Delay Tolerance for Stable Stochastic Systems and Extensions

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***Abstract*—This article establishes a "robustness" type result, namely, delay tolerance for stable stochastic systems under suitable conditions. We study the delay tolerance for stable stochastic systems and delayed feedback controls of such systems, where the delay can be state-dependent or induced by the sampling-data. First, we consider systems with global Lipschitz continuous coefficients and show that when the original stochastic system without delay is pth moment exponentially stable, the system with small delays is still pth moment exponentially stable. In particular, when the pth moment exponential stability is based on Lyapunov conditions, we can obtain explicit delay bounds for moment exponential stability. Then, we consider a class of stochastic systems with nonglobal Lipschitz conditions and find a delay bound for almost sure and mean square exponential stability. As extension of the stability tolerance criteria, consensus, and tracking control of multiagent systems with measurement noises and nonuniform delays are studied.**

***Index Terms*—Delay, multiagent system, stability, stochastic system.**

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I. Introduction

THIS article presents an effort to address the important question: How much delay can a stochastic system endures so that the stability is still achieved. This, in fact, can be thought of as a robustness consideration. The range of allowable delays can be thought of as a "stability margin" in certain sense. Our consideration stems from the fact that stochastic systems are used frequently in the real-world applications ranging from numerous dynamic systems in engineering, computer science, and network science to population biology, epidemiology, and economics [1]. While the study of stochastic systems has a long history, one of the central issues drawing much attention in the literature is stability [2]–[5], which offers challenges and opportunities to the automatic control theory [6]–[9]. With stochastic perturbations, stability analysis involves various stochastic stability concepts such as stability in probability, stability in distribution, almost sure stability, and moment stability [3], [10]. This is different from the systems with deterministic disturbances in Hu *et al.* [11]. For many applications, delay is often unavoidable. This together with stochastic perturbations leads to the consideration of stochastic differential delay systems (SDDSs) [3], [12], whose future state depends not only on the present but also on the past history. Because stochastic stability of such SDDSs is vital, much effort has been devoted to the study; both Razumikhin methods and Lyapunov function (or functional) methods are used for the stability analysis for SDDSs. Using Razumikhin methods, moment asymptotic and/or exponential stability were obtained [13]–[17], whereas using Lyapunov function (or functional) methods, not only the moment stability but also the almost sure stability were obtained. By the Lyapunov function method, Mao [3] gave delay-independent pth moment stability and obtained almost sure stability from the moment exponential stability under linear growth condition. Rakkiyappan *et al.* [18] established the moment stability conditions in terms of linear matrix inequalities (LMIs) for uncertain stochastic neural networks with delays. Gershon *et al.* [19] studied H_∞ state-feedback control of stochastic delay systems using Lyapunov functions and LMIs. Shaikhet [20] also introduced many Lyapunov functionals to examine the stochastic stability of different SDDSs. Using suitable Lyapunov functionals, Fei *et al.* [21] established the delay-dependent moment stability of the highly nonlinear hybrid stochastic system.

Although many important results have been obtained to date, little is known about the delay tolerance for stable stochastic systems. We consider the following stochastic system

$$dy(t) = f(y(t))dt + \sum_{i=1}^{d} g_i(y(t))dw_i(t), t \geq t_0 \qquad (1)$$

with the initial data $y(t_0) = y_0$, $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = g_i(0) = 0$, $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))^T$ is a d-dimensional standard Brownian motion. We first assume that the stochastic system (1) (its trivial solution) is pth ($p \geq 2$) moment exponentially stable or almost surely exponentially stable. That is, there are constants $M > 0$ and $\gamma > 0$, such that

$$\mathbb{E}|y(t)|^p \leq M\mathbb{E}|y_0|^p e^{-\gamma t}, \text{ or } \limsup_{t\to\infty} \frac{\log|y(t)|}{t} \leq -\gamma \text{ a.s.}$$

The fundamental question is under what conditions the following delay system is still pth moment (or almost surely) exponentially stable

$$dx(t) = \sum_{i=1}^{d_1} f_i(\vartheta_i(x_t))dt + \sum_{i=1}^{d} g_i(\theta_i(x_t))dw_i(t) \qquad (2)$$

where the sequence $\{f_i\}_{i=1}^{d_1}$ is a decomposition of f with $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $f_i(0) = 0$, and $f(x) = \sum_{i=1}^{d_1} f_i(x)$, $x_t = \{x(t+u) : u \in [-\tau, 0]\}$, $\vartheta_i, \theta_i : C([-\tau_i, 0], \mathbb{R}^n) \to \mathbb{R}^n$ satisfies $|\vartheta_i(\varphi)| \vee |\theta_i(\varphi)| \leq \sup_{s\in[-\tau_i,0]} |\varphi(s)|$ for $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, $\tau_i \geq 0$, $\tau = \max_{i=1}^{d_1\vee d}\{\tau_i\}$. Here, delay terms can have various forms like the state-dependent delay studied in [22], the deterministic delay (fixed or time-varying), and random delay [23].

Considering the issue above leads to delay tolerance criteria.

1) First, it is important in the stability analysis of SDDSs. In fact, Lyapunov functional is not unique for solving the stability of SDDSs. To choose an appropriate Lyapunov functional is a difficult work. Based on the delay tolerance criteria, one can find the delay bound directly without wasting time on trying to find suitable Lyapunov functionals.
2) Second, it facilitates the design of delayed feedback control for stochastic systems. With the delay tolerance criteria, we do not need to change the control gain when the delay appears and falls in an allowed region. In fact, one only requires the stability of the delay-free system and the structure of the delays.
3) Finally, it can produce the stability of a class of semidiscrete stochastic systems. Most importantly, with the delay tolerance criteria, we can obtain the design of the sampled-data control and the explicit relationship between the sampling period and the systems parameters.

Prior to this article, the delay tolerance has been considered mainly for the almost surely stable stochastic systems. Mohammed and Scheutzow [24] and Scheutzow [25] considered the almost sure exponential stability of the linear scalar stochastic delay equation with pure diffusion $dx(t) = \sigma\theta_1(x_t)dw_1(t), \sigma > 0$, and showed that the delay system is still almost sure stable for sufficient small delay. Most recently, Mao *et al.* have engaged in the study of almost sure stability of the general SDDSs [26] and the associated switching cases [27]. These works are important since they showed that almost surely stable system can be tolerant with a small delay, where the small delay bound has been revealed in [26] and [27] for the almost sure stability. One can easily see from the references above, if the diffusion is not degenerate, then its delay can still contribute to the almost sure stability, but this is not the case for degenerated diffusions. Hence, one has to consider the stability without taking the positive role of noises into consideration. A representative work in this direction is Mao [28], where the fixed delay tolerance issues for the mean square stable stochastic systems were investigated under the global Lipschitz conditions. However, stability tolerance under the general delays (time-vary or state-dependent) and general Lyapunov conditions have not been examined. Moreover, the issues under the nonglobal Lipschitz conditions have not been well understood. This article fills in these gaps.

Inspired by the idea in [28], we study the delay tolerance for moment or almost surely stable stochastic systems. According to the different information about the delay-free systems, different delay tolerance results are obtained. Under the global Lipschitz condition, by assuming that we only know that the trivial solution of the delay-free systems is pth moment exponentially stable, we obtain a tolerance delay bound for the pth moment and almost sure exponential stability. We derive a weaker delay bound if the trivial solution of the delay-free system is moment exponentially stable based on a Lyapunov condition. For such results, the concrete form of the delay need not be known, where the delays can be deterministic, random, or state-dependent. Under nonglobal Lipschitz conditions, we consider the case with time-varying delays. It will be proved that if the time-varying delays are differentiable (or differentiable except at a sequence), then exponentially stable stochastic system can still be tolerant to time-varying delays with small derivatives. These results can solve many control problems with delayed feedback control without the global Lipschitz and linear growth conditions.

As an extension, the control of multiagent systems with noises and time-varying nonuniform delays is investigated. The delay tolerance results aim to solve the consensus and tracking problem of multiagent systems under the multiplicative noises and the nonuniform delays. This can be considered as a further extension of our recent works [29], [30] from the uniform fixed delays to the time-varying nonuniform delays. In fact, the nonuniform delays for multiagent consensus have been investigated intensively for deterministic models (see [31]–[35] for example). The Lyapunov functionals with derivatives of the states and characteristic methods are two important tools for designing the consensus control and establishing the consensus conditions. However, the two methods fail in the presence of noises since the state of stochastic system is not differentiable and the characteristic methods are difficult to be applied. To date, little is known about the models with measurement noises and nonuniform delays. With the delay tolerance results obtained in this article, we have the ability to overcome the difficulty induced by nonuniform delays and obtain the design of the consensus and tracking protocol.

The rest of the article is arranged as follows. Section II addresses the delay tolerance under the global Lipschitz assumption, where the exponential stability condition and Lyapunov condition are studied, respectively. Section III examines the delay tolerance for the mean square and almost sure exponential stability under the nonglobal Lipschitz and local linear growth conditions. Section IV applies the delay tolerance idea to study consensus and tracking control of multiagent systems with nonuniform time-varying delays and measurement noises. Section V concludes the paper with further remarks.

Notation: We work with the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $|\cdot|$. For a vector or a matrix A, its transpose is denoted by A^T. For a matrix A, denote its trace norm by $|A| = \sqrt{\text{trace}(A^T A)}$; for a symmetric matrix A with real entries, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the largest and smallest eigenvalues, respectively. Use $a \vee b$ to denote $\max\{a, b\}$ and $a \wedge b$ to denote $\min\{a, b\}$. For $\tau > 0$, denote by

$C([-\tau,0];\mathbb{R}^n)$ the family of all $\mathbb{R}^n$-valued continuous functions on the interval $[-\tau,0]$ with the norm $\|\varphi\|_C=\sup_{t\in[-\tau,0]}|\varphi(t)|$. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions. That is, it is right continuous and increasing while $\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets.

II. Delay Tolerance Under Global Lipschitz Conditions

In this section, we assume that the coefficients of (1) are Lipschitz continuous.

Assumption II.1: Assume that f,g satisfy $f(0)=g(0)=0$ and there exist positive constants K_1 and K_{2i}, such that

$$|f(x)-f(y)|\leq K_1|x-y|,\ |g_i(x)-g_i(y)|\leq K_{2i}|x-y|,$$

for all $x,y\in\mathbb{R}^n$.

It is known that under Assumption II.1, the delay-free stochastic system (1) admits a unique global solution. We also assume that for each i, f_i in the decomposition of f [see (2)] is Lipschitz continuous.

Assumption II.2: Assume that the decomposition $f(x)=\sum_{i=1}^{d_1}f_i(x)$ is a Lipschitz decomposition, that is, each component $f_i(x)$ satisfies $|f_i(x)-f_i(y)|\leq K_{1i}|x-y|$ with $K_{1i}\geq 0$.

Remark II.1: Assumption II.2 under Assumption II.1 is easy to verify. For example, if we consider the plant $\dot{x}=Ax+Bu$ with $u=Kx$, then the closed-loop system has the form $\dot{x}=Ax+BKx$. So we can let $f(x)=Ax+BKx$, which falls into the case of Assumption II.2. The corresponding issue can be considered as that if the timely control can make the system stable, then how about for the delayed control.

For the purpose of stability, assume that $f(0)=f_i(0)=0, g_j(0)=0$ for all i,j. This implies that each of the two stochastic systems (1) and (2) admits a trivial solution, respectively. Note that under the global Lipschitz assumptions above, the moment exponential stability implies the almost sure exponential stability [3]. So in this section, we only focus on the pth moment exponential stability. We consider the case $p\geq 2$. For the case $p\in(0,2)$, we can use the inequality $(\mathbb{E}|x(t)|^p)^{\frac{1}{p}}\leq(\mathbb{E}|x(t)|^2)^{\frac{1}{2}}$ to obtain the pth ($p\in(0,2)$) moment exponential stability.

Remark II.2: The diffusion term (multiplicative noise) may work positively for almost sure and pth ($0<p<1$) moment stability (see [3], [36]) when the diffusion is nondegenerate. Considering the models with nondegenerate diffusion excludes many real models (for example, the double-integrator stochastic systems in [37]). As a remedy, we consider more general models without the requirement of diffusions to be nondegenerate. As a result, our results are not related to that of the positive role played by the diffusion for almost sure and pth ($0<p<2$) moment stability in the existing literature. That is also the reason for us to obtain the pth-moment exponential stability with $0<p<2$ from pth-moment exponential stability with $p\geq 2$. It is possible to obtain small moment stability under weaker conditions, which is more difficult to deal with and is to be considered in our future work.

For the convenience of the reader, we recall the Itô formula. Let $L^p(\mathbb{R}_+;\mathbb{R}^n)$ denote the family of all $\mathbb{R}^n$-valued measurable $\{\mathfrak{F}_t\}$-adapted processes $f=\{f(t)\}_{t\geq 0}$, such that $\int_0^T|f(s)|^pds<\infty$ a.s. for every $T>0$. Consider the following Itô process

$$dX(t)=F(t)dt+\sum_{i=1}^d G_i(t)dw_i(t)$$

where $F(t)\in L^1(\mathbb{R}_+;\mathbb{R}^n)$ and $F(t)\in L^2(\mathbb{R}_+;\mathbb{R}^n)$. Then for any $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$, we have the following Itô formula [3]

$$V(X(t),t)=V(X(t_0),t_0)+\int_{t_0}^t\mathcal{S}V(X(s),s)ds+\sum_{i=1}^d\int_{t_0}^t V_x(X(s),s)G_i(s)dw_i(s)$$

where $\mathcal{S}V(x,t)=\frac{\partial V(x,t)}{\partial t}+\frac{1}{2}\sum_{i=1}^d G_i(t)^TV_{xx}(x,t)G_i(t)+V_x(x,t)F(t)$,$V_x(x,t)=(\frac{\partial V(x,t)}{\partial x_1},\ldots,\frac{\partial V(x,t)}{\partial x_n})$, and $V_{xx}(x,t)=(\frac{\partial^2V(x)}{\partial x_j\partial x_l})$. Here, we need to remark that $\mathcal{S}$ is not the usual operator associated with the stochastic differential equations but is merely a symbol; likewise $\mathcal{S}V(x,t)$ is just a notation. However, for stochastic system (1), we can obtain the true operator $\mathbb{L}:C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)\to\mathbb{R}_+$ associated with the Itô diffusion

$$\mathbb{L}V(x,t)=\frac{\partial V(x,t)}{\partial t}+V_x(x,t)f(x)+\frac{1}{2}\sum_{i=1}^d g_i^T(x)V_{xx}(x,t)g_i(x). \tag{3}$$

In what follows, we first consider the delay tolerance based on the moment exponential stability. In this case, we may obtain a delay bound for the delay system (2) to ensure moment exponential stability. However, this delay bound might be too conservative since we only know that the delay-free system (1) is moment exponentially stable. To proceed, we study the delay tolerance issue under Lyapunov conditions imposed on (3), which may guarantee the moment exponential stability of delay-free systems. In this case, we can get a relaxed delay bound for system (2) to ensure moment exponential stability.

A. Delay Tolerance Based on the Moment Exponential Stability

Define $D(\varphi)=\sup_{-\tau\leq u\leq 0}|\varphi(u)-\varphi(0)|$. The following assumption is required and is easily verifiable. For example, the functional $\vartheta_i(\varphi)=\theta_j(\varphi)=\varphi(-\tau)$ falls in this assumption.

Assumption II.3: Assume that for $i=1,2,|\vartheta_i(\varphi)-\vartheta_i(\phi)|\vee|\theta_i(\varphi)-\theta_i(\phi)|\leq\|\varphi-\phi\|$ and $|\theta_i(\varphi)-\varphi(0)|\leq D(\varphi)$.

Before presenting the delay bound for the stability of the stochastic delay system (2), we present some lemmas. The first lemma provides boundedness estimates of the solution to stochastic delay system (2).

Lemma II.1: Let Assumptions II.1, II.2, and II.3 hold, $\xi\in L^p_{\mathfrak{F}_0}(\Omega;C([-\tau,0];\mathbb{R}^n))$, and write $x(t)=x(t;t_0,\xi)$. Then we have the following estimates

$$\mathbb{E}(\sup_{t_0\leq u\leq t}|x(u)|^p)\leq 3\mathbb{E}\|x_{t_0}\|^pe^{\gamma_1(t-t_0)} \tag{4}$$

and

$$\mathbb{E}|D(x_{t+\tau})|^p\leq M_1(\tau)\mathbb{E}\|x_{t_0}\|^pe^{\gamma_1(t+\tau-t_0)} \tag{5}$$

where $\gamma_1 = 2p(\sum_{i=1}^{d_1} K_{1i} + \sum_{i=1}^{d} \frac{(32d+1)p-1}{2} K_{2i}^2), M_1(\tau) = 2^{2p+1}(\tau^p d_1^{p-1} \sum_{i=1}^{d_1} K_{1i}^p + d^{p-1} \sum_{i=1}^{d} K_{2i}^p \tau^{p/2} p_0), p_0 = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$.

Proof: By the Itô formula, we have

$$|x(t)|^p = |x(t_0)|^p + p\int_{t_0}^t [|x(s)|^{p-2}x(s)^T \sum_{i=1}^{d_1} f_i(\vartheta_i(x_s)) + p\sum_{i=1}^{d}\int_{t_0}^t \frac{p-2}{2}|x(s)|^{p-4}|x(s)^T g_i(\theta_i(x_s))|^2 ds + \frac{1}{2}|x(s)|^{p-2}\sum_{i=1}^{d}|g_i(\theta_i(x_s))|^2]ds + M(t) \quad (6)$$

where $M(t) = \sum_{i=1}^{d} m_i(t)$, $m_i(t) = \int_{t_0}^t |x(s)|^{p-2}x(s)^T g_i(\theta_i(x_s))dw_i(s)$. Let $H_{t_0}^t = \mathbb{E}(\sup_{t_0\le u\le t}|x(u)|^p)$. Then, from Assumptions II.1–II.3 and the inequality $x^\alpha y^\beta \le \frac{\alpha}{\alpha+\beta}x^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}y^{\alpha+\beta}$ for $x, y, \alpha, \beta > 0$, we have

$$\begin{aligned} H_{t_0}^t &\le \mathbb{E}|x(t_0)|^p + \mathbb{E}(\sup_{t_0\le s\le t} M(s)) \\ &\quad + p\int_{t_0}^t \left[\sum_{i=1}^{d_1} K_{1i}\mathbb{E}(|x(s)|^{p-1}|\vartheta_i(x_s)|) + \sum_{i=1}^{d} K_{2i}^2 \frac{p-1}{2}\mathbb{E}(|x(s)|^{p-2}|\theta_i(x_s)|^2)\right] ds \\ &\le \mathbb{E}|x(t_0)|^p + C_1\int_{t_0}^t H_{t_0-\tau}^s ds + \sum_{i=1}^{d}\mathbb{E}(\sup_{t_0\le s\le t} m_i(s)) \end{aligned} \quad (7)$$

where $C_1 = p(\sum_{i=1}^{d_1} K_{1i} + \frac{1}{2}\sum_{i=1}^{d} K_{2i}^2(p-1))$. By the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}(\sup_{t_0\le s\le t} m_i(s)) \\ &\le p4\sqrt{2}\left(\int_{t_0}^t |x(s)|^{2p-4}|x(s)^T g_i(\theta_i(x_s))|^2 ds\right)^{1/2} \\ &\le 4\sqrt{2}pK_{2i}^2\mathbb{E}\left(\sup_{t_0\le u\le t}|x(u)|^p\int_{t_0}^t |x(s)|^{p-2}|\theta_i(x_s)|^2 ds\right)^{1/2} \\ &\le 0.5\frac{1}{d}\mathbb{E}(\sup_{t_0\le u\le t}|x(u)|^p) + 16p^2dK_{2i}^2\int_{t_0}^t H_{t_0-\tau}^s ds. \end{aligned}$$

Substituting this into (7) yields $H_{t_0}^t \le 2\mathbb{E}|x(t_0)|^p + \gamma_1\int_{t_0}^t H_{t_0-\tau}^s ds$. Note that $H_{t_0-\tau}^t \le H_{t_0-\tau}^{t_0} + H_{t_0}^t \le 3\mathbb{E}\|x_{t_0}\|^p + \gamma_1\int_{t_0}^t H_{t_0-\tau}^s ds$. The Gronwall inequality yields $\mathbb{E}(\sup_{t_0-\tau\le u\le t}|x(u)|^p) \le 3\mathbb{E}\|x_{t_0}\|^p e^{\gamma_1(t-t_0)}$. Hence, the desired assertion (4) follows. By the Hölder inequality and the Burkholder–Davis–Gundy inequality, we get

$$\begin{aligned} &\mathbb{E}(\sup_{0\le u\le\tau}|x(t+u) - x(t)|^p) \\ &\le (2d_1)^{p-1}\tau^{p-1}\sum_{i=1}^{d_1} K_{1i}^p\int_t^{t+\tau}\mathbb{E}|\vartheta_i(x_s)|^p ds \\ &\quad + (2d)^{p-1}p_0\sum_{i=1}^{d}\mathbb{E}\left(\int_t^{t+\tau}|g_i(\theta_i(x_s))|^2 ds\right)^{p/2} \\ &\le C_2\int_t^{t+\tau}\mathbb{E}(\sup_{s-\tau\le u\le s}|x(u)|^p)ds \\ &\le 3\tau C_2\mathbb{E}\|x_{t_0}\|^p e^{\gamma_1(t+\tau-t_0)} \end{aligned} \quad (8)$$

where $C_2 = (2d_1)^{p-1}\tau^{p-1}\sum_{i=1}^{d_1} K_{1i}^p + (2d)^{p-1}\tau^{p/2-1}p_0\sum_{i=1}^{d} K_{2i}^p$, $p_0 = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$. This together with the definition of $D(x_t)$ gives

$$\begin{aligned} \mathbb{E}|D(x_{t+\tau})|^p &= \mathbb{E}(\sup_{-\tau\le u\le 0}|x(t+\tau+u) - x(t+\tau)|^p) \\ &\le 2^{p-1}\mathbb{E}|x(t+\tau) - x(t)|^p \\ &\quad + 2^{p-1}\mathbb{E}(\sup_{0\le u\le\tau}|x(t+u) - x(t)|^p) \\ &\le M_1(\tau)\mathbb{E}\|x_{t_0}\|^p e^{\gamma_1(t+\tau-t_0)}. \end{aligned}$$

That is, the desired assertion (5) follows. ■

In this lemma, we obtain the moment estimate of $\mathbb{E}(\sup_{t_0-\tau\le u\le t}|x(u)|^p)$ with the exponent γ_1, and then based on this estimate, we get estimate (5) of $\mathbb{E}|D(x_{t+\tau})|^p$. Here, we remark that the exponent γ_1 for $\mathbb{E}|D(x_{t+\tau})|^p$ can be estimated more accurately if the functionals $\{\vartheta_i, \theta_i\}_{i=1,2}$ satisfy $\mathbb{E}|\vartheta_i(\varphi)|^p \vee \mathbb{E}|\theta_i(\varphi)|^p \le \sup_{-\tau\le u\le 0}\mathbb{E}|\varphi(u)|^p$ for all $\varphi \in L_{\mathfrak{F}_0}^p(\Omega; C([-\tau, 0]; \mathbb{R}^n))$(for example $\theta_i(\varphi) = \varphi(-\tau)$). This is summarized in the following lemma.

Lemma II.2: Let Assumptions II.1 and II.2 hold, $\xi \in L_{\mathfrak{F}_0}^p(\Omega; C([-\tau, 0]; \mathbb{R}^n))$, write $x(t) = x(t; t_0, \xi)$, and assume $\mathbb{E}|\vartheta_i(\varphi)|^p \vee \mathbb{E}|\theta_i(\varphi)|^p \le \sup_{-\tau\le u\le 0}\mathbb{E}|\varphi(u)|^p$ for all $\varphi \in L_{\mathfrak{F}_0}^p(\Omega; C([-\tau, 0]; \mathbb{R}^n))$. Then we have the following estimates $\mathbb{E}|x(t)|^p \le 2\mathbb{E}\|x_{t_0}\|^p e^{\gamma_2(t-t_0)}$, $\mathbb{E}|D(x_{t+\tau})|^p \le M_1(\tau)\mathbb{E}\|x_{t_0}\|^p e^{\gamma_2(t+\tau-t_0)}$, where $\gamma_2 = p(\sum_{i=1}^{d_1} K_{1i} + \frac{p-1}{2}\sum_{i=1}^{d} K_{2i}^2)$.

Proof: From (6), we can obtain that

$$\begin{aligned} \mathbb{E}|x(t)|^p &\le \mathbb{E}|x(t_0)|^p + C_3\int_{t_0}^t \mathbb{E}|x(s)|^p ds \\ &\quad + \sum_{i=1}^{d_1} K_{1i}\int_{t_0}^t \mathbb{E}|\theta_i(x_s)|^p ds \\ &\quad + \frac{p-1}{2}\sum_{i=1}^{d} K_{2i}^2\int_{t_0}^t \mathbb{E}|\theta_2(x_s)|^p ds \\ &\le \mathbb{E}|x(t_0)|^p + C_4\int_{t_0}^t \sup_{t_0-\tau\le u\le s}\mathbb{E}|x(u)|^p ds \end{aligned}$$

where $C_3 = (p-1)(\sum_{i=1}^{d_1} K_{1i} + \frac{1}{2}\sum_{i=1}^{d} K_{2i}^2(p-2))$ and $C_4 = p(\sum_{i=1}^{d_1} K_{1i} + \frac{p-1}{2}\sum_{i=1}^{d} K_{2i}^2)$. Therefore, the Gronwall inequality leads to the desired assertion. ■

The following lemma produces the estimate of the error $e(t) := x(t) - y(t)$.

Lemma II.3: Let Assumptions II.1, II.2, and II.3 hold, $\xi \in L_{\mathfrak{F}_0}^p(\Omega; C([-\tau, 0]; \mathbb{R}^n))$, and write $x(t) = x(t; t_0, \xi)$. Then, we have

$$\mathbb{E}|e(t)|^p \le J(\tau, t-t_0)\mathbb{E}\|x_{t_0}\|^p$$

where $J(\tau,z) = (\sum_{i=1}^{d_1} K_{1i} + 2(p-1)\sum_{i=1}^{d} K_{2i}^2)\ e^{M_3(z-\tau)} \frac{M_1}{\gamma_1}[e^{\gamma_1 z} - e^{\gamma_1 \tau}]$ with $M_1(\tau)$ and γ_1 defined in Lemma II.1, $M_3 = 2(p-1)^2\sum_{i=1}^{d} K_{2i}^2 + (2p-1)\sum_{i=1}^{d_1} K_{1i}$.

Proof: It is easy to see that $de(t) = \sum_{i=1}^{d_1}[f_i(\vartheta_i(x_t))) - f_i(y(t))]dt + \sum_{i=1}^{d}[g_i(\theta_i(x_t)) - g_i(y(t))]dw(t)$. Apply the Itô formula and Assumptions II.1 and II.2 yields

$$\begin{aligned}\mathbb{E}|e(t)|^p &\le p\sum_{i=1}^{d_1} K_{1i}\int_{t_0+\tau}^{t}\mathbb{E}(|e(s)|^{p-1}|\vartheta_i(x_s) - y(s)|)ds\\ &\quad + \frac{p(p-1)}{2}\sum_{i=1}^{d} K_{2i}^2\\ &\quad \times\int_{t_0+\tau}^{t}\mathbb{E}(|e(s)|^{p-2}|\theta_i(x_s) - y(s)|^2)ds\\ &\le p\left(\sum_{i=1}^{d_1} K_{1i} + (p-1)\sum_{i=1}^{d} K_{2i}^2\right)\int_{t_0+\tau}^{t}\mathbb{E}|e(s)|^p ds\\ &\quad + p\sum_{i=1}^{d_1} K_{1i}\int_{t_0+\tau}^{t}\mathbb{E}(|e(s)|^{p-1}|\vartheta_i(x_s) - x(s)|)ds\\ &\quad + p(p-1)\sum_{i=1}^{d} K_{2i}^2\\ &\quad \times\int_{t_0+\tau}^{t}\mathbb{E}(|e(s)|^{p-2}|\theta_i(x_s) - x(s)|^2)ds\\ &\le M_3\int_{t_0+\tau}^{t}\mathbb{E}|e(s)|^p ds + \sum_{i=1}^{d_1} K_{1i}\\ &\quad \times\int_{t_0+\tau}^{t}\mathbb{E}|\vartheta_i(x_s) - x(s)|^p ds\\ &\quad + 2(p-1)\sum_{i=1}^{d} K_{2i}^2\int_{t_0+\tau}^{t}\mathbb{E}|\theta_i(x_s) - x(s)|^p ds.\end{aligned}$$

Applying the Gronwall inequality and Lemma II.1 lead to

$$\begin{aligned}\mathbb{E}|e(t)|^p &\le e^{M_3(t-t_0-\tau)}\left(\sum_{i=1}^{d_1} K_{1i}\int_{t_0+\tau}^{t}\mathbb{E}|\vartheta_i(x_s) - x(s)|^p ds\right.\\ &\quad \left. + 2(p-1)\sum_{i=1}^{d} K_{2i}^2\int_{t_0+\tau}^{t}\mathbb{E}|\theta_i(x_s) - x(s)|^p ds\right)\\ &\le J(\tau, t-t_0)\mathbb{E}\|x_{t_0}\|^p.\end{aligned}$$

■

Theorem II.1: Let Assumptions II.1, II.2, and II.3 hold. Assume that the original system (1) is pth moment exponentially stable with $\mathbb{E}|y(t)|^p \le M\mathbb{E}|y_0|^p e^{-\gamma t}$ for $M, \gamma > 0$. Then there is a positive number τ^*, such that the delay system (2) is pth moment exponentially stable for any $\tau < \tau^*$. In fact, τ^* can be determined using $\tau^* = \sup_{\varepsilon\in(0,1)}\{\tau > 0 | 2^{p-1}M_1(\tau)e^{\gamma_1(\tau+h)} + \varepsilon e^{\gamma_1\tau} + 4^{p-1}J(\tau,\tau+h))) - 1 = 0\}$, where $h = h(\varepsilon) = \log(\frac{\varepsilon}{3\cdot 4^{p-1}M})$ for $\varepsilon \in (0,1)$.

Proof: We first prove that τ^* is well defined. For any fixed $\varepsilon \in (0,1)$, define

$$\begin{aligned}Q(\tau) &:= 2^{p-1}M_1(\tau)e^{\gamma_1(\tau+h)} + \varepsilon e^{\gamma_1\tau}\\ &\quad + 4^{p-1}J(\tau,\tau+h))) - 1.\end{aligned}$$

Note that $Q(-\frac{\log\varepsilon}{\gamma_1}) > 0$, $\lim_{\tau\to 0} = \varepsilon - 1 < 0$, and $Q'(\tau) > 0$. Hence, there is a unique root $\tau(\varepsilon) > 0$ such that $Q(\tau^*) = 0$, and $\tau^* = \sup_{\varepsilon\in(0,1)}\tau(\varepsilon)$ is well defined.

Let $\varepsilon_0 = arc\sup_{\varepsilon\in(0,1)}\tau(\varepsilon)$, $h = h(\varepsilon_0)$, and $\tau < \tau^*$. Write $x(t;t_0,\xi) := x(t)$ for all $t \ge t_0$ and $y(t_0+\tau+h; t_0+\tau, x(t_0+\tau)) = y(t_0+\tau+h)$. Bear in mind that

$$\mathbb{E}|y(t_0+\tau+h)|^p \le M\mathbb{E}|x(t_0+\tau)|^p e^{-\gamma h}.$$

This together with Lemma II.1 implies $\mathbb{E}|y(t_0+\tau+h)|^p \le 3M\mathbb{E}\|\xi\|^p e^{\gamma_1\tau-\gamma h}$. It follows from Lemma II.3 that

$$\begin{aligned}\mathbb{E}|x(t_0+\tau+h)|^p &\le 2^{p-1}\mathbb{E}|x(t_0+\tau+h) - y(t_0+\tau+h)|^p\\ &\quad + 2^{p-1}\mathbb{E}|y(t_0+\tau+h)|^p\\ &\le 2^{p-1}(3Me^{\gamma_1\tau-\gamma h} + J(\tau,\tau+h))\mathbb{E}\|\xi\|^p.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\|x_{t_0+\tau+h}\|^p &\le 2^{p-1}\mathbb{E}|D(x_{t_0+\tau+h}|^p\\ &\quad + 2^{p-1}\mathbb{E}|x(t_0+\tau+h)|^p\\ &\le J_0(\tau)\mathbb{E}\|\xi\|^p \quad (9)\end{aligned}$$

where $J_0(\tau) = 2^{p-1}M_1(\tau)e^{\gamma_1(\tau+h)} + 4^{p-1}(3Me^{\gamma_1\tau-\gamma h} + J(\tau,\tau+h)))$. From the definitions of τ and h, $J_0(\tau) < 1$. Denote $\gamma_0 = -\frac{\log J_0(\tau)}{\tau+h}$. Then, it follows from (9) that $\mathbb{E}\|x_{t_0+\tau+h}\|^p \le e^{-\gamma_0(\tau+h)}\mathbb{E}\|\xi\|^p$. Then similar to the derivation in [27], we can obtain that for $k = 1, 2, \ldots$,

$$\mathbb{E}\|x_{t_0+k(\tau+h)}\|^p \le e^{-\gamma_0 k(\tau+h)}\mathbb{E}\|\xi\|^p.$$

This implies the pth moment exponential stability of the delay system (2). ■

Theorem II.1 indicates that the moment exponentially stable stochastic system can be tolerant with a small delay with a bound τ^*. It is not required for us to know the exact stability conditions on the coefficients $f(x)$ and $\{g_i(x)\}_{i=1}^d$. The more information about the systems we have, the more accurate results we would get. So it is a natural question that if we know some additional information about the coefficients f, $\{g_i(x)\}_{i=1}^d$, and their decomposition, can we improve the delay bound obtained in Theorem II.1? For example, all the information about the coefficients is available for the following linear system

$$dy(t) = -\mu y(t)dt + \sigma y(t)dw(t), \mu, \sigma > 0. \quad (10)$$

Most importantly for the pth moment stability analysis, one can resort to the Lyapunov function like $V(y) = |y|^p$ and derive

$$\mathbb{L}V(y) = -p(\mu - \frac{p-1}{2}\sigma^2)|y|^p \quad (11)$$

which is a key in moment stability analysis and produces $\mathbb{E}|y(t)|^p = |y(0)|^p e^{-p(\mu-\frac{p-1}{2}\sigma^2)t}$. That is, equation (10) is pth moment exponentially stable for $p < (2\mu+\sigma^2)/\sigma^2$. We hope to find a delay bound τ better than that in Theorem II.1, such that the following delay system is still pth moment exponentially stable

$$dx(t) = -\mu\theta_1(x_t)dt + \sigma\theta_2(x_t)dw(t), x_0 = \xi \quad (12)$$

where $w(t)$ is a scalar Brownian motion. That is the attention of the following subsection.

B. Delay Tolerance Based on Lyapunov Conditions

In this section, we assume the Lyapunov stability condition like (11) as a precondition and examine the delay tolerance. The following Lyapunov stability result is classical (see [3]).

Theorem II.2: If there exist a function $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ and two positive constants c_1 and c_2, such that

1) $c_1|x|^p \le V(x) \le c_2|x|^p$
2) $\mathbb{L}V(x) \le -\lambda V(x)$ for some fixed $\lambda > 0$.

Then stochastic system (1) is pth moment exponentially stable.

To examine the delay tolerance under the above Lyapunov conditions, the following integral version of the Halanay inequality [38] will be used. Then the delay tolerance under the Lyapunov function is given in Theorem II.3.

Lemma II.4: Consider

$$v(t) \le \|v_{t_0}\| e^{-\gamma_0(t-t_0)} f(t) + K\int_{t_0}^{t} e^{-\gamma_0(t-s)} \sup_{u\in[s-\tau,s]} v(u)ds,$$

where $t \ge t_0, \gamma_0 > 0, K > 0$, and $f(t)$ is a nondecreasing positive function. If $\gamma_0 > K$, then there exist positive constants γ and K, such that

$$v(t) \le \|v_{t_0}\| K e^{-\gamma(t-t_0)} f(t)$$

where γ is the unique root of the equation $\gamma = -\gamma_0 + Ke^{\gamma\tau}$.

Theorem II.3: Let Assumptions II.1 and II.2 hold, $\mathbb{E}|\vartheta_i(\varphi) - x|^2 \vee \mathbb{E}|\theta_i(\varphi) - x|^2 \le \sup_{u\in[0,\tau]} \mathbb{E}|x - \varphi(-u)|^2$ for $\varphi \in L^p_{\mathfrak{F}_0}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$ and $x \in \mathbb{R}^n$. Assume that there exist a function $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ and constants $p \ge 2, \{c_i\}_{i=1}^4$ satisfying conditions 1), 2), and

3) $|V_x(x)| \le c_3|x|^{p-1}$
4) $|V_{xx}(x)| \le c_4|x|^{p-2}$.

Then the delay system (2) is pth moment exponentially stable if

$$C_5(\tau) < \lambda \tag{13}$$

where $C_5(\tau) = 2\frac{\sqrt{p-1}}{pc_1}2^{\frac{p-1}{2}}(\tau^p d_1^{p-1}\sum_{i=1}^{d_1} K_{1i}^p + \tau^{p/2}d^{p-1}p_0 \sum_{i=1}^{d} K_{2i}^p)^{1/2}(c_3\sum_{i=1}^{d_1} K_{1i} + c_4\sum_{i=1}^{d} K_{2i}^2)$, $p_0 = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$.

Proof: Applying the Itô formula yields

$$\begin{aligned} de^{\gamma t}V(x(t)) &= \gamma e^{\gamma t}V(x(t))dt + \sum_{i=1}^{d_1} e^{\gamma t}V_x(x(t))f_i(\vartheta_i(x_t))dt + dM(t) \\ &\quad + \frac{1}{2}\sum_{i=1}^{d} g_i(\theta_i(x_t))^T V_{xx}(x(t))g_i(\theta_i(x_t))dt \end{aligned} \tag{14}$$

where $M(t) = \sum_{i=1}^d \int_0^t e^{\gamma s}V_x(x(s))g_i(\theta_i(x_s)dw_i(s)$ is a martingale with $\mathbb{E}M(t) = 0$ because of the global Lipschitz conditions. Note that

$$\begin{aligned} &V_x(x(s))f_i(\vartheta_i(x_s)) \\ &\quad = V_x(x(s))f(x(s)) + V_x(x(s))[f_i(\vartheta_i(x_s)) - f_i(x(s))] \end{aligned} \tag{15}$$

and

$$\begin{aligned} &g_i(\theta_i(x_s))^T V_{xx}(x(s))g_i(\theta_i(x_s)) \\ &\quad = g_i(x(s))^T V_{xx}(x(s))g_i(x(s)) \\ &\qquad + g_i(\theta_i(x_s))^T V_{xx}(x(s))[g(\theta_i(x_s)) - g(x(s))] \\ &\qquad + [g_i(\theta_i(x_s)) - g(x(s))]^T V_{xx}(x(s))g_i(x(s)). \end{aligned} \tag{16}$$

Then, substituting (15) and (16) into (14) and taking expectations, we get from condition 2)

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(x(t)) &\le \mathbb{E}V(x(0)) + (\gamma - \lambda)\int_0^t e^{\gamma s}\mathbb{E}V(x(s))ds \\ &\quad + \int_0^t e^{\gamma s}\mathbb{E}\left[\sum_{i=1}^{d_1}\Theta_{1i}(s) + \sum_{i=1}^{d}\Theta_{2i}(s)\right]dt \end{aligned} \tag{17}$$

where $\Theta_{1i}(s) = V_x(x(s))[f_i(\vartheta_i(x_s)) - f_i(x(s))]$ and $\Theta_{2i}(s) = 0.5g_i(\theta_i(x_s))^T V_{xx}(x(s))[g(\theta_i(x_s)) - g_i(x(s))] + 0.5[g_i(\theta_i(x_s)) - g_i(x(s))]^T V_{xx}(x(s))g_i(x(s))$. It follows that $\alpha(xy^{p-2}\frac{z}{\alpha}) \le \frac{1}{p}\alpha(2x^p + (p-2)y^p) + \frac{1}{\alpha^{p-1}}z^p$, for any $x, y, z, \alpha > 0$, and we have

$$\begin{aligned} \mathbb{E}\Theta_{1i}(s) &\le c_3K_{1i}\mathbb{E}(|x(s)|^{p-1}|\vartheta_i(x_s) - x(s)|) \\ &\le c_3K_{1i}\left(\alpha\frac{p-1}{p}\mathbb{E}|x(s)|^p + \frac{1}{p\alpha}\mathbb{E}|\vartheta_i(x_s) - x(s)|^p\right) \end{aligned} \tag{18}$$

and

$$\begin{aligned} \mathbb{E}\Theta_{2i}(t) &\le \frac{c_4}{2}K_{2i}^2\left(\alpha\frac{1}{p}\mathbb{E}|\theta_i(x_s)|^p + \alpha\frac{p-2}{p}\mathbb{E}|x(s)|^p \right. \\ &\qquad \left. + \frac{1}{p\alpha}\mathbb{E}|\theta_i(x_s) - x(s)|^p\right) \\ &\quad + \frac{c_4}{2}K_{2i}^2\left(\alpha\frac{p-1}{p}\mathbb{E}|x(s)|^p \right. \\ &\qquad \left. + \frac{1}{p\alpha}\mathbb{E}|\theta_i(x_s) - x(s)|^p\right). \end{aligned} \tag{19}$$

Note that $\mathbb{E}|\vartheta_i(x_s) - x(s)|^2 \vee \mathbb{E}|\theta_i(x_s) - x(s)|^2 \le \sup_{u\in[0,\tau]} \mathbb{E}|x(s) - x(s-u)|^2$. Similarly to (8), we have that for any $u \in [0, \tau]$

$$\begin{aligned} &\mathbb{E}|x(t+u) - x(t)|^p \\ &\quad \le (2d_1\tau)^{p-1}\sum_{i=1}^{d_1}K_{1i}^p\int_t^{t+u}\mathbb{E}|\vartheta_i(x_s)|^p ds \\ &\qquad + (2d)^{p-1}\tau^{p/2-1}p_0\sum_{i=1}^{d}K_{2i}^p\int_t^{t+u}\mathbb{E}|\theta_i(x_s)|^p ds \\ &\quad \le C_6(\tau)\sup_{u\in[t-2\tau,t]}\mathbb{E}|x(u)|^p \end{aligned} \tag{20}$$

where $C_6(\tau) = 2^{p-1}(\tau^p d_1^{p-1}\sum_{i=1}^{d_1} K_{1i}^p + \tau^{p/2}d^{p-1}p_0 \sum_{i=1}^{d} K_{2i}^p)$. Let $\alpha = \sqrt{\frac{C_6(\tau)}{p-1}}$. Combining (18), (19), and (20), we have

$$\begin{aligned} \mathbb{E}\Theta_{1i}(t) &\le \frac{1}{p}c_3K_{1i}\left[(p-1)\alpha + \frac{C_6(\tau)}{\alpha}\right]\sup_{u\in[t-2\tau,t]}\mathbb{E}|x(u)|^2 \\ &\le 2\frac{\sqrt{p-1}}{pc_1}\sqrt{C_6(\tau)}c_3K_{1i}\sup_{u\in[t-2\tau,t]}\mathbb{E}V(x(u)) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\Theta_{2i}(t) &\le \frac{1}{p}c_4K_{2i}^2\left[(p-1)\alpha + \frac{C_6(\tau)}{\alpha}\right]\sup_{u\in[t-2\tau,t]}\mathbb{E}|x(u)|^2 \\ &\le 2\frac{\sqrt{p-1}}{pc_1}\sqrt{C_6(\tau)}c_4K_{2i}^2\sup_{u\in[t-2\tau,t]}\mathbb{E}V(x(u)). \end{aligned}$$

This together with (17) yields

$$e^{\gamma t}\mathbb{E}V(x(t)) \le \mathbb{E}V(x(0)) + (\gamma-\lambda)\int_0^t e^{\gamma s}\mathbb{E}V(x(s))ds + C_5(\tau)\int_0^t e^{\gamma s}\sup_{u\in[s-3\tau,s]}\mathbb{E}V(x(u))ds, \tag{21}$$

where $C_5(\tau)=2\frac{\sqrt{p-1}}{pc_1}\sqrt{C_6(\tau)}(c_3\sum_{i=1}^{d_1}K_{1i}+c_4\sum_{i=1}^{d}K_{2i}^2)$. Let $\gamma=\lambda$. Then the inequality (21) can be rewritten as $\mathbb{E}V(x(t)) \le e^{-\lambda t}\mathbb{E}V(x(0)) + C_5(\tau)\int_0^t e^{-\lambda(t-s)} \sup_{u\in[s-3\tau,s]}\mathbb{E}V(x(u))ds$. Note that condition (13) implies $\lambda > C_5(\tau)$. Then by the integral version of Halanay inequality (Lemma II.4), there exist M_0 and η, such that

$$\mathbb{E}|x(t)|^p \le M_0 \sup_{\theta\in[-\tau,0]}\mathbb{E}|x(\theta)|^2 e^{-\eta t}$$

where η is the unique root of the following equation $\eta=-\lambda+C_5(\tau)e^{3\eta\tau}$. Therefore, the desired assertion follows. ■

In Theorem II.3, we add two additional conditions 3) and 4) to get the delay tolerance. One may question whether there exists a Lyapunov function satisfying the two conditions. In fact, the classical Lyapunov function $V(x)=(x^TPx)^{p/2}$ with $P>0$ satisfies these conditions with $c_1=\lambda_{\min}(P)^{p/2}$ and $c_i=|P|^{p/2}$ for $i=2,3,4$, and condition 2) in Theorem II.2 has the form

$$\begin{aligned}\mathbb{L}V(x) &= p(x^TPx)^{p/2-1}\left[x^Tf(x)+\frac{1}{2}\sum_{i=1}^{d}g_i(x)^TPg_i(x)\right] \\ &\quad + p\left(\frac{p}{2}-1\right)(x^TPx)^{p/2-1}\sum_{i=1}^{d}|x^TPg_i(x)|^2 \\ &\le -\lambda(x^TPx)^{p/2}. \end{aligned}\tag{22}$$

Moreover, we have the following corollary based on (22) for $p=2$.

Corollary II.1: Assume that there exists a positive definite matrix P, such that (22) holds for $p=2$. Then delay system (2) is mean square exponentially stable if $\sqrt{2(\tau^2d_1\sum_{i=1}^{d_1}K_{1i}^2+\tau dc_2\sum_{i=1}^{d}K_{2i}^2)}(\sum_{i=1}^{d_1}K_{1i}+\sum_{i=1}^{d}K_{2i}^2)\frac{|P|}{\lambda_{\min}(P)}<\lambda$.

Now, for the delay tolerance concerning (10) and (12), we can obtain from Theorem II.3 that if τ satisfies

$$2\frac{\sqrt{p-1}}{p}\sqrt{2\tau^p\mu^p+4\tau^{p/2}\sigma^p}(\mu+\sigma^2)<p\left(\mu-\frac{p-1}{2}\sigma^2\right),$$

then delay system (12) is still pth moment exponentially stable. Especially, if $p=2$, then the mean square stable system (10) can tolerate a delay

$$\tau<\frac{\sqrt{4\sigma^4+2\mu^2(\frac{2\mu-\sigma^2}{\mu+\sigma^2})^2}-2\sigma^2}{\mu^2}$$

such that the delay system (12) is still mean square exponentially stable. Hence, Theorem II.3 under Lyapunov stability conditions is more powerful in solving delay tolerance.

Remark II.3: Theorems II.1 and II.3 can be extended to the case with Markovian switching within a finite number of states. In fact, for the hybrid case, we only need to change the Lyapunov function $V(x)$ to the switched Lyapunov function $V(x,i)$. In this case, the term $\sum_{j=1}^{m}\gamma_{ij}V(x,j)$ will be added into the operator $\mathbb{L}V$, where m is the number of states of a Markov chain $r(t)$ with the state space $\mathbb{S}=\{1,2\ldots,m\}$, and $\Gamma=[\gamma_{ij}]_{m\times m}$ is generator of the Markov chain $r(t)$. It is also interesting to extend current results to the semi-Markov and singular stochastic systems in [39], which made a good contribution to extend the slide model control from deterministic systems to the semi-Markov and singular stochastic versions.

Remark II.4: Note that the above results do not require the concrete form of the delay map ϑ_i and θ_i. In fact, these maps can include many types of delays. For example, the delay can be state-dependent like $\theta_i(x_t)=x(t-\tau\frac{|x(t-\tau)|^2}{1+|x(t-\tau)|^2})$, and state-independent like $\theta_i(x_t)=x(t-\tau_i(t))$, where $\tau_i(t)$ is allowed to be not smooth. The nonsmooth delay can be used to describe the delay induced by the sampled data, where the delay $\tau_i(t)=t-t_k$ for $t\in[t_k,t_{k+1})$, $\{t_k\}_{k=1}^{\infty}$ are the sampling times. In this case, we can see that the sampled-data control problem falls in the delay tolerance issue. For deterministic systems, many works have contributed to this issue [40]. For stochastic systems, only a few works have been achieved due to the nondifferentiability of the solution leading to the failure of many methods for the deterministic systems. Mao and his coauthors [41], [42] proved that the sampled-data control for stochastic system is applicable for sufficient small sampling period. You *et al.* [43] extended these results to get a better delay bound under the constant period sampling. The results obtained in this article remove the assumption of constant period sampling and introduce some explicit conditions on time-varying sampling period.

For deterministic delay systems, the affine Bessel–Legendre inequality [44] and Wirtinger's inequality [45] can be used to get the stability analysis. However, these methods cannot be extended to the stochastic version since these methods involve the derivatives of the state and the state of stochastic system is not differentiable. Note that for stochastic systems, there seems to be no evidence that the Lyapunov functional is better than Lyapunov function. Moreover, one cannot design a Lyapunov functional without knowledge of the form of the delays. So the above analysis takes the Lyapunov function rather than Lyapunov functional for the stability analysis for the delay system (2). Here, the Lyapunov function is from the stability analysis of the delay-free systems.

Note that all the delay tolerance results are based on the global Lipschitz conditions. One may question whether the global Lipschitz assumption can be relaxed. At this point, we do not have a way to relax this assumption for the case above without the knowledge of the concrete form and property of the delays. However, if the delay terms have the forms $\vartheta_i(x_t)=\theta_i(x_t)=x(t-\tau_i(t))$ and $\{\tau_i(t)\}_i$ are differentiable except for the discrete time sequence $\{t_i\}_{i=1}^{\infty}$, $t_i\le t_{i+1}$, $\lim_{i\to\infty}t_i=\infty$, the delay tolerance still holds under a class of non-Lipschitz conditions, which is the focus of the next section.

III. Delay Tolerance Under Nonglobal Lipschitz Conditions

In this section, we study the delay tolerance for the exponential stability under nonglobal Lipschitz condition.

Assumption III.1: Assume that f,g satisfy $f(0)=g_i(0)=0$ and for each $j>0$, there exist positive constants H_j, such that

$$|f(x)-f(y)|\vee|g_i(x)-g_i(y)|\le H_j|x-y|$$

for all $x,y\in\mathbb{R}^n$ with $|x|\vee|y|<j$.

We also assume that the drift f contains a non-Lipschitz part but the delay must not appear in non-Lipschitz term. That is, under the decomposition $f(x) = \sum_{i=0}^{d} f_i(x)$ with $f_0(\cdot)$ being non-Lipschitz continuous, we consider the following delay system

$$dx(t) = \left[f_0(x(t)) + \sum_{i=1}^{d_1} f_i(x(t-\tau_i(t)))\right]dt + \sum_{i=1}^{d} g_i(x(t-\tau_i(t)))dw_i(t) \quad (23)$$

where $\{\tau_i(t)\}$ are time-varying delays and $f_i, g_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. We make the following assumptions on these delays and functions.

Assumption III.2: Each $\tau_i(t) \in [0, \tau]$ is differentiable on $[t_k, t_{k+1})$ for $k = 1, 2, \ldots$ with the derivative $|\dot{\tau}_i(t)| \le \kappa < 1$ for certain $\kappa > 0$.

Assumption III.3: Assume that all the coefficients in (23) are local Lipschitz continuous, and there exist a matrix $P > 0$ and some constants $\mu_0, K_1, \{K_{1i}\} > 0$, $p > 1$, and β, β_1, $\{\sigma_{1i}\}$, $\{\sigma_{2i}\} \ge 0$, such that

$$\begin{aligned} 2x^T P f(x) &\le -\mu_0|x|^2 - \beta|x|^{p+1} \\ g_i(x)^T P g_i(x) &\le \sigma_{1i}|x|^2 + \sigma_{2i}|x|^{p+1}, i = 1, \ldots, d \\ |f(x)| &\le K_1|x| + \beta_1|x|^p \\ |f_i(x)| &\le K_{1i}|x|, i = 1, \ldots, d_1. \end{aligned}$$

Remark III.1: Note that the general diffusion without the nondegenerate property may not contribute to the stability. In this case, high-order term producing $-\beta|x|^{p+1}$ in the drift is required to suppress the growth induced by the high-order term in the diffusion. Hence, high-order terms in the drift have to contribute positively to the stability. In fact, without such condition, the corresponding system may be unstable. To illustrate, consider the deterministic system $dx(t) = (-x(t) + x^3(t))dt$, which satisfies $2x^T f(x) = -2|x|^2 + 2|x|^4$. Then it is easy to see that this system cannot tend to the trivial solution. Moreover, the similar conditions in Assumption III.3 were frequently used (see, for example [21]).

Let $\lambda = \mu_0 - \sum_{i=1}^{d} \sigma_{1i} > 0$ and $\lambda_1 = \beta - \sum_{i=1}^{d} \sigma_{2i} > 0$. Note that Assumption III.3 implies $2x^T P f(x) + \sum_{i=1}^{d} g_i^T(x) P g_i(x) \le -\lambda|x|^2 - \lambda_1|x|^{1+p}$. It follows from [3] that the trivial solution of the delay-free system is almost surely and mean square exponentially stable if $\lambda > 0, \lambda_1 > 0$. Before giving the stability analysis of the delay system, we first examine the regularity of the delay system (23).

Theorem III.1: Let Assumptions III.1, III.2, and III.3 hold with $\lambda > 0$ and $\lambda_1 > 0$. If

$$\sum_{i=1}^{d_1} K_{1i}\tau < 1 \quad (24)$$

$$J_{10} := \lambda - |P|\left(\sum_{i=1}^{d_1} K_{1i}^2 + d_1 K_1^2\right)\tau + H(\kappa)\kappa > 0 \quad (25)$$

$$J_{20} := \lambda_1 - \left(2|P|\beta_1\tau\sum_{i=1}^{d_1} K_{1i} + \frac{\kappa}{1-\kappa}\sum_{i=1}^{d} \sigma_{2i}\right) > 0 \quad (26)$$

where $H(\kappa) = |P|(d_1 \sum_{i=1}^{d_1} K_{1i}^2 + d_1 + \frac{1+d_1\tau}{1-\kappa}\sum_{i=1}^{d_1} K_{1i}^2) + \frac{1}{1-\kappa}\sum_{i=1}^{d} \sigma_{1i}$. Then delay system (23) admits a unique global solution.

Proof: Note that the delay system (23) admits a unique local solution on $t \in [-\tau, \rho_e)$ under Assumption III.3, where ρ_e is the explosion time for the solution $x(t)$. To show that this solution is in fact global, we need only prove $\rho_e = \infty$ a.s. Let $z(t) = x(t) + \sum_{i=1}^{d_1} \int_{t-\tau_i(t)}^{t} f_i(x(s))ds$. Then we have

$$\begin{aligned} dz(t) &= \left[f(x(t)) + \sum_{i=1}^{d_1} \dot{\tau}_i(t) f_i(x(t-\tau_i(t)))\right]dt \\ &\quad + \sum_{i=1}^{d} g_i(x(t-\tau_i(t)))dw_i(t) \\ &=: F(t)dt + \sum_{i=1}^{d} G_i(t)dw_i(t). \end{aligned} \quad (27)$$

Hence, $z(t)$ can be considered as an Itô process. We first prove that the explosion times for $z(t)$ and $x(t)$ are equal. Note that $\sum_{i=1}^{d_1} K_{1i}\tau < 1$ and

$$\begin{aligned} |x(t)| &\le |z(t)| + \sum_{i=1}^{d_1} \int_{t-\tau_i(t)}^{t} |f_i(x(s))|ds \\ &\le |z(t)| + \tau\sum_{i=1}^{d_1} K_{1i} \sup_{t-\tau\le s\le t} |x(s)| \end{aligned}$$

which implies

$$|x(t)| \le \sup_{0\le s\le t} |x(s)| \le \frac{1}{1-\tau\sum_{i=1}^{d_1} K_{1i}} \sup_{0\le s\le t} |z(s)| + C_0$$

where $C_0 = \frac{\tau\sum_{i=1}^{d_1} K_{1i}}{1-\tau\sum_{i=1}^{d_1} K_{1i}} \sup_{-\tau\le s\le 0} |x(s)|$. If we have an explosion time for $z(t)$, denoted by ρ_{ze}, then we must have $\rho_{ze} \le \rho_e$. Note also that

$$\begin{aligned} |z(t)| &\le |x(t)| + \tau\sum_{i=1}^{d_1} K_{1i} \sup_{t-\tau\le s\le t} |x(s)| \\ &\le \left(1 + \tau\sum_{i=1}^{d_1} K_{1i}\right) \sup_{t-\tau\le s\le t} |x(s)|. \end{aligned}$$

This implies $\rho_{ze} \ge \rho_e$. Hence, $\rho_{ze} = \rho_e$. For each $k > |z(0)|$, define the stopping time $\rho_k = \inf\{t \in [0, \rho_e) : |z(t)| \ge k\}$. Clearly, ρ_k is increasing as $k \to \infty$ and $\rho_k \to \rho_\infty \le \rho_{ze} = \rho_e$ a.s. If we can show $\rho_\infty = \infty$ a.s., then $\rho_e = \infty$, which implies that the solution $x(t)$ is global. Define $\theta_i(x_t) = x(t - \tau_i(t))$. Consider the Itô process (27) and introduce a Lyapunov function

$$V_1(x) = x^T P x.$$

Applying the Itô formula, we have

$$dV_1(z(t)) = \mathcal{S}V_1(z(t), t)dt + dM(t) \quad (28)$$

where $M(t) = \sum_{i=1}^{d} \int_0^t 2z^T(s) P g_i(\theta_i(x_s))dw_i(s)$ and

$$\begin{aligned} \mathcal{S}V_1(z(t), t) &= 2z(t)^T P\left(f(x(t)) + \sum_{i=1}^{d_1} \dot{\tau}_i(t) f_i(\theta_i(x_t))\right) \\ &\quad + \sum_{j=1}^{d} g_j^T(\theta_j(x_t)) P g_j^T(\theta_j(x_t)) \end{aligned}$$

$$= 2x(t)^T Pf(x(t)) + \sum_{j=1}^{d} g_j^T(\theta_j(x_t))Pg_j^T(\theta_j(x_t))$$
$$+ 2x(t)^T P \sum_{i=1}^{d_1} \dot{\tau}_i(t) f_i(\theta_i(x_t))$$
$$+ 2\sum_{i=1}^{d_1} \int_{t-\tau_i(t)}^{t} f_i(x(s))^T ds P f(x(t))$$
$$+ 2\sum_{i=1}^{d_1} \int_{t-\tau_i(t)}^{t} f_i(x(s))^T ds P \sum_{i=1}^{d} \dot{\tau}_i(t) f_i(\theta_i(x_t))$$
$$=: 2x(t)^T Pf(x(t)) + J_1(t) + J_2(t) + J_3(t) + J_4(t) \quad (29)$$

where $t \in [t_i, t_{i+1})$, $i = 0, 1, 2\ldots$ By the inequality $2x^T y \le \varepsilon|x|^2 + \frac{1}{\varepsilon}|y|^2$, $\varepsilon > 0$, $x \in \mathbb{R}^n$, we can obtain

$$J_2(t) \le |P| \sum_{i=1}^{d_1} \dot{\tau}_i(t) \left(|x(t)|^2 + K_{1i}^2 |x(t-\tau_i(t))|^2\right)$$
$$\le |P|\kappa d_1 |x(t)|^2 + |P|\kappa \sum_{i=1}^{d_1} K_{1i}^2 |x(t-\tau_i(t))|^2.$$

By the inequality $x^p y^q \le \frac{p}{p+q} x^{p+q} + \frac{q}{p+q} y^{p+q}$, for $p, q, x, y > 0$, we have

$$J_3(t) \le 2|P| \sum_{i=1}^{d_1} |f(x(t))| \int_{t-\tau}^{t} |f_i(x(s))| ds$$
$$\le 2|P| \sum_{i=1}^{d_1} (K_1|x(t)| + \beta_1 |x(t)|^p) \int_{t-\tau}^{t} |f_i(x(s))| ds$$
$$\le |P| d_1 K_1^2 \tau |x(t)|^2 + |P| \sum_{i=1}^{d_1} K_{1i}^2 \int_{t-\tau}^{t} |x(s)|^2 ds$$
$$+ 2|P|\beta_1 \sum_{i=1}^{d_1} K_{1i} (\frac{p}{p+1} \tau |x(t)|^{p+1}$$
$$+ \frac{1}{p+1} \int_{t-\tau}^{t} |x(s)|^{p+1} ds)$$

and

$$J_4(t) \le |P|\kappa d_1 \sum_{i=1}^{d_1} K_{1i}^2 \int_{t-\tau}^{t} |x(s)|^2 ds$$
$$+ |P| d_1 \kappa \tau \sum_{i=1}^{d_1} K_{1i}^2 |x(t-\tau_i(t))|^2.$$

Substituting the three inequalities above into (29) yields

$$\mathcal{S}V_1(z(t), t)$$
$$\le 2x(t)^T Pf(x(t)) + c\kappa \sum_{i=1}^{d_1} K_{1i}^2 |x(t-\tau_i(t))|^2$$
$$+ J_1(t) + b|x(t)|^2 + 2|P|\beta_1 \frac{p\tau}{p+1} \sum_{i=1}^{d_1} K_{1i} |x(t)|^{p+1}$$
$$+ \frac{2|P|\beta_1}{p+1} \sum_{i=1}^{d_1} K_{1i} \int_{t-\tau}^{t} |x(s)|^{p+1} ds + a \int_{t-\tau}^{t} |x(s)|^2 ds$$

where $a = |P|(1 + d_1\kappa) \sum_{i=1}^{d_1} K_{1i}^2$, $b = |P| d_1 (\kappa + \tau K_1^2)$, and $c = |P|(1 + d_1\tau)$. Define $V(z(t), t) = V_1(z(t)) + V_2(t)$, where

$$V_2(t) = \frac{1}{1-\kappa} \sum_{j=1}^{d} \int_{t-\tau_i(t)}^{t} g_j^T(x(s)) P g_j^T(x(s)) ds$$
$$+ a \int_{-\tau}^{0} \int_{t+s}^{t} |x(\theta)|^2 d\theta ds$$
$$+ \frac{c\kappa}{1-\kappa} \sum_{i=1}^{d_1} K_{1i}^2 \int_{t-\tau_i(t)}^{t} |x(s)|^2 ds$$
$$+ \frac{2|P|\beta_1}{p+1} \sum_{i=1}^{d_1} K_{1i} \int_{-\tau}^{0} \int_{t+s}^{t} |x(\theta)|^{p+1} d\theta ds.$$

Note that $V_2(t)$ is differentiable and $z(t)$ is an Itô process. Then, applying the Itô formula again yields that for $t \in [t_i, t_{i+1})$

$$dV(z(t), t) = \mathcal{S}V(z(t), t)dt + dM(t) \quad (30)$$

where

$$\mathcal{S}V(z(t), t)$$
$$= \mathcal{S}V_1(z(t), t) + \dot{V}_2(t)$$
$$\le 2x(t)^T Pf(x(t)) + \sum_{j=1}^{d} g_j^T(x(t)) P g_j^T(x(t))$$
$$+ \left[a\tau + b + \frac{\kappa}{1-\kappa}\left(c\sum_{i=1}^{d_1} K_{1i}^2 + \sum_{i=1}^{d} \sigma_{1i}^2\right)\right] |x(t)|^2$$
$$+ \left(2|P|\beta_1\tau \sum_{i=1}^{d_1} K_{1i} + \frac{\kappa}{1-\kappa} \sum_{i=1}^{d} \sigma_{2i}^2\right) |x(t)|^{p+1}$$
$$\le h_1(\tau)|x(t)|^2 + h_2(\tau)|x(t)|^{p+1} \quad (31)$$

and $h_1(\tau) = -\lambda + [a\tau + b + \frac{\kappa}{1-\kappa}(c\sum_{i=1}^{d_1} K_{1i}^2 + \sum_{i=1}^{d} \sigma_{1i})]$ and $h_2(\tau) = -\lambda_1 + (2|P|\beta_1\tau \sum_{i=1}^{d_1} K_{1i} + \frac{\kappa}{1-\kappa} \sum_{i=1}^{d} \sigma_{2i})$.

Let $U(t) = \mathbb{E}V(z(t), t)$. Then for any $k > |z(0)|$, we have from (30) that for $t \in [t_i, t_{i+1}]$

$$U(t \wedge \rho_k) = U(t_i \wedge \rho_k) + \mathbb{E} \int_{t_i \wedge \rho_k}^{t \wedge \rho_k} \mathcal{S}V(z(s), s) ds$$
$$\le U(t_i \wedge \rho_k) - h_1(\tau)\mathbb{E} \int_{t_i \wedge \rho_k}^{t \wedge \rho_k} |x(s)|^2 ds$$
$$- h_2(\tau)\mathbb{E} \int_{t_i \wedge \rho_k}^{t \wedge \rho_k} |x(s)|^{p+1} ds$$
$$\le U(t_{i-1} \wedge \rho_k) - h_1(\tau)\mathbb{E} \int_{t_{i-1} \wedge \rho_k}^{t \wedge \rho_k} |x(s)|^2 ds$$
$$- h_2(\tau)\mathbb{E} \int_{t_{i-1} \wedge \rho_k}^{t \wedge \rho_k} |x(s)|^{p+1} ds \le \cdots$$
$$\le U(0) - h_1(\tau)\mathbb{E} \int_{0}^{t \wedge \rho_k} |x(s)|^2 ds$$
$$- h_2(\tau)\mathbb{E} \int_{0}^{t \wedge \rho_k} |x(s)|^{p+1} ds.$$

This together with the definition of $V(z(t),t)$ implies

$$\mathbb{E}V_1(z(t\wedge\rho_k)) \le U(t\wedge\rho_k) \le U(0).$$

Note that

$$\mathbb{E}V_1(z(t\wedge\rho_k))) \ge \mathbb{E}(V_1(z(t\wedge\rho_k))1_{\{\rho_k\le t\}}) \ge k^2\mathbb{P}\{\rho_k \le t\}.$$

Hence, $k^2\mathbb{P}\{\rho_k\le t\}\le U(0)$. Then, for any $t>0$, $\lim_{k\to\infty}\mathbb{P}\{\rho_k\le t\}=0$, which together with the arbitrariness of t implies $\rho_\infty=\infty$ a.s. Therefore, the solution $x(t)$ is global. ■

Theorem III.1 shows that if the delay bound τ and its derivative bound κ satisfy conditions (24), (25), and (26), then the solution of the delay system is still global. In other words, for the nonglobal Lipschitz stochastic system, small delays do not affect the existence of the global solution. In what follows, we also show that small delays also do not affect the mean square and almost sure exponential stability of the solutions. The following lemma is important in the almost sure convergence analysis [3].

Lemma III.1. Semimartingale Convergence Theorem: Let $A_1(t)$ and $A_2(t)$ be two $\mathfrak{F}_t$-adapted increasing processes on $t\ge 0$ with $A_1(0)=A_2(0)=0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0)=0$ a.s. and ζ be a nonnegative $\mathfrak{F}_0$-measurable random variable. Assume that $Y(t)$ is nonnegative and

$$Y(t)=\zeta+A_1(t)-A_2(t)+M(t),\ t\ge 0.$$

If $\lim_{t\to\infty}A_1(t)<\infty$ a.s., then for almost all $\omega\in\Omega$,

$$\lim_{t\to\infty}Y(t)<\infty \text{ and } \lim_{t\to\infty}A_2(t)<\infty.$$

Theorem III.2: Let Assumptions III.1, III.2, and III.3 hold with $\lambda>0$ and $\lambda_1>0$. If conditions (24), (25), and (26) hold, then the trivial solution for the delay system (23) is almost surely and mean square exponentially stable, that is, there is a constant $\gamma>0$, such that

$$\limsup_{t\to\infty} e^{\gamma s}|x(s)|^2<\infty\ a.s.$$

and

$$\limsup_{t\to\infty} e^{\gamma s}\mathbb{E}|x(s)|^2<\infty.$$

Proof: We first show that the trivial solution is almost surely exponentially stable. Note that

$$\begin{aligned} &V(z(t),t)\\ &\le (d_1+1)|x(t)|^2+(d_1+1)\sum_{i=1}^{d_1}K_{1i}^2\tau\int_{t-\tau}^t|x(u)|^2du\\ &\quad+\frac{|P|}{1-\kappa}\sum_{j=1}^{d}\sigma_{1i}\vee\sigma_{2i}\left(\int_{t-\tau}^t|x(s)|^2ds+\int_{t-\tau}^t|x(s)|^{p+1}ds\right)\\ &\quad+a\tau\int_{t-\tau}^t|x(s)|^2ds+\frac{c\kappa}{1-\kappa}\sum_{i=1}^{d}K_{1i}^2\int_{t-\tau}^t|x(s)|^2ds\\ &\quad+2|P|\beta_1\frac{1}{p+1}\sum_{i=1}^{d_1}K_{1i}\tau\int_{t-\tau}^t|x(s)|^{p+1}ds\\ &=C_8|x(t)|^2+C_9\int_{t-\tau}^t|x(u)|^2du+C_{10}\int_{t-\tau}^t|x(u)|^{p+1}du \end{aligned}\tag{32}$$

where $C_8=(d_1+1)$, $C_9=(d_1+1)\sum_{i=1}^{d_1}K_{1i}^2\tau+a\tau+\frac{1}{1-\kappa}|P|\sum_{j=1}^{d}\sigma_{1i}\vee\sigma_{2i}+\frac{c\kappa}{1-\kappa}\sum_{i=1}^{d}K_{1i}^2$, $C_{10}=\frac{1}{1-\kappa}|P|\sum_{j=1}^{d}\sigma_{1i}\vee\sigma_{2i}+2|P|\beta_1\frac{1}{p+1}\sum_{i=1}^{d_1}K_{1i}\tau$. Applying the Itô formula to $e^{\gamma t}V_1(z(t))$ and using $V(z(t),t)=V_1(z(t))+V_2(t)$, we have from (31) and (32) that for any $\gamma>0, t\in[t_i,t_{i+1}]$

$$\begin{aligned} &e^{\gamma t}V(z(t),t)\\ &=e^{\gamma t_i}V(z(t_i),t_i)+\int_{t_i}^t e^{\gamma s}\gamma V(z(s),s)ds\\ &\quad+\int_{t_i}^t e^{\gamma s}\mathcal{S}V(z(s),s)ds+\int_{t_i}^t e^{\gamma s}dM(s)\\ &\le e^{\gamma t_i}V(z(t_i),t_i)+(C_8\gamma+h_1(\tau))\int_{t_i}^t e^{\gamma s}|x(s)|^2ds\\ &\quad+h_2(\tau)\int_{t_i}^t e^{\gamma s}|x(s)|^{p+1}ds\\ &\quad+\gamma C_9\int_{t_i}^t e^{\gamma s}\int_{s-\tau}^s|x(u)|^2duds\\ &\quad+\gamma C_{10}\int_{t_i}^t e^{\gamma s}\int_{s-\tau}^s|x(u)|^{p+1}duds+\int_{t_i}^t e^{\gamma s}dM(s)\\ &\le\cdots\\ &\le V(z(0),0)+(h_1(\tau)+\gamma C_8)\int_0^t e^{\gamma s}|x(s)|^2ds\\ &\quad+\gamma C_9\int_0^t e^{\gamma s}\int_{s-\tau}^s|x(u)|^2duds\\ &\quad+h_2(\tau)\int_0^t e^{\gamma s}|x(s)|^{p+1}ds+\int_0^t e^{\gamma s}dM(s)\\ &\quad+\gamma C_{10}\int_0^t e^{\gamma s}\int_{s-\tau}^s|x(u)|^{p+1}duds \end{aligned}\tag{33}$$

where $M(t)$ is defined in (28). Note that for $q=2, p+1$, $\int_0^t e^{\gamma s}\int_{s-\tau}^s|x(u)|^q duds\le\tau e^{\gamma\tau}\int_{-\tau}^0|x(u)|^q du+\tau e^{\gamma\tau}\int_0^t|x(u)|^q du$. Hence, from (33), we get

$$\begin{aligned} e^{\gamma t}V_1(z(t))&\le C_4+J_1(\gamma)\int_0^t e^{\gamma s}|x(s)|^2ds+\int_0^t e^{\gamma s}dM(s)\\ &\quad+J_2(\gamma)\int_0^t e^{\gamma s}|x(s)|^{p+1}ds \end{aligned}\tag{34}$$

where $C_{11}=V(z(0),0)+\gamma C_9\tau e^{\gamma\tau}\int_{-\tau}^0|x(u)|^2du+\gamma C_{10}\tau e^{\gamma\tau}\int_{-\tau}^0|x(u)|^2du$, $J_1(\gamma)=h_1(\tau)+\gamma C_8+\gamma C_9\tau e^{\gamma\tau}$ and $J_2(\gamma)=h_2(\tau)+\gamma C_{10}\tau e^{\gamma\tau}$. It can be observed that $J_i(0)=h_i(\tau)<0$ and $J_i'(\gamma)>0$ for any $\gamma>0$, $i=1,2$. Note that $J_1(\frac{\lambda}{C_8})>0$ and $J_2(\frac{\lambda_1}{C_{10}\tau})>0$. Hence, there are two positive constants γ_1 and γ_2, such that $J_i(\gamma_i)=0$ and $J_i(\gamma)<0$, $i=1,2$, for $\gamma<\gamma^*:=\gamma_1\wedge\gamma_2$. Let $Y(t)=C_{11}+\int_0^t e^{\gamma s}dM(s)$. Note that for $\gamma<\gamma^*$, $-J_1(\gamma)\int_0^t e^{\gamma s}|x(s)|^2ds-J_2(\gamma)\int_0^t e^{\gamma s}|x(s)|^{p+1}ds<Y(t)$. Therefore, the semimartingale convergence theorem yields

$\limsup_{t\to\infty} Y(t) < \infty, a.s.$ and then

$$-J_1(\gamma)\int_0^t e^{\gamma s}|x(s)|^2 ds - J_2(\gamma) \times \int_0^t e^{\gamma s}|x(s)|^{p+1} ds < \infty, \text{a.s.}$$

It is easy to see $\limsup_{s\to\infty} e^{\gamma s}|x(s)|^2 < \infty$ a.s., and then the almost sure exponential stability follows.

It remains to establish the mean square exponential stability. Let ρ_k be defined in Theorem III.1. Then from (34), we have that for $\gamma < \gamma^*$

$$\mathbb{E}e^{\gamma(t\wedge\rho_k)} V(z(t\wedge\rho_k), t\wedge\rho_k) \leq \mathbb{E}C_{11} + J_1(\gamma)\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s)|^2 ds + J_2(\gamma)\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s)|^{p+1} ds.$$

Noting that $\rho_\infty = \infty$ a.s. was proved in Theorem III.1, applying Fatou's lemma yields $-J_1(\gamma)\int_0^t e^{\gamma s}\mathbb{E}|x(s)|^2 ds \leq \mathbb{E}C_{11}, \forall\, t > 0$. This produces the mean square exponential stability. ■

Remark III.2: In Theorem III.1, the delay term in the drift is assumed to be Lipschitz. To date, we have not found a way to handle non-Lipschitz delay terms like $dx(t) = (-\mu x(t) - x(t-\tau)^3)dt + \sigma x(t)dw(t)$. The previous work [46] tells us that the non-Lipschitz delay term may not affect the boundedness of the solution in any finite time. But it is still unclear how the non-Lipschitz delay term affects the stability property for stochastic systems.

From Theorem III.2, we can obtain the following delay tolerance when the non-Lipschitz term $f_0(x)$ vanishes.

Corollary III.1: Let Assumptions III.1, III.2, and III.3 hold with $\lambda > 0$ and $\beta = \beta_1 = \sigma_{2i} = 0$. If conditions (24) and (25) hold, then delay system (23) is stable for small delays $\{\tau_i(t)\}$ with small derivative. In addition, delay system (23) is also almost surely and mean square exponentially stable.

Especially, if the delay is fixed, and the delay system has the following form

$$dx(t) = f(x(t-\tau))dt + \sum_{i=1}^{d} g_i(x(t-\tau))dw_i(t) \quad (35)$$

we have the following corollary.

Corollary III.2: Assume that there exist a function $V(x) \in C^2(\mathbb{R}^n;\mathbb{R}_+)$ and constants $p=2$, $\{c_i\}_{i=1}^4$ satisfying conditions 1), 2), and $V_{xx}(x) \equiv P$ for certain matrix $P > 0$. If $2|P|K_1^2\tau < \lambda$, then the delay system (35) is almost surely and mean square exponentially stable.

Remark III.3: Li and Mao [47] recently contributed an important work on delay feedback stabilization of hybrid stochastic system. If we consider the degenerate system like $dx(t) = u(t)dt + x(t)dw(t)$ with $u(t) = -x(t-\tau)$, our delay tolerance results provide a large delay bound since the stability rule (Rule 3.5 in [47]) requires $\tau < 1/4$, but our Corollary III.2 only requires $\tau < 1/2$.

Now, we continue to consider the stability tolerance of (10) and (12) with $\theta_i(x_t) = x(t-\tau_i)$ based on Theorem III.1. In fact, from Theorem III.1, we can easily obtain that the mean square stable system (10) can tolerate a fixed delay $\tau < \frac{2\mu-\sigma^2}{2\mu^2}$, such that the delay system (12) is still mean square exponentially stable. We can see that if $\mu > \sqrt{2}$, then $\frac{\sqrt{4\sigma^4+2\mu^2(\frac{2\mu-\sigma^2}{\mu^2+\sigma^2})^2}-2\sigma^2}{\mu^2} < \frac{2\mu-\sigma^2}{2\mu^2}$. That is, Corollary III.2 can provide a large delay bound for the case $\mu > \sqrt{2}$. But for some special cases, we can obtain a tight delay bound. To this end, we consider $p=2$ in view of characteristic equation. By Itô formula for (12), we have

$$\frac{d\mathbb{E}|x(t)|^2}{dt} = -2\mu\mathbb{E}[x(t)x(t-\tau_1)]ds + \sigma^2\mathbb{E}|x(t-\tau_2)|^2.$$

Let $Z(t,s) = \mathbb{E}[x(t)x(s)]$. Then we have

$$\dot{Z}(t,t) = -2\mu Z(t, t-\tau_1) + \sigma^2 Z(t-\tau_2, t-\tau_2). \quad (36)$$

To examine the stability of the equation above, we assume that its solution has the form

$$Z(t,s) = e^{\gamma t}e^{\gamma s}. \quad (37)$$

We now examine the conditions on γ, such that (37) is indeed a solution to (36). In fact, substituting (37) into (36) admits the following characteristic equation

$$2\gamma = -2\mu e^{-\gamma\tau_1} + \sigma^2 e^{-2\gamma\tau_2}. \quad (38)$$

That is, the delay equation (12) with $\theta_i(x_t) = x(t-\tau_i)$ is mean square exponentially stable if and only if all the infinitely many characteristic roots of the characteristic equation (38) have negative real parts. Especially, a) if $\tau_1 = 2\tau_2$, by solving the characteristic equation (38), we know that the trivial solution of (10) is mean square exponentially stable if and only if $0 < (\mu - \frac{\sigma^2}{2})\tau_1 < \frac{\pi}{2}$. That is, the mean square stable stochastic system (10) can tolerate the delay $\tau_1 < \frac{0.5\pi}{\mu-\frac{\sigma^2}{2}}$. b) if $\tau_1 = 0$, then solving (38) yields that the mean square stable stochastic system (10) can tolerate any bounded delay τ_2 in the diffusion.

The delay that the linear system (10) can tolerate established above is better than that in Theorem II.1. But the characteristic equation above cannot be used for the nonlinear system and the general pth moment stability. So it is still an important direction to find a refined delay tolerance criterion in the future.

Remark III.4: Razumikhin theorem is another important technique in examining the stability of stochastic delay systems and was developed in [15]–[17] and [48] for different type stochastic systems. In this article, delay tolerance criteria are discussed based on Lyapunov function and functional because these issues are a class of delay-term dominated and delay-dependent stability, which cannot be solved by Razumikhin methods directly. The skills developed in this article would be helpful for us to extend output feedback control studied in [49] to delay output feedback control.

IV. Extensions to Consensus and Tracking Control of Multiagent Systems Under Multiplicative Noises and Nonuniform Delays

In this section, we aim to use the delay tolerance idea to study the multiagent consensus and tracking under multiplicative noises and time-varying nonuniform delays. This is an important extension from the case with the uniform fixed time delays in our previous works [30], [50] to the case with time-varying uninform time delays.

We consider N agents with the information flow structures among different agents being modeled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ denotes the set of undirected edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly. The Laplacian matrix of $\mathcal{G}$ is defined as $L = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\deg_1, \ldots, \deg_N)$ and $\deg_i = \sum_{j=1}^N a_{ij}$, $i = 1, \ldots, N$. Note that $\mathcal{G}$ is undirected. We denote $\widetilde{Q} = [\phi_2, \ldots, \phi_N]$, where ϕ_i is the unit eigenvector of L associated with the eigenvalue $\lambda_i = \lambda_i(L)$, that is, $\phi_i^T L = \lambda_i \phi_i^T$, $\|\phi_i\| = 1$, $i = 2, \ldots, N$. Then $Q = (\frac{1}{\sqrt{N}}\mathbf{1}_N, \widetilde{Q})$ is an orthogonal matrix. Let $\Lambda := \mathrm{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$.

We consider a network of agents with the first-order dynamics

$$\dot{x}_i(t) = u_i(t),\ i = 1, 2, \ldots, N,\ t \in \mathbb{R}_+ \tag{39}$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state of i's agent and $u_i(t) \in \mathbb{R}^n$ is the corresponding input control to be designed. Denote $x(t) = [x_1^T(t), \ldots, x_N^T(t)]^T$ and $u(t) = [u_1^T(t), \ldots, u_N^T(t)]^T$. We consider that the measurements of relative states by agent i have the following form $z_{ji}(t) = \Delta_{xji}(t - \tau_{ji}(t)) + g_{ji}(\Delta_{xji}(t - \tau_{ji}(t)))\xi_{ji}(t)$, $j \in N_i$, where $\Delta_{xji}(t) = x_j(t) - x_i(t)$, $\tau_{ji}(t) \in [0, \tau]$ is the time-varying delay depending the communication channel and satisfying $\tau_{ji}(t) = \tau_{ij}(t)$, N_i denotes the neighbors of the ith agent, $\xi_{ji}(t) \in \mathbb{R}$ denotes the measurement noises and g_{ji} is the noise intensity. We need the following assumptions.

Assumption IV.1: The noise process $\xi_{ji}(t) \in \mathbb{R}$ satisfies $\int_0^t \xi_{ji}(s)ds = w_{ji}(t)$, $t \geq 0, j \in N_i, i = 1, 2, \ldots, N$, where $\{w_{ji}(t), j \in N_i, i = 1, 2, \ldots, N\}$ are scalar independent Brownian motions.

Assumption IV.2: For each (j, i), $g_{ji}(0) = 0$ and there exists a positive constant σ_{ji}, such that $|g_{ji}(x) - g_{ji}(y)| \leq \sigma_{ji}|x - y|$, for all $x, y \in \mathbb{R}^n$.

Based on the measurement $z_{ji}(t)$, we aim to find a control u, such that the multiagent system (39) achieves the mean square consensus or almost sure consensus, whose definitions are as follows.

Definition IV.1: We say that the control $u(t)$ solves mean square (or almost sure) weak consensus if it makes the agents have the property that for any initial data φ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}|x_i(t) - x_j(t)|^2 = 0$ [or $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ almost surely (a.s.)]. If, in addition, there is a random vector $x^* \in \mathbb{R}^n$, such that $\mathbb{E}\|x^*\|^2 < \infty$ and $\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x^*\|^2 = 0$ (or $\mathbb{P}\{\|x^*\| < \infty\} = 1$ and $\lim_{t\to\infty} \|x_i(t) - x^*\| = 0$, a.s.), $i = 1, 2, \ldots, N$, then we say that the control $u(t)$ solves mean square (or almost sure) strong consensus.

We consider the following consensus control

$$u_i(t) = u_i^K(t) = K \sum_{j \in N_i} z_{ji}(t),\ i = 1, 2, \ldots, N \tag{40}$$

where K is the symmetrical control gain matrix to be designed.

Consensus control under undirected topologies. Under control (40), system (39) has the form $dx_i(t) = \sum_{j=1}^N a_{ij}\Delta_{xji}(t - \tau_{ji}(t))dt + \sum_{j=1}^N a_{ij} g_{ji}(\Delta_{xji}(t - \tau_{ji}(t)))dw_{ji}(t)$, which can be rewritten as

$$\begin{aligned} dx(t) = &-\sum_{k=1}^r (L_k \otimes K)x(t - \tau_k(t))dt \\ &+ \sum_{i,j=1}^N [\eta_{N,i} \otimes K g_{ji}(\Delta_{xji}(t - \tau_{ji}(t)))]dw_{ji}(t) \end{aligned} \tag{41}$$

where $r \leq N(N-1)/2$, $\tau_k(\cdot) \in \{\tau_{ij}(\cdot) : i, j = 1, \ldots, N\}$ for $k = 1, \ldots r$, $\eta_{N,i}$ denotes the N-dimensional column vector with the ith element being 1 and others being zero, and $L_k = [l_{kij}] \in \mathbb{R}^{N\times N}$ with

$$l_{kji} = \begin{cases} -a_{ij},\ j \neq i,\ \tau_k(\cdot) = \tau_{ij}(\cdot) \\ 0,\ j \neq i,\ \tau_k(\cdot) \neq \tau_{ij}(\cdot) \\ -\sum_{p\neq i} l_{kip},\ j = i. \end{cases}$$

It can be observed that L_k is symmetric and $\sum_{k=1}^r L_k = L$ since $\tau_{ij}(\cdot) = \tau_{ji}(\cdot)$. Moreover, we can see that each row sum of the matrix L_k is zero, that is, $1_N^T L_k = 0$ for all $k = 1, \ldots, N$. Define $\delta(t) = [(I_N - J_N) \otimes I_n]x(t)\widetilde{\delta}(t) = (Q^{-1} \otimes I_n)\delta(t) = [\widetilde{\delta}_1^T(t), \ldots, \widetilde{\delta}_N^T(t)]^T$, and $\bar{\delta}(t) = [\widetilde{\delta}_2^T(t), \ldots, \widetilde{\delta}_N^T(t)]^T$, $\widetilde{\delta}_i(t) \in \mathbb{R}^n$. Then, by the definition of Q^{-1}, we have $\widetilde{\delta}_1(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N^T \otimes I_n)\delta(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N^T(I_N - J_N) \otimes I_n)y(t) = 0$ and

$$\begin{aligned} d\bar{\delta}(t) = &-\sum_{k=1}^r (\Lambda_k \otimes K)\bar{\delta}(t - \tau_k(t))dt \\ &+ \sum_{i,j=1}^N G_{ij}(t - \tau_{ij}(t))dw_{ji}(t) \end{aligned} \tag{42}$$

where $\Lambda_k = \widetilde{Q}^T L_k \widetilde{Q}$, and $G_{ij}(t) = a_{ij}\widetilde{Q}^T(I_N - J_N)\eta_{N,i} \otimes (K g_{ji}(\Delta_{xji}(t)))$ with $\eta_{N,i}$ denoting the N-dimensional column vector with the ith element being 1 and others being zero. It is easy to see that $\sum_{k=1}^r \Lambda_k = \Lambda$. Note that $\delta_i(t) = x_i - \sum_{k=1}^N x_k(t)/N$, $i = 1, \ldots, N$. Hence, mean square (or almost sure) weak consensus equals $\lim_{t\to\infty} \mathbb{E}|\bar{\delta}(t)|^2 = 0$ (or $\lim_{t\to\infty} |\bar{\delta}(t)| = 0$, a.s.) for any initial data. Note that $|G_{ji}(t)|^2 \leq \frac{N-1}{N}|K|^2\sigma_{ji}^2 a_{ij}|\delta_j(t) - \delta_i(t)|^2$. Considering the Lyapunov function $V(x) = |x|^2$, we obtain for $\tau_{ij}(t) = 0$

$$\begin{aligned} \mathbb{L}V(\bar{\delta}(t)) &= 2\bar{\delta}(t)^T(\Lambda \otimes \frac{K^T + K}{2})\bar{\delta}(t) + \sum_{i,j=1}^N |G_{ij}(t)|^2 \\ &\leq -2\lambda_{\min}(\Phi_K)|\bar{\delta}(t)|^2 \end{aligned} \tag{43}$$

where $\Phi_K = \Lambda \otimes \frac{K+K^T}{2} - \frac{N-1}{N}|K|^2\bar{\sigma}^2(\Lambda \otimes I_n)$, $\bar{\sigma} = \max_{i,j}\sigma_{ji}$. That is, if $\Phi_K > 0$, then the system (42) without time delays is mean square and almost surely exponentially stable [3]. And then the control $u(t)$ defined by (40) solves mean square and almost sure weak consensus for multiagent systems without time delays, which was also proved in [36]. Note that $|G_{ji}(t)|^2 \leq \frac{N-1}{N}|K|^2\sigma_{ji}^2 a_{ij}|\bar{\delta}(t)|^2$. Let $C_{12}(\tau) = |K|^2\sqrt{2}(\tau^2 r\sum_{i=1}^r |\Lambda_i|^2 + \tau N(N-1)\sum_{i,j=1}^N a_{ij}\sigma_{ji}^2)^{1/2}(\sum_{i=1}^r |\Lambda_i| + \frac{N-1}{N}|K|\sum_{i,j=1}^N a_{ij}\sigma_{ji}^2)$. By Corollary II.1, if

$$C_{12}(\tau) < 2\lambda_{\min}(\Phi_K) \tag{44}$$

delay system (42) is mean square exponentially stable, and then the control $u(t)$ defined by (40) solves mean square weak consensus with an exponential rate. Note that all the coefficients in (42) are Lipschitz continuous. Then the mean square exponential stability implies the almost sure exponential stability. That is, under condition (44), the control $u(t)$ defined by (40) also solves almost sure weak consensus with an exponential rate. Hence, from Lemma 4.1 in [30], we have the following theorem.

Theorem IV.1: Let Assumptions IV.1 and IV.2 hold. If (44) holds, then the control $u(t)$ defined by (40) solves mean square and almost sure strong consensus.

From (44), we can see that if the delay-free multiagent system with the condition $\Phi_K > 0$ can achieve the mean square consensus, then a small delay is allowed since the left side of (44) tends to zero as $\tau \to 0$. That is, if the time-delay τ is small and satisfies (44), we do not have to change the control gain K designed for the delay-free case. Then one may ask how about the case with large time delays. In fact, for large delays, we can adjust the control gain to guarantee the mean square consensus. To see it clearly, we choose the control gain $K = kI_n$ with $k > 0$, and then (44) can be rewritten as

$$kb_1\left(\sum_{i=1}^{r}|\Lambda_i| + \frac{N-1}{N}k\sum_{i,j=1}^{N}a_{ij}\sigma_{ji}^2\right) < 2\lambda_2\left(1-\frac{N-1}{N}k\bar{\sigma}^2\right) \tag{45}$$

where $b_1 = \sqrt{2}(\tau^2 r\sum_{i=1}^{r}|\Lambda_i|^2 + \tau N(N-1)\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}^2)^{1/2}$. Let $a = \frac{N-1}{N}b_1\sum_{i,j=1}^{N}a_{ij}\sigma_{ji}^2$, $b = b_1\sum_{i=1}^{r}|\Lambda_i| + \frac{N-1}{N}\lambda_2\bar{\sigma}^2$. Then (45) can be guaranteed by the choice $k \in (0, \bar{k})$, where $\bar{k} = \frac{-b+\sqrt{b^2+8a\lambda_2}}{2a}$.

Tracking Control Under Leader-Following Topologies. We aim to design the control, such that all the N agents can track a leader denoted by 0. The state of the leader is assumed to be a constant denoted by x_0. For the ith follower, the dynamic is described by (39) with $u_i(t)$ defined by (40). Note that this is different from the above since for each agent i, its neighbor set N_i may contain the leader 0. Assumptions IV.1 and IV.2 are also deemed to include the leader 0. Considering the information flow from the leader to the followers, we denote the topology graph by $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{V}}, \widetilde{\mathcal{A}}\}$ with $\widetilde{\mathcal{V}} = \{0,1,2,\ldots,N\}$ and $\widetilde{\mathcal{A}} = \begin{pmatrix} 0 & 0_{1\times N} \\ a_0 & \mathcal{A}\end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$, where $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, $a_0 = [a_{10},\ldots,a_{N0}]^T$, $a_{i0} = 1$ if $0 \in N_i$, otherwise $a_{i0} = 0$. Let $B = diag(a_{10},\ldots,a_{N0})$ and $B_i = diag(0,\ldots,a_{i0},\ldots)$. We use $\mathcal{G} = (\mathcal{V},\mathcal{A})$ to represent the subgraph formed by the N followers, where $\mathcal{V} = \widetilde{\mathcal{V}}\setminus\{0\}$.

Definition IV.2: We say that the control $u(t)$ solves mean square (or almost sure) tracking problem if it makes the $N+1$ agents have the property that for any initial data φ and all $i \in \mathcal{V}$, $\lim_{t\to\infty}\mathbb{E}|x_i(t)-x_0|^2 = 0$ (or $\lim_{t\to\infty}|x_i(t)-x_0| = 0$ a.s.).

We impose the following assumption on the graph $\widetilde{\mathcal{G}}$ and its subgraph $\mathcal{G}$.

Assumption IV.3: Assume that the graph $\widetilde{\mathcal{G}}$ contains a spanning tree and its subgraph $\mathcal{G}$ is undirected.

Let $L_0 = L + B$. Under Assumption IV.3, we know that $\mathcal{L}_0$ is symmetric, and all eigenvalues of the matrix $\mathcal{L}_0$ are positive ([51]), denoted by $\{\lambda_{0i}\}_{i=1}^N$. Hence, there exists an unitary matrix Φ, such that $\Phi^T L_0\Phi = diag(\lambda_{01},\ldots,\lambda_{0N}) =: \Lambda_0$. Without loss of generality, we assume $0 < \lambda_{01} \leq \ldots \leq \lambda_{0N}$.

Let $\delta_i(t) = x_i(t) - x_0$ for $i = 1,\ldots,N$. Define $\delta(t) = [\delta_1^T(t),\ldots,\delta_N^T(t)]^T$. Let $\bar{\delta}(t) = \Phi^T\delta(t)$, $\Phi(i) = \Phi^T\eta_{N,i}$. Then we get $d\delta_i(t) = \sum_{j=0}^{N}a_{ij}\Delta_{\delta ji}(t-\tau_{ji}(t))dt + \sum_{j=0}^{N}a_{ij}g_{ji}(\Delta_{\delta ji}(t-\tau_{ji}(t)))dw_{ji}(t)$. Similarly to (41), we have

$$d\delta(t) = -\sum_{k=1}^{r+N}(L_k\otimes K)\delta(t-\tau_k(t))dt + \sum_{i=1}^{N}\sum_{j=0}^{N}G_{ij}(t)dw_{ji}(t) \tag{46}$$

where L_k for $k = 1,\ldots,r$ are defined above, $L_{r+i} = a_{0i}\eta_{N,i}$, $\tau_{r+i}(t) = \tau_{i0}(t)$, $i = 1,\ldots,N$, and $G_{ij}(t) = a_{ij}[\eta_{N,i}\otimes Kg_{ji}(\Delta_{\delta ji}(t-\tau_{ji}(t)))]$. Hence, mean square (or almost sure) tracking equals $\lim_{t\to\infty}\mathbb{E}|\delta(t)|^2 = 0$ (or $\lim_{t\to\infty}|\delta(t)| = 0$, a.s.) for any initial data. It can be proved that $\sum_{k=1}^{r+N}L_k = L + B$, and

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=0}^{N}a_{ij}|\eta_{N,i}\otimes Kg_{ji}(\Delta_{\delta ji}(t))|^2 &\leq |K|^2\sum_{i=1}^{N}\sum_{j=0}^{N}a_{ij}\sigma_{ji}^2|\Delta_{\delta ji}(t)|^2 \\ &\leq |K|^2\bar{\sigma}^2\delta(t)^T((L+B)\otimes I_n)\delta(t).\end{aligned}$$

Considering the Lyapunov function $V(x) = |x|^2$, we obtain for $\tau_{ij}(t) = 0$

$$\begin{aligned}\mathbb{L}V(\delta(t)) &= 2\delta(t)^T((L+B)\otimes K)\delta(t) + \sum_{i,j=1}^{N}|G_{ij}(t)|^2 \\ &\leq -2\lambda_{\min}(\Phi_K)|\bar{\delta}(t)|^2\end{aligned}$$

where $\Phi_K = (L+B)\otimes\frac{K+K^T}{2} - |K|^2\bar{\sigma}^2((L+B)\otimes I_n)$. That is, if $\Phi_K > 0$, then the system (46) without time delays is mean square exponentially stable. And then the control $u(t)$ solves mean square tracking for the first-order multiagent systems without time delays. Note that $|G_{ji}(t)|^2 \leq |K|^2\sigma_{ji}^2a_{ij}|\bar{\delta}(t-\tau_{ji}(t))|^2$. Let $C_{13}(\tau) = |K|\sqrt{2\tau}(\tau r\sum_{i=1}^{r+N}|L_i|^2 + N \quad (N+1)\sum_{i=1}^{N}\sum_{j=0}^{N}a_{ij}\sigma_{ji}^2)^{1/2}(\sum_{i=1}^{r+N}|L_i| + |K|\sum_{i=1}^{N}\sum_{j=0}^{N}a_{ij}\sigma_{ji}^2)$. By Corollary II.1, if

$$C_{13}(\tau) < 2\lambda_{\min}(\Phi_K), \tag{47}$$

then the delay system (46) is mean square exponentially stable, and then the protocol $u(t)$ solves mean square tracking.

Theorem IV.2: Let Assumptions IV.1 and IV.2 hold. If (47) holds, then the control $u(t)$ defined by (40) solves mean square and almost sure tracking.

Similarly, we can choose the control gain $K = kI_n$ with $k > 0$, and then (47) can be rewritten as

$$kb_1\left(\sum_{i=1}^{r+N}|L_i| + k\sum_{i=1}^{N}\sum_{j=0}^{N}a_{ij}\sigma_{ji}^2\right) < 2\lambda_{01}(1-k\bar{\sigma}^2), \tag{48}$$

where $b_1 = \sqrt{2\tau}(\tau r\sum_{i=1}^{r}|L_i|^2 + N(N+1)\sum_{i,j=1}^{N}a_{ij}\sigma_{ji}^2)^{1/2}$. Let $a = b_1\sum_{i,j=1}^{N}a_{ij}\sigma_{ji}^2$, $b = b_1\sum_{i=1}^{r}|L_i| + \lambda_{01}\bar{\sigma}^2$. Then (48) can be guaranteed by the choice $k \in (0,\bar{k})$, where $\bar{k} = \frac{-b+\sqrt{b^2+8a\lambda_2}}{2a}$.

Remark IV.1: Especially, if the leader-following topology is a star, then the corresponding closed-loop system is decoupled, that is, $d\delta_i(t) = -k\delta_i(t-\tau_{0i}(t))dt + kg_{0i}(-\delta_{0i}(t-\tau_{0i}(t)))dw_{0i}(t)$. Applying Corollary II.1, we obtain that the above system is mean square exponentially stable if $k\sqrt{2\tau^2+4\tau\bar{\sigma}^2}(1+k\bar{\sigma}^2) < 2-k\bar{\sigma}^2$.

Let us consider a scalar four-agent example under the topology graph $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{A}\}$, where $\mathcal{V} = \{1,2,3,4\}$, $\mathcal{E} = \{(1,2),(2,1),(2,3),(3,4),(4,3),(3,2)\}$ and $\mathcal{A} = [a_{ij}]_{4\times 4}$ with $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = 1$ and other being zero. Moreover, we can obtain $\lambda_2 = 0.5858$, $\lambda_3 = 2$, and $\lambda_4 = 3.4142$. The initial state is given by $x(0) = [-7,4,3,-8]^T$.

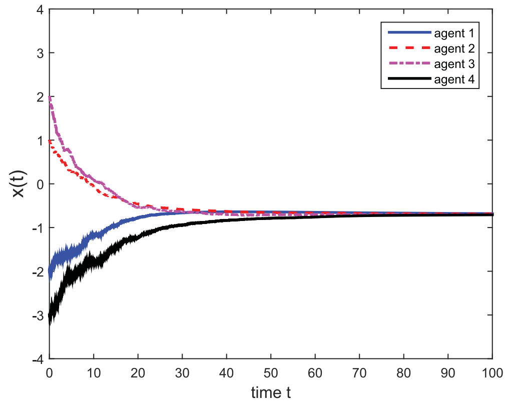

Fig. 1. Asymptotic behaviors of the four agents: $k = 0.05$ and $\tau = 0$.

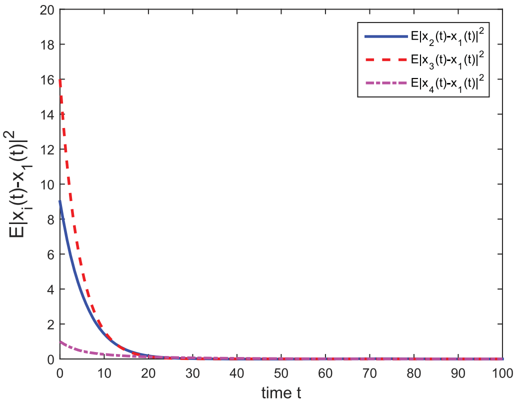

Fig. 2. Mean square errors of $x_i(t) - x_1(t)| : k = 0.05$ and $\tau = 0$.

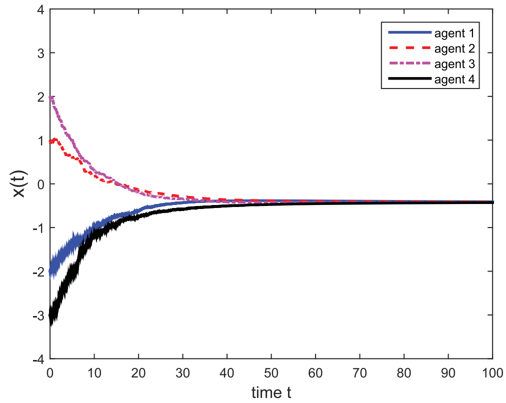

Fig. 3. Asymptotic behaviors of the four agents: $k = 0.05$, $\tau_1 = 0.2$, $\tau_1 = 0.1, \tau_1 = 0.15$.

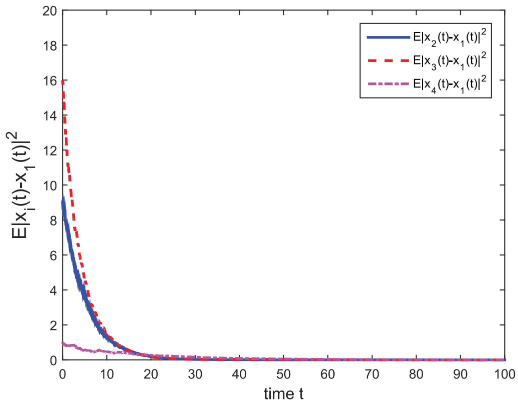

Fig. 4. Mean square errors of $|x_i(t) - x_1(t)| : k = 0.05$, $\tau_1 = 0.2$, $\tau_1 = 0.1, \tau_1 = 0.15$.

Assume $f_{ji}(x) = \sigma_{ji}$ with $\sigma_{ji} = 0.5$, $i, j = 1, 2, 3, 4$, $\bar{\sigma} = \max_{i,j} \sigma_{ji}$.

For the delay-free case, we choose $k = 0.05$, and then it can be seen that $0.01875 = 2\frac{N-1}{N}k\bar{\sigma}^2 < 2\lambda_2 = 0.1716$. That is, Φ_k defined in (43) is positive definite. Hence, the four agents achieve the mean square and almost sure strong consensus. The almost sure strong consensus is revealed in Fig. 1 and the mean square weak consensus is simulated in Fig. 2 by taking 10^3 samples to approximate $\mathbb{E}|x_i(t) - x_1(t)|^2$.

Then we will reveal that a small delay is tolerated for stochastic consensus. Let $\tau_1 = \tau_{12} = \tau_{21} = 0.2$, $\tau_2 = \tau_{23} = \tau_{32} = 0.1$ and $\tau_2 = \tau_{34} = \tau_{43} = 0.15$. The initial data is $x(t) = [-2, 1, 2, -3]^T$ for $t \in [-\tau, 0]$, $\tau = \max_i \tau_i$. In this case, $r = 3$

$$L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then it can be seen that $0.9614 = kb_1(\sum_{i=1}^{r} |\Lambda_i| + \frac{N-1}{N}k\sum_{i,j=1}^{N} \sigma_{ji}^2) < 2\lambda_2(1 - \frac{N-1}{N}k\bar{\sigma}^2) = 1.1606$. That is, condition (45) holds. Hence, by Theorem IV.1, almost sure and mean square strong consensus can be achieved. In fact, Fig. 3 shows that all the agents will tend to a common value. That is, the almost sure strong consensus is solved. Similarly, taking 10^3 samples to approximate $\mathbb{E}|x_i(t) - x_1(t)|^2$ will produce Fig. 4, which depicts the mean square weak consensus.

V. Conclusion

This article establishes an important "robustness" type result, namely, delay tolerance for stable stochastic systems under different Lipschitz type conditions. Under global Lipschitz conditions, we first show that if the delay-free stochastic system is pth moment exponentially stable, the corresponding stochastic delay version is still pth moment exponentially stable for sufficiently small delay. Then we prove that if the moment exponential stability for delay-free system is based on Lyapunov conditions, a large delay is allowed for the delay system to be moment exponentially stable. Without the global Lipschitz conditions, we studied the delay tolerance issues for mean square stable stochastic systems under a class of one-sided linear growth condition on the drift and polynomial growth condition on the diffusion. We also considered applications of delay tolerance to the control design of multiagent systems with multiplicative noises and nonuniform delays.

The results obtained can be used to examine the stochastic systems with G-Brownians and improve the delay bound as in

Ren *et al.* [52]. It can also be extended to the discrete-time case. Based on this article, many related issues can be dealt with. For example, the delay tolerance for the boundedness and ergodicity of stochastic systems.

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