(t) + d_2(t).$$

It follows from (11) that

$$\bar{\phi}(t+1) = (A^* + b^*k_p(\bar{\theta}^T(t) - \bar{\theta}^{*T}))\bar{\phi}(t) + b^*k_p\theta_3(t)y^*(t+n^*) + d_2(t)$$

with $\bar{\theta}(t) = [\theta_1^T(t), \theta_2^T(t)]^T$ and $\bar{\theta}^* = [\theta_1^{*T}, \theta_2^{*T}]^T$. The above equation implies that

$$\|\bar{\phi}(t+1)\| \leq c_5\|\bar{\phi}(t)\| + c_6 + c_7\Delta(t) \tag{A.19}$$

for some positive constants c_i , $i = 5, 6, 7$, independent of $\Delta(t)$. Note that $\Delta(t)$ grows at most exponentially, indicating that $\bar{\phi}(t)$ grows at most exponentially.

Next, we assume that $\bar{\phi}(t)$ grows unboundedly. Then, for any given $\delta_0 > 0$ and $t_4 > 0$, property (A.19) guarantees that we can find $\delta \in (0, \delta_0]$ and $t_3 > 0$ such that

$$\|\bar{\phi}(t)\| \geq \frac{1}{\delta_0}, \quad t \in \{t_3 - n^*, \dots, t_3 - 1\}, \tag{A.20}$$

$$\|\bar{\phi}(t)\| = \frac{1}{\delta}, \quad t = t_3, \tag{A.21}$$

$$\|\bar{\phi}(t)\| \geq \frac{1}{\delta}, \quad t \in \{t_3 + 1, \dots, t_3 + t_4 + 1\}. \tag{A.22}$$

Based on the inequalities $\frac{x_1+\dots+x_p}{p} \leq \sqrt{\frac{x_1^2+\dots+x_p^2}{p}}$, $\frac{x_1+\dots+x_p}{p} \geq (x_1 \dots x_p)^{1/p}$, and $e^p \geq p + 1$, in addition to (A.18), (A.20)–(A.22), we derive

$$\prod_{t=t_3}^{t_3+j} (c_0 + c_1\omega(t)) \leq \left(c_0 + \frac{c_1}{j+1} \sum_{t=t_3}^{t_3+j} \omega(t) \right)^{j+1} \leq \left(c_0 + c_1\delta_0\Delta\sqrt{c_3(n^*+1)} + c_1\frac{\sqrt{c_4}}{\sqrt{j+1}} \right) \leq \left(c_0 + c_1\delta_0\Delta\sqrt{c_3(n^*+1)} \right)^{j+1} e^{\frac{c_1\sqrt{c_4}}{c_0+c_1\delta_0\Delta\sqrt{c_3(n^*+1)}}\sqrt{j+1}}. \tag{A.23}$$

From (A.23), we can choose

$$\delta_0 < \frac{1 - c_0}{c_1\Delta\sqrt{c_3(n^*+1)}} \tag{A.24}$$

so that

$$c_0 + c_1\delta_0\Delta\sqrt{c_3(n^*+1)} < 1.$$

Note that the last term of (A.23) decays to zero with respect to j . Thus, there exists some $j_1 \geq 0$ such that

$$\prod_{t=t_3}^{t_3+j} (c_0 + c_1\omega(t)) < \frac{1}{2}, \quad \forall j \geq j_1. \tag{A.25}$$

Based on (A.17) and the boundedness of $\omega(t)$, we have

$$\begin{aligned} \|\bar{\phi}(t_3 + j + 1)\| &\leq \prod_{t=t_3}^{t_3+j} (c_0 + c_1\omega(t)) \|\bar{\phi}(t_3)\| \\ &\quad + c_8\Delta(t) + c_9 \end{aligned} \quad (\text{A.26})$$

for some positive constants c_8 and c_9 independent of $\Delta(t)$. From (A.24) and (A.26), we choose δ_0 such that

$$\delta_0 \in \left(0, \min\left\{ \frac{1 - c_0}{c_1\Delta(t)\sqrt{c_3(n^* + 1)}}, \frac{1}{2(c_8\Delta(t) + c_9)} \right\} \right).$$

Then, for such a δ_0 and $t_4 \geq j_1$, it follows from (A.25) and (A.26) that

$$\|\bar{\phi}(t_3 + j + 1)\| < \frac{1}{\delta}, \quad j \in \{t_3 + j_1, \dots, t_3 + t_4 + 1\}$$

which contradicts (A.22). Thus, we conclude that $\bar{\phi}(t)$ is bounded, and (34) follows from (13)–(14) and (27)–(28). \square

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