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General lemmas for stability analysis of linear continuous-time systems with slowly time-varying parameters

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This note investigates stability conditions for some dynamic systems with slowly time-varying parameters or with unknown internal or external disturbances. The models and conditions are motivated by adaptive control problems. The results presented here provide some sufficient conditions for the exponential convergence of the state sequence of systems without system disturbance, and for the uniform boundedness of the state sequence of such systems when they are subject to uniformly bounded system disturbances.

1. Introduction

Over the last two decades, adaptive control problems have been the subject of intensive study in systems and control theory (see for example, the texts by Åström and Wittenmark 1989, Caines 1988, Chen and Guo 1991, and the papers by Goodwin *et al.* 1991, Ortega and Tang 1989). The strategies to construct adaptive control laws are often quite different, but they usually have one common aspect; that is, they combine some estimation procedure with the 'certainty equivalence principle'. This often leads to a closed-loop system which has time-varying parameters. Therefore, an analysis of the stability of the corresponding closed-loop system is equivalent to an analysis of the stability of a corresponding time-varying system.

When the system parameter varies arbitrarily, it is hard to find a general method to tell us whether or not the system is stable. For the case where the time-varying parameters are differentiable and the derivatives of the parameters are sufficiently small, an early work of Desoer (1969) showed that the systems in question are exponentially stable. For the case where the derivatives of the parameters are not sufficiently small, a counter-example shows that the system may be unstable (see for example, Rosenbrock 1963). Moreover, in many situations of interest, the time-variation of the parameters is not differentiable (see for example, Ezzine and Haddad 1989, Feng *et al.* 1992, Sworder and Chou 1985).

Example 1: Suppose the time-varying system is described by $\dot{x}(t) = A(t)x(t)$. Then if

$$A(t) = \begin{cases} A_1, \ t \in [2i, 2i+1), & i = 0, 1, \ldots \\ A_2, \ t \in [2i+1, 2i+2), & i = 0, 1, \ldots \end{cases} \tag{1}$$

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where

$$A_1 = A_2^\tau = \begin{bmatrix} -\log 2 & 2 \\ 0 & -\log 2 \end{bmatrix}$$

then all the eigenvalues of $A(t)$ are $-\log 2$ and $-\log 2$ for all $t \geqslant 0$. But, for the initial condition

$$x(0) = \frac{\sqrt{2}}{4}[\sqrt{2} - 1,\ 1]^\tau$$

the solution is

$$x(t) = \begin{cases} \begin{bmatrix} \dfrac{2 - \sqrt{2} + 2\sqrt{2}(t - 2k)}{2^{t-2k+2}} \\[2ex] \dfrac{\sqrt{2}}{2^{t-2k+2}} \end{bmatrix} \left(\dfrac{3}{4} + \dfrac{\sqrt{2}}{2}\right)^k, & \forall t \in [2k, 2k+1) \\[6ex] \begin{bmatrix} \dfrac{2 + \sqrt{2}}{2^{t-2k+2}} \\[2ex] \dfrac{2(2 + \sqrt{2})(2 - 2k - 1) + \sqrt{2}}{2^{t-2k+2}} \end{bmatrix} \left(\dfrac{3}{4} + \dfrac{\sqrt{2}}{2}\right)^k, & \forall t \in [2k+1, 2k+2) \end{cases}$$

From this we get

$$\|x(t)\|^2 \geqslant \frac{1}{8}\left(\frac{3}{4} + \frac{\sqrt{2}}{2}\right)^{t-2}, \quad \forall t \geqslant 2$$

which means that $\|x(t)\|$ exponentially goes to infinity.

However, if

$$A(t) = \begin{cases} A_1, \ t \in [6i, 6i + 3), & i = 0, 1, \ldots \\ A_2, \ t \in [6i + 3, 6i + 6), & i = 0, 1, \ldots \end{cases} \tag{2}$$

then we can show that $\|x(t)\|$ exponentially converges to zero for any initial value $x(0)$.

It is easy to see that $A(t)$, given by (1) or (2), is piecewise constant. The only difference is that $A(t)$ in (1) jumps faster than $A(t)$ in (2). From this, and the analysis mentioned above, we observe that if the parameters jump 'too' fast, then the system may be unstable even if all the eigenvalues of $A(t) \leqslant -\varepsilon < 0$. However, if the parameters jump rarely enough in some sense, then $A(t) \leqslant -\varepsilon < 0$ can ensure the stability of the system. □

This note is a continuous-time version of the previous work of Giri *et al.* (1990), which presents a general lemma for investigating the uniform boundedness of the solutions of a class of discrete-time systems. The purpose of this note is to provide some sufficient conditions on the time-varying parameters, which may be continuous or discontinuous functions of time t, in order to guarantee the exponential convergence of the state sequence of the corresponding autonomous systems (i.e. without disturbance), and to guarantee the uniform boundedness of the state sequence of the involved systems with uniformly bounded disturbances.

2. Main results

Similar to Giri *et al.* (1990), we introduce the following definitions, which are suitable to measure the degree of variation of parameters which are continuously or discontinuously varying in time.

Definition 1: Let $\mu \in \mathbb{R}$. A real process $\{s(t)\}$ is said to be *μ-asymptotically small in the mean* if

$$\limsup_{l\to\infty} \limsup_{k\to\infty} \frac{1}{l}\int_k^{k+l} s(t)\,\mathrm{d}t \leqslant \mu \tag{3}$$

□

Definition 2: Let $\nu \in \mathbb{R}$ and $\{S(t)\}$ be real matrix process. If there is a real number sequence $\{t_k\}$ satisfying

$$t_k < t_{k+1} \underset{k\to\infty}{\longrightarrow} \infty \text{ and } \sup_{k\geqslant 0} (t_{k+1} - t_k) \triangleq T < \infty,$$

such that

$$\limsup_{l\to\infty} \limsup_{k\to\infty} \frac{1}{l}\int_{t_k}^{t_k+l} \|S(t) - S(t_k)\|\,\mathrm{d}t \leqslant \nu \tag{4}$$

then $\{S(t)\}$ said to be *ν-asymptotically slowly varying in the mean (ν-ASVM) with respect to $\{t_k\}$.* □

For a real number μ, the set of all μ-asymptotically-small-in-the-mean processes is denoted by $S_a(\mu)$. Similarly, for a real number ν and a sequence $\{t_k\}$, the set of all the matrix processes of ν-ASVM with respect to $\{t_k\}$ is denoted by $V_a(\nu, \{t_k\})$.

Lemma 1: *Suppose that $\{F(t)\} \in V_a(\nu, \{t_k\})$ is an $n \times n$ matrix-valued process, and satisfies*

$$\sup_{k\geqslant 1} \|F(t_k)\| \leqslant C < \infty \text{ and } \max \{\Re\lambda_i(F(t_k)),\ i = 1, \ldots, n;\ k = 1, 2, \ldots\} \leqslant -\alpha \tag{5}$$

where $\{t_k\}$ is the same as in Definition 2, C and $\alpha > 0$ are constants, $\lambda_i(F(t_k))$ denotes the ith eigenvalue of matrix $F(t_k)$, and $\Re\lambda_i(F(t_k))$ denotes the real part of $\lambda_i(F(t_k))$. If $\{x(t)\}$ is generated by

$$\dot{x}(t) = F(t)x(t) \tag{6}$$

then there exists a real number ν_0 such that for any $\nu \in [0, \nu_0)$, $\|x(t)\|$ converges to zero exponentially.

Proof: Define the fundamental matrix $\Phi_{t,u}$ of $F(t)$, i.e.

$$\frac{\mathrm{d}\Phi_{t,0}}{\mathrm{d}t} = F(t)\Phi_{t,0},\ \Phi_{0,0} = I,\ \Phi_{t,u} = \Phi_{t,0}\Phi_{u,0}^{-1}, \quad \forall t, u \geqslant 0 \tag{7}$$

It is easy to see that in order to prove the lemma, it suffices to show that there exists a real number ν_0 such that for any $\nu \in [0, \nu_0)$

$$\|\Phi_{t,u}\| \leqslant C_1\beta^{t-u}, \quad \forall t \geqslant u \geqslant 0 \tag{8}$$

where $C_1 < \infty$ and $\beta \in (0, 1)$ are constants.

By (5) and the argument used by Desoer (1970) we may conclude that there exist constants $M \geqslant 1$ and $\rho > 0$ such that

$$\exp\{F(t_k)(t-s)\} \leqslant M e^{-\rho(t-s)}, \quad \forall t \geqslant s \geqslant 0$$

Noticing that $\{F(t)\} \in V_a(v, \{t_k\})$, we see that there exists $\widetilde{T} \geqslant 2\rho^{-1}\log M$ such that

$$\int_{t_k}^{t} \|F(s) - F(t_k)\| \, ds \leqslant v(t - t_k), \quad \forall t_k \geqslant \widetilde{T}, \quad \forall t - t_k \geqslant \widetilde{T}$$

Set

$$T_0 = \max_{i \geqslant 0} \{t_i \colon t_i \in [t_0 + \widetilde{T}, t_0 + \widetilde{T} + T]\}$$

and

$$T_{k+1} = \max_{i \geqslant 0} \{t_i \colon t_i \in (T_k + \widetilde{T}, T_k + \widetilde{T} + T]\}, \quad \forall k \geqslant 0$$

where T is defined in Definition 2.

Then it is clear that

$$\widetilde{T} \leqslant T_{k+1} - T_k \leqslant \widetilde{T} + T \tag{9}$$

$$\exp\{F(T_k)(t-s)\} \leqslant M e^{-\rho(t-s)}, \quad \forall t \geqslant s \geqslant 0 \tag{10}$$

$$\int_{T_k}^{T_{k+1}} \|F(s) - F(T_k)\| \, ds \leqslant v(\widetilde{T} + T) \tag{11}$$

For any fixed T_k, from (7) we get

$$\frac{d\Phi_{t,u}}{dt} = F(T_k)\Phi_{t,u} + [F(t) - F(T_k)]\Phi_{t,u}, \quad \Phi_{u,u} = I, \quad \forall t \geqslant T_k \geqslant u \geqslant T_0 \tag{12}$$

which implies that

$$\Phi_{t,u} = e^{F(T_k)(t-T_k)}\Phi_{T_k,u} + \int_{T_k}^{t} e^{F(T_k)(t-s)}[F(s) - F(T_k)]\Phi_{s,u} \, ds \tag{13}$$

From this and (10) it follows that

$$\|\Phi_{t,u}\| \leqslant M e^{-\rho(t-T_k)}\|\Phi_{T_k,u}\| + M\int_{T_k}^{t} e^{-\rho(t-s)}\|F(s) - F(T_k)\| \cdot \|\Phi_{s,u}\| \, ds$$

i.e.

$$m(t) \leqslant M m(T_k) + M\int_{T_k}^{t} \|F(s) - F(T_k)\| m(s) \, ds \tag{14}$$

where $m(t) \triangleq e^{\rho t}\|\Phi_{t,u}\|$.

From the Bellman–Gronwall lemma (see for example, Desoer and Vidyasagar 1975) it follows from (14) that

$$m(t) \leqslant M m(T_k) \exp\left\{M\int_{T_k}^{t} \|F(s) - F(T_k)\| \, ds\right\}$$

i.e.

$$\|\Phi_{t,u}\| \leqslant M \exp\left\{-\rho(t - T_k) + M\int_{u}^{t} \|F(s) - F(T_k)\| \, ds\right\} \times \|\Phi_{T_k,u}\| \tag{15}$$

Combining this with (11) leads to

$$\|\Phi_{T_{k+1},u}\| \leq \|\Phi_{T_k,u}\| \cdot M e^{-(\rho - Mv)(\tilde{T}+T)} \tag{16}$$

for any fixed $v \in [0, M^{-1})$. Therefore, if we take

$$v_0 = \min\left\{\frac{\rho}{4M}, \frac{1}{M}\right\} \leq \min\left\{\frac{1}{2M}\left(\rho - \frac{\log M}{\tilde{T} + T}\right), \frac{1}{M}\right\}$$

then for any fixed $v \in [0, v_0)$

$$\alpha \triangleq M e^{-(\rho - Mv)(\tilde{T}+T)} < 1 \quad \text{and} \quad \alpha^{(\tilde{T}+T)^{-1}} \leq e^{-\rho/4}$$

Hence, (16) implies that

$$\|\Phi_{T_{k+1},u}\| \leq \alpha \|\Phi_{T_k,u}\| \tag{17}$$

For any fixed $t \geq u \geq T_0$, it is easy to see that there exist k_t and k_u such that $k_t \geq k_u$, $t \in [T_{k_t}, T_{k_t+1})$ and $u \in [T_{k_u}, T_{k_u+1})$.

By (7) and $\|\Phi_{u,u}\| = 1$ we get

$$\|\Phi_{t,u}\| \leq 1 + \int_u^t \|F(s)\| \cdot \|\Phi_{s,u}\| \, ds, \quad \forall t \geq u \geq T_0$$

This, together with the Bellman–Gronwall lemma (see, for example, Desoer and Vidyasagar 1975) yields

$$\|\Phi_{t,u}\| \leq \exp\left\{\int_u^t \|F(s)\| \, ds\right\}, \quad \forall t \geq u \geq T_0$$

which, combined with (11) and the condition $\sup_{k\geq 0} \|F(t_k)\| \leq C$, implies that

$$\|\Phi_{T_{k_u+1},u}\| \leq \exp\left\{\int_{T_{k_u}}^{T_{k_u+1}} \|F(s)\| \, ds\right\} \leq \exp\{(v + C)(\tilde{T} + T)\}$$

Therefore, from (15), (11) and (17) it follows that

$$\|\Phi_{t,u}\| \leq M \exp\{(Mv + v + C)(\tilde{T} + T)\} \alpha^{k_t - k_u - 1} \tag{18}$$

for any $t \geq u \geq T_0$.

By (9) and the definitions of k_t and k_u we see that $t - u \leq T_{k_t+1} - T_{k_u} \leq (k_t - k_u + 1)(\tilde{T} + T)$. Thus, we have

$$k_t - k_u \geq (\tilde{T} + T)^{-1}(t - u) - 1$$

Substituting this into (18) results in

$$\|\Phi_{t,u}\| \leq M \exp\{(Mv + v + C)(\tilde{T} + T)\} \alpha^{(\tilde{T}+T)^{-1}(t-u)-2}$$

which together with $\alpha^{(\tilde{T}+T)^{-1}} \leq e^{-\rho/4}$ implies

$$\|\Phi_{t,u}\| \leq M \exp\{(Mv + v + C)(\tilde{T} + T)\} \alpha^{-2} e^{-\rho(t-u)/4}, \quad \forall t \geq u \geq T_0$$

From this it is not difficult to see that (8) holds for some large enough constant C_1 and $\beta = e^{-\rho/4} \in (0, 1)$. □

Corollary 1: *Suppose that $F(0)$ is stable, $F(t)$ satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|F(s) - F(0)\| \, ds \leq v$$