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Mean field linear–quadratic control: Uniform stabilization and social optimality☆

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ABSTRACT

This paper is concerned with uniform stabilization and social optimality for general mean field linear–quadratic control systems, where subsystems are coupled via individual dynamics and costs, and the state weight is *not assumed with the definiteness condition*. For the finite-horizon problem, we first obtain a set of forward–backward stochastic differential equations (FBSDEs) from variational analysis, and construct a feedback-type control by decoupling the FBSDEs. For the infinite-horizon problem, by using solutions to two Riccati equations, we design a set of decentralized control laws, which is further proved to be asymptotically social optimal. Some equivalent conditions are given for uniform stabilization of the systems in different cases, respectively. Finally, the proposed decentralized controls are compared to the asymptotic optimal strategies in previous works.

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1. Introduction

Mean field games have drawn increasing attention in many fields including system control, applied mathematics and economics (Bensoussan, Frehse, & Yam, 2013; Caines, Huang, & Malhamé, 2017; Gomes & Saude, 2014). The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By combining mean field approximations and individual's best response, the dimensionality difficulty is overcome. Mean field games and control have found wide applications, including smart grids (Chen, Busic, Busic, & Meyn, 2017; Li, Ma, Li, Chen, & Gu, 2019; Ma, Callaway, & Hiskens, 2013), finance, economics (Chan & Sircar, 2015; Guéant, Lasry, & Lions, 2011; Wang & Huang, 2019), and social sciences (Bauso, Tembine, & Basar, 2016), etc.

By now, mean field games have been intensively studied in the LQ (linear–quadratic) framework (Bensoussan, Sung, Yam, & Yung, 2016; Elliott, Li, & Ni, 2013; Huang, Caines, & Malhamé, 2007; Li & Zhang, 2008; Moon & Basar, 2017; Wang & Zhang, 2012b). Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an ϵ-Nash equilibrium for mean field LQ games with discount costs by the NCE approach (Huang et al., 2007). The NCE approach was then applied to the cases with long run average costs (Li & Zhang, 2008) and with Markov jump parameters (Wang & Zhang, 2012b), respectively. The works (Bensoussan et al., 2016; Carmona & Delarue, 2013) employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of the equilibrium strategy over a finite horizon. For other aspects of mean field games, readers are referred to Carmona and Delarue (2013), Huang, Malhamé, and Caines (2006), Lasry and Lions (2007) and Yin, Mehta, Meyn, and Shanbhag (2012) for nonlinear mean field games, Weintraub, Benkard, and Van Roy (2008) for oblivious equilibrium in dynamic games, Huang (2010) and Wang and Zhang (2012a) for mean field games with major players, Huang and Huang (2017) and Moon and Basar (2017) for robust mean field games.

Besides noncooperative games, social optima in mean field models have also attracted much interest. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual costs, which is a type of team decision problem (Ho, 1980). Huang et al. considered social optima in mean field LQ control, and provided an asymptotic

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team-optimal solution (Huang, Caines, & Malhamé, 2012). Wang and Zhang (2017) investigated the mean field social optimal problem where the Markov jump parameter appears as a common source of randomness. For further literature, see Huang and Nguyen (2016) for social optima in mixed games, Arabneydi and Mahajan (2015) for team-optimal control with finite population and partial information.

Most previous results on mean field games and control were given by using the fixed-point method (Cardaliaguet, 2012; Carmona & Delarue, 2013; Huang et al., 2007, 2012; Li & Zhang, 2008; Wang & Zhang, 2012a, 2017). However, the fixed-point analysis (e.g., from the contraction mapping theorem) is sometimes conservative, particularly for high-dimensional systems. In this paper, we solve the problem by decoupling directly high-dimensional forward–backward stochastic differential equations (FBSDEs). In recent years, some progress has been made for study of the optimal LQ control by tackling the FBSDEs. See Sun, Li, and Yong (2016), Zhang, Qi, and Fu (2019), Zhang and Xu (2017) and Yong (2013) for details.

This paper investigates uniform stabilization and social optimality for linear–quadratic mean field control systems, where subsystems (agents) are coupled via dynamics and individual costs. The state weight Q is not limited to positive semi-definite. This model can be taken as a generation of robust mean field control problems (Huang & Huang, 2017; Moon & Basar, 2017; Wang & Huang, 2017). Since the weight Q in the cost functional is indefinite, the prior boundedness of the state is not implied directly by the finiteness of the cost, which brings about additional difficulty to show the social optimality of decentralized control.

For the finite-horizon social control problem, we first obtain a set of FBSDEs by examining the variation of the social cost, and give centralized feedback-type control laws by decoupling the FBSDEs. With mean field approximations, we design a set of decentralized control laws. By exploiting the uniform convexity property of the problem, the decentralized controls are further shown to have asymptotic social optimality. For the infinite-horizon case, we design a set of decentralized control laws by using solutions of two Riccati equations, which is shown to be asymptotically social optimal. Some equivalent conditions are further given for uniform stabilization of all the subsystems when the state weight Q is positive semi-definite or only symmetric. Furthermore, the explicit expressions of optimal social costs are given in terms of the solutions to two Riccati equations, and the proposed decentralized control laws are compared to the feedback strategies in previous works. Finally, some numerical examples are given to illustrate the effectiveness of the proposed control laws.

The main contributions of the paper are summarized as follows.

- We first obtain necessary and sufficient existence conditions of finite-horizon centralized optimal control by variational analysis, and then design a feedback-type decentralized control by tackling FBSDEs with mean field approximations.
- In the case $Q \geq 0$, the necessary and sufficient conditions are given for uniform stabilization of the systems with the help of the system's observability and detectability.
- In the case that Q is *indefinite*, the necessary and sufficient conditions are given for uniform stabilization of the systems using the Hamiltonian matrices.
- The asymptotically optimal decentralized controls are obtained under very basic assumptions (without verifying the fixed-point condition). The corresponding social costs are explicitly given by virtue of the solutions to two Riccati equations.

The organization of the paper is as follows. In Section 2, the socially optimal control problem is formulated. In Section 3, we construct asymptotically optimal decentralized control laws by tackling FBSDEs for the finite-horizon case. In Section 4, for the infinite-horizon case, the asymptotically optimal controls are designed and analyzed, and some equivalent conditions are further given for uniform stabilization in different cases. In Section 5, some numerical examples are given to show the effectiveness of the proposed control laws. Section 6 concludes the paper.

The following notation will be used throughout this paper. $\|\cdot\|$ denotes the Euclidean vector norm or Frobenius matrix norm. For a vector z and a matrix Q, $\|z\|_Q^2 = z^TQz$, $tr(Q)$ is the trace of the matrix Q, and $Q > 0$ ($Q \geq 0$) means that Q is positive definite (positive semidefinite). For two vectors x, y, $\langle x, y\rangle = x^Ty$. $C([0,T],\mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued continuous functions defined on $[0,T]$, and $C_{\rho/2}([0,\infty),\mathbb{R}^n)$ is a subspace of $C([0,\infty),\mathbb{R}^n)$ which is given by $\{f|\int_0^\infty e^{-\rho t}\|f(t)\|^2dt < \infty\}$. $L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ is the space of all $\mathcal{F}$-adapted $\mathbb{R}^k$-valued processes $x(\cdot)$ such that $\mathbb{E}\int_0^T\|x(t)\|^2dt < \infty$. For convenience of presentation, we use $C, C_1, C_2, \ldots$ to denote generic positive constants, which may vary from place to place.

2. Problem description

Consider a large population system with N agents. Agent i evolves by the following stochastic differential equation:

$$dx_i(t) = [Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f(t)]dt + \sigma(t)dW_i(t), \quad 1 \leq i \leq N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and input of the ith agent. $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t), f, \sigma \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. $\{W_i(t), 1 \leq i \leq N\}$ are a sequence of independent 1-dimensional Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. The cost function of agent i is given by

$$J_i(u) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_i(t) - \Gamma x^{(N)}(t) - \eta(t)\right\|_Q^2 + \|u_i(t)\|_R^2\right\}dt, \tag{2}$$

where $\rho > 0$ and Q, R are symmetric matrices with appropriate dimensions. Q is allowed to be *indefinite*. $R > 0$, and $\eta \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. Denote $u = \{u_1,\ldots,u_N\}$. The decentralized control set is given by

$$\mathcal{U}_{d,i} = \left\{u_i \;\middle|\; u_i(t) \text{ is adapted to } \sigma(x_i(s), 0 \leq s \leq t), \mathbb{E}\int_0^\infty e^{-\rho t}\|u_i(t)\|^2dt < \infty\right\}.$$

For comparison, define the centralized control sets as

$$\mathcal{U}_{c,i} = \left\{u_i \middle| u_i(t) \text{ is adapted to } \sigma\{\bigcup_{i=1}^N \mathcal{F}_t^i\}, \mathbb{E}\int_0^\infty e^{-\rho t}\|u_i(t)\|^2dt < \infty\right\},$$

and $\mathcal{U}_c = \{(u_1,\ldots,u_N)|u_i \text{ belongs to } \mathcal{U}_{c,i}, 1 \leq i \leq N\}$, where $\mathcal{F}_t^i = \sigma(x_i(0), W_i(s), 0 \leq s \leq t), i = 1,\ldots,N$.

In this paper, we mainly study the following problem.

(P). Seek a set of decentralized control laws to optimize social cost for the system (1)–(2), i.e., $\inf_{u_i\in\mathcal{U}_{d,i}} J_{\rm soc}$, where $J_{\rm soc} = \sum_{i=1}^N J_i(u)$.

Remark 2.1. The related results can be extended to the case of multidimensional Brownian motions trivially. Here we consider that $\sigma(t)$ is time-varying and satisfies some growth rate. For convenience of the statement, we assume W_i is scalar and $\sigma \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. For the finite-horizon problem, our results still hold for the case that the matrices $A, B, G, \ldots$ depend on t.

Assume

(A1) The initial states of agents $x_i(0), i = 1,\ldots,N$ are mutually independent and have the same mathematical expectation. $x_i(0) = x_{i0}$, $\mathbb{E}x_i(0) = \bar{x}_0$, $i = 1,\ldots,N$. There exists a constant C_0 (independent of N) such that $\max_{1\le i\le N}\mathbb{E}\|x_i(0)\|^2 < C_0$.

3. The finite-horizon problem

For the convenience of design, we first consider the following finite-horizon problem.

$$(\mathbf{P1})\quad \inf_{u\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nr})} J^{\mathrm{F}}_{\mathrm{soc}}(u),$$

where $J^{\mathrm{F}}_{\mathrm{soc}}(u) = \sum_{i=1}^N J^{\mathrm{F}}_i(u)$ and $\mathcal{F}_t = \sigma\{\bigcup_{i=1}^N \mathcal{F}^i_t\}$. Here

$$J^{\mathrm{F}}_i(u) = \mathbb{E}\int_0^T e^{-\rho t}\Big\{\big\|x_i(t) - \Gamma x^{(N)}(t) - \eta(t)\big\|^2_Q + \|u_i(t)\|^2_R\Big\}dt. \tag{3}$$

We first give equivalent conditions for the convexity of (P1).

Proposition 3.1. *(i) Problem (P1) is convex in u if and only if for any $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^r)$, $i = 1,\ldots,N$,*

$$\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big\{\big\|y_i(t) - \Gamma y^{(N)}(t)\big\|^2_Q + \|u_i(t)\|^2_R\Big\}dt \ge 0,$$

where $y^{(N)} = \sum_{j=1}^N y_j/N$ and y_i satisfies

$$dy_i(t) = [Ay_i(t) + Gy^{(N)}(t) + Bu_i(t)]dt,$$
$$y_i(0) = 0,\ i = 1,2,\ldots,N. \tag{4}$$

(ii) Problem (P1) is uniformly convex in u if and only if for any $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^r)$, there exists $\gamma > 0$ such that

$$\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big\{\big\|y_i(t) - \Gamma y^{(N)}(t)\big\|^2_Q + \|u_i(t)\|^2_R\Big\}dt \ge \gamma\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\|u_i(t)\|^2 dt.$$

Proof. Let x_i and $\acute{x}_i$ be the state processes of agent i with the control υ and $\acute{\upsilon}$, respectively. Take any $\lambda_1 \in [0,1]$ and let $\lambda_2 = 1-\lambda_1$. Then

$$\begin{aligned}&\lambda_1 J^{\mathrm{F}}_{\mathrm{soc}}(\upsilon) + \lambda_2 J^{\mathrm{F}}_{\mathrm{soc}}(\acute{\upsilon}) - J^{\mathrm{F}}_{\mathrm{soc}}(\lambda_1\upsilon + \lambda_2\acute{\upsilon})\\ =&\lambda_1\lambda_2\sum_{i=1}^N \mathbb{E}\int_0^T\{\|x_i(t) - \acute{x}_i(t) - \Gamma(x^{(N)}(t) - \acute{x}^{(N)}(t))\|^2_Q\\ &+\|\upsilon_i(t) - \acute{\upsilon}_i(t)\|^2_R\}dt.\end{aligned}$$

Denote $u = \upsilon - \acute{\upsilon}$, and $y_i = x_i - \acute{x}_i$. Thus, y_i satisfies (4). By the definition of (uniform) convexity, the lemma follows. □

By examining the variation of $J^{\mathrm{F}}_{\mathrm{soc}}$, we obtain the necessary and sufficient conditions for the existence of centralized optimal control of (P1). To simplify the presentation later, we denote

$$\begin{cases}\Xi \triangleq \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma,\\ \bar{\eta} \triangleq Q\eta - \Gamma^T Q\eta.\end{cases}$$

Theorem 3.1. *Suppose $R > 0$. Then (P1) has a set of optimal control laws if and only if Problem (P1) is convex in u and the following equation system admits a set of solutions $(x_i, p_i, \beta^j_i, i,j = 1,\ldots,N)$:*

$$\begin{cases}dx_i(t) = \big(Ax_i(t) - BR^{-1}B^T p_i(t) + Gx^{(N)}(t) + f(t)\big)dt\\ \qquad + \sigma(t)dW_i(t),\\ dp_i(t) = -\big[(A-\rho I)^T p_i(t) + G^T p^{(N)}(t) + Qx_i(t)\big]dt\\ \qquad + \big[\Xi x^{(N)}(t) + \bar{\eta}(t)\big]dt + \sum_{j=1}^N \beta^j_i(t)dW_j(t),\\ x_i(0) = x_{i0},\quad p_i(T) = 0,\quad i = 1,\ldots,N,\end{cases} \tag{5}$$

where $p^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N p_i(t)$, and furthermore the optimal control is given by $\check{u}_i(t) = -R^{-1}B^T p_i(t)$.

Proof. Suppose that $\check{u}_i = -R^{-1}B^T p_i$, where $(p_i, \beta^j_i, i,j = 1,\cdots,N)$ is a set of solutions to the second equation in (5). Denote by $\check{x}_i$ the state of agent i under the control $\check{u}_i$. For any $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^r)$ and $\theta \in \mathbb{R}$ $(\theta \ne 0)$, let $u^\theta_i = \check{u}_i + \theta u_i$. Denote by x^θ_i the solution of the following perturbed state equation

$$\begin{aligned}dx^\theta_i(t) =& \Big[Ax^\theta_i(t) + B(\check{u}_i(t) + \theta u_i(t)) + f(t)\\ &+\frac{G}{N}\sum_{i=1}^N x^\theta_i(t)\Big]dt + \sigma(t)dW_i(t),\\ x^\theta_i(0) =& x_{i0},\ i = 1,2,\ldots,N.\end{aligned}$$

Let $y_i = (x^\theta_i - \check{x}_i)/\theta$. It can be verified that y_i satisfies (4). Then by Itô's formula, for any $i = 1,\ldots,N$,

$$\begin{aligned}0 =&\mathbb{E}[\langle e^{-\rho T}p_i(T), y_i(T)\rangle - \langle p_i(0), y_i(0)\rangle]\\ =&\mathbb{E}\int_0^T e^{-\rho t}\Big[\langle -\big[(A-\rho I)^T p_i(t) + G^T p^{(N)}(t) + Qx_i(t)\big]\\ &+\Xi x^{(N)}(t) + \bar{\eta}(t), y_i(t)\rangle + \langle p_i(t), (A-\rho I)y_i(t)\\ &+Gy^{(N)}(t) + Bu_i(t)\rangle\Big]dt,\end{aligned}$$

which implies

$$\begin{aligned}0 =& \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big[\langle -\big(G^T p^{(N)}(t) + Qx_i(t)\big)\\ &+\Xi x^{(N)}(t) + \bar{\eta}(t), y_i(t)\rangle\\ &+\langle G^T p^{(N)}(t), y_i(t)\rangle + \langle B^T p_i(t), u_i(t)\rangle\Big]dt.\end{aligned} \tag{6}$$

From (3), we have

$$J^{\mathrm{F}}_{\mathrm{soc}}(\check{u} + \theta u) - J^{\mathrm{F}}_{\mathrm{soc}}(\check{u}) = 2\theta I_1 + \theta^2 I_2 \tag{7}$$

where $\check{u} = (\check{u}_1,\ldots,\check{u}_N)$, and

$$\begin{aligned}I_1 \triangleq& \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big[\big\langle Q\big(\check{x}_i(t) - (\Gamma\check{x}^{(N)}(t) + \eta)\big),\\ &y_i(t) - \Gamma y^{(N)}(t)\big\rangle + \langle R\check{u}_i(t), u_i(t)\rangle\Big]dt,\\ I_2 \triangleq& \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big[\big\|y_i(t) - \Gamma y^{(N)}(t)\big\|^2_Q + \|u_i(t)\|^2_R\Big]dt.\end{aligned}$$

Note that (suppressing the time t)

$$\begin{aligned}&\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\big\langle Q\big(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)\big), \Gamma y^{(N)}\big\rangle dt\\ =&\mathbb{E}\int_0^T e^{-\rho t}\Big\langle \Gamma^T Q\sum_{i=1}^N\big(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)\big), \frac{1}{N}\sum_{j=1}^N y_j\Big\rangle dt\end{aligned}$$

$$=\sum_{j=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big\langle\frac{\Gamma^TQ}{N}\sum_{i=1}^{N}\big(\check{x}_i-(\Gamma\check{x}^{(N)}+\eta)\big),y_j\Big\rangle dt$$
$$=\sum_{j=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\big\langle\Gamma^TQ\big((I-\Gamma)\check{x}^{(N)}-\eta\big),y_j\big\rangle dt.$$

From (6), one can obtain that

$$\begin{aligned}I_1=&\mathbb{E}\sum_{i=1}^{N}\int_0^T e^{-\rho t}\Big[\big\langle Q\big(\check{x}_i-(\Gamma\check{x}^{(N)}+\eta)\big),\\&y_i-\Gamma y^{(N)}\big\rangle+\langle R\check{u}_i+B^Tp_i,u_i\rangle\Big]dt\\&+\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big[\big\langle-\big(G^Tp^{(N)}(t)+Qx_i(t)\big)\\&+\Xi x^{(N)}(t)+\bar{\eta}(t)+(A-\rho I)^Tp_i+G^Tp^{(N)},y_i\big\rangle dt\\=&\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\langle R\check{u}_i+B^Tp_i,u_i\rangle dt.\end{aligned}\tag{8}$$

From (7), $\check{u}$ is a minimizer to Problem (P1) if and only if $I_2\geq 0$ and $I_1=0$. By Proposition 3.1, $I_2\geq 0$ if and only if (P1) is convex. $I_1=0$ is equivalent to

$$\check{u}_i=-R^{-1}B^Tp_i.$$

Thus, we have the optimality system (5). This implies that (5) admits a solution $(\check{x}_i,\check{p}_i,\check{\beta}_i^j,i,j=1,\dots,N)$.

On other hand, if the equation system (5) admits a solution $(\check{x}_i,\check{p}_i,\check{\beta}_i^j,i,j=1,\dots,N)$. Let $\check{u}_i=-R^{-1}B^T\check{p}_i$. If (P1) is convex, then $\check{u}$ is a minimizer to Problem (P1). □

It follows from (5) that

$$\begin{cases}dx^{(N)}(t)=\big[(A+G)x^{(N)}(t)-BR^{-1}B^Tp^{(N)}(t)+f(t)\big]dt\\\qquad+\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma(t)dW_i(t),\\dp^{(N)}(t)=-\Big[(A+G-\rho I)^Tp^{(N)}(t)\\\qquad+(I-\Gamma)^TQ(I-\Gamma)x^{(N)}(t)-\bar{\eta}(t)\Big]dt\\\qquad+\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N}\beta_i^j(t)dW_j(t),\\x^{(N)}(0)=\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}x_{i0},\quad p^{(N)}(T)=0.\end{cases}\tag{9}$$

Let $p_i(t)=P(t)x_i(t)+K(t)x^{(N)}(t)+s(t)$, $t\geq 0$. Then by (5), (9) and Itô's formula (suppressing the time t),

$$\begin{aligned}dp_i=&\dot{P}x_idt+P\Big[\big(Ax_i-BR^{-1}B^T(Px_i+Kx^{(N)}+s)\\&+Gx^{(N)}+f\big)dt+\sigma dW_i\Big]+(\dot{s}+\dot{K}x^{(N)})dt\\&+K\Big\{\Big[(A+G)x^{(N)}-BR^{-1}B^T((P+K)x^{(N)}\\&+s)+f\Big]dt+\frac{1}{N}\sum_{i=1}^{N}\sigma dW_i\Big\}\\=&-\Big[(A-\rho I)^T(Px_i+Kx^{(N)}+s)\\&+G^T((P+K)x^{(N)}+s)\\&+Qx_i-\Xi x^{(N)}-\bar{\eta}\Big]dt+\sum_{j=1}^{N}\beta_i^jdW_j.\end{aligned}$$

This implies $\beta_i^i=\frac{1}{N}K\sigma+P\sigma$, $\beta_i^j=\frac{1}{N}K\sigma$, $j\neq i$,

$$\begin{aligned}\rho P(t)=&\dot{P}(t)+A^TP(t)+P(t)A+Q\\&-P(t)BR^{-1}B^TP(t),\ P(T)=0,\end{aligned}\tag{10}$$
$$\begin{aligned}\rho K(t)=&\dot{K}(t)+(A+G)^TK(t)+K(t)(A+G)+G^TP(t)\\&+P(t)G-(P(t)+K(t))BR^{-1}B^T(P(t)+K(t))\\&+P(t)BR^{-1}B^TP(t)-\Xi,\ K(T)=0,\end{aligned}\tag{11}$$
$$\begin{aligned}\rho s(t)=&\dot{s}(t)+[A+G-BR^{-1}B^T(P+K)]^Ts(t)\\&+(P+K)f(t)-\bar{\eta}(t),\ s(T)=0.\end{aligned}\tag{12}$$

Remark 3.1. Note that (11) is not a standard Riccati equation. Its solvability may be referred to Abou-Kandil, Freiling, Ionescu, and Jank (2003). In particular, by Theorem 4.3 in Ma and Yong (1999, Chapter 2), if $\det\Big\{[0,\ I]e^{\mathcal{A}t}\begin{bmatrix}0\\I\end{bmatrix}\Big\}>0$ with $\mathcal{A}=\begin{bmatrix}A-\frac{\rho}{2}I & -BR^{-1}B^T\\-Q & -A^T+\frac{\rho}{2}I\end{bmatrix}$, then we have

$$P(t)=\Big\{[0,\ I]e^{\mathcal{A}t}\begin{bmatrix}0\\I\end{bmatrix}\Big\}^{-1}\Big\{[0,\ I]e^{\mathcal{A}t}\begin{bmatrix}I\\0\end{bmatrix}\Big\}.$$

Remark 3.2. Denote $\Pi=P+K$. Then from (10) and (11), Π satisfies

$$\begin{aligned}&\rho\Pi(t)=\dot{\Pi}(t)+(A+G)^T\Pi(t)+\Pi(t)(A+G)\\&-\Pi(t)BR^{-1}B^T\Pi(t)+(I-\Gamma)^TQ(I-\Gamma).\end{aligned}\tag{13}$$

with $\Pi(T)=0$. By Sun et al. (2016, Theorem 4.5), the solvability of (10) and (11) is equivalent to the uniform convexity of two optimal control problems. Particularly, if $Q\geq 0$, then (10) and (11) admit a unique solution, respectively.

Theorem 3.2. *Assume (A1) holds, and (10)–(11) admit a solution, respectively. Then (P1) has an optimal control*

$$\check{u}_i(t)=-R^{-1}B^T[P(t)x_i(t)+K(t)x^{(N)}(t)+s(t)],$$

where P,K and s are determined by (10)–(12).

To prove Theorem 3.2, we first provide a lemma, which plays a key role in the later analysis.

Lemma 3.1. *If (10) and (11) admit a solution, respectively, then Problem (P1) is uniformly convex.*

Proof. By (10), (13), and direct calculations, we have

$$\begin{aligned}&\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\big\|y_i(t)-\Gamma y^{(N)}(t)\big\|_Q^2+\|u_i(t)\|_R^2\Big)dt\\=&\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\|y_i(t)\|_Q^2-\|y^{(N)}(t)\|_{\Xi}^2+\|u_i(t)\|_R^2\Big)dt\\=&\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\|y_i(t)-y^{(N)}(t)\|_Q^2+\|y^{(N)}(t)\|_{Q-\Xi}^2\\&+\|u_i(t)-u^{(N)}(t)\|_R^2+\|u^{(N)}(t)\|_R^2\Big)dt\\=&\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\big\|u_i(t)-u^{(N)}(t)\\&+R^{-1}B^TP(t)(y_i(t)-y^{(N)}(t))\big\|_R^2\\&+\big\|u^{(N)}(t)+R^{-1}B^T\Pi(t)y^{(N)}(t)\big\|_R^2\Big)dt\end{aligned}$$

$$\geq \sum_{i=1}^{N} \mathbb{E} \int_0^T e^{-\rho t} \Big(\big\| u_i(t) + R^{-1}B^T P(t)(y_i(t) - y^{(N)}(t)) + R^{-1}B^T \Pi(t) y^{(N)}(t) \big\|_R^2 \Big) dt$$

$$\geq \gamma \sum_{i=1}^{N} \mathbb{E} \int_0^T e^{-\rho t} \|u_i(t)\|^2 dt,$$

where the last line follows by Sun et al. (2016, Lemma 2.3). From Theorem 3.1, the lemma follows. □

Proof of Theorem 3.2. Since (10) and (11) have a solution, respectively, then by Ma and Yong (1999, Chapter 2, §4), (9) admits a unique solution. Thus, the FBSDE (5) is decoupled and the existence of a solution follows. From Lemma 3.1, (P1) is uniformly convex. By Theorem 3.1, (P1) has an optimal control given by $\breve{u}_i(t) = -R^{-1}B^T[P(t)x_i(t) + K(t)x^{(N)}(t) + s(t)]$, $t \geq 0$, where P, K and s are determined by (10)–(12). □

As an approximation to $x^{(N)}$ in (9), we obtain

$$\frac{d\bar{x}}{dt} = (A+G)\bar{x}(t) - BR^{-1}B^T(\Pi(t)\bar{x}(t) + s(t)) + f(t), \ \bar{x}(0) = \bar{x}_0. \tag{14}$$

Then, by Theorem 3.2, the decentralized control law for agent i may be taken as

$$\hat{u}_i(t) = -R^{-1}B^T[P(t)\hat{x}_i(t) + K(t)\bar{x}(t) + s(t)], \quad 0 \leq t \leq T, \ i = 1, \ldots, N, \tag{15}$$

where P, K, and s are determined by (10)–(12), and $\bar{x}$ and $\hat{x}_i$ respectively satisfy (14) and

$$d\hat{x}_i(t) = \Big[(A - BR^{-1}B^T P(t))\hat{x}_i(t) + G\hat{x}^{(N)}(t) + f(t) - BR^{-1}B^T(K(t)\bar{x}(t) + s(t)) \Big] dt + \sigma(t) dW_i(t). \tag{16}$$

Remark 3.3. Here, we firstly obtain the centralized open-loop solution by variational analysis. By tackling the coupled FBSDEs combined with mean field approximations, the decentralized control laws are designed. Note that in this case s and $\bar{x}$ are fully decoupled and no fixed-point equation is needed.

Theorem 3.3. *Assume that (A1) holds, and (10)–(11) admit a solution, respectively. The set of decentralized control laws $\{\hat{u}_1, \ldots, \hat{u}_N\}$ in (15) has asymptotic social optimality, i.e.,*

$$\Big| \frac{1}{N} J_{\rm soc}^{\rm F}(\hat{u}) - \frac{1}{N} \inf_{u \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nr})} J_{\rm soc}^{\rm F}(u) \Big| = O(\frac{1}{\sqrt{N}}),$$

and the corresponding social cost is given by

$$J_{\rm soc}^{\rm F}(\hat{u}) = \sum_{i=1}^{N} \mathbb{E}\Big\{ \|x_{i0} - x^{(N)}(0)\|_P^2 + \|x^{(N)}(0)\|_\Pi^2 + 2s^T(0)x^{(N)}(0) \Big\} + Nq_T + N\epsilon_T, \tag{17}$$

where

$$q_T = \int_0^T e^{-\rho t} \Big[\|\sigma(t)\|_{P(t)}^2 + \|\sigma(t)\|_{\Pi(t)}^2 - \|B^T s(t)\|_{R^{-1}}^2 + 2s^T(t)f(t) \Big] dt, \tag{18}$$

$$\epsilon_T = \mathbb{E} \int_0^T e^{-\rho t} \|B^T K(t)(x^{(N)}(t) - \bar{x}(t))\|_{R^{-1}}^2 dt. \tag{19}$$

Proof. See Appendix A. □

4. The infinite-horizon problem

Based on the analysis in Section 3, we may design the following decentralized control laws for Problem (P):

$$\hat{u}_i(t) = -R^{-1}B^T[P\hat{x}_i(t) + (\Pi - P)\bar{x}(t) + s(t)], \quad t \geq 0, \ i = 1, \ldots, N, \tag{20}$$

where P and Π are maximal solutions[1] to the equations

$$\rho P = A^T P + PA - PBR^{-1}B^T P + Q, \tag{21}$$

$$\rho \Pi = (A+G)^T \Pi + \Pi(A+G) - \Pi BR^{-1}B^T \Pi + Q - \Xi, \tag{22}$$

and $s, \bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ are determined by

$$\rho s(t) = \dot{s}(t) + (A + G - BR^{-1}B^T \Pi)^T s(t) + \Pi f(t) - \bar{\eta}(t), \tag{23}$$

$$\dot{\bar{x}}(t) = (A+G)\bar{x}(t) - BR^{-1}B^T(\Pi\bar{x}(t) + s(t)) + f(t), \ \bar{x}(0) = \bar{x}_0. \tag{24}$$

Here $s(0)$ is to be determined, and the existence conditions of P, Π, s and $\bar{x}$ need to be investigated further.

4.1. Uniform stabilization of subsystems

We now list some basic assumptions for reference:
(A2) The system $(A - \frac{\rho}{2}I, B)$ is stabilizable, and $(A + G - \frac{\rho}{2}I, B)$ is stabilizable. Particularly, $\bar{A} + G - \frac{\rho}{2}I$ is Hurwitz, where $\bar{A} \triangleq A - BR^{-1}B^T P$.
(A3) $Q \geq 0$, $(A - \frac{\rho}{2}I, \sqrt{Q})$ is observable, and $(A + G - \frac{\rho}{2}I, \sqrt{Q}(I - \Gamma))$ is observable.
Assumptions (A2) and (A3) are basic in the study of the LQ optimal control problem. We will show that under some conditions, (A2) is also necessary for uniform stabilization of multiagent systems. In many cases, (A3) may be weakened to the following assumption.
(A3′) $Q \geq 0$, $(A - \frac{\rho}{2}I, \sqrt{Q})$ is detectable, and $(A + G - \frac{\rho}{2}I, \sqrt{Q}(I - \Gamma))$ is detectable.

Lemma 4.1. *Under (A2)–(A3), (21) and (22) admit unique solutions $P > 0, \Pi > 0$, respectively, and (23)–(24) admits a set of unique solutions $s, \bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$.*

Proof. From (A2)–(A3) and (Anderson & Moore, 1990), (21) and (22) admit unique solutions $P > 0, \Pi > 0$ such that $A - BR^{-1}B^T P - \frac{\rho}{2}I$ and $A + G - BR^{-1}B^T \Pi - \frac{\rho}{2}I$ are Hurwitz, respectively. From an argument in Wang and Zhang (2012a, Appendix A), we obtain $s \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ if and only if

$$s(t) = \int_t^\infty e^{-(A+G-BR^{-1}B^T \Pi - \rho I)(t-\tau)} (\Pi f(\tau) - \bar{\eta}(\tau)) d\tau. \quad \square$$

Lemma 4.2. *Let (A1)–(A3) hold. Then for Problem (P),*

$$\mathbb{E} \int_0^\infty e^{-\rho t} \|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 dt = O(\frac{1}{N}), \tag{25}$$

where $\hat{x}^{(N)} = \sum_{i=1}^N \hat{x}_i$, and $\bar{x}$ satisfies (24).

Proof. See Appendix B. □

It is shown that the decentralized control laws (15) uniformly stabilize the systems (1) .

[1] For a Riccati equation (e.g., (21)), P is called a maximal solution if for any solutions P', $P - P' \geq 0$.

Theorem 4.1. *Let (A1)–(A3) hold. Then for any N,*

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty. \tag{26}$$

Proof. See Appendix B. □

We now give two equivalent conditions for uniform stabilization of multiagent systems.

Theorem 4.2. *Let (A3) hold. Assume that (21)–(22) admit symmetric solutions. Then for Problem (P) the following statements are equivalent:*
(i) For any initial condition $(\hat{x}_1(0),\ldots,\hat{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty. \tag{27}$$

(ii) Eqs. (21) and (22) admit unique maximal solutions such that $P>0$, $\Pi>0$, and $\bar{A}+G-\frac{\rho}{2}I$ is Hurwitz.
(iii) (A2) holds.

Proof. See Appendix C. □

For $G=0$, we have a simplified version of Theorem 4.2.

Corollary 1. *Assume that (A3) holds and $G=0$. Assume that (21)–(22) admit symmetric solutions. Then the following statements are equivalent:*
(i) For any $(\hat{x}_1(0),\ldots,\hat{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty.$$

(ii) Eqs. (21) and (22) admit unique maximal solutions such that $P>0$, $\Pi>0$, respectively.
(iii) The system $(A-\frac{\rho}{2}I,B)$ is stabilizable.

When (A3) is weakened to (A3′), we have the following equivalent conditions of uniform stabilization.

Theorem 4.3. *Let (A3′) hold. Assume that (21)–(22) admit solutions. Then the following are equivalent:*
(i) For any initials $(\hat{x}_1(0),\ldots,\hat{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty.$$

(ii) Eqs. (21) and (22) admit unique maximal solutions $P\geq 0$, $\Pi\geq 0$, and $\bar{A}+G-\frac{\rho}{2}I$ is Hurwitz.
(iii) (A2) holds.

Proof. See Appendix C. □

For the more general case that Q are indefinite, we have the following equivalent conditions for uniform stabilization of all the subsystems. Assume
(**A3**″) both M_1 and M_2 have no eigenvalues on the imaginary axis, where

$$M_1=\begin{bmatrix}A-\frac{\rho}{2}I & BR^{-1}B^T\\ Q & -A^T+\frac{\rho}{2}I\end{bmatrix},$$

$$M_2=\begin{bmatrix}A+G-\frac{\rho}{2}I & BR^{-1}B^T\\ Q-\Xi & -(A+G)^T+\frac{\rho}{2}I\end{bmatrix}.$$

Theorem 4.4. *Assume that (A3″) holds, and (21)–(22) admit solutions. Then the following are equivalent:*
(i) For any $(\hat{x}_1(0),\ldots,\hat{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty.$$

(ii) Eqs. (21) and (22) admit unique ρ-stabilizing solutions[2] (which are also the maximal solutions), and $\bar{A}+G-\frac{\rho}{2}I$ is Hurwitz.
(iii) (A2) holds.

Remark 4.1. M_1 and M_2 are Hamiltonian matrices. The Hamiltonian matrix plays a significant role in studying general algebraic Riccati equations. See more details of the property of Hamiltonian matrices in Abou-Kandil et al. (2003) and Molinari (1977).

Remark 4.2. For the case $Q=0$ and $G=0$, the Hamiltonian matrices reduce to

$$M_1=M_2=\begin{bmatrix}A-\frac{\rho}{2}I & BR^{-1}B^T\\ 0 & -A^T+\frac{\rho}{2}I\end{bmatrix}.$$

Then it follows from Theorem 4.4 that if $A-\frac{\rho}{2}I$ have no eigenvalues on the imaginary axis, the decentralized controls (15) uniformly stabilize the systems (1) if and only if $(A-\frac{\rho}{2}I,B)$ is stabilizable. Since $Q=0$ and $A-\frac{\rho}{2}I$ is not Hurwitz necessarily, the system $(A-\frac{\rho}{2}I,\sqrt{Q})$ is not detectable, which implies that the assumptions of Theorem 4.3 in Huang et al. (2012) do not hold.

To show Theorem 4.4, we need two lemmas. The first lemma is copied from Molinari (1977, Theorem 6).

Lemma 4.3. *Eqs. (21) and (22) admit unique ρ-stabilizing solutions (which are also the maximal solutions) if and only if (A2) and (A3″) hold.*

Lemma 4.4. *Let (A1) hold. Assume that (21) and (22) admit ρ-stabilizing solutions, respectively, and $\bar{A}+G-\frac{\rho}{2}I$ is Hurwitz. Then*

$$\sum_{i=1}^{N}\mathbb{E}\int_0^{\infty} e^{-\rho t}\left(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2\right)dt<\infty.$$

Proof. From the definition of ρ-stabilizing solutions, $A-BR^{-1}B^TP-\frac{\rho}{2}I$ and $A+G-BR^{-1}B^T\Pi-\frac{\rho}{2}I$ are Hurwitz. By the argument in the proof of Theorem 4.1, the lemma follows. □

Proof of Theorem 4.4. By using Lemmas 4.3 and 4.4 together with a similar argument in the proof of Theorem 4.1, the theorem follows. □

Example 1. Consider a scalar system with $A=a$, $B=b$, $G=g$, $Q=q$, $\Gamma=\gamma$, $R=r>0$. Then

$$M_1=\begin{bmatrix}a-\rho/2 & b^2/r\\ q & -a+\rho/2\end{bmatrix},$$

$$M_2=\begin{bmatrix}a+g-\rho/2 & b^2/r\\ q(1-\gamma)^2 & -(a+g-\rho/2)\end{bmatrix}.$$

By direct computations, neither M_1 nor M_2 has eigenvalues in the imaginary axis if and only if

$$(a-\frac{\rho}{2})^2+\frac{b^2}{r}q>0, \tag{28}$$

$$(a+g-\frac{\rho}{2})^2+\frac{b^2}{r}(1-\gamma)^2q>0. \tag{29}$$

[2] For a Riccati equation (21), P is called a ρ-stabilizing solution if P satisfies (21) and all the eigenvalues of $A-BR^{-1}B^TP-\frac{\rho}{2}I$ are in left half-plane.

Note that if $q>0$ (or $a-\rho/2<0$, $q=0$), i.e., $(a-\rho/2, \sqrt{q})$ is observable (detectable), then (28) holds, and if $(1-\gamma)^2q>0$ $(a+g-\rho/2<0, q=0)$, i.e., $(a+g-\rho/2, \sqrt{q}(1-\gamma))$ is observable (detectable), then (29) holds.

For this model, the Riccati equation (21) is written as

$$\frac{b^2}{r}p^2-(2a-\rho)p-q=0. \tag{30}$$

Let $\Delta=4[(a-\rho/2)^2+b^2q/r]$. If (28) holds then $\Delta>0$, which implies (30) admits two solutions. If $q>0$ then (30) has a unique positive solution such that $a-b^2p/r-\rho/2=-\sqrt{\Delta}/2<0$. If $q=0$ and $a-\rho/2<0$ then (30) has a unique non-negative solution $p=0$ such that $a-b^2p/r-\rho/2=a-\rho/2<0$.

Assume that (28) and (29) hold. By Theorem 4.4, the system is uniformly stable if and only if $(a-\rho/2, b)$ is stabilizable (i.e., $b\neq 0$ or $a-\rho/2<0$), and $a-b^2p/r-\rho/2+g<0$. Note that $a-b^2p/r-\rho/2<0$. When $g\leq 0$, we have $a-b^2p/r-\rho/2+g<0$.

Example 2. We further consider the model in Example 1 for the case that $a+g=\rho/2$ and $\gamma=1$ (i.e., (29) does not hold). In this case, the Riccati equation (22) admits a unique solution $\Pi=0$. (23) becomes $\rho s(t)=\dot{s}(t)+\frac{\rho}{2}s(t)$ and has a unique solution $s(t)\equiv 0$ in $C_{\rho/2}([0,\infty),\mathbb{R})$. Thus, $\bar{x}$ satisfies

$$\frac{d\bar{x}}{dt}=\frac{\rho}{2}\bar{x}(t)+f(t). \tag{31}$$

Assume that f is a constant. Then (31) does not admit a solution in $C_{\rho/2}([0,\infty),\mathbb{R})$ unless $\bar{x}(0)=-2f/\rho$.

4.2. Asymptotic social optimality

Now we are in a position to state the asymptotic optimality of the decentralized control.

Theorem 4.5. *Let (A1)–(A3) hold. For Problem (P), the set of decentralized control laws $\{\hat{u}_1,\dots,\hat{u}_N\}$ given by (20) has asymptotic social optimality, i.e.,*

$$\left|\frac{1}{N}J_{\rm soc}(\hat{u})-\frac{1}{N}\inf_{u\in\mathcal{U}_c}J_{\rm soc}(u)\right|=O(1/\sqrt{N}).$$

Proof. We first prove that for $u\in\mathcal{U}_c$, $J_{\rm soc}(u)<NC_1$ implies that

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}(\|x_i(t)\|^2+\|u_i(t)\|^2)dt<NC_2, \tag{32}$$

for all $i=1,\dots,N$. From $J_{\rm soc}(u)<NC_1$, we have $\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\|u_i(t)\|^2dt<NC$ and

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\left\|x_i(t)-\Gamma x^{(N)}(t)\right\|_Q^2dt<NC, \tag{33}$$

which further implies that

$$\begin{aligned}&\mathbb{E}\int_0^\infty e^{-\rho t}\left\|(I-\Gamma)x^{(N)}(t)\right\|_Q^2\\ \leq&\frac{1}{N}\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\left\|x_i(t)-\Gamma x^{(N)}(t)\right\|_Q^2dt<C.\end{aligned} \tag{34}$$

By (1) we have

$$\begin{aligned}dx^{(N)}(t)=&\left[(A+G)x^{(N)}(t)+Bu^{(N)}(t)+f(t)\right]dt\\&+\frac{1}{N}\sum_{i=1}^N\sigma(t)dW_i(t),\end{aligned}$$

which leads to for any $r\in[0,1]$,

$$\begin{aligned}x^{(N)}(t)=&e^{(A+G)r}x^{(N)}(t-r)\\&+\int_{t-r}^t e^{(A+G)(t-\tau)}[Bu^{(N)}(\tau)+f(\tau)]d\tau\\&+\frac{1}{N}\sum_{i=1}^N\int_{t-r}^t e^{(A+G)(t-\tau)}\sigma(\tau)dW_i(\tau).\end{aligned} \tag{35}$$

By $J_{\rm soc}(u)<C_1$ and basic SDE estimates, we can find a constant C such that

$$\mathbb{E}\int_r^\infty e^{-\rho t}\left\|\int_{t-r}^t e^{(A+G)(t-\tau)}Bu^{(N)}(\tau)d\tau\right\|^2dt\leq C.$$

From (34) and (35) we obtain

$$\begin{aligned}\mathbb{E}\int_r^\infty& e^{-\rho t}[x^{(N)}(t-r)]^Te^{(A+G)^Tr}(I-\Gamma)^TQ(I-\Gamma)\\&\cdot e^{(A+G)r}x^{(N)}(t-r)dt\leq C,\end{aligned}$$

which implies that for any $r\in[0,1]$,

$$\begin{aligned}\mathbb{E}\int_0^\infty& e^{-\rho t}[x^{(N)}(\tau)]^Te^{(A+G)^Tr}(I-\Gamma)^TQ(I-\Gamma)\\&\cdot e^{(A+G)r}x^{(N)}(\tau)d\tau\leq C.\end{aligned}$$

By taking integration with respect to $r\in[0,1]$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^\infty& e^{-\rho t}[x^{(N)}(\tau)]^T\left[\int_0^1 e^{(A+G)^Tr}(I-\Gamma)^TQ(I-\Gamma)\right.\\&\left.\cdot e^{(A+G)r}dr\right]x^{(N)}(\tau)d\tau\leq C.\end{aligned}$$

This together with (A3) leads to

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|x^{(N)}(t)\|^2dt<C, \tag{36}$$

which with (33) further gives

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\|x_i(t)\|_Q^2dt<NC. \tag{37}$$

By (1), we have

$$\begin{aligned}x_i(t)=&e^{Ar}x_i(t-r)\\&+\int_{t-r}^t e^{A(t-\tau)}[Bu_i(\tau)+f(\tau)+Gx^{(N)}(\tau)]d\tau\\&+\int_{t-r}^t e^{A(t-\tau)}\sigma(\tau)dW_i(\tau).\end{aligned} \tag{38}$$

It follows from (36) that

$$\begin{aligned}&\mathbb{E}\int_r^\infty e^{-\rho t}\left\|\int_{t-r}^t e^{A(t-\tau)}Gx^{(N)}(\tau)d\tau\right\|^2dt\\ \leq&\mathbb{E}\int_0^\infty e^{-\rho\tau}\|Gx^{(N)}(\tau)\|^2\int_0^r\left\|e^{(A-\frac{\rho}{2}I)v}\right\|^2dvd\tau\leq C.\end{aligned}$$

From (37) and (38), we obtain that

$$\sum_{i=1}^N\mathbb{E}\int_r^\infty e^{-\rho t}x_i^T(t-r)e^{A^Tr}Qe^{Ar}x_i(t-r)dt\leq NC.$$

This together with (A3) implies that

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\|x_i(t)\|^2dt<NC,$$

which gives (32). From this with Theorem 4.1,

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\tilde{x}_i(t)\|^2+\|\tilde{u}_i(t)\|^2\right)dt<NC.$$

By a similar argument to the proof of Theorem 3.3 combined with Lemma 4.2, the conclusion follows. □

If (A3) is replaced by (A3′), the decentralized control (20) still has asymptotic social optimality.

Corollary 2. *Assume that (A1)–(A2), (A3′) hold. The decentralized control (20) is asymptotically social optimal.*

Proof. Without loss of generality, we simply assume $A+G=\mathrm{diag}\{\mathbb{A}_1,\mathbb{A}_2\}$, where $\mathbb{A}_1-(\rho/2)I$ is Hurwitz, and $-(\mathbb{A}_2-(\rho/2)I)$ is Hurwitz (if necessary, we may apply a nonsingular linear transformation as in the proof of Theorem 4.3). Write $x^{(N)}=[z_1^T,z_2^T]$ and $Q^{1/2}(I-\Gamma)=[S_1,S_2]$ such that $\big\|(I-\Gamma)x^{(N)}(t)\big\|_Q^2=\|S_1z_1(t)+S_2z_2(t)\|^2$, and $(\mathbb{A}_2-(\rho/2)I,S_2)$ is observable. By the proof of Theorem 4.1 or (Huang, 2010), $\mathbb{E}\int_0^\infty e^{-\rho t}\|u^{(N)}(t)\|^2dt<\infty$ implies $\mathbb{E}\int_0^\infty e^{-\rho t}\|z_1(t)\|^2dt<\infty$, which together with (34) gives $\mathbb{E}\int_0^\infty e^{-\rho t}\|S_2z_2(t)\|^2dt<\infty$. This and the observability of $(A_2-(\rho/2)I,S_2)$ leads to $\mathbb{E}\int_0^\infty e^{-\rho t}\|z_2(t)\|^2dt<\infty$. Thus, $\mathbb{E}\int_0^\infty e^{-\rho t}\|x^{(N)}(t)\|^2dt<\infty$. The other parts of the proof are similar to that of Theorem 4.5. □

For the case that Q are indefinite, we have the following result of asymptotic optimality.

Theorem 4.6. *Let (A1)–(A2), (A3″) hold. Assume (21)–(22) admit negative definite solutions $P^-<0$ and $\Pi^-<0$, respectively. Then, the set of decentralized control in (20) is asymptotically socially optimal. Furthermore, if $\{x_{i0}\}$ have the same variance, then the asymptotic average social optimum is given by*

$$\lim_{N\to\infty}\frac{1}{N}J_{\rm soc}(\hat u)=\mathbb{E}\big[\|x_{i0}-\bar x_0\|_P^2+\|\bar x_0\|_\Pi^2+2s^T(0)\bar x_0\big]+q_\infty,$$

where

$$q_\infty=\int_0^\infty e^{-\rho t}\big[\|\sigma(t)\|_P^2+\|\sigma(t)\|_\Pi^2 -\|B^Ts(t)\|_{R^{-1}}^2+2s^T(t)f(t)\big]dt. \qquad (39)$$

Proof. From the above assumptions and Theorem 4.4, the Riccati equation (21) admits a ρ-stabilizing solution P and a negative definite solution P^-; (22) has a ρ-stabilizing solution Π and a negative definite solution Π^-. By a similar argument in the proof of Lemma 3.1, we obtain for any $u\in\mathcal{U}_c$,

$$\begin{aligned}
&J_{\rm soc}(u)\\
=&\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\Big(\|x_i-x^{(N)}\|_Q^2+\|x^{(N)}\|_{Q-\Xi}^2+\|\eta\|_Q^2\\
&-2\eta^TQ(I-\Gamma)x_i+\|u_i-u^{(N)}\|_R^2+\|u^{(N)}\|_R^2\Big)dt\\
=&\sum_{i=1}^N\mathbb{E}\Big[\|x_{i0}-x^{(N)}(0)\|_{P^-}^2+\|x^{(N)}(0)\|_{\Pi^-}^2\\
&+2s^T(0)x^{(N)}(0)\Big]-\lim_{T\to\infty}\sum_{i=1}^N e^{-\rho T}\mathbb{E}\big[\|x^{(N)}(T)\|_{\Pi^-}^2\\
&+\|x_i(T)-x^{(N)}(T)\|_{P^-}^2+2s^T(T)x^{(N)}(T)\big]\\
&+\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\Big(\big\|u^{(N)}+R^{-1}B^T\Pi^-x^{(N)}\big\|_R^2\\
&+\big\|u_i-u^{(N)}+R^{-1}B^TP^-(x_i-x^{(N)})\big\|_R^2\Big)dt+q_\infty.
\end{aligned}$$

By Willems (1971, Theorem 8), the centralized optimal control exists and the optimal state is ρ-stable. Thus, we only need to consider the following control set

$$\mathcal{U}_c'=\Big\{(u_1,\dots,u_N)|u_i(t)\text{ is adapted to }\mathcal{F}_t,\ \mathbb{E}\int_0^\infty e^{-\rho t}\|x_i(t)\|^2dt<\infty,\forall i\Big\}.$$

For any $u\in\mathcal{U}_c'$ satisfying $J_{\rm soc}(u)\le NC$, we have

$$\begin{aligned}
&J_{\rm soc}(u)\\
=&\sum_{i=1}^N\mathbb{E}\big[\|x_{i0}-x^{(N)}(0)\|_P^2+\|x^{(N)}(0)\|_\Pi^2+2s^T(0)\bar x_0\big]\\
&+\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}\Big(\big\|u_i-u^{(N)}+R^{-1}B^TP(x_i-x^{(N)})\big\|_R^2\\
&+\big\|u^{(N)}+R^{-1}B^T\Pi x^{(N)}\big\|_R^2\Big)dt+q_\infty\le NC. \qquad (40)
\end{aligned}$$

Denote $v^{(N)}=u^{(N)}+R^{-1}B^T\Pi x^{(N)}$. From (1),

$$\begin{aligned}
dx^{(N)}(t)=&(A+G-BR^{-1}B^T\Pi)x^{(N)}(t)dt\\
&+Bv^{(N)}(t)dt+\frac{1}{N}\sum_{i=1}^N\sigma(t)dW_i(t).
\end{aligned}$$

By Huang (2010), there exist $C_1,C_2>0$ such that

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|x^{(N)}\|^2dt\le C_1\mathbb{E}\int_0^\infty e^{-\rho t}\|v^{(N)}\|^2+C_2.$$

This together with (40) gives

$$\begin{aligned}
&\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}(\|x^{(N)}\|^2+\|u^{(N)}\|^2)dt\\
=&N\mathbb{E}\int_0^\infty e^{-\rho t}(\|x^{(N)}\|^2+\|v^{(N)}-R^{-1}B^T\Pi x^{(N)}\|^2)dt\\
\le&NC_3\mathbb{E}\int_0^\infty e^{-\rho t}\|v^{(N)}\|^2+NC_4\le NC. \qquad (41)
\end{aligned}$$

Similarly, we have

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}(\|x_i-x^{(N)}\|^2+\|u_i-u^{(N)}\|^2)dt\le NC.$$

From this and (41),

$$\sum_{i=1}^N\mathbb{E}\int_0^\infty e^{-\rho t}(\|x_i\|^2+\|u_i\|^2)dt\le NC.$$

The remainder of the proof can follow by that of Theorem 3.3. For the case that $\{x_{i0}\}$ have the same variance, from (17), the asymptotic average social optimum ($\lim_{N\to\infty}\frac{1}{N}J_{\rm soc}(\hat u)$) is given by $\mathbb{E}\big[\|x_{i0}-\bar x_0\|_P^2+\|\bar x_0\|_\Pi^2+2s^T(0)\bar x_0\big]+q_\infty$. □

Remark 4.3. The work Huang et al. (2012) investigated mean field LQ problem (P) with $Q\ge 0$. To obtain asymptotic social optimality, they need $Q>0$ and $I-\Gamma$ is nonsingular. In Corollary 2, we have loosed the assumption to (A3′), i.e., $(A-(\rho/2)I,\sqrt{Q})$ and $(A-(\rho/2)I,\sqrt{Q}(I-\Gamma))$ are detectable. In Theorem 4.6, we further give the condition for the case of indefinite Q. Particularly, for the scalar case, the condition is equivalent to (28)–(29). It can be verified that the assumption $Q>0$ and $I-\Gamma$ is nonsingular implies (28)–(29), but the converse is not true.

4.3. Comparison to previous solutions

In this section, we compare the proposed decentralized control laws with the feedback decentralized strategies in previous works.

We first introduce a definition from Basar and Olsder (1982).

Definition 4.1. For a control problem with an admissible control set $\mathcal{U}$, a control law $u \in \mathcal{U}$ is said to be a representation of another control $u^* \in \mathcal{U}$ if
(i) they both generate the same unique state trajectory, and
(ii) they both have the same open-loop value on this trajectory.

For Problem (P), let $f = 0$, and $G = 0$. In Huang et al. (2012, Theorem 4.3), the decentralized control laws are given by

$$\breve{u}_i(t) = -R^{-1}B^T(Px_i(t) + \bar{s}(t)), \quad i = 1, \ldots, N, \tag{42}$$

where P is the stabilizing solution of (21), and $\bar{s} = \bar{K}x^\dagger + \phi$. Here $\bar{K}$ satisfies

$$\rho\bar{K} = \bar{K}\bar{A} + \bar{A}^T\bar{K} - \bar{K}BR^{-1}B^T\bar{K}^T - \Xi,$$

and $x^\dagger, \phi \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ are given by

$$\frac{d\bar{x}^\dagger}{dt} = \bar{A}\bar{x}^\dagger(t) - BR^{-1}B^T(\bar{K}\bar{x}^\dagger(t) + \phi(t)), \bar{x}^\dagger(0) = \bar{x}_0,$$
$$\frac{d\phi}{dt} = -[A - BR^{-1}B^T(P + \bar{K}) - \rho I]\phi(t) + \bar{\eta}(t),$$

in which $\bar{A} = A - BR^{-1}B^TP$ and $\phi(0)$ is to be determined by $\phi \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$. By comparing this with (22)–(24), one can obtain that $\bar{K} = \Pi - P$, $\bar{x} = \bar{x}^\dagger$ and $\phi = s$. From the above discussion, we have the equivalence of the two sets of decentralized control laws.

Proposition 4.1. *The set of decentralized control laws* $\{\hat{u}_1, \ldots, \hat{u}_N\}$ *in* (20) *is a representation of* $\{\breve{u}_1, \ldots, \breve{u}_N\}$ *given by* (42).

Remark 4.4. The work Huang et al. (2012) studied the problem (P) with $Q \geq 0$ by the fixed-point approach. In Theorem 4.3, they have shown that the fixed-point equation admits a unique solution, when $(A - (\rho/2)I, \sqrt{Q})$ is detectable and $\Xi = \Gamma^TQ + Q\Gamma - \Gamma^TQ\Gamma \leq 0$. In fact, the above assumption is merely a sufficient condition to ensure (A3′) $(A - (\rho/2)I, \sqrt{Q - \Xi})$ is detectable.

Remark 4.5. The work Huang and Zhou (2020) investigated asymptotic solvability of mean field LQ games by the re-scaling method. They considered (1)–(2) with $Q \geq 0$ and derived a low-dimensional ordinary differential equation system by dynamic programming. Actually, the method proposed in this paper can be viewed as a type of direct approach. Different from Huang and Zhou (2020), we tackle directly high-dimensional FBSDEs along the line of maximum principle.

5. Numerical examples

Now, two numerical examples are given to illustrate the effectiveness of the proposed decentralized control.

We first consider a scalar system with 30 agents in Problem (P). Take $A = 0.8, B = R = 1, Q = -0.1, G = -0.2, f(t) = 1, \sigma(t) = 0.2, \rho = 0.6, \Gamma = 0.2, \eta = 5$ in (1)–(2). The initial states of 50 agents are taken independently from a normal distribution $N(5, 0.3)$. Note that $B \neq 0$, and $\bar{A} + G - \frac{\rho}{2}I = -0.5873 < 0$. Then (A1)–(A2) hold. Since $M_1 = \begin{bmatrix} 0.5 & 1 \\ -0.1 & -0.5 \end{bmatrix}, M_2 = \begin{bmatrix} 0.3 & 1 \\ -0.064 & -0.3 \end{bmatrix}$ have no eigenvalues on the imaginary axis, (A3″) also holds. Under the control law (20), the trajectories of $\bar{x}$ and $\hat{x}^{(N)}$ in Problem (P) are shown in Fig. 1. It can be seen that $\bar{x}$ and $\hat{x}^{(N)}$ coincide well, which illustrate the consistency of mean field approximations.

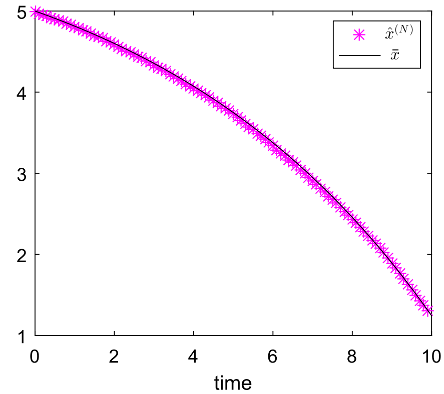

Fig. 1. Curves of $\bar{x}$ and $\hat{x}^{(N)}$.

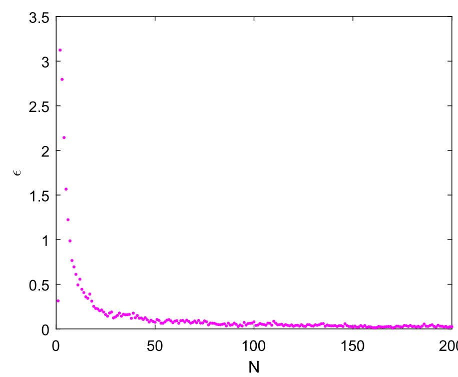

Fig. 2. Curves of ϵ with resect to N.

Denote $\epsilon = \left|\frac{1}{N}J_{\rm soc}(\hat{u}) - \frac{1}{N}\inf_{u\in\mathcal{U}_c} J_{\rm soc}(u)\right|$. By Theorems 3.3 and 4.6, we obtain $\epsilon = \int_0^\infty e^{-\rho t}\|B^TK(x^{(N)}(t) - \bar{x}(t))\|^2_{R^{-1}}dt$. The cost gap ϵ is demonstrated in Fig. 2 where the agent number N grows from 1 to 200.

Finally, we consider the 2-dimensional case of Problem (P). Take parameters as follows: $A = \begin{bmatrix} 0.1 & 0 \\ -1 & 0.2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $G = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\eta = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$, $f = [1 \quad 1]^T$ and $\sigma = [0.5 \quad 0.5]^T$. Denote $\hat{x}_i(t) = [\hat{x}_i^1(t) \ \hat{x}_i^2(t)]^T$. Both of $\hat{x}_i^1(0)$ and $\hat{x}_i^2(0)$ are taken independently from a normal distribution $N(5, 0.5)$. Under the control laws (20), the trajectories of $\hat{x}_i^1$ and $\hat{x}_i^2$, $i = 1, \ldots, N$ are shown in Figs. 3 and 4, respectively. The curves of $\hat{x}_i^1, i = 1, \ldots, 30$ soon converge to 0 with some fluctuation. The curves of $\hat{x}_i^2, i = 1, \ldots, 30$ first decrease and then grow up before the time 40. After a period of time, they converge to a constant, which verify the ρ-stability of agents.

6. Concluding remarks

In this paper, we have considered uniform stabilization and social optimality for mean field LQ multiagent systems. For finite- and infinite-horizon problems, we design the decentralized control laws by decoupling FBSDEs, respectively, which are further shown to be asymptotically optimal. Some equivalent conditions are further given for uniform stabilization of the systems in

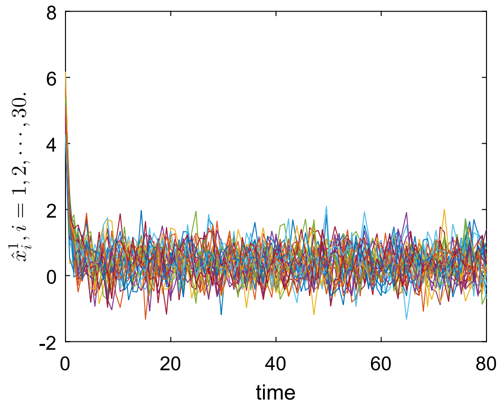

Fig. 3. Curves of $\hat{x}_i^1$, $i = 1, \ldots, 30$.

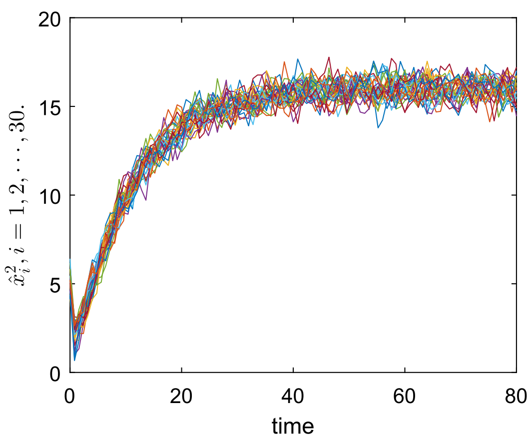

Fig. 4. Curves of $\hat{x}_i^2$, $i = 1, \ldots, 30$.

different cases. Finally, we compare such decentralized control laws with the asymptotic optimal strategies in previous works.

An interesting generalization is to consider mean field LQ control systems with heterogeneous coefficients by the direct approach (He et al., 2015). Also, the variational analysis may be applied to construct decentralized control laws for the nonlinear social control model.

Appendix A. Proof of Theorem 3.3

To prove Theorem 3.3, we need a lemma.

Lemma A.1. *Let (A1) hold. Assume that (10) and (11) admit a solution, respectively. Under the control (15), we have*

$$\max_{0\le t\le T} \mathbb{E}\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 = O(1/N). \tag{A.1}$$

Proof. It follows by (16) that

$$d\hat{x}^{(N)}(t) = \Big[(\bar{A}(t)+G)\hat{x}^{(N)}(t) - BR^{-1}B^T(K(t)\bar{x}(t) + s(t)) + f(t) \Big] dt + \frac{1}{N}\sum_{i=1}^N \sigma(t)dW_i(t).$$

where $\bar{A}(t) = A - BR^{-1}B^TP(t)$. From (14), we have

$$\hat{x}^{(N)}(t) - \bar{x}(t) = \Phi(t)[\hat{x}^{(N)}(0) - \bar{x}(0)] + \frac{1}{N}\sum_{i=1}^N \int_0^t \Phi(t,\tau)\sigma(\tau)dW_i(\tau), \tag{A.2}$$

where Φ satisfies $\frac{d}{dt}\Phi(t,\tau) = (\bar{A}(t)+G)\Phi(t,\tau),\ \Phi(\tau,\tau) = I$. By (A1), one can obtain

$$\begin{aligned}\mathbb{E}\big\|\hat{x}^{(N)}(t) - \bar{x}(t)\big\|^2 \le& 2\big\|\Phi(t,0)\big\|^2 \Big\{ \mathbb{E}\big\|\hat{x}^{(N)}(0) - \bar{x}_0\big\|^2 \\ &+ \frac{2}{N}\int_0^t \big\|\Phi(t,\tau)\sigma(\tau)\big\|^2 d\tau \Big\} \\ \le& \frac{2}{N}\big\|\Phi(t,0)\big\|^2 \max_{1\le i\le N}\mathbb{E}\|\hat{x}_{i0}\|^2 \\ &+ \frac{2}{N}\int_0^T \big\|\Phi(t,\tau)\big\|^2\big\|\sigma(\tau)\big\|^2 d\tau,\end{aligned} \tag{A.3}$$

which completes the proof. □

Proof of Theorem 3.3. Note that $\inf_{u\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nr})} J^{\rm F}_{\rm soc}(u) \le J^{\rm F}_{\rm soc}(\hat{u})$. To obtain

$$\frac{1}{N}J^{\rm F}_{\rm soc}(\hat{u}) \le \frac{1}{N}\inf_{u\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nr})} J^{\rm F}_{\rm soc}(u) + O(\frac{1}{\sqrt{N}}),$$

we only need to prove for any $u \in \mathcal{U}' \triangleq \{u \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nr}) : J^{\rm F}_{\rm soc}(u) \le J^{\rm F}_{\rm soc}(\hat{u})\}$, the following holds:

$$\frac{1}{N}J^{\rm F}_{\rm soc}(\hat{u}) \le \frac{1}{N}J^{\rm F}_{\rm soc}(u) + O(\frac{1}{\sqrt{N}}).$$

We now show that for $u \in \mathcal{U}'$, $\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}(\|x_i(t)\|^2 + \|u_i(t)\|^2)dt < NC_2$. By Lemma 3.1, (P1) is uniformly convex which gives there exists $\delta_0 > 0$ such that

$$\delta_0 \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\|u_i(t)\|^2 dt - C \le J^{\rm F}_{\rm soc}(u).$$

Since $J^{\rm F}_{\rm soc}(\hat{u}) < NC_1$, we have $J^{\rm F}_{\rm soc}(u) < NC_1$, which implies $\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\|u_i(t)\|^2 dt < NC$. This leads to

$$\mathbb{E}\int_0^T e^{-\rho t}\|u^{(N)}\|^2 dt \le \frac{1}{N}\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\|u_i\|^2 dt < C,$$

where $u^{(N)} = \frac{1}{N}\sum_{i=1}^N u_i$. By (1),

$$dx^{(N)}(t) = \Big[(A+G)x^{(N)}(t) + Bu^{(N)}(t) + f(t)\Big]dt + \frac{1}{N}\sum_{i=1}^N \sigma(t)dW_i(t),$$

which implies $\max_{0\le t\le T}\mathbb{E}\|x^{(N)}(t)\|^2 \le C$. Note that

$$x_i(t) = e^{At}x_{i0} + \int_0^t e^{A(t-\tau)}\sigma(\tau)dW_i(\tau) + \int_0^t e^{A(t-\tau)}[Gx^{(N)}(\tau) + Bu_i(\tau) + f(\tau)]d\tau.$$

We have

$$\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\|x_i(t)\|^2dt$$
$$\leq C\Big(\sum_{i=1}^{N}\mathbb{E}\|x_{i0}\|^2+N\max_{0\leq t\leq T}\mathbb{E}\|x^{(N)}(t)\|^2 \tag{A.4}$$
$$+\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\|u_i(t)\|^2dt+NC_1\Big)< NC_2.$$

By (14) and (16), we obtain that

$$\mathbb{E}\int_0^T e^{-\rho t}\big(\|\hat{x}_i(t)\|^2+\|\hat{u}_i(t)\|^2+\|\bar{x}(t)\|^2\big)dt< C. \tag{A.5}$$

Let $\tilde{x}_i=x_i-\hat{x}_i$, $\tilde{u}_i=u_i-\hat{u}_i$ and $\tilde{x}^{(N)}=\frac{1}{N}\sum_{i=1}^N\tilde{x}_i$. Then by (1) and (16),

$$d\tilde{x}_i(t)=(A\tilde{x}_i(t)+G\tilde{x}^{(N)}(t)+B\tilde{u}_i(t))dt,\ \ \tilde{x}_i(0)=0. \tag{A.6}$$

From (3), $J_{\rm soc}^{\rm F}(u)=\sum_{i=1}^N(J_i^{\rm F}(\hat{u})+\tilde{J}_i^{\rm F}(\tilde{u})+\mathcal{I}_i)$, where

$$\tilde{J}_i^{\rm F}(\tilde{u})\stackrel{\Delta}{=}\mathbb{E}\int_0^T e^{-\rho t}\big[\|\tilde{x}_i(t)-\Gamma\tilde{x}^{(N)}(t)\|_Q^2+\|\tilde{u}_i(t)\|_R^2\big]dt,$$
$$\mathcal{I}_i=2\mathbb{E}\int_0^T e^{-\rho t}\Big[\big(\hat{x}_i(t)-\Gamma\hat{x}^{(N)}(t)-\eta(t)\big)^TQ$$
$$\times\big(\tilde{x}_i(t)-\Gamma\tilde{x}^{(N)}(t)\big)+\hat{u}_i^T(t)R\tilde{u}_i(t)\Big]dt.$$

By Lemma 3.1 and Proposition 3.1, $\tilde{J}_i^{\rm F}(\tilde{u})\geq 0$. We only need to prove $\frac{1}{N}\sum_{i=1}^N\mathcal{I}_i=O(\frac{1}{\sqrt{N}})$. By direct computations, one can obtain

$$\sum_{i=1}^{N}\mathcal{I}_i=\sum_{i=1}^{N}2\mathbb{E}\int_0^T e^{-\rho t}\Big\{\tilde{x}_i^T\big[Q(\hat{x}_i-\Gamma\bar{x}-\eta)$$
$$-\Gamma^TQ((I-\Gamma)\bar{x}-\eta)\big]+\sum_{i=1}^{N}\hat{u}_i^TR\tilde{u}_i\Big\}dt \tag{A.7}$$
$$+\sum_{i=1}^{N}2\mathbb{E}\int_0^T e^{-\rho t}(\hat{x}^{(N)}-\bar{x})^T\Xi\tilde{x}_idt.$$

By (10)–(12), (A.6) and Itô's formula,

$$0=\mathbb{E}\int_0^T\sum_{i=1}^{N}e^{-\rho t}\Big\{-\tilde{x}_i^T\big[Q\hat{x}_i-Q(\Gamma\bar{x}+\eta)$$
$$-\Gamma^TQ((I-\Gamma)\bar{x}-\eta)\big]-\hat{u}_i^TR\tilde{u}_i\Big\}dt$$
$$+N\mathbb{E}\int_0^T e^{-\rho t}(\hat{x}^{(N)}-\bar{x})^T(G^TP+PG)\tilde{x}^{(N)}dt.$$

From this and (A.7), we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\mathcal{I}_i=2\mathbb{E}\int_0^T e^{-\rho t}(\hat{x}^{(N)}(t)-\bar{x}(t))^T$$
$$\times(\Xi+G^TP+PG)\tilde{x}^{(N)}(t)dt.$$

By Lemma A.1, (A.4) and (A.5), we obtain

$$\Big|\frac{1}{N}\sum_{i=1}^{N}\mathcal{I}_i\Big|^2\leq C\mathbb{E}\int_0^T e^{-\rho t}\|\hat{x}^{(N)}(t)-\bar{x}(t)\|^2dt$$
$$\times\mathbb{E}\int_0^T e^{-\rho t}\|\tilde{x}^{(N)}(t)\|^2dt,$$

which implies $|\frac{1}{N}\sum_{i=1}^N\mathcal{I}_i|=O(1/\sqrt{N})$.

Moreover, by (10), (13) and direct calculations,

$$J_{\rm soc}^{\rm F}(\hat{u})$$
$$=\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\big\|\hat{x}_i-\Gamma\hat{x}^{(N)}-\eta\big\|_Q^2+\|\hat{u}_i\|_R^2\Big)dt$$
$$=\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\|\hat{x}_i\|_Q^2+\|\hat{x}^{(N)}\|_\Xi^2+\|\eta\|_Q^2$$
$$-2\eta^TQ(I-\Gamma)\hat{x}_i+\|\hat{u}_i\|_R^2\Big)dt$$
$$=\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\|\hat{x}_i-\hat{x}^{(N)}\|_Q^2+\|\hat{x}^{(N)}\|_{Q-\Xi}^2+\|\eta\|_Q^2$$
$$-2\eta^TQ(I-\Gamma)\hat{x}_i+\|\hat{u}_i-\hat{u}^{(N)}\|_R^2+\|\hat{u}^{(N)}\|_R^2\Big)dt$$
$$=\sum_{i=1}^{N}\mathbb{E}\big[\|x_{i0}-x^{(N)}(0)\|_P^2+\|x^{(N)}(0)\|_\Pi^2+2s^T(0)x^{(N)}(0)\big]$$
$$+\sum_{i=1}^{N}\mathbb{E}\int_0^T e^{-\rho t}\Big(\big\|\hat{u}_i-\hat{u}^{(N)}+R^{-1}B^TP(\hat{x}_i-\hat{x}^{(N)})\big\|_R^2$$
$$+\big\|\hat{u}^{(N)}+R^{-1}B^T\Pi\hat{x}^{(N)}\big\|_R^2\Big)dt+q_T$$
$$=\sum_{i=1}^{N}\mathbb{E}\big[\|x_{i0}-x^{(N)}(0)\|_P^2+\|x^{(N)}(0)\|_\Pi^2$$
$$+2s^T(0)x^{(N)}(0)\big]+Nq_T+N\epsilon_T,$$

where q_T and ϵ_T are given by (18)–(19). □

Appendix B. Proofs of Lemma 4.2 and Theorem 4.1

Proof of Lemma 4.2. From (A.2), we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}^{(N)}(t)-\bar{x}(t)\|^2dt$$
$$\leq 2\mathbb{E}\int_0^\infty\Big\|e^{(\bar{A}+G-\frac{\rho}{2}I)t}\Big\|^2\big\|\hat{x}^{(N)}(0)-\bar{x}(0)\big\|^2dt$$
$$+2\mathbb{E}\int_0^\infty e^{-\rho t}\frac{1}{N}\Big\|\int_0^t e^{(\bar{A}+G)(t-\tau)}\sigma dW_i(\tau)\Big\|^2dt$$
$$\leq\frac{2}{N}\int_0^\infty\Big\|e^{(\bar{A}+G-\frac{\rho}{2}I)t}\Big\|^2\mathbb{E}\big\|\max_{1\leq i\leq N}\hat{x}_i(0)\big\|^2dt$$
$$+\frac{C}{N}\mathbb{E}\int_0^\infty e^{-\rho\tau}\|\sigma\|^2\int_\tau^\infty\big\|e^{(\bar{A}+\bar{G}-\frac{\rho}{2}I)(t-\tau)}\big\|^2dtd\tau$$
$$\leq O(1/N).\ \ \square$$

Proof of Theorem 4.1. By (A1)–(A3), Lemmas 4.1 and 4.2, we obtain that $\bar{x}\in C_{\rho/2}([0,\infty),\mathbb{R}^n)$ and

$$\mathbb{E}\int_0^\infty e^{-\rho t}\Big(\big\|\hat{x}^{(N)}(t)-\bar{x}(t)\big\|^2\Big)dt=O(\frac{1}{N}),$$

which further gives that $\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}^{(N)}(t)\|^2dt<\infty$. Denote $g\stackrel{\Delta}{=}-BR^{-1}B^T((\Pi-P)\bar{x}+s)+Gx^{(N)}+f$. Then

$$\hat{x}_i(t)=e^{\bar{A}t}\hat{x}_{i0}+\int_0^t e^{\bar{A}(t-\tau)}g(\tau)d\tau+\int_0^t e^{\bar{A}(t-\tau)}\sigma dW_i(\tau). \tag{B.1}$$

Note that $\bar{A}-\frac{\rho}{2}I$ is Hurwitz. By Schwarz's inequality,

$$\begin{aligned}
&\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}_i(t)\|^2dt\\
\leq &3\mathbb{E}\int_0^\infty \left\|e^{(\bar{A}-\frac{\rho}{2}I)t}\right\|^2\|\hat{x}_{i0}\|^2dt\\
&+3\mathbb{E}\int_0^\infty e^{-\rho t}t\int_0^t\left\|e^{\bar{A}(t-\tau)}g(\tau)\right\|^2d\tau dt\\
&+3\mathbb{E}\int_0^\infty e^{-\rho t}\int_0^t tr[e^{\bar{A}^T(t-\tau)}\sigma^T(\tau)\sigma(\tau)e^{\bar{A}(t-\tau)}]d\tau dt\\
\leq &C+3\mathbb{E}\int_0^\infty e^{-\rho\tau}\|g(\tau)\|^2\int_\tau^\infty t\left\|e^{(\bar{A}-\frac{\rho}{2}I)(t-\tau)}\right\|^2dtd\tau\\
&+3C\mathbb{E}\int_0^\infty e^{-\rho\tau}\|\sigma(\tau)\|^2\int_\tau^\infty\left\|e^{(\bar{A}-\frac{\rho}{2}I)(t-\tau)}\right\|^2dtd\tau\\
\leq &C_1
\end{aligned}$$

This with (20) completes the proof. □

Appendix C. Proofs of Theorems 4.2 and 4.3

Proof of Theorem 4.2. (i)⇒ (ii). By (16),

$$\begin{aligned}\frac{d\mathbb{E}[\hat{x}_i]}{dt}=&\bar{A}\mathbb{E}[\hat{x}_i(t)]-BR^{-1}B^T((\Pi-P)\bar{x}(t)+s(t))\\
&+G\mathbb{E}[\hat{x}^{(N)}(t)]+f(t),\quad \mathbb{E}[\hat{x}_i(0)]=\bar{x}_0.\end{aligned}\tag{C.1}$$

It follows from (A1) that $\mathbb{E}[\hat{x}_i(t)]=\mathbb{E}[\hat{x}_j(t)]=\mathbb{E}[\hat{x}^{(N)}(t)]$, $j\neq i$. By comparing (24) and (C.1), we obtain $\mathbb{E}[\hat{x}_i(t)]=\bar{x}(t)$. Note $\|\bar{x}(t)\|^2=\left\|\mathbb{E}\hat{x}_i(t)\right\|^2\leq\mathbb{E}\|\hat{x}_i(t)\|^2$. It follows from (27) that

$$\int_0^\infty e^{-\rho t}\|\bar{x}(t)\|^2dt<\infty.\tag{C.2}$$

By (24), we have

$$\begin{aligned}\bar{x}(t)=&e^{(A+G-BR^{-1}B^T\Pi)t}\Big[\bar{x}_0\\
&+\int_0^t e^{-(A+G-BR^{-1}B^T\Pi)\tau}h(\tau)d\tau\Big],\end{aligned}$$

where $h=-BR^{-1}B^Ts+f$. By the arbitrariness of $\bar{x}_0$ with (C.2) we obtain that $A+G-BR^{-1}B^T\Pi-\frac{\rho}{2}I$ is Hurwitz. That is, $(A+G-\frac{\rho}{2}I,B)$ is stabilizable. By Anderson and Moore (1990), (22) admits a unique solution such that $\Pi>0$. Note that $\mathbb{E}[x^{(N)}(t)]^2\leq\frac{1}{N}\sum_{i=1}^N\mathbb{E}[\hat{x}_i^2(t)]$. Then from (27) we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left\|\hat{x}^{(N)}(t)\right\|^2dt<\infty.\tag{C.3}$$

This leads to $\mathbb{E}\int_0^\infty e^{-\rho t}\|g(t)\|^2dt<\infty$, where $g=-BR^{-1}B^T((\Pi-P)\bar{x}+s)+G\hat{x}^{(N)}+f$. By (B.1), we obtain

$$\begin{aligned}\mathbb{E}\|\hat{x}_i(t)\|^2=&\mathbb{E}\left\|e^{\bar{A}t}\left(x_{i0}+\int_0^t e^{-\bar{A}\tau}g(\tau)d\tau\right)\right\|^2\\
&+\mathbb{E}\int_0^t tr\big[\sigma^T(\tau)e^{(\bar{A}^T+\bar{A})(t-\tau)}\sigma(\tau)\big]d\tau.\end{aligned}$$

By (27) and the arbitrariness of x_{i0} we obtain that $\bar{A}-\frac{\rho}{2}I$ is Hurwitz, i.e., $(A-\frac{\rho}{2}I,B)$ is stabilizable. By Anderson and Moore (1990), (21) admits a unique solution such that $P>0$.

From (C.2) and (C.3),

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left\|\hat{x}^{(N)}(t)-\bar{x}(t)\right\|^2dt<\infty.\tag{C.4}$$

On the other hand, (A.2) gives

$$\begin{aligned}\mathbb{E}\left\|\hat{x}^{(N)}(t)-\bar{x}(t)\right\|^2=&\mathbb{E}\left\|e^{(\bar{A}+G)t}[\hat{x}^{(N)}(0)-\bar{x}_0]\right\|^2\\
&+\frac{1}{N}\int_0^t tr\big[\sigma^T(\tau)e^{(\bar{A}^T+G^T+\bar{A}+G)(t-\tau)}\sigma(\tau)\big]d\tau.\end{aligned}$$

By (C.4) and the arbitrariness of x_{i0}, $i=1,\ldots,N$, we obtain that $\bar{A}+G-\frac{\rho}{2}I$ is Hurwitz.

(ii)⇒(iii). Let $V(t)=e^{-\rho t}\bar{y}^T(t)\Pi\bar{y}(t)$, where $\bar{y}$ satisfies

$$\dot{\bar{y}}(t)=(A+G)\bar{y}(t)+B\bar{u}(t),\quad \bar{y}(0)=\bar{y}_0.$$

Denote V by V^* when $\bar{u}=\bar{u}^*=-R^{-1}B^T\Pi\bar{y}$. By (22),

$$\begin{aligned}\frac{dV^*}{dt}=&\bar{y}^T(t)\big[-\rho\Pi+(A+G-BR^{-1}B^T\Pi)^T\Pi\\
&+\Pi(A+G-BR^{-1}B^T\Pi)\big]\bar{y}(t)\\
=&\bar{y}^T(t)\big[-(Q-\Xi)-\Pi BR^{-1}B^T\Pi\big]\bar{y}(t)\leq 0.\end{aligned}$$

Note that $V^*\geq 0$. Then $\lim_{t\to\infty}V^*(t)$ exists, which implies

$$\lim_{t_0\to\infty}[V^*(t_0)-V^*(t_0+T)]=0.\tag{C.5}$$

Rewrite $\Pi(t)$ in (13) by $\Pi_T(t)$. Then we have $\Pi_{T+t_0}(t_0)=\Pi_T(0)$. By (13),

$$\begin{aligned}&\int_{t_0}^{T+t_0}e^{-\rho t}(\|\bar{y}(t)\|^2_{Q-\Xi}+\|\bar{u}(t)\|^2_R)dt\\
=&e^{-\rho t_0}\bar{y}^T(t_0)\Pi_{T+t_0}(t_0)\bar{y}(t_0)\\
&+\int_0^T e^{-\rho t}\left\|\bar{u}(t)+R^{-1}B^T\Pi_{T+t_0}(t_0)\bar{y}(t)\right\|^2_Rdt\\
\geq&e^{-\rho t_0}\left\|\bar{y}(t_0)\right\|^2_{\Pi_{T+t_0}(t_0)}=e^{-\rho t_0}\left\|\bar{y}(t_0)\right\|^2_{\Pi_T(0)}.\end{aligned}$$

This with (C.5) implies

$$\begin{aligned}&\lim_{t_0\to\infty}e^{-\rho t_0}\left\|\bar{y}(t_0)\right\|^2_{\Pi_T(0)}\\
\leq&\lim_{t_0\to\infty}\int_{t_0}^{T+t_0}e^{-\rho t}(\|\bar{y}(t)\|^2_{Q-\Xi}+\|\bar{u}^*(t)\|^2_R)dt\\
=&\lim_{t_0\to\infty}[V^*(t_0)-V^*(t_0+T)]=0.\end{aligned}$$

By (A3), one can get that there exists $T>0$ such that $\Pi_T(0)>0$ (see e.g. Zhang et al. (2019) and Zhang, Zhang, and Chen (2008)). Thus, we have $\lim_{t\to\infty}e^{-\rho t}\left\|\bar{y}(t)\right\|^2=0$, which implies that $(A+G-\frac{\rho}{2}I,B)$ is stabilizable. Similarly, we can show $(A-\frac{\rho}{2}I,B)$ is stabilizable.

(iii)⇒(i). This part has been proved in Theorem 4.1. □

Proof of Theorem 4.3. (iii)⇒(i). From Anderson and Moore (1990), (21) and (22) admit unique solutions $P\geq 0,\Pi\geq 0$ such that $A-BR^{-1}B^TP-\frac{\rho}{2}I$ and $A-BR^{-1}B^T\Pi-\frac{\rho}{2}I$ are Hurwitz, respectively. Thus, there exists a unique $s(0)$ such that $s\in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. It is straightforward that $\bar{x}\in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. By the argument in the proof of Theorem 4.1, (i) follows. (i)⇒(ii). The proof of this part is similar to that of (i)⇒(ii) in Theorem 4.2.

(ii)⇒(iii). Since $\Pi\geq 0$, there exists an orthogonal U such that $U^T\Pi U=\begin{bmatrix}0&0\\0&\Pi_2\end{bmatrix}$, where $\Pi_2>0$. From (21),

$$\begin{aligned}\rho U^T\Pi U=&(U^T\bar{\mathbb{A}}U)^TU^T\Pi U+U^T\Pi UU^T\bar{\mathbb{A}}U\\
&+U^T\bar{\mathbb{Q}}U,\end{aligned}\tag{C.6}$$

where $\bar{\mathbb{A}}\overset{\Delta}{=}A+G-BR^{-1}B^T\Pi$, $\bar{\mathbb{Q}}=Q-\Xi+\Pi BR^{-1}B^T\Pi$. Denote

$$U^T\bar{\mathbb{A}}U=\begin{bmatrix}\bar{\mathbb{A}}_{11}&\bar{\mathbb{A}}_{12}\\\bar{\mathbb{A}}_{21}&\bar{\mathbb{A}}_{22}\end{bmatrix},\quad U^T\bar{\mathbb{Q}}U=\begin{bmatrix}\bar{\mathbb{Q}}_{11}&\bar{\mathbb{Q}}_{12}\\\bar{\mathbb{Q}}_{21}&\bar{\mathbb{Q}}_{22}\end{bmatrix}.$$

By pre- and post-multiplying by ξ^T and ξ where $\xi = [\xi_1^T, 0]^T$, it follows that

$$0 = \rho \xi^T U^T \Pi U \xi = \xi^T U^T \bar{Q} U \xi.$$

From the arbitrariness of ξ_1, we obtain $\bar{Q}_{11} = 0$. Since $\bar{Q}$ is semi-positive definite, then $\bar{Q}_{12} = \bar{Q}_{21} = 0$, and $\bar{Q}_{22} \geq 0$. By comparing each block matrix of both sides of (C.6), we obtain $\bar{\mathbb{A}}_{21} = 0$. It follows from (C.6) that

$$\rho \Pi_2 = \Pi_2 \bar{\mathbb{A}}_{22} + \bar{\mathbb{A}}_{22}^T \Pi_2 + \bar{Q}_{22}. \quad \text{(C.7)}$$

Let $\zeta = [\zeta_1^T, \zeta_2^T]^T = U^T \bar{y}^*$, where $\bar{y}^*$ satisfies $\dot{\bar{y}}^* = \bar{\mathbb{A}}\bar{y}^*$. Then we have

$$\dot{\zeta}_1 = \bar{\mathbb{A}}_{11}\zeta_1 + \bar{\mathbb{A}}_{12}\zeta_2,$$
$$\dot{\zeta}_2 = \bar{\mathbb{A}}_{22}\zeta_2.$$

By Lemma 4.1 of Wonham (1968), the detectability of $(A+G, (Q-\Xi)^{1/2})$ implies the detectability of $(\bar{\mathbb{A}}, \bar{Q}^{1/2})$. Take $\zeta(0) = \xi = [\xi_1^T, 0]^T$. Then $\bar{Q}^{1/2}\bar{y} = \bar{Q}^{1/2}U\zeta = 0$, which together with the detectability of $(\bar{\mathbb{A}}, \bar{Q}^{1/2})$ implies $\zeta_1 \to 0$ and $\bar{\mathbb{A}}_{11}$ is Hurwitz. Denote $S(t) = e^{-\rho t}\zeta_2^T \Pi_2 \zeta_2$. By (C.7),

$$S(T) - S(0) = -\int_0^T \zeta_2^T(t)\bar{Q}_{22}\zeta_2(t)dt \leq 0,$$

which implies $\lim_{t\to\infty} S(t)$ exists. By a similar argument with the proof of Theorem 4.2, we obtain $\lim_{t_0\to\infty} e^{-\rho t_0}\|\zeta_2(t_0)\|^2_{\Pi_{2,T}(0)} = 0$ and $\Pi_{2,T}(0) > 0$, which gives $\zeta_2 \to 0$ and $\bar{\mathbb{A}}_{22}$ is Hurwitz. This with the fact that $\bar{\mathbb{A}}_{11}$ is Hurwitz gives that ζ is stable, which leads to (iii). □

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