Equilibrium Solutions of Multiperiod Mean-Variance Portfolio Selection

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***Abstract*—This is a companion paper of [Mixed equilibrium solution of time-inconsistent stochastic linear-quadratic problem, *SIAM J. Control Optim.*, vol. 57, no. 1, 533–569, 2019], where general theory has been established to characterize the open-loop equilibrium control, feedback equilibrium strategy and mixed equilibrium solution for a time-inconsistent stochastic linear-quadratic problem. This note is, on the one hand, to test the developed theory of that paper and on the other hand to push the solvability of multiperiod mean-variance portfolio selection. A nondegenerate assumption, which is popular in the existing literature about multiperiod mean-variance portfolio selection, has been removed in this note; and neat conditions have been obtained to characterize the existence of equilibrium solutions.**

***Index Terms*—Multiperiod mean-variance portfolio selection, stochastic linear-quadratic (LQ) control, time inconsistency.**

I. Introduction

Recently, a notion named mixed equilibrium solution is introduced in [12] for the time-inconsistent stochastic linear-quadratic (LQ, for short) optimal control; it contains two different parts: a pure-feedback-strategy part and an open-loop-control part, which together constitute a time-consistent solution. It is shown that the open-loop-control part will be of the feedback form of the equilibrium state. If we let the pure-feedback-strategy part be zero or let the open-loop-control part be independent of the initial state, then the mixed equilibrium solution will reduce to the open-loop equilibrium control and the (linear) feedback equilibrium strategy, respectively, both of which have been extensively studied in the existing literature [1], [2], [4], [5], [7], [8], [13], [14], [18], [20], [21]. Furthermore, the mixed equilibrium solution is not a hollow concept, whose study will give us more flexibility to deal with the time-inconsistent problems.

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The multiperiod mean-variance portfolio selection is a particular example of time-inconsistent problem. In fact, the recent developments in time-inconsistent problems and the revisits of multiperiod mean-variance portfolio selection [1], [2], [4]–[8] are mutually stimulated. The (single-period) mean-variance formulation initiated by Markowitz [10] is the cornerstone of modern portfolio theory and is widely used in both academic and financial industry. The multiperiod mean-variance portfolio selection, which has been extensively studied, is the natural extension of [10]. Until 2000 and for the first time, Li–Ng [9] and Zhou–Li [22] reported the analytical precommitment optimal policies for the discrete-time case and the continuous-time case, respectively.

To proceed, consider a capital market consisting of one riskless asset and m risky assets over a finite time horizon N. Let $s_k (> 1)$ be a given deterministic return of the riskless asset at time period k and $e_k = (e_k^1, \dots, e_k^m)^T$ the vector of random returns of the m risky assets at period k. We assume that vectors $e_k, k = 0, 1, \dots, N-1$, are statistically independent and the only information known about the random return vector e_k is its first two moments: its mean $\mathbb{E}(e_k) = (\mathbb{E}e_k^1, \mathbb{E}e_k^2, \dots, \mathbb{E}e_k^m)^T$ and its covariance $\mathrm{Cov}(e_k) = \mathbb{E}[(e_k - \mathbb{E}e_k)(e_k - \mathbb{E}e_k)^T]$. Clearly, $\mathrm{Cov}(e_k)$ is nonnegative definite, i.e., $\mathrm{Cov}(e_k) \succeq 0$.

Let $X_k \in \mathbb{R}$ be the wealth of the investor at the beginning of the kth period, and let u_k^i be the amount invested in the ith risky asset at period k, $i = 1, 2, \dots, m$. Then, $X_k - \sum_{i=1}^m u_k^i$ is the amount invested in the riskless asset at period k, and the wealth at the beginning of the $(k+1)$th period [9] is given by

$$X_{k+1} = \sum_{i=1}^m e_k^i u_k^i + \left(X_k - \sum_{i=1}^m u_k^i \right) s_k = s_k X_k + O_k^T u_k \quad (1)$$

where O_k is the excess return vector of risky assets [9] defined as $O_k = (O_k^1, O_k^2, \dots, O_k^m)^T = (e_k^1 - s_k, e_k^2 - s_k, \dots, e_k^m - s_k)^T$. In this note, we consider the case that short selling is allowed, i.e., $u_k^i, i = 1, \dots, m$, take values in $\mathbb{R}$.

Let $\mathcal{F}_k = \sigma(e_\ell, \ell = 0, 1, \dots, k-1)$, $k \in \{0, \dots, N-1\} \equiv \mathbb{T}$ with $\mathcal{F}_0$ given by $\{\emptyset, \Omega\}$. For $t \in \mathbb{T}$, denote

$$l_{\mathcal{F}}^2(t; \mathbb{R}) = \left\{ \nu_t \middle| \nu_t \in \mathbb{R} \text{ is } \mathcal{F}_t\text{-measurable}, \ \mathbb{E}|\nu_t|^2 < \infty \right\}$$

$$l_{\mathcal{F}}^2(\mathbb{T}_t; \mathbb{R}^m) = \left\{ \{\nu_k, k \in \mathbb{T}_t\} \middle| \begin{array}{l} \nu_k \in \mathbb{R}^m \text{ is } \mathcal{F}_k\text{-measurable}, \\ \mathbb{E}|\nu_k|^2 < \infty, \ k \in \mathbb{T}_t \end{array} \right\}$$

where $\mathbb{T}_t = \{t, \dots, N-1\}$. Then, a time-inconsistent version of multiperiod mean-variance problem [9] is formulated as follows.

Problem (MV): For $t \in \mathbb{T}$ and $x \in l_{\mathcal{F}}^2(t; \mathbb{R})$, find $u^* \in l_{\mathcal{F}}^2(\mathbb{T}_t; \mathbb{R}^m)$ such that

$$J(t, x; u^*) = \inf_{u \in l_{\mathcal{F}}^2(\mathbb{T}_t; \mathbb{R}^m)} J(t, x; u).$$

Here

$$J(t, x; u) = \mathbb{E}_t (X_N - \mathbb{E}_t X_N)^2 - (\mu_1 x + \mu_2) \mathbb{E}_t X_N$$

which is subject to

$$\begin{cases} X_{k+1} = s_k X_k + O_k^T u_k, & k \in \mathbb{T}_t \\ X_t = x \end{cases}$$

with $\mu_1, \mu_2 > 0$ being the tradeoff parameters between the mean and variance of the terminal wealth. Furthermore, $\mathbb{E}_t(\cdot)$ above is the conditional mathematical expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$.

It should be mentioned that Problem (MV) above has two unconventional features: the term $\mu_1 x \mathbb{E}_t X_N$ makes $J(t, x; u)$ a state-dependent (or rank-dependent) utility, and the cost functional $J(t, x; u)$ involves nonlinear term of the conditional expectation. It is known that any of the above two features will ruin the time consistency of optimal control, namely, Bellman's principle of optimality will no longer work for Problem (MV). Note that the above model is more general than that of [13], where the case without $\mu_1 x \mathbb{E}_t X_N$ is dealt with.

In [9], realizing the time inconsistency (called nonseparability there), Li and Ng derived an optimal policy of multiperiod mean-variance portfolio selection by using an embedding scheme. The optimal policy of [9] is with respect to the initial pair, i.e., it is optimal only when viewed at the initial time. This scheme is called the precommitted optimal solution now. By applying a precommitted optimal control (for an initial pair), we find that it will no longer be an optimal control for the intertemporal initial pairs. Though the precommitted optimal solution is of some practical and theoretical values, it neglects and has not really addressed the time inconsistency.

In recent years, there is a surge to study the time-inconsistent optimal control together with the revisit to multiperiod mean-variance portfolio selection [1], [2], [4], [5], [7], [8], [13], [14], [18], [20], [21]. Two kinds of time-consistent equilibrium solutions are investigated in these papers including the open-loop equilibrium control and closed-loop equilibrium strategy. To compare, open-loop formulation is to find an open-loop equilibrium "control," while the "strategy" is the object of closed-loop formulation. Strotz's equilibrium solution [16] is essentially a closed-loop equilibrium strategy, which is further elaborately developed by Yong to the LQ optimal control [18], [21] as well as the nonlinear optimal control [17], [19], [20]. In contrast, open-loop equilibrium control is extensively studied by Hu *et al.* [7], [8], Yong [21], Ni–Zhang–Krstic [13], and Qi–Zhang [14]. In particular, the closed-loop formulation can be viewed as the extension of Bellman's dynamic programing, and the corresponding equilibrium strategy (if it exists) is derived by a backward procedure [18]–[21]. Differently, the open-loop equilibrium control is characterized via the maximum-principle-like methodology [7], [8], [13].

It is noted that some nondegenerate assumptions are posed in [1], [2], [4], [5], and [7]–[9]. Specifically, the volatilities of the stocks in [1], [2], [7], and [8] and the return rates of the risky securities in [4], [5], and [9] are assumed to be nondegenerate, i.e., $\mathrm{Cov}(e_k) \succ 0, k \in \mathbb{T}$. To make the formulation more practical, it is natural to consider, at least in theory, how to generalize these results to the case where degeneracy is allowed. In fact, mean-variance portfolio selection problems with degenerate covariance matrices may date back to the 1970s. In [3] or the "corrected" version [15], Buser proposes the single-period version with possibly singular covariance matrix. Clearly, such class of problems are more general than the classical ones [10], and more consistent with the reality.

In this note, we do not pose the nondegenerate assumption and want to find the conditions such that the time-consistent equilibrium solutions of Problem (MV) exist. This can be done by using the theory developed by [12]. The rest of this paper is organized as follows. Section II gives the definitions of equilibrium solutions, whose existence is investigated in Section III. In Section IV, an example of [9] is revisited.

II. Equilibrium Solutions

In the following, we introduce three equilibrium solutions for Problem (MV), which are the open-loop equilibrium control, feedback equilibrium strategy and mixed equilibrium solution. Note that the following notions are consistent with those of [12]. Throughout this note, Problem (MV) for the initial pair (t, x) will be simply denoted as Problem (MV)$_{tx}$.

Definition 2.1: A control $u^{t,x} \in l^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$ is called an open-loop equilibrium control of Problem (MV)$_{tx}$, if

$$J\left(k, \widehat{X}_k^{t,x,*}; u^{t,x}|_{\mathbb{T}_k}\right) \le J\left(k, \widehat{X}_k^{t,x,*}; \left(u_k, u^{t,x}|_{\mathbb{T}_{k+1}}\right)\right) \quad (2)$$

for any $k \in \mathbb{T}_t$ and any $u_k \in l^2_{\mathcal{F}}(k; \mathbb{R}^m)$. Here, $u^{t,x}|_{\mathbb{T}_k}$ and $u^{t,x}|_{\mathbb{T}_{k+1}}$ are the restrictions of $u^{t,x}$ on $\mathbb{T}_k$ and $\mathbb{T}_{k+1}$, respectively; and $\widehat{X}^{t,x,*}$ is given by

$$\begin{cases} \widehat{X}_{k+1}^{t,x,*} = s_k \widehat{X}_k^{t,x,*} + O_k^T u_k^{t,x}, & k \in \mathbb{T}_t \\ \widehat{X}_t^{t,x,*} = x. \end{cases}$$

Definition 2.2: i) At stage $k \in \mathbb{T}_t$, a function $f_k(\cdot)$ is called an admissible feedback strategy (or simply a feedback strategy) if for any $\zeta \in l^2_{\mathcal{F}}(k; \mathbb{R})$, $f_k(\zeta) \in l^2_{\mathcal{F}}(k; \mathbb{R}^m)$ with

$$l^2_{\mathcal{F}}(k; \mathbb{R}^m) = \left\{\nu_k | \nu_k \in \mathbb{R}^m \text{ is } \mathcal{F}_k\text{-measurable}, \ \mathbb{E}|\nu_k|^2 < \infty.\right\}.$$

The set of such type of $f_k' s$ is denoted by $\mathbb{F}_k$, and $\mathbb{F}_t \times \cdots \times \mathbb{F}_{N-1}$ is denoted by $\mathbb{F}_{\mathbb{T}_t}$.

ii) Let $f = (f_t, \ldots, f_{N-1}) \in \mathbb{F}_{\mathbb{T}_t}$. For $k \in \mathbb{T}_t$ and $\zeta \in l^2_{\mathcal{F}}(k; \mathbb{R}^n)$, $f_k(\zeta)$ can be divided into two parts, namely, $f_k(\zeta) = f_k^c + f_k^p(\zeta)$, where $f_k^c = f_k(0)$ is the inhomogeneous part and the remainder $f_k^p(\cdot)$ is the pure-feedback-strategy part of f_k. Furthermore, $(f_t^p, \ldots, f_{N-1}^p)$ is called a pure-feedback strategy.

Definition 2.3: i) A strategy $\psi \in \mathbb{F}_{\mathbb{T}_t}$ is called a feedback equilibrium strategy of Problem (MV)$_{t,x}$, if the following two points hold:

a) ψ does not depend on x.

b) For any $k \in \mathbb{T}_t$ and any $u_k \in l^2_{\mathcal{F}}(k; \mathbb{R}^m)$, it holds that

$$\begin{aligned} & J\left(k, \widetilde{X}_k^{t,x,*}; \left(\psi \cdot \widetilde{X}^{t,x,*}\right)|_{\mathbb{T}_k}\right) \\ & \le J\left(k, \widetilde{X}_k^{t,x,*}; \left(u_k, \left(\psi \cdot X^{k,u_k}\right)|_{\mathbb{T}_{k+1}}\right)\right). \end{aligned} \quad (3)$$

In (3), $(\psi \cdot X^{t,x,*})|_{\mathbb{T}_k}$ and $(\psi \cdot X^{k,u_k})|_{\mathbb{T}_{k+1}}$ (with $\mathbb{T}_k = \{k, \ldots, N-1\}$, $\mathbb{T}_{k+1} = \{k+1, \ldots, N-1\}$) are given by

$$\begin{aligned} \left(\psi \cdot \widetilde{X}^{t,x,*}\right)|_{\mathbb{T}_k} &= \left(\psi_k\left(\widetilde{X}_k^{t,x,*}\right), \ldots, \psi_{N-1}\left(\widetilde{X}_{N-1}^{t,x,*}\right)\right) \\ (\psi \cdot X^{k,u_k})|_{\mathbb{T}_{k+1}} &= (\psi_{k+1}(X_{k+1}^{k,u_k}), \ldots, \psi_{N-1}(X_{N-1}^{k,u_k})) \end{aligned}$$

where $\widetilde{X}^{t,x,*}, X^{k,u_k}$ are as follows:

$$\begin{cases} \widetilde{X}_{k+1}^{t,x,*} = s_k \widetilde{X}_k^{t,x,*} + O_k^T \psi_k\left(\widetilde{X}_k^{t,x,*}\right), & k \in \mathbb{T}_t \\ \widetilde{X}_t^{t,x,*} = x \end{cases}$$

$$\begin{cases} X_{\ell+1}^{k,u_k} = s_\ell X_\ell^{k,u_k} + O_\ell^T \psi_\ell\left(X_\ell^{k,u_k}\right), & \ell \in \mathbb{T}_{k+1} \\ X_{k+1}^{k,u_k} = s_k X_k^{k,u_k} + O_k^T u_k \\ X_k^{k,u_k} = \widetilde{X}_k^{t,x,*}. \end{cases}$$

ii) Let $(\Psi, \gamma) \in l^2(\mathbb{T}_t; \mathbb{R}^m) \times l^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$ with $l^2(\mathbb{T}_t; \mathbb{R}^m) = \{\{\nu_k, k \in \mathbb{T}_t\} | \nu_k \in \mathbb{R}^m, |\nu_k|^2 < \infty, k \in \mathbb{T}_t\}$. If Ψ and γ do not depend on x, and ψ of i) equals to (Ψ, γ), namely

$$\psi_k(\xi) = \Psi_k \xi + \gamma_k, \quad k \in \mathbb{T}_t, \quad \xi \in l^2_{\mathcal{F}}(k; \mathbb{R}^n)$$

then (Ψ,γ) is called a linear feedback equilibrium strategy of Problem $(\text{MV})_{tx}$.

Definition 2.4: i) $(\Phi, v^{t,x}) \in l^2(\mathbb{T}_t;\mathbb{R}^m)\times l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ is called a mixed equilibrium solution of Problem $(\text{MV})_{tx}$, if the following two points hold:

a) Φ does not depend on x, and $v^{t,x}$ depends on x.

b) For any $k\in\mathbb{T}_t$ and any $u_k\in l^2_{\mathcal{F}}(k;\mathbb{R}^m)$, it holds that

$$\begin{aligned} &J\left(k, X_k^{t,x,*}; \left(\Phi\cdot X^{t,x,*}+v^{t,x}\right)|_{\mathbb{T}_k}\right)\\ &\le J\left(k, X_k^{t,x,*}; \left(u_k, \left(\Phi\cdot X^{k,u_k}+v^{t,x}\right)|_{\mathbb{T}_{k+1}}\right)\right). \end{aligned} \tag{4}$$

In (4), $(\Phi\cdot X^{t,x,*}+v^{t,x})|_{\mathbb{T}_k}$ and $(\Phi\cdot X^{k,u_k}+v^{t,x})|_{\mathbb{T}_{k+1}}$ are given, respectively, by

$$\left(\Phi_k X_k^{t,x,*}+v_k^{t,x},\dots,\Phi_{N-1}X_{N-1}^{t,x,*}+v_{N-1}^{t,x}\right)$$
$$\left(\Phi_{k+1}X_{k+1}^{k,u_k}+v_{k+1}^{t,x},\dots,\Phi_{N-1}X_{N-1}^{k,u_k}+v_{N-1}^{t,x}\right)$$

where $X^{t,x,*}$ and X^{k,u_k} are defined by

$$\begin{cases} X_{k+1}^{t,x,*}=\left(s_k+O_k^T\Phi_k\right)X_k^{t,x,*}+O_k^T v_k^{t,x}, & k\in\mathbb{T}_t\\ X_t^{t,x,*}=x \end{cases}$$

$$\begin{cases} X_{\ell+1}^{k,u_k}=\left(s_k+O_k^T\Phi_k\right)X_k^{k,u_k}+O_k^T v_k^{t,x}, & \ell\in\mathbb{T}_{k+1}\\ X_{k+1}^{k,u_k}=s_k X_k^{k,u_k}+O_k^T u_k\\ X_k^{k,u_k}=X_k^{t,x,*}. \end{cases}$$

ii) Φ and $v^{t,x}$ in i) are called, respectively, the pure-feedback-strategy part and the open-loop-control part of the mixed equilibrium solution $(\Phi, v^{t,x})$.

iii) Letting $\Phi=0$ in i), the corresponding $v^{t,x}$ satisfying (2.4) is called an open-loop equilibrium control of Problem $(\text{MV})_{tx}$.

iv) If $(\Phi, v^{t,x})$ does not depend on x, then it is a linear feedback equilibrium strategy of Problem $(\text{MV})_{tx}$.

Remark 2.1: By the definition, a mixed equilibrium solution $(\Phi, v^{t,x})$ is time consistent in the sense that $(\Phi, v^{t,x})|_{\mathbb{T}_k}$ is a mixed equilibrium solution for the initial pair $(k, X_k^{t,x,*})$. Since $(\Phi\cdot X^{k,\Phi}+v^{t,x})|_{\mathbb{T}_k}=(\Phi_k X_k^{k,\Phi}+v_k^{t,x},(\Phi\cdot X^{k,\Phi}+v^{t,x})|_{\mathbb{T}_{k+1}})$, we obtain $(u_k,(\Phi\cdot X^{k,u_t,\Phi}+v^{t,x})|_{\mathbb{T}_{k+1}})$ from $(\Phi\cdot X^{k,\Phi}+v^{t,x})|_{\mathbb{T}_k}$ by replacing not only $\Phi_k X_k^{k,\Phi}+v_k^{t,x}$ with u_k but also $X^{k,\Phi}$ with $X^{k,u_k,\Phi}$. Furthermore, it is valuable to mention that $v^{t,x}$'s in both sides of (4) are the same. This is why we call Φ the pure-feedback-strategy part and $v^{t,x}$ the open-loop-control part.

III. Characterization on the Equilibrium Solutions

To solve Problem $(\text{MV})_{tx}$, we shall transform (1) into a linear controlled system with multiplicative noises so that the general theory [12] can work. Precisely, define

$$\begin{cases} w_k^i=e_k^i-s_k-\mathbb{E}(e_k^i-s_k)\\ D_k^i=(0,\dots,0,1,0,\dots,0)\\ i=1,\dots,m, \quad k=0,1,\dots,N-1 \end{cases}$$

where the ith entry of D_k^i is 1. Then, $\{w_k=(w_k^1,\dots,w_k^m)^T, k\in\mathbb{T}\}$ is a martingale difference sequence as $e_k, k=0,\dots,N-1$, are statistically independent. Furthermore,

$$\mathbb{E}_k[w_k w_k^T]=\mathbb{E}[w_k w_k^T]=\text{Cov}(e_k)=(\delta_k^{ij})_{m\times m}.$$

This leads to

$$\begin{cases} X_{k+1}=(s_k X_k+(\mathbb{E}O_k)^T u_k)+\sum_{i=1}^m D_k^i u_k w_k^i\\ X_t=x, \quad k\in\mathbb{T}_t. \end{cases} \tag{5}$$

We first characterize the open-loop equilibrium portfolio control of Problem $(\text{MV})_{tx}$.

Theorem 3.1: For any $t\in\mathbb{T}$ and any $x\in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem $(\text{MV})_{tx}$ admits an open-loop equilibrium control if and only if $\mathbb{E}O_k\in\text{Ran}(\text{Cov}(O_k))$, $k\in\mathbb{T}$. In addition

$$u_k^{t,x}=-\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k X_k^{t,x,*}-\widehat{\mathcal{O}}_k^\dagger\widehat{\theta}_k, \quad k\in\mathbb{T}_t$$

is an open-loop equilibrium control of Problem $(\text{MV})_{tx}$ with

$$\begin{cases} \widehat{X}_{k+1}^{t,x,*}=(s_k-O_k^T\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k)\widehat{X}_k^{t,x,*}-O_k^T\widehat{\mathcal{O}}_k^\dagger\widehat{\theta}_k, & k\in\mathbb{T}_t\\ \widehat{X}_t^{t,x,*}=x \end{cases}$$

where $\widehat{\mathcal{O}}_k^\dagger$ is the Moore–Penrose generalized inverse of $\widehat{\mathcal{O}}_k$ with

$$\begin{cases} \widehat{\mathcal{O}}_k=(\widehat{S}_{k+1}+\widehat{T}_{k+1})\text{Cov}(O_k)\\ \widehat{\mathcal{L}}_k=-\frac{\mu_1}{2}s_{k+1}\cdots s_{N-1}\mathbb{E}O_k\\ \widehat{\theta}_k=-\frac{\mu_2}{2}s_{k+1}\cdots s_{N-1}\mathbb{E}O_k, & k\in\mathbb{T}_t \end{cases} \tag{6}$$

$$\begin{cases} \widehat{S}_k+\widehat{T}_k=(\widehat{S}_{k+1}+\widehat{T}_{k+1})s_k^2\\ \qquad\qquad -s_k(\widehat{S}_{k+1}+\widehat{T}_{k+1})(\mathbb{E}O_k)^T\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k\\ \widehat{S}_N+\widehat{T}_N=1, \quad k\in\mathbb{T}_t \end{cases}$$

and for $k=N-2,N-1$, $s_{k+1}\cdots s_{N-1}$ in (6) is understood as

$$s_{k+1}\cdots s_{N-1}=\begin{cases} 1, & k=N-1\\ s_{N-1}, & k=N-2. \end{cases}$$

Proof: This result is proved according to [12, Th. 3.11]. In this case, [12, Th. 3.11, eqs. (3.21)–(3.23)] become to

$$\begin{cases} \widehat{S}_k=s_k^2\widehat{S}_{k+1}, \quad \widehat{\mathcal{S}}_k=s_k^2\widehat{\mathcal{S}}_{k+1}\equiv 0, \quad \widehat{S}_N=1, \quad \widehat{\mathcal{S}}_N=0\\ \widehat{\mathbb{O}}_k=(\mathbb{E}O_k)\widehat{\mathcal{S}}_{k+1}(\mathbb{E}O_k)^T+\sum_{i,j=1}^m\delta_k^{ij}(D_k^i)^T\widehat{S}_{k+1}D_k^j\\ \quad =\widehat{S}_{k+1}\text{Cov}(O_k)\succeq 0, \quad k\in\mathbb{T} \end{cases} \tag{7}$$

$$\begin{cases} \widehat{T}_k=s_k^2\widehat{T}_{k+1}-s_k(\widehat{S}_{k+1}+\widehat{T}_{k+1})(\mathbb{E}O_k)^T\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k\\ \widehat{\mathcal{T}}_k=\widehat{\mathcal{T}}_{k+1}\left[s_k^2-s_k(\mathbb{E}O_k)^T\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k\right]\equiv 0\\ \widehat{T}_N=0, \quad \widehat{\mathcal{T}}_N=0, \ \widehat{\mathcal{O}}_k\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k-\widehat{\mathcal{L}}_k=0, \quad k\in\mathbb{T} \end{cases} \tag{8}$$

and

$$\widehat{\pi}_k=s_k\widehat{\pi}_{k+1}, \ \widehat{\pi}_N=-\frac{\mu_2}{2}, \quad \widehat{\mathcal{O}}_k\widehat{\mathcal{O}}_k^\dagger\widehat{\theta}_k-\widehat{\theta}_k=0, \quad k\in\mathbb{T} \tag{9}$$

where

$$\begin{cases} \widehat{\mathcal{O}}_k=\sum_{i,j=1}^m\delta_k^{ij}(D_k^i)^T(\widehat{S}_{k+1}+\widehat{T}_{k+1})D_k^j\\ \quad =(\widehat{S}_{k+1}+\widehat{T}_{k+1})\text{Cov}(O_k)\\ \widehat{\mathcal{L}}_k=\widehat{U}_{k+1}\mathbb{E}O_k, \quad \widehat{\theta}_k=\widehat{\pi}_{k+1}\mathbb{E}O_k, \quad k\in\mathbb{T} \end{cases}$$

with $\widehat{U}_k=s_k\widehat{U}_{k+1}$, $\widehat{U}_N=-\frac{\mu_1}{2}$, $k\in\mathbb{T}$. From (7)–(9), we have

$$\begin{cases} \widehat{S}_k+\widehat{T}_k=(\widehat{S}_{k+1}+\widehat{T}_{k+1})s_k^2\\ \qquad\qquad -s_k(\widehat{S}_{k+1}+\widehat{T}_{k+1})(\mathbb{E}O_k)^T\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k\\ \widehat{S}_N+\widehat{T}_N=1\\ \widehat{\mathcal{O}}_k\widehat{\mathcal{O}}_k^\dagger\widehat{\mathcal{L}}_k-\widehat{\mathcal{L}}_k=0, \ k\in\mathbb{T}. \end{cases}$$

By some calculations, we have

$$\begin{aligned} \widehat{S}_k+\widehat{T}_k&=(\widehat{S}_{k+1}+\widehat{T}_{k+1})s_k^2+s_k s_{k+1}\cdots s_{N-1}\frac{\mu_1}{2}\\ &\quad\times(\widehat{S}_{k+1}+\widehat{T}_{k+1})(\widehat{S}_{k+1}+\widehat{T}_{k+1})^\dagger(\mathbb{E}O_k)^T\\ &\quad\times\left[\text{Cov}(O_k)\right]^\dagger\mathbb{E}O_k \end{aligned}$$

where $(\widehat{S}_{k+1}+\widehat{T}_{k+1})^\dagger$ equals to $(\widehat{S}_{k+1}+\widehat{T}_{k+1})^{-1}$ if $\widehat{S}_{k+1}+\widehat{T}_{k+1}\neq 0$; otherwise, it will be 0. Therefore

$$(\widehat{S}_{k+1}+\widehat{T}_{k+1})(\widehat{S}_{k+1}+\widehat{T}_{k+1})^\dagger=\begin{cases}1, & \widehat{S}_{k+1}+\widehat{T}_{k+1}\neq 0\\ 0, & \widehat{S}_{k+1}+\widehat{T}_{k+1}=0.\end{cases}$$

As $s_k>1,\mu_1>0,\mathrm{Cov}(O_k)\succeq 0, k\in\mathbb{T}$, and $\widehat{S}_N+\widehat{T}_N=1$, it follows that $\widehat{S}_k+\widehat{T}_k>0, k\in\mathbb{T}$. This together with $\widehat{\pi}_k\neq 0, \widehat{U}_k\neq 0, k\in\mathbb{T}$ implies that the solvability of (7)–(9) is equivalent to the property $\mathbb{E}O_k\in\mathrm{Ran}(\mathrm{Cov}(O_k)), k\in\mathbb{T}$. Then, the proof is completed by following [12, Th. 3.11]. ■

Corollary 3.1: Let $\mathrm{Cov}(O_k)\succ 0, k\in\mathbb{T}$. Then, for any $t\in\mathbb{T}$ and any $x\in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem $(\mathrm{MV})_{tx}$ admits a unique open-loop equilibrium control, which is given by

$$u_k^{t,x}=-\widehat{\mathcal{O}}_k^{-1}\widehat{\mathcal{L}}_k X_k^{t,x,*}-\widehat{\mathcal{O}}_k^{-1}\widehat{\theta}_k,\quad k\in\mathbb{T}_t$$

with

$$\begin{cases}\widehat{X}_{k+1}^{t,x,*}=(s_k-O_k^T\widehat{\mathcal{O}}_k^{-1}\widehat{\mathcal{L}}_k)\widehat{X}_k^{t,x,*}-O_k^T\widehat{\mathcal{O}}_k^{-1}\widehat{\theta}_k\\ \widehat{X}_t^{t,x,*}=x,\quad k\in\mathbb{T}_t\end{cases}$$

and $\widehat{\mathcal{O}}_k,\widehat{\mathcal{L}}_k,\widehat{\theta}_k, k\in\mathbb{T}_t$, are given in (6).

Proof: It follows from [12, Th. 3.9] and Theorem 3.1. ■

The following result is based on [12, Th. 3.13], which is about the feedback equilibrium strategy of Problem $(\mathrm{MV})_{tx}$.

Theorem 3.2: For any $t\in\mathbb{T}$ and any $x\in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem $(\mathrm{MV})_{tx}$ admits a feedback equilibrium strategy if and only if the following difference equations:

$$\begin{cases}\widetilde{S}_k=\widetilde{S}_{k+1}\Big[s_k^2-2s_k(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\\ \qquad+\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\mathbb{E}(O_kO_k^T)\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\Big]\\ \widetilde{\mathcal{S}}_k=s_k^2\widetilde{\mathcal{S}}_{k+1}-2s_k\widetilde{\mathcal{S}}_{k+1}(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\\ \qquad+\widetilde{\mathcal{S}}_{k+1}\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\mathbb{E}O_k(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\\ \qquad+\widetilde{S}_{k+1}\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\mathrm{Cov}(O_k)\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\\ \widetilde{S}_N=1,\quad\widetilde{\mathcal{S}}_N=0\\ \widetilde{\mathbb{O}}_k\succeq 0,\quad\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k=\widetilde{\mathbb{L}}_k,\quad k\in\mathbb{T}\end{cases}\tag{10}$$

and

$$\begin{cases}\widetilde{\pi}_k=-\widetilde{\beta}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\theta}_k+(s_k-(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k)^T\widetilde{\pi}_{k+1}\\ \pi_N=-\frac{\mu_2}{2},\quad\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\theta}_k=\widetilde{\theta}_k,\quad k\in\mathbb{T}\end{cases}\tag{11}$$

are solvable, namely

$$\widetilde{\mathbb{O}}_k\succeq 0,\quad\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k=\widetilde{\mathbb{L}}_k,\quad\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\theta}_k=\widetilde{\theta}_k,\quad k\in\mathbb{T}$$

where

$$\begin{cases}\widetilde{\mathbb{O}}_k=\widetilde{\mathcal{S}}_{k+1}\mathbb{E}O_k(\mathbb{E}O_k)^T+\widetilde{S}_{k+1}\mathrm{Cov}(O_k)\\ \widetilde{\mathbb{L}}_k=(s_k\widetilde{\mathcal{S}}_{k+1}+\widetilde{U}_{k+1})\mathbb{E}O_k\\ \widetilde{\theta}_k=\widetilde{\pi}_{k+1}\mathbb{E}O_k,\quad k\in\mathbb{T}\end{cases}\tag{12}$$

with

$$\widetilde{U}_k=(s_k-(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k)\widetilde{U}_{k+1},\quad\widetilde{U}_N=-\frac{\mu_1}{2},\quad k\in\mathbb{T}$$

and

$$\widetilde{\beta}_k=\ s_k\widetilde{\mathcal{S}}_{k+1}(\mathbb{E}O_k)^T-\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\big[\widetilde{\mathcal{S}}_{k+1}\mathbb{E}O_k(\mathbb{E}O_k)^T+\widetilde{S}_{k+1}\mathrm{Cov}(O_k)\big],\quad k\in\mathbb{T}.$$

Theorem 3.3: Let $\mathbb{E}O_k\in\mathrm{Ran}(\mathrm{Cov}(O_k)), k\in\mathbb{T}$. Then, (10) and (11) are solvable; for any $t\in\mathbb{T}$ and $x\in l^2_{\mathcal{F}}(t;\mathbb{R})$, (Φ^t,v^t) with

$$\Phi^t=\{-\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k,\quad k\in\mathbb{T}_t\},\quad v^t=\{-\widetilde{\mathbb{O}}_k^\dagger\widetilde{\theta}_k,\quad k\in\mathbb{T}_t\}$$

is a feedback equilibrium strategy of Problem $(\mathrm{MV})_{tx}$, where $\widetilde{\mathbb{O}}_k,\widetilde{\mathbb{L}}_k,\widetilde{\theta}_k, k\in\mathbb{T}_t$, are given in (12).

Proof: Note that (10) can be equivalently rewritten as

$$\begin{cases}\widetilde{S}_k=\widetilde{S}_{k+1}\Big[(s_k-(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k)^2\\ \qquad+\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\mathrm{Cov}(O_k)\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\Big]\\ \widetilde{\mathcal{S}}_k=\widetilde{\mathcal{S}}_{k+1}(s_k-(\mathbb{E}O_k)^T\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k)^2\\ \qquad+\widetilde{S}_{k+1}\widetilde{\mathbb{L}}_k^T\widetilde{\mathbb{O}}_k^\dagger\mathrm{Cov}(O_k)\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k\\ \widetilde{S}_N=1,\quad\widetilde{\mathcal{S}}_N=0\\ \widetilde{\mathbb{O}}_k\succeq 0,\quad\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k=\widetilde{\mathbb{L}}_k,\quad k\in\mathbb{T}.\end{cases}\tag{13}$$

Clearly, $\widetilde{S}_k\geq\widetilde{\mathcal{S}}_k\geq 0, k\in\mathbb{T}$.

Let $\mathbb{E}O_k\in\mathrm{Ran}(\mathrm{Cov}(O_k)), k\in\mathbb{T}$. We now prove that (10) and (11) are solvable. For a generic $k\in\mathbb{T}$, we prove the conclusion by the following two cases.

Case 1: $\widetilde{S}_{k+1}>0$. As $\mathbb{E}O_k\in\mathrm{Ran}(\mathrm{Cov}(O_k))$, there exists $\xi\in\mathbb{R}^n$ such that $\mathrm{Cov}(O_k)\xi=\mathbb{E}O_k$. Furthermore, $\xi^T\mathrm{Cov}(O_k)\xi=\xi^T\mathbb{E}O_k\geq 0$. Then

$$\begin{aligned}&\widetilde{\mathbb{O}}_k\frac{s_k\widetilde{\mathcal{S}}_{k+1}+\widetilde{U}_{k+1}}{\widetilde{\mathcal{S}}_{k+1}\xi^T\mathbb{E}O_k+\widetilde{S}_{k+1}}\xi\\ &=\frac{s_k\widetilde{\mathcal{S}}_{k+1}+\widetilde{U}_{k+1}}{\widetilde{\mathcal{S}}_{k+1}\xi^T\mathbb{E}O_k+\widetilde{S}_{k+1}}\\ &\quad\times(\widetilde{\mathcal{S}}_{k+1}\mathbb{E}O_k(\mathbb{E}O_k)^T\xi+\widetilde{S}_{k+1}\mathrm{Cov}(O_k)\xi)\\ &=(s_k\widetilde{\mathcal{S}}_{k+1}+\widetilde{U}_{k+1})\mathbb{E}O_k=\widetilde{\mathbb{L}}_k.\end{aligned}$$

Hence, $\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k=\widetilde{\mathbb{L}}_k$, and (13) is solvable at k. Similarly

$$\widetilde{\mathbb{O}}_k\frac{\widetilde{\pi}_{k+1}}{\widetilde{\mathcal{S}}_{k+1}\xi^T\mathbb{E}O_k+\widetilde{S}_{k+1}}\xi=\widetilde{\pi}_{k+1}\mathbb{E}O_k=\widetilde{\theta}_k$$

which implies the solvability of (11) at k.

Case 2: $\widetilde{S}_{k+1}=0$. If $\widetilde{S}_{k+2}>0$, we have $\widetilde{\mathbb{O}}_k=0$ and

$$\begin{aligned}&s_{k+1}-(\mathbb{E}O_{k+1})^T\widetilde{\mathbb{O}}_{k+1}^\dagger\widetilde{\mathbb{L}}_{k+1}\\ &=\widetilde{\mathbb{L}}_{k+1}^T\widetilde{\mathbb{O}}_{k+1}^\dagger\mathrm{Cov}(O_{k+1})\widetilde{\mathbb{O}}_{k+1}^\dagger\widetilde{\mathbb{L}}_{k+1}=0\end{aligned}\tag{14}$$

which further implies $\widetilde{U}_{k+1}=0$ and $\widetilde{\mathbb{L}}_k=0$; hence, (13) is solvable at k. Furthermore, (14) implies $\widetilde{\mathbb{L}}_{k+1}^T\widetilde{\mathbb{O}}_{k+1}^\dagger\mathrm{Cov}(O_{k+1})=0$. From this and (14), we have $\widetilde{\beta}_{k+1}=0$ and $\widetilde{\pi}_{k+1}=0$, and hence (11) is solvable at k under the condition of $\widetilde{S}_{k+1}=0,\widetilde{S}_{k+2}>0$. Finally, if $\widetilde{S}_{k+2}=0$, we must have some $\tau>k+1$ such that $\widetilde{S}_\tau=0,\widetilde{S}_{\tau+1}>0$. Similar to the comments below (14), we have $\widetilde{U}_\tau=\widetilde{\beta}_\tau=\widetilde{\pi}_\tau=0$, which implies $\widetilde{U}_{k+1}=0$ and the solvability of (13) at k. As $\widetilde{\mathbb{O}}_\ell=0,\ell\in\{k,\ldots,\tau-1\}$, it follows that $\widetilde{\pi}_{k+1}=(s_{k+1}-(\mathbb{E}O_{k+1})^T\widetilde{\mathbb{O}}_{k+1}^\dagger\widetilde{\mathbb{L}}_{k+1})^T\cdots(s_{\tau-1}-(\mathbb{E}O_{\tau-1})^T\widetilde{\mathbb{O}}_{\tau-1}^\dagger\widetilde{\mathbb{L}}_{\tau-1})^T\widetilde{\pi}_\tau=0$. Hence, (11) is solvable at k.

In summary, for a generic $k\in\mathbb{T}$, we have proved the solvability of (11) and (13) at k, namely, $\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\mathbb{L}}_k=\widetilde{\mathbb{L}}_k$ and $\widetilde{\mathbb{O}}_k\widetilde{\mathbb{O}}_k^\dagger\widetilde{\theta}_k=\widetilde{\theta}_k$. From [12, Th. 3.13] and Theorem 3.2, we can complete the proof. ■

Theorem 3.4: Let $\mathrm{Cov}(O_k)\succ 0, k\in\mathbb{T}$. Then, for any $t\in\mathbb{T}$ and any $x\in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem $(\mathrm{MV})_{tx}$ admits a unique feedback equilibrium strategy, which is given by (Φ^t,v^t) with

$$\Phi^t=\{-\widetilde{\mathbb{O}}_k^{-1}\widetilde{\mathbb{L}}_k,\quad k\in\mathbb{T}_t\},v^t=\{-\widetilde{\mathbb{O}}_k^{-1}\widetilde{\theta}_k,\quad k\in\mathbb{T}_t\}.$$

Here, $\widetilde{\mathbb{O}}_k, \widetilde{\mathbb{L}}_k, \widetilde{\theta}_k, k \in \mathbb{T}_t$, are given in (12).

Proof: In this case, (10) and (11) are solvable. From (13), we know that $\widetilde{S}_k > 0, k \in \mathbb{T}$. In fact, suppose $\widetilde{S}_{k_0} = 0$ and $\widetilde{S}_{k_0+1} \neq 0$ for some $k_0 \in \mathbb{T}$, then

$$s_{k_0} - (\mathbb{E}O_{k_0})^T \widetilde{\mathbb{O}}_{k_0}^{\dagger} \widetilde{\mathbb{L}}_{k_0} = \widetilde{\mathbb{L}}_{k_0}^T \widetilde{\mathbb{O}}_{k_0}^{\dagger} \mathrm{Cov}(O_{k_0}) \widetilde{\mathbb{O}}_{k_0}^{\dagger} \widetilde{\mathbb{L}}_{k_0} = 0.$$

As $\mathrm{Cov}(O_{k_0}) \succ 0$, it follows $\widetilde{\mathbb{O}}_{k_0}^{\dagger} \widetilde{\mathbb{L}}_{k_0} = 0$, which implies $0 = s_{k_0} - (\mathbb{E}O_{k_0})^T \widetilde{\mathbb{O}}_{k_0}^{\dagger} \widetilde{\mathbb{L}}_{k_0} = s_{k_0}$. This is impossible, and thus $\widetilde{S}_k > 0, k \in \mathbb{T}$. Furthermore, we have $\widetilde{\mathbb{O}}_k \succ 0, k \in \mathbb{T}$. ■

We now consider the mixed equilibrium portfolio solution. In this case, [12, eq. (3.26)–(3.28)] read as

$$\begin{cases} S_k = S_{k+1}\Big[(s_k + (\mathbb{E}O_k)^T\Phi_k)^2 + \Phi_k^T \mathrm{Cov}(O_k)\Phi_k\Big] \\ \mathcal{S}_k = \mathcal{S}_{k+1}(s_k + (\mathbb{E}O_k)^T\Phi_k)^2 \\ \qquad + S_{k+1}\Phi_k^T\mathrm{Cov}(O_k)\Phi_k \\ S_N = 1, \quad \mathcal{S}_N = 0 \\ \mathbb{O}_k = \mathcal{S}_{k+1}\mathbb{E}O_k(\mathbb{E}O_k)^T + S_{k+1}\mathrm{Cov}(O_k) \succeq 0 \\ k \in \mathbb{T} \end{cases} \tag{15}$$

$$\begin{cases} T_k = S_{k+1}\Big[(s_k + (\mathbb{E}O_k)^T\Phi_k)(\mathbb{E}O_k)^T \\ \qquad + \Phi_k^T\mathrm{Cov}(O_k)\Big](-\mathcal{O}_k^{\dagger}\mathcal{L}_k - \Phi_k) \\ \qquad + T_{k+1}\Big[(s_k + (\mathbb{E}O_k)^T\Phi_k) \\ \qquad \times (s_k - (\mathbb{E}O_k)^T\mathcal{O}_k^{\dagger}\mathcal{L}_k) \\ \qquad - \Phi_k^T\mathrm{Cov}(O_k)\mathcal{O}_k^{\dagger}\mathcal{L}_k\Big] \\ \mathcal{T}_k = \Big[\mathcal{S}_{k+1}(s_k + (\mathbb{E}O_k)^T\Phi_k)(\mathbb{E}O_k)^T \\ \qquad + S_{k+1}\Phi_k^T\mathrm{Cov}(O_k)\Big](-\mathcal{O}_k^{\dagger}\mathcal{L}_k - \Phi_k) \\ \qquad + \Big[\mathcal{T}_{k+1}(s_k + (\mathbb{E}O_k)^T\Phi_k) \\ \qquad \times (s_k - (\mathbb{E}O_k)^T\mathcal{O}_k^{\dagger}\mathcal{L}_k) \\ \qquad - T_{k+1}\Phi_k^T\mathrm{Cov}(O_k)\mathcal{O}_k^{\dagger}\mathcal{L}_k\Big] \\ T_N = 0, \quad \mathcal{T}_N = 0 \\ \mathcal{O}_k\mathcal{O}_k^{\dagger}\mathcal{L}_k = \mathcal{L}_k, \quad k \in \mathbb{T} \end{cases} \tag{16}$$

and

$$\begin{cases} \pi_k = -\beta_k\mathcal{O}_k^{\dagger}\theta_k + (s_k + (\mathbb{E}O_k)^T\Phi_k)^T\pi_{k+1} \\ \pi_N = -\frac{\mu_2}{2} \\ \mathcal{O}_k\mathcal{O}_k^{\dagger}\theta_k = \theta_k, \quad k \in \mathbb{T} \end{cases} \tag{17}$$

where

$$\begin{cases} \mathcal{O}_k = (\mathcal{S}_{k+1} + \mathcal{T}_{k+1})\mathbb{E}O_k(\mathbb{E}O_k)^T \\ \qquad + (S_{k+1} + T_{k+1})\mathrm{Cov}(O_k) \\ \mathcal{L}_k = s_k(\mathcal{S}_{k+1} + \mathcal{T}_{k+1})\mathbb{E}O_k + U_{k+1}\mathbb{E}O_k \\ \theta_k = \pi_{k+1}\mathbb{E}O_k, \quad k \in \mathbb{T} \end{cases}$$

$$\begin{cases} U_k = (s_k + (\mathbb{E}O_k)^T\Phi_k)U_{k+1} \\ U_N = -\frac{\mu_1}{2}, \quad k \in \mathbb{T} \end{cases}$$

and

$$\beta_k = (\mathcal{S}_{k+1} + \mathcal{T}_{k+1})(s_k + \Phi_k^T\mathbb{E}O_k)(\mathbb{E}O_k)^T + (S_{k+1} + T_{k+1})\Phi_k^T\mathrm{Cov}(O_k), \quad k \in \mathbb{T}.$$

From (15) and (16), we obtain

$$\begin{cases} S_k + T_k = (S_{k+1} + T_{k+1})\Big[(s_k + (\mathbb{E}O_k)^T\Phi_k) \\ \qquad \times (s_k - (\mathbb{E}O_k)^T\mathcal{O}_k^{\dagger}\mathcal{L}_k) \\ \qquad - \Phi_k^T\mathrm{Cov}(O_k)\mathcal{O}_k^{\dagger}\mathcal{L}_k\Big] \\ \mathcal{S}_k + \mathcal{T}_k = (\mathcal{S}_{k+1} + \mathcal{T}_{k+1})(s_k + (\mathbb{E}O_k)^T\Phi_k) \\ \qquad \times (s_k - (\mathbb{E}O_k)^T\mathcal{O}_k^{\dagger}\mathcal{L}_k) - (S_{k+1} \\ \qquad + T_{k+1})\Phi_k^T\mathrm{Cov}(O_k)\mathcal{O}_k^{\dagger}\mathcal{L}_k, \quad k \in \mathbb{T}. \end{cases} \tag{18}$$

Letting $S_k + T_k - \mathcal{S}_k - \mathcal{T}_k = \Delta_k, k \in \widetilde{\mathbb{T}} = \{0, \ldots, N\}$, then

$$\begin{cases} \Delta_k = \Delta_{k+1}(s_k + (\mathbb{E}O_k)^T\Phi_k)(s_k - (\mathbb{E}O_k)^T\mathcal{O}_k^{\dagger}\mathcal{L}_k) \\ \Delta_N = 1, \quad k \in \mathbb{T}. \end{cases} \tag{19}$$

For the case $\mathrm{Cov}(O_k) \succ 0, k \in \mathbb{T}$, we have shown in the proof of Theorem 3.4 that $\widetilde{S}_k > 0, k \in \mathbb{T}$. Noting (13), (18), and (19), we could select Φ by the continuity such that $S_k + T_k > 0, \Delta \geq 0, k \in \widetilde{\mathbb{T}}$, and hence $\mathcal{O}_k, k \in \mathbb{T}$ are invertible. As for any $k \in \mathbb{T}$, $\mathbb{O}_k \succeq 0$, we know that (15) is solvable. By [12, Th. 3.14], the following result is straightforward.

Proposition 3.1: Let $\Phi \in l^2(\mathbb{T}; \mathbb{R}^m)$ such that (16) and (17) are solvable, and let

$$v_k^{t,x} = -(\mathcal{O}_k^{\dagger}\mathcal{L}_k + \Phi_k)X_k^{t,x,*} - \mathcal{O}_k^{\dagger}\theta_k, \quad k \in \mathbb{T}_t$$

where

$$\begin{cases} X_{k+1}^{t,x,*} = (s_k - O_k^T\mathcal{O}_k^{\dagger}\mathcal{L}_k)X_k^{t,x,*} - O_k^T\mathcal{O}_k^{\dagger}\theta_k \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases}$$

Then, $(\Phi|_{\mathbb{T}_t}, v^{t,x})$ is a mixed equilibrium solution of Problem $(\mathrm{MV})_{tx}$.

Remark 3.1: Mixed equilibrium solution is studied in this paper, by which we can investigate the open-loop equilibrium control and the linear feedback equilibrium strategy in a unified way. Importantly, the mixed equilibrium solution is not a hollow concept. In an example of [12], it is shown that neither the open-loop equilibrium control nor the feedback equilibrium strategy exists for the initial pair (t, x) with $t = 0, 1$ and $x \in l_{\mathcal{F}}^2(t; \mathbb{R}^2)$, although we are able to construct ten mixed equilibrium solutions. Therefore, it is necessary to study the mixed equilibrium solution, which gives us more flexibility to deal with the time-inconsistent control.

IV. An Example

Consider a multiperiod mean-variance portfolio selection problem. A capital market consists of one riskless asset and three risky assets over a finite time horizon $N = 4$, and the parameters of the model are as follows:

$$x = 10, \quad s_k = 1.04, \quad \mathbb{E}e_k^1 = 1.162, \quad \mathbb{E}e_k^2 = 1.246$$
$$\mathbb{E}e_k^3 = 1.228, \quad k = 0, 1, 2, 3$$

and the covariance of $e_k = (e_k^1, e_k^2, e_k^3)^T$ is

$$\mathrm{Cov}(e_k) = \begin{bmatrix} 0.2920 & 0.3740 & 0.2900 \\ 0.3740 & 1.7080 & 0.2080 \\ 0.2900 & 0.2080 & 0.5780 \end{bmatrix} \succ 0, \quad k = 0, 1, 2, 3.$$

In this paper, we assume $\mu_1 = \mu_2 = 0.2$. Clearly

$$\mathbb{E}O_k = (0.1220, \quad 0.2060, \quad 0.1880)^T, \quad k = 0, 1, 2, 3.$$

As $\mathrm{Cov}(O_k) = \mathrm{Cov}(e_k) \succ 0, k = 0, 1, 2, 3$, Problem (MV) has a unique open-loop equilibrium control and a unique feedback equilibrium strategy. In what follows, we will compute the equilibrium solutions, respectively.

Open-loop equilibrium control

By (6) and some calculations, we have

$$-\widehat{\mathcal{O}}_3^{-1}\widehat{\mathcal{L}}_3 = \begin{bmatrix}0.0047\\0.0077\\0.0274\end{bmatrix}, \quad -\widehat{\mathcal{O}}_2^{-1}\widehat{\mathcal{L}}_2 = \begin{bmatrix}0.0045\\0.0073\\0.0261\end{bmatrix}$$

$$-\widehat{\mathcal{O}}_1^{-1}\widehat{\mathcal{L}}_1 = \begin{bmatrix}0.0043\\0.0070\\0.0250\end{bmatrix}, \quad -\widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{L}}_0 = \begin{bmatrix}0.0041\\0.0067\\0.0239\end{bmatrix}$$

$$-\widehat{\mathcal{O}}_3^{-1}\widehat{\theta}_3 = \begin{bmatrix}0.0047\\0.0077\\0.0274\end{bmatrix}, \quad -\widehat{\mathcal{O}}_2^{-1}\widehat{\theta}_2 = \begin{bmatrix}0.0045\\0.0073\\0.0261\end{bmatrix}$$

$$-\widehat{\mathcal{O}}_1^{-1}\widehat{\theta}_1 = \begin{bmatrix}0.0043\\0.0070\\0.0250\end{bmatrix}, \quad -\widehat{\mathcal{O}}_0^{-1}\widehat{\theta}_0 = \begin{bmatrix}0.0041\\0.0067\\0.0239\end{bmatrix}.$$

Then, the unique open-loop equilibrium portfolio control for the initial pair $(0, x)$ is given by

$$u_k^{0,x} = -\widehat{\mathcal{O}}_k^{-1}\widehat{\mathcal{L}}_k X_k^{0,x,*} - \widehat{\mathcal{O}}_k^{-1}\widehat{\theta}_k \tag{20}$$

with

$$\begin{cases} \widehat{X}_{k+1}^{0,x,*} = (s_k - O_k^T\widehat{\mathcal{O}}_k^{-1}\widehat{\mathcal{L}}_k)\widehat{X}_k^{0,x,*} - O_k^T\widehat{\mathcal{O}}_k^{-1}\widehat{\theta}_k \\ \widehat{X}_0^{0,x,*} = x, \quad k = 0,1,2,3. \end{cases}$$

Feedback equilibrium strategy

By (10), (11), and some calculations, we have

$$-\widetilde{\mathbb{O}}_3^{-1}\widetilde{\mathbb{L}}_3 = \begin{bmatrix}0.0047\\0.0077\\0.0274\end{bmatrix}, -\widetilde{\mathbb{O}}_2^{-1}\widetilde{\mathbb{L}}_2 = \begin{bmatrix}0.0045\\0.0073\\0.0259\end{bmatrix} \tag{21}$$

$$-\widetilde{\mathbb{O}}_1^{-1}\widetilde{\mathbb{L}}_1 = \begin{bmatrix}0.0043\\0.0069\\0.0246\end{bmatrix}, -\widetilde{\mathbb{O}}_0^{-1}\widetilde{\mathbb{L}}_0 = \begin{bmatrix}0.0040\\0.0065\\0.0233\end{bmatrix} \tag{22}$$

$$-\widetilde{\mathbb{O}}_3^{-1}\widetilde{\theta}_3 = \begin{bmatrix}0.0047\\0.0077\\0.0274\end{bmatrix}, -\widetilde{\mathbb{O}}_2^{-1}\widetilde{\theta}_2 = \begin{bmatrix}0.0045\\0.0073\\0.0259\end{bmatrix} \tag{23}$$

$$-\widetilde{\mathbb{O}}_1^{-1}\widetilde{\theta}_1 = \begin{bmatrix}0.0043\\0.0069\\0.0276\end{bmatrix}, -\widetilde{\mathbb{O}}_0^{-1}\widetilde{\theta}_0 = \begin{bmatrix}0.0040\\0.0026\\0.0233\end{bmatrix}. \tag{24}$$

Then, the unique feedback equilibrium strategy is given by $\{(-\widetilde{\mathbb{O}}_k^{-1}\widetilde{\mathbb{L}}_k, -\widetilde{\mathbb{O}}_k^{-1}\widetilde{\theta}_k), k = 0,1,2,3\}$.

Mixed equilibrium solution

Frankly speaking, up to now we have not yet got the quantitative result about what kind of $\Phi' s$ will make (16) and (17) be solvable, but for any given Φ we can check the solvability of (16) and (17). So, here we would like to select Φ by using the command "randn" of MATLAB and check whether it is the pure-feedback-strategy part of a mixed equilibrium solution. Specifically, use "randn" to generate 4 $\Phi' s$ with $\Phi = \{\Phi_k \in \mathbb{R}^{3\times 1}, k = 0,1,2,3\}$ and perform the iterations (15)–(17) for four times. Denoting $\phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$, the followings are the four ϕ's and the eigenvalues of the corresponding $\mathcal{O}_k, k = 0,1,2,3$:

$$\begin{cases} \phi = \begin{bmatrix} -2.5864 & -0.7152 & 0.3583 & 0.6190 \\ -0.6164 & -0.9901 & 0.6117 & -1.8179 \\ 0.2276 & -0.3816 & 0.7348 & -1.3685 \end{bmatrix} \\ \begin{cases} 0.0358,\ 0.2787,\ 0.8133: & \text{eigenvalues of } \mathcal{O}_0 \\ 0.0549,\ 0.4294,\ 1.2536: & \text{eigenvalues of } \mathcal{O}_1 \\ 0.0366,\ 0.2861,\ 0.8349: & \text{eigenvalues of } \mathcal{O}_2 \\ 0.0813,\ 0.6370,\ 1.8598: & \text{eigenvalues of } \mathcal{O}_3 \end{cases} \end{cases}$$

$$\begin{cases} \phi = \begin{bmatrix} 0.4841 & 1.1024 & 1.2733 & 0.9204 \\ -0.9765 & -0.3161 & -0.2800 & 0.9836 \\ -0.7177 & -2.4514 & 1.1265 & 0.2306 \end{bmatrix} \\ \begin{cases} 0.1148,\ 0.9002,\ 2.6286: & \mathcal{O}_0 \\ 0.1745,\ 1.3695,\ 3.9993: & \mathcal{O}_1 \\ 0.1219,\ 0.9564,\ 2.7927: & \mathcal{O}_2 \\ 0.0813,\ 0.6370,\ 1.8598: & \mathcal{O}_3 \end{cases} \end{cases}$$

$$\begin{cases} \phi = \begin{bmatrix} 0.0945 & -1.1274 & 0.7550 & -0.1835 \\ -0.7431 & -1.5599 & -0.3098 & 1.3948 \\ 0.8976 & -0.0299 & -0.1677 & -0.3083 \end{bmatrix} \\ \begin{cases} 0.0663,\ 0.5188,\ 1.5146: & \mathcal{O}_0 \\ 0.1169,\ 0.9164,\ 2.6758: & \mathcal{O}_1 \\ 0.1078,\ 0.8455,\ 2.4688: & \mathcal{O}_2 \\ 0.0813,\ 0.6370, 1.8598: & \mathcal{O}_3 \end{cases} \end{cases}$$

$$\begin{cases} \phi = \begin{bmatrix} -0.8463 & 0.9086 & -0.5227 & 0.1135 \\ -0.2754 & -0.8210 & -0.4079 & -1.3856 \\ -0.6649 & -1.8209 & 0.6685 & -0.4858 \end{bmatrix} \\ \begin{cases} 0.0356,\ 0.2771,\ 0.8087: & \mathcal{O}_0 \\ 0.0583,\ 0.4563,\ 1.3320: & \mathcal{O}_1 \\ 0.0547,\ 0.4278,\ 1.2487: & \mathcal{O}_2 \\ 0.0813,\ 0.6370,\ 1.8598: & \mathcal{O}_3. \end{cases} \end{cases}$$

For each ϕ, all the eigenvalues of $\mathcal{O}_k, k = 0,1,2,3$, are nonzero; this means that $\mathcal{O}_k, k = 0,1,2,3$, are all invertible. Therefore, the corresponding (16) and (17) are solvable. In addition, (15) is clearly solvable. From Proposition 3.1, we know that for all the above four cases the mixed equilibrium solutions exist. For example, with the first ϕ above, the mixed equilibrium solution is as follows. Let

$$\Phi_0 = \begin{bmatrix}-2.5864\\-0.6164\\0.2276\end{bmatrix}, \ \Phi_1 = \begin{bmatrix}-0.7152\\-0.9901\\-0.3816\end{bmatrix} \tag{25}$$

$$\Phi_2 = \begin{bmatrix}0.3583\\0.6117\\0.7348\end{bmatrix}, \ \Phi_3 = \begin{bmatrix}0.6190\\-1.8179\\-1.3685\end{bmatrix} \tag{26}$$

and

$$v_k^{0,x} = -(\mathcal{O}_k^{-1}\mathcal{L}_k + \Phi_k)X_k^{0,x,*} - \mathcal{O}_k^{-1}\theta_k \tag{27}$$

with

$$\begin{cases} X_{k+1}^{0,x,*} = (s_k - O_k^T\mathcal{O}_k^{-1}\mathcal{L}_k)X_k^{0,x,*} - O_k^T\mathcal{O}_k^{-1}\theta_k \\ X_0^{0,x,*} = x, \quad k = 0,1,2,3 \end{cases}$$

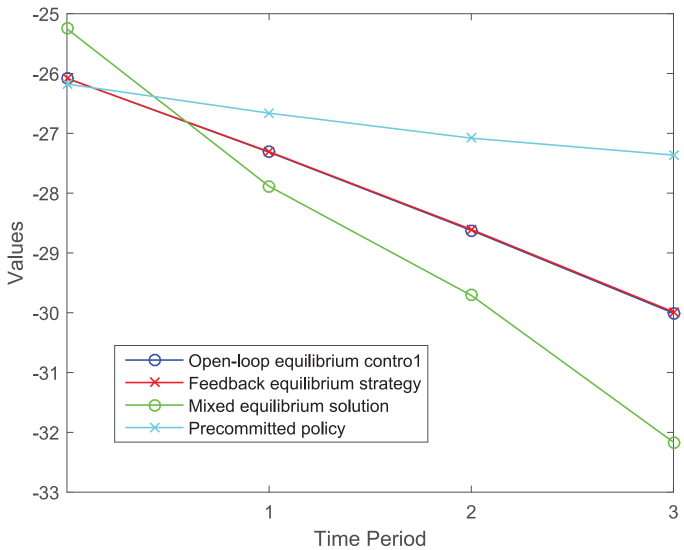

Fig. 1. Curves of $V_o(k), V_f(k), V_m(k), V_p(k)$, $k = 0, 1, 2, 3$.

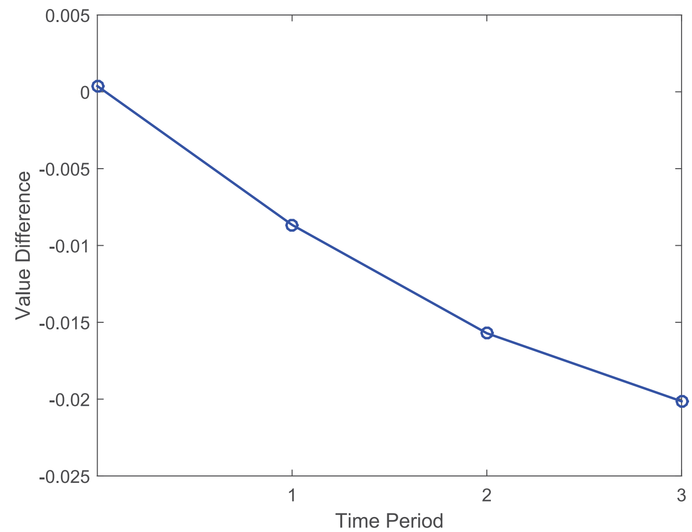

Fig. 2. Curve of $V_o(k) - V_f(k)$, $k = 0, 1, 2, 3$.

and

$$-\mathcal{O}_0^{-1}\mathcal{L}_0 = \begin{bmatrix} 0.0132 \\ 0.0215 \\ 0.0765 \end{bmatrix}, \; -\mathcal{O}_1^{-1}\mathcal{L}_1 = \begin{bmatrix} 0.0080 \\ 0.0130 \\ 0.0461 \end{bmatrix}$$

$$-\mathcal{O}_2^{\dagger}\mathcal{L}_2 = \begin{bmatrix} 0.0113 \\ 0.0183 \\ 0.0651 \end{bmatrix}, \; -\mathcal{O}_3^{-1}\mathcal{L}_3 = \begin{bmatrix} 0.0047 \\ 0.0077 \\ 0.0274 \end{bmatrix}$$

$$-\mathcal{O}_0^{-1}\theta_0 = \begin{bmatrix} 0.0124 \\ 0.0202 \\ 0.0719 \end{bmatrix}, \; -\mathcal{O}_1^{-1}\theta_1 = \begin{bmatrix} 0.0077 \\ 0.0125 \\ 0.0444 \end{bmatrix}$$

$$-\mathcal{O}_2^{-1}\theta_2 = \begin{bmatrix} 0.0110 \\ 0.0179 \\ 0.0637 \end{bmatrix}, \; -\mathcal{O}_3^{-1}\theta_3 = \begin{bmatrix} 0.0047 \\ 0.0077 \\ 0.0274 \end{bmatrix}.$$

Then, $(\Phi, v^{0,x})$ is a mixed equilibrium portfolio solution of Problem (MV) for $(0, x)$, where $\Phi = \{\Phi_k, k = 0, 1, 2, 3\}$.

Furthermore, by results of [9] and [11], we have the following pre-committed optimal control $u_p^{0,x}$:

$$\begin{aligned} u_{p,k}^{0,x} &= -W_k^{\dagger} H_k (X_k - \mathbb{E}_0 X_k) + \frac{1}{2}\mathcal{W}_k^{\dagger}\mathbb{E}O_k s_{k+1}\cdots s_{N-1}\mu_1 x \\ &\quad + \frac{1}{2}\mathcal{W}_k^{\dagger}\mathbb{E}O_k s_{k+1}\ldots s_{N-1}\mu_2, \quad k = 0, 1, 2, 3 \end{aligned} \tag{28}$$

where

$$\begin{cases} W_k = P_{k+1}\mathbb{E}(O_k O_k^T) \\ H_k = s_k P_{k+1}\mathbb{E}O_k \\ \mathcal{W}_k = P_{k+1}\text{Cov}(O_k) \\ k = 0, 1, 2, 3 \end{cases}$$

with

$$\begin{cases} P_k = s_k^2 P_{k+1}[1 - (\mathbb{E}O_k)^T[\mathbb{E}(O_k O_k^T)]^{\dagger}\mathbb{E}O_k] \\ P_N = 1, \quad k = 0, 1, 2, 3. \end{cases}$$

For $k = 0, 1, 2, 3$, define the expected values under the four policies

$$\begin{aligned} V_o(k) &= \mathbb{E}\left[J(k, X_k; u_o^{0,x}|_{\mathbb{T}_k})\right] \\ V_f(k) &= \mathbb{E}\left[J(k, X_k; u_f^{0,x}|_{\mathbb{T}_k})\right] \\ V_m(k) &= \mathbb{E}\left[J(k, X_k; u_m^{0,x}|_{\mathbb{T}_k})\right] \\ V_p(k) &= \mathbb{E}\left[J(k, X_k; u_p^{0,x}|_{\mathbb{T}_k})\right] \end{aligned}$$

where $u_o^{0,x}$ is the open-loop equilibrium control (20); and $u_f^{0,x}, u_m^{0,x}$ are the controls generated by the feedback equilibrium strategy (21)–(24) and mixed equilibrium solution (25)–(27). Then, we have the curves of the expected values corresponding to the four policies; see Fig. 1 and Fig. 2.

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