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Stability of stochastic functional differential systems using degenerate Lyapunov functionals and applications☆

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ABSTRACT

Motivated by the seminal work of Dupire (2009) on functional Itô formulas, this work investigates asymptotic properties of systems represented by stochastic functional differential equations (SFDEs). Stability of general delay-dependent SFDEs is investigated using degenerate Lyapunov functionals, which are only positive semi-definite rather than positive definite as used in the classical work. This paper first establishes boundedness and regularity of SFDEs by using degenerate Lyapunov functionals. Then moment and almost sure exponential stabilities are obtained based on degenerate Lyapunov functionals and the semi-martingale convergence theorem. As an application of the stability criteria, consentability of stochastic multi-agent systems with nonlinear dynamics is studied.

1. Introduction

In his seminal work (Dupire, 2009), Dupire extended the Itô formula to a functional setting by using a pathwise functional derivative. This work has substantially eased the difficulties in finding solutions for non-Markovian processes due to time delays, that have defied bona fide operators and functional Itô formulas in the past. The current paper aims to use the newly developed functional Itô formula to examine stability and related issues of stochastic functional differential systems.

Many real systems, including population biology, epidemiology, economics, neural networks, and control of mechanical and electrical systems, inevitably involve delays, leading to delay differential equations (DDEs) and more general functional differential equations (FDEs) (Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999; Luo, Gong, & Jia, 2017). Some of their important properties have been established (Bejarano & Zheng, 2014; Bresch-Pietri & Krstic, 2014).

Although small delays may occasionally enhance stability in some systems (Yu, Chen, Cao, & Ren, 2013), delays are typically a source of instability, poor performance, and difficulty for analysis and control design. For linear time invariant systems, eigenvalues of system matrices may be used to deduce stability (Hale & Lunel, 1993). More complex functional equations are often treated by using Lyapunov functional and Razumikhin methods (Hale & Lunel, 1993; Kolmanovskii & Nosov, 1986). Combined with stochastic disturbances, delay systems become stochastic delay differential equations (SDDEs) and stochastic functional differential equations (SFDEs) (Mao, 1997; Mohammed, 1986), resulting in the consideration of moment stability, almost sure stability, and stability in probability. It is extremely difficult to establish necessary and sufficient conditions of mean square and almost sure stabilities for SFDEs. At present, Razumikhin methods and Lyapunov functionals are two main tools for establishing sufficient stability conditions. The stochastic versions of Razumikhin methods were developed in Janković, Randjelović, and Jovanović (2009), Mao (1996), Wu and Hu (2012) and Wu, Yin, and Wang (2015) for studying moment asymptotic stability. However, unlike its deterministic counterpart, the stochastic Razumikhin methods have limited success (Hale & Lunel, 1993; Teel, 1998).

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While sufficient conditions on delay-independent stability of SFDEs are often obtained by using Lyapunov functions, studies on delay-dependent stability mainly employ Lyapunov functionals and linear matrix inequalities (LMIs). Mao (1997) established the delay-independent pth-moment stability by using Lyapunov functions and obtained the almost sure stability from the moment exponential stability and a linear growth condition. Huang and Mao (2009a) studied the mean square exponential stability of neutral SDDEs by using a Lyapunov functional, and certain stability conditions were given in terms of LMIs. Caraballo, Real, and Taniguchi (2007) studied the almost sure exponential stability and ultimate boundedness of solutions for a class of neutral stochastic semi-linear partial delay differential equations. Rakkiyappan, Balasubramaniam, and Lakshmanan (2008) used Lyapunov functionals to derive LMI-type stability conditions for uncertain stochastic neural networks. Gershon, Shaked, and Berman (2007) investigated H_∞ state-feedback control of stochastic delay systems by Lyapunov functions and LMIs. Shaikhet (2013) introduced different Lyapunov functionals to examine stochastic stabilities of different SFDEs. These results were given under specific Lyapunov functionals. A fundamental stability theory related to Lyapunov functionals has not been developed.

Regarding delayed feedback stabilization problems of stochastic systems, Verriest and Florchinger (1995) gave delay-independent conditions on delayed feedback stabilization of linear SDDEs by Riccati-type equations in which multi-matrices must be determined. Huang and Mao (2009b) and Mao (2002) obtained delay-dependent exponential stability conditions for linear SDDEs in terms of LMIs. Karimi (2011) investigated robust mode-dependent delayed state feedback control for a class of uncertain delay systems in terms of LMIs.

All of the above references are based on positive definitive Lyapunov functionals. In this paper, using degenerate Lyapunov functionals, we develop several fundamental stability theorems, including moment asymptotic stability, exponential stability, and almost sure exponential stability. Degenerate Lyapunov functionals being positive semi-definite were well investigated in Kolmanovskii and Myshkis (1999) for deterministic neutral FDEs. Some specific degenerate Lyapunov functionals were also applied to examine the moment stability of stochastic functional differential equations with global Lipschitz conditions (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997; Shaikhet, 2013). Due to lack of bona fide operators and Itô formulas, whether degenerate Lyapunov functionals can be used to examine stability of SFDEs with nonlinear dynamics, nonlinear growth, and non-Lipschitz coefficients remains an open issue.

This paper fills in the gap. Our results reveal that appropriate degenerate Lyapunov functionals can be used to simplify stability analysis and control design. The results are then applied to multi-agent consentability and consensus problems, which are motivated by highway autonomous vehicles, unmanned aerial vehicles, team robots, among other applications (Cheng, Hou, & Tan, 2014; Li, Fu, Xie, & Zhang, 2011; Zong, Li, Yin, Wang, & Zhang, 2017; Zong, Li, & Zhang, 2017). While nonlinear multi-agent systems have attracted much attention (Strogatz, 2001; Zhu, Xie, Han, Meng, & Teo, 2017), stochastic multi-agent consentability and consensus with nonlinear dynamics, even for the delay-free case, are still not resolved. The existing LMI-type conditions are not explicit for consentability. The results of this paper resolve the issue. Assuming that the nonlinear terms satisfy a Lipschitz condition with uncertainty, we give sufficient stochastic consentability conditions, and demonstrate explicitly that the multi-agent consentability depends on the delay in feedback, noise intensities, balanced graphs, and the Lipschitz constant of the nonlinear dynamics.

The rest of the paper is structured as follows. Section 2 introduces the notation and develops the functional Itô formula for SFDEs. Section 3 establishes the fundamental degenerate Lyapunov functional theorem to produce the regularity and moment boundedness of SFDEs. Section 4 presents asymptotic moment estimates, moment stability, and almost sure stability based on the semi-martingale convergence theorem. In particular, mean square stability and almost sure stability are established for stochastic systems represented by semi-linear SFDEs. Section 5 applies the theoretical results to examine stochastic consentability of multi-agent systems with nonlinear dynamics. Section 6 concludes the paper with some remarks.

2. Functional Itô formula

We work with the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $|\cdot|$. For a vector or a matrix A, its transpose is denoted by A^T. For a matrix A, denote its trace norm by $|A| = \sqrt{\text{trace}(A^TA)}$. For a symmetric matrix A with real entries, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the largest and smallest eigenvalues, respectively. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. Use $a \vee b$ to denote $\max\{a, b\}$ and $a \wedge b$ to denote $\min\{a, b\}$. For $\tau > 0$, denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathbb{R}^n$-valued continuous functions on the interval $[-\tau, 0]$ with the norm $\|\varphi\|_C = \sup_{t\in[-\tau,0]}|\varphi(t)|$. Let $D([-\tau, 0]; \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued functions on $[-\tau, 0]$ that are right continuous and have left-hand limits endowed with the Skorokhod topology. The metric in $D([-\tau, 0]; \mathbb{R}^n)$ is given by $d_D(x, y) = \inf\{\epsilon, \sup_{t\in[-\tau,0]}|t - \lambda(t)| < \varepsilon$ and $\sup_{t\in[-\tau,0]}|x(t) - y(\lambda(t))| < \varepsilon$ for some $\lambda(t) \in \Lambda\}$, where Λ is the space of the continuous increasing functions from $[-\tau, 0]$ onto $[-\tau, 0]$. Under this metric, D becomes a complete and separable metric space (Kushner, 1984). We work with $(\Omega, \mathfrak{F}, \mathbb{P})$, a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))^T$ be a d-dimensional standard Brownian motion. Consider the following system given by a stochastic functional differential equation (SFDE)

$$dy(t) = f(y_t, t)dt + g(y_t, t)dw(t), \tag{1}$$

where $y_t = \{y(t + \theta) : \theta \in [-\tau, 0]\}$, $f : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^{n\times d}$ satisfy the following condition throughout the paper.

Assumption 2.1 (*Local Lipschitz Condition*). For each $j > 0$, there exists a constant $C_j > 0$ such that $|f(\psi, t) - f(\phi, t)| \vee |g(\psi, t) - g(\phi, t)| \leq C_j\|\psi - \phi\|_C$ for all $t \geq 0$ and $\psi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ with $\|\psi\|_C \vee \|\phi\|_C < j$.

We now develop the functional Itô formulation to the semi-martingale y_t determined by (1). For each $\varphi \in D([-\tau, 0]; \mathbb{R}^n)$, $\varphi_t(\theta) = \varphi(t + \theta), \theta \in [-\tau, 0]$. Motivated (Cont & Fournié, 2013; Dupire, 2009), we define the horizontal extension $\delta\varphi$ of φ: $\delta\varphi(\theta) = \varphi(\theta + \delta), \theta \in [-\tau, -\delta)$, and $\delta\varphi(\theta) = \varphi(0), \theta \in [-\delta, 0]$; and the vertical perturbation φ^v of φ: $\varphi^v(\theta) = \varphi(\theta), \theta \in [-\tau, 0)$, and $\varphi^v(0) = \varphi(0) + v, v \in \mathbb{R}^n$, Let $\{e_i, i = 1, \ldots, n\}$ denote the canonical basis in $\mathbb{R}^n$. Then for $\varphi \in D[-\tau, 0]$, we define

$$\mathcal{D}V(\varphi, t) := \lim_{\delta\to 0^+} \frac{V(\delta\varphi, t + \delta) - V(\varphi, t)}{\delta},$$

$$\nabla_x V(\varphi, t) := \{\partial_i V(\varphi, t), i = 1, \ldots, n\},$$

$\partial_i V(\varphi, t) := \lim_{h\to 0} \frac{V(\varphi^{he_i}, t) - V(\varphi, t)}{h}$ and

$$\nabla_{xx} V(\varphi, t) := \{\partial_{i,j} V(\varphi, t), i, j = 1, \ldots, n\},$$

$\partial_{i,j} V(\varphi, t) := \lim_{h\to 0} \frac{\partial_j V(\varphi^{he_i}, t) - \partial_j V(\varphi, t)}{h}$, where the limits are assumed to exist. We use $\mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R})$ to denote the class of functions that have continuous derivatives up to the second order w.r.t. x and continuous derivative w.r.t. t.

Remark 2.1. If $V(\varphi,t) = U(\varphi(0),t)$, where $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R};\mathbb{R})$, then $\mathcal{D}V(\varphi,t) = \frac{\partial}{\partial t}U(\varphi(0),t)$, $\nabla_x V(\varphi,t) = \frac{\partial}{\partial x}U(\varphi(0),t)$, and $\nabla_{xx}V(\varphi,t) = \frac{\partial^2}{\partial x^2}U(\varphi(0),t)$. If $V(\varphi,t) = \int_{-\tau}^0 |\varphi(\theta)|^2\,d\theta$, then $\mathcal{D}V(\varphi,t) = |\varphi(0)|^2 - |\varphi(-\tau)|^2$, and $\nabla_x V(\varphi,t) = 0$. Then we obtain the following functional Itô formula, whose proof is outlined in the Appendix.

Theorem 2.1. *Let $y(t)$ be a continuous semi-martingale with finite quadratic variation $[y](t)$. Then the functional Itô formula holds for $V \in \mathcal{C}^{2,1}(D([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+;\mathbb{R})$,*

$$V(y_t,t) = V(y_0,0) + \int_0^t \mathcal{D}V(y_s,s)ds + \int_0^t \nabla_x V(y_s,s)dy(s) + \int_0^t \frac{1}{2}\text{trace}(\nabla_{xx}V(y_s,s)d[y](s)). \quad (2)$$

Assume that SFDE (1) has a unique solution. Then based on the functional Itô formula, we have

$$V(y_t,t) = V(y_s,s) + \int_s^t \mathfrak{L}V(y_u,u)du + \int_s^t \nabla_x V(y_u,u)g(y_u,u)dw(u), \quad (3)$$

where the operator $\mathfrak{L}$ is defined by $\mathfrak{L}V(\varphi,t) = \mathcal{D}V(\varphi,t) + \nabla_x V(\varphi,t)f(\varphi,t) + \frac{1}{2}\text{trace}[g^T(\varphi,t)\nabla_{xx}V(\varphi,t)g(\varphi,t)]$. We assumed the existence and uniqueness above, which will be guaranteed by Theorems in Sections 3 and 4.

Remark 2.2. The functional Itô formula of Cont and Fournié (2013) and Dupire (2009) for the path-dependent stochastic systems enlarges the domains applicable for the Itô formula. In this paper, the new functional Itô formula (3) allows a constructive asymptotic analysis of SFDEs, provides a procedure to compute $\mathfrak{L}V$ for general functional $V(\varphi)$, and forms the foundation of the fundamental stability criteria (Theorems 4.1–4.3 below) which also weakens the stability conditions in Kolmanovskii and Myshkis (1999, Theorem 2.1, p. 389) (see Remark 4.2). It is well known that for stochastic systems without delay, Lyapunov function play an important role in stability analysis, where conditions on $\mathfrak{L}V$ for the function $V(x)$ under the classical Itô formula are required (see Mao, 1997). However, for the general functional $V(\varphi)$, the classical Itô formula cannot be used to compute $\mathfrak{L}V$ and the fundamental Lyapunov functional method based on $\mathfrak{L}V$ cannot be established under the classical Itô formula.

3. Regularity and moment boundedness

Theorem 3.1. *Assume that there exist a functional $V \in \mathcal{C}^{2,1}(D([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+;\mathbb{R}_+)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$, $\eta(t) \geq 0$ with $\int_0^t \eta(s)ds < \infty$ for any $t \in \mathbb{R}_+$ and some constants $c_0 > c_1 \geq 0$, $p > 0$, $\alpha \geq \beta \geq 0$, $\lambda \in \mathbb{R}$ such that*

(I) $c_0|\varphi(0)|^p - c_1\int_{-\tau}^0 |\varphi(s)|^p d\mu_1(s) \leq V(\varphi,t)$;
(II) $\mathfrak{L}V(\varphi,t) \leq \eta(t) - U_1(\varphi(0)) + \int_{-\tau}^0 U_2(\varphi(s))d\mu_2(s) - \alpha U_3(\varphi(0)) + \beta\int_{-\tau}^0 U_3(\varphi(s))d\mu_3(s)$;
(III) $U_2(x) - U_1(x) \leq -\lambda|x|^p$, *for all* $x \in \mathbb{R}^n$,

where $\{\mu_i\}_{i=1}^3$ are probability measures defined on $[-\tau,0]$. Then SFDE (1) admits a unique global solution for any initial data $y_0 \in C([-\tau,0];\mathbb{R}^n)$. Especially, if $\int_0^\infty \eta(s)ds < \infty$ and $\lambda \geq 0$, then the solution to SFDE (1) is bounded in the sense of pth moment, that is, $\sup_{s\geq 0}\mathbb{E}|y(s)|^p < \infty$.

Remark 3.1. The conditions in Theorem 3.1 extend the classical Lyapunov theory for stochastic systems without delay. In fact, if $\tau = 0$, $\{\mu_i\}_{i=1}^3$ become the Dirac measure δ_0 and the functional $V(\varphi,t)$ is a certain function $V(x,t)$, then condition (I) reduces to $(c_0 - c_1)|x|^p \leq V(x,t)$, which is frequently used in moment stability analysis, and which justifies the requirement $c_0 > c_1$. In the non-delay case, conditions (II) and (III) become $\mathfrak{L}V(x,t) \leq \eta(t) - \lambda|x|^p$, which is a classical condition in the Lyapunov asymptotic analysis of stochastic systems (Mao, 1997). Intuitively, conditions (II)–(III) imply that the functional can grow since λ can be negative. But condition (I) is added to avoid exploding in finite time. In fact, if $\eta(t) = 0$, the growth rate is no more than the exponential rate, see (9) below. However, if $\lambda > 0$, then conditions (II)–(III) imply that the functional cannot grow forever. In fact, if $\eta(t) = 0$, the overall solution tends to be bounded. Such conditions are often required for Lyapunov function stability of SFDEs (see Luo, Mao, & Shen, 2006; Song & Shen, 2013). Hence, the lower bound condition (I) is not required if $\lambda > 0$ and $\int_0^\infty \eta(t)dt < \infty$. More details will be given in Theorem 3.2.

Also, we require $V(\cdot,t) \geq 0$, but not $c_0|\varphi(0)|^p - c_1\int_{-\tau}^0 |\varphi(s)|^p d\mu_1(s) \geq 0$. For the case $d\mu_i(s) = ds$, $\tau = 1$, $\varphi(s) = (2+s)^{-1/2}$, $c_0 = 0.11$ and $c_1 = 0.1$, $c_0|\varphi(0)|^p - c_1\int_{-\tau}^0 |\varphi(s)|^p d\mu_1(s) < 0$. This differs from the classical Lyapunov functionals where $c_0|\varphi(0)|^p \leq V(\varphi,t)$ is required (Huang & Mao, 2009b). Hence, the functional in this paper is a degenerate Lyapunov functional. Sections 4.2 and 4.3 give some specific examples of Lyapunov functionals for the stability analysis.

Proof. For any given initial data $y_0 \in C([-\tau,0],\mathbb{R}^n)$, by Assumption 2.1, there exists a unique maximal local strong solution $y(t)$ to (1) on $t \in [-\tau,\rho_e)$, where ρ_e is the explosion time (Mao, 1997). To prove that this solution is global, we need only show that $\rho_e = \infty$ a.s. Define the stopping time $\rho_k = \inf\{t \in [0,\rho_e) : |y(t)| \geq k\}$. Clearly, ρ_k is increasing as $k \to \infty$ and $\rho_k \to \rho_\infty \leq \tau_e$ a.s. If we can show $\rho_\infty = \infty$ a.s., then $\tau_e = \infty$, which implies that the solution $y(t)$ is actually global. Let $\bar{V}^k(t) := \mathbb{E}V(y_{t\wedge\rho_k}, t\wedge\rho_k)$. By the generalized Itô formula (3) and condition (II), we obtain that for any $k > 0$ and $t \geq 0$,

$$\begin{aligned}\bar{V}^k(t) &= \mathbb{E}V(y_0,0) + \mathbb{E}\int_0^{t\wedge\rho_k} \mathfrak{L}V(y_s,s)ds \\ &\leq \mathbb{E}V(y_0,0) + \mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^0 U_2(y(s+\theta))d\mu_2(\theta)ds \\ &\quad - \mathbb{E}\int_0^{t\wedge\rho_k} U_1(y(s))ds - \alpha\mathbb{E}\int_0^{t\wedge\rho_k} U_3(y(s))ds \\ &\quad + \beta\mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^0 U_3(y(s+\theta))d\mu_3(\theta)ds + \int_0^t \eta(s)ds. \end{aligned} \quad (4)$$

By the Fubini theorem, we get that for $i = 2, 3$,

$$\begin{aligned}&\mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^0 U_i(y(s+\theta))d\mu_i(\theta)ds \\ &= \mathbb{E}\int_{-\tau}^0\int_0^{t\wedge\rho_k} U_i(y(s+\theta))dsd\mu_i(\theta) \\ &\leq \int_{-\tau}^0 \mathbb{E}U_i(y(s))ds + \mathbb{E}\int_0^{t\wedge\rho_k} U_i(y(s))ds. \end{aligned} \quad (5)$$

Note that $\alpha \geq \beta$ and $U_2(x) - U_1(x) \leq -\lambda|x|^p$. Hence, substituting the inequalities above into (4) yields

$$\begin{aligned}\bar{V}^k(t) &\leq \mathbb{E}V(y_0,0) + \mathbb{E}\int_{-\tau}^0 U_2(y(\theta))d\theta + \beta\mathbb{E}\int_{-\tau}^0 U_3(y(\theta))d\theta \\ &\quad + (-\lambda\vee 0)\mathbb{E}\int_0^t |y(s\wedge\rho_k)|^p ds + \int_0^t \eta(s)ds \\ &\leq (1 + \int_0^t \eta(s)ds)C_1 + (-\lambda\vee 0)\mathbb{E}\int_0^t |y(s\wedge\rho_k)|^p ds, \end{aligned} \quad (6)$$

where $C_1 = 1 + \mathbb{E}V(y_0, r(0)) + \mathbb{E}\int_{-\tau}^0 U_2(y(\theta))d\theta + \beta\mathbb{E}\int_{-\tau}^0 U_3(y(\theta))d\theta$. By condition (I), we have that for any $s > 0$,

$$c_0\mathbb{E}|y(s\wedge\rho_k)|^p \le \bar{V}^k(s) + c_1 \sup_{u\in[s-\tau,s]}\mathbb{E}|y(u\wedge\rho_k)|^p \le \bar{V}^k(s) + c_1 \sup_{u\in[-\tau,s]}\mathbb{E}|y(u\wedge\rho_k)|^p. \quad (7)$$

Note that $\rho_k \ge 0$. Then we get

$$\sup_{s\in[-\tau,t]}\mathbb{E}|y(s\wedge\rho_k)|^p \le \bar{C}(y_0) + \sup_{s\in[0,t]}\mathbb{E}|y(s\wedge\rho_k)|^p \le \bar{C}(y_0) + \frac{1}{c_0}\sup_{s\in[0,t]}\bar{V}^k(s) + \frac{c_1}{c_0}\sup_{s\in[-\tau,t]}\mathbb{E}|y(s\wedge\rho_k)|^p, \quad (8)$$

which together with $c_1 < c_0$ gives $\sup_{s\in[-\tau,t]}\mathbb{E}|y(s\wedge\rho_k)|^p \le \frac{c_0}{c_0-c_1}\bar{C}(y_0) + \frac{\sup_{s\in[0,t]}\bar{V}^k(s)}{c_0-c_1}$, where $\bar{C}(y_0) = \sup_{s\in[-\tau,0]}\mathbb{E}|y(s)|^p$. Substituting the inequality above into (6) yields $\mathbb{E}|y(t\wedge\rho_k)|^p \le (1 + \int_0^t \eta(s)ds)C_2 + C_3\int_0^t \mathbb{E}|y(s\wedge\rho_k)|^p ds$, where $C_2 = \frac{1}{c_0-c_1}(c_0\bar{C}(y_0) + C_1)$, $C_3 = \frac{-\lambda\vee 0}{c_0-c_1}$. Applying the Gronwall inequality produces

$$\mathbb{E}|y(t\wedge\rho_k)|^p \le (1+\int_0^t \eta(s)ds)C_2 + C_3\int_0^t e^{C_3(t-s)}(1+\int_0^s \eta(u)du)ds \le (1+\int_0^t \eta(s)ds)C_2 e^{C_3 t} =: C_4(t). \quad (9)$$

Note that $\mathbb{E}|y(t\wedge\rho_k)|^p \ge \mathbb{E}[|y(t\wedge\rho_k)|^p 1_{\{\rho_k\le t\}}] \ge k^p\mathbb{P}\{\rho_k \le t\}$. Hence, $k^p\mathbb{P}\{\rho_k \le t\} \le C_4(t)$. Then, for any $t > 0$, $\lim_{k\to\infty}\mathbb{P}\{\rho_k \le t\} = 0$, which together with the arbitrariness of t implies $\rho_\infty = \infty$ a.s. Therefore, the solution $y(t)$ is global. Especially, if $\int_0^\infty \eta(s)ds < \infty$ and $\lambda \ge 0$, then we can see that (9) is still true with $C_3 = 0$. Hence, we get from (9) that for any $t \ge 0$, $\mathbb{E}|y(t)|^p = \mathbb{E}[\lim_{k\to\infty}|y(t\wedge\rho_k)|^p] \le \liminf_{k\to\infty}\mathbb{E}|y(t\wedge\rho_k)|^p \le (1 + \int_0^\infty \eta(s)ds)C_2$, where we used Fatou's lemma and $\rho_\infty = \infty$. □

Theorem 3.2. *Assume that there exist a functional $V \in \mathcal{C}^{2,1}(D([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+;\mathbb{R}_+)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$, and two nonnegative constants $\alpha \ge \beta \ge 0$ such that conditions (II)–(III) hold with $\int_0^\infty \eta(t)dt < \infty$ and $\lambda > 0$. Then SFDE (1) admits a unique global solution for any initial data $y_0 \in C([-\tau,0];\mathbb{R}^n)$. Moreover, the solution to SFDE (1) is bounded in the sense of pth moment.*

Proof. Note that $\alpha \ge \beta$ and $U_2(x) - U_1(x) \le -\lambda|x|^p$, $\lambda > 0$. Then, substituting (5) in (4) yields

$$\bar{V}^k(t) \le \mathbb{E}V(y_0,0) + \int_{-\tau}^0 \mathbb{E}U_2(y(\theta))d\theta + \int_0^t \eta(s)ds + \beta\int_{-\tau}^0 \mathbb{E}U_3(y(\theta))d\theta - \lambda\int_0^t \mathbb{E}|y(s\wedge\rho_k)|^p ds. \quad (10)$$

Hence, for any $t \ge 0$, $\lambda\int_0^t \mathbb{E}|y(s\wedge\rho_k)|^p ds < C_5 + \int_0^\infty \eta(s)ds$, where $C_5 = \mathbb{E}V(y_0,0) + \mathbb{E}\int_{-\tau}^0 U_2(y(\theta))d\theta + \beta\mathbb{E}\int_{-\tau}^0 U_3(y(\theta))d\theta$. This also implies $\mathbb{E}|y(s\wedge\rho_k)|^p < C_6$ for certain $C_6 > 0$. Then, we use similar methods as above to prove that $\rho_\infty = \infty$ a.s. Therefore, the solution $y(t)$ is global. Similarly to the proof of Theorem 3.1, we obtain the moment boundedness of the solution by using Fatou's lemma. □

4. Asymptotic analysis and stability

4.1. Asymptotic analysis of nonlinear SFDEs

Theorem 4.1. *Assume that there exist a functional $V \in \mathcal{C}^{2,1}(D([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+;\mathbb{R}_+)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$, $\eta(t) \ge 0$ and some nonnegative constants $c_0 > c_1$, $c_2c_3 \ne 0$, $\alpha > \beta$, such that*

(I′) $c_0|\varphi(0)|^p - c_1\int_{-\tau}^0 |\varphi(s)|^p d\mu_1(s) \le V(\varphi,t) \le c_2|\varphi(0)|^p + c_3\int_{-\tau}^0 |\varphi(s)|^p d\mu_4(s)$;

(II) $\mathfrak{L}V(\varphi,t) \le \eta(t) - U_1(\varphi(0)) + \int_{-\tau}^0 U_2(\varphi(s))d\mu_2(s) - \alpha U_3(\varphi(0)) + \beta\int_{-\tau}^0 U_3(\varphi(s))d\mu_3(s)$;

(III′) $U_2(x) - U_1(x) \le 0$ *and there exists a* $\gamma_0 > 0$ *such that* $\gamma_0(c_2 + c_3e^{\gamma_0\tau})|x|^p - U_1(x) + e^{\gamma_0\tau}U_2(x) \le 0$ *for all* $x \in \mathbb{R}^n$,

where $\{\mu_i\}_{i=1}^4$ are probability measures defined on $[-\tau,0]$. Then the unique global solution to SFDE (1) satisfies

(a) $\lim_{t\to\infty}\mathbb{E}|y(t)|^p \le \frac{\eta_0}{\gamma(c_0-c_1e^{\gamma\tau})}$ *if* $\eta(t) \le \eta_0$ *for any* $t > 0$;

(b) $\lim_{t\to\infty}\mathbb{E}|y(t)|^p = 0$ *if* $\lim_{t\to\infty}\int_0^t e^{-\gamma(t-s)}\eta(s)ds = 0$;

(c) $\mathbb{E}|y(t)|^p \le Ce^{-\gamma t}$ *for some* $C > 0$ *if* $\int_0^\infty e^{\gamma t}\eta(t)dt < \infty$,

where $\gamma \in (0,\gamma^*)$, $\gamma^* = \gamma_0\wedge\gamma_1\wedge\gamma_2$, $\gamma_1 = \frac{1}{\tau}\log(\frac{\alpha}{\beta})$, $\gamma_2 = \frac{1}{\tau}\log(\frac{c_0}{c_1})$.

Proof. Note that $U_2(x) - U_1(x) \le 0$ implies $U_2(x) - U_1(x) \le |\lambda|\,|x|^p$. Then, condition (II) holds. Based on Theorem 3.1, the solution is regular. Letting $Z^k(t) = e^{\gamma(t\wedge\rho_k)}V(y_{t\wedge\rho_k}, t\wedge\rho_k)$ and applying Itô's formula (2), we obtain that

$$\begin{aligned}Z^k(t) &= V(y_0,0) + \int_0^{t\wedge\rho_k} e^{-\gamma s}\Delta_x V(y_s,s)g(y_s,s)dw(s) + \int_0^{t\wedge\rho_k} e^{\gamma s}[\gamma V(y_s,s) + \mathfrak{L}V(y_s,s)]ds \\ &\le V(y_0,0) + \int_0^{t\wedge\rho_k} e^{-\gamma s}\Delta_x V(y_s,s)g(y_s,s)dw(s) + \int_0^{t\wedge\rho_k} e^{\gamma s}\eta(s)ds + c_2\gamma\int_0^{t\wedge\rho_k} e^{\gamma s}|y(s)|^p ds \\ &\quad + c_3\gamma\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^0 |y(u+s)|^p d\mu_4(u)ds - \alpha\int_0^{t\wedge\rho_k} e^{\gamma s}U_3(y(s))ds - \int_0^{t\wedge\rho_k} e^{\gamma s}U_1(y(s))ds \\ &\quad + \beta\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^0 U_3(y(u+s))d\mu_3(u)ds + \int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^0 U_2(y(u+s))d\mu_2(u)ds. \quad (11)\end{aligned}$$

Note that for $i = 2, 3, 4$, and $h : \mathbb{R}^n \to \mathbb{R}_+$, by interchanging the order of integration,

$$\int_0^t e^{\gamma s}\int_{-\tau}^0 h(y(u+s))d\mu_i(u)ds \le e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}h(y(s))ds + e^{\gamma\tau}\int_0^t e^{\gamma s}h(y(s))ds. \quad (12)$$

Substituting (12) into (11), we get

$$\begin{aligned}\mathbb{E}Z^k(t) &\le C_7(\gamma) + \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\eta(s)ds + (c_2 + c_3e^{\gamma\tau})\gamma\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|y(s)|^p ds \\ &\quad + (\beta e^{\gamma\tau} - \alpha)\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U_3(y(s))ds - \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}(U_1(y(s)) - e^{\gamma\tau}U_2(y(s)))ds, \quad (13)\end{aligned}$$

where $C_7(\gamma) = \mathbb{E}[V(y_0, 0)] + c_3\gamma e^{\gamma\tau}\mathbb{E}\int_{-\tau}^{0} e^{\gamma s}|y(s)|^p ds + \beta e^{\gamma\tau}\mathbb{E}\int_{-\tau}^{0} e^{\gamma s}U_3(y(s))ds + e^{\gamma\tau}\mathbb{E}\int_{-\tau}^{0} e^{\gamma s} U_2(y(s))ds$. From $\alpha > \beta$, we know that $\beta e^{\gamma\tau} - \alpha < 0$ for any $\gamma < \gamma_1 := \frac{1}{\tau}\log(\frac{\alpha}{\beta})$. Then we get from (13) that for $\gamma < \gamma_1$,

$$\mathbb{E}Z^k(t) \le C_7(\gamma) + \int_0^t e^{\gamma s}\eta(s)ds + \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}h(\gamma, y(s))ds, \tag{14}$$

where $h(\gamma, x) = \gamma(c_2 + c_3e^{\gamma\tau})|x|^p - U_1(x) + e^{\gamma\tau}U_2(x)$. Hence, from condition (III′), $h(\gamma, x) \le 0$ for any $\gamma \in (0, \gamma_0 \wedge \gamma_1)$. Letting $k \to \infty$ in (14), we have

$$e^{\gamma t}\mathbb{E}[V(y_t, t)] \le C_7(\gamma) + \int_0^t e^{\gamma s}\eta(s)ds. \tag{15}$$

Let $Y(s) = \sup_{u\in[-\tau,s]} e^{\gamma u}\mathbb{E}|y(u)|^p$. By condition (I′), we have that for any $s > 0$,

$$\begin{aligned} c_0e^{\gamma s}\mathbb{E}|y(s)|^p &\le e^{\gamma s}\mathbb{E}V(y_s, s) + c_1e^{\gamma\tau}\sup_{u\in[s-\tau,s]} e^{\gamma u}\mathbb{E}|y(u)|^p \\ &\le e^{\gamma s}\mathbb{E}V(y_s, s) + c_1e^{\gamma\tau}Y(s). \end{aligned}$$

Then for any $\gamma > 0$,

$$\begin{aligned} c_0Y(t) &\le c_0Y(0) + c_0Y(t) \\ &\le c_0Y(0) + \sup_{s\in[0,t]} e^{\gamma s}\mathbb{E}V(y_s, s) + c_1e^{\gamma\tau}Y(t). \end{aligned} \tag{16}$$

From $c_1 < c_0$, we know that $c_0 - c_1e^{\gamma\tau} > 0$ for any $\gamma < \gamma_3 := \frac{1}{\tau}\log(\frac{c_0}{c_1})$. This together with (15) and (16) implies that for any $\gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge \gamma_3)$,

$$\begin{aligned} Y(t) &= \frac{c_0Y(0)}{c_0 - c_1e^{\gamma\tau}} + \frac{\sup_{s\in[0,t]} e^{\gamma s}\mathbb{E}V(y_s, s)}{c_0 - c_1e^{\gamma\tau}} \\ &\le C_8(\gamma) + \frac{1}{c_0 - c_1e^{\gamma\tau}}\int_0^t e^{\gamma s}\eta(s)ds, \end{aligned} \tag{17}$$

where $C_8(\gamma) = \frac{1}{c_0-c_1e^{\gamma\tau}}(c_0Y(0) + C_7(\gamma))$. Therefore, for any $t > 0$, $e^{\gamma t}\mathbb{E}|y(t)|^p \le C_8(\gamma) + \frac{1}{c_0-c_1e^{\gamma\tau}}\int_0^t e^{\gamma s}\eta(s)ds$. That is, $\mathbb{E}|y(t)|^p \le C_8(\gamma)e^{-\gamma t} + \frac{\int_0^t e^{-\gamma(t-s)}\eta(s)ds}{c_0-c_1e^{\gamma\tau}}$. Then the desired assertions can be easily obtained. □

Remark 4.1. Note that $\lim_{t\to\infty}\int_0^t e^{-\gamma(t-s)}\eta(s)ds = 0$ is guaranteed by $\eta \in L^1([0, \infty); \mathbb{R}_+)$ or η is continuous and satisfies $\lim_{t\to\infty}\eta(t) = 0$. In fact, if η is continuous and satisfies $\lim_{t\to\infty}\eta(t) = 0$, we use L'Hôpital's rule to deduce $\lim_{t\to\infty}\int_0^t e^{-\gamma(t-s)}\eta(s)ds = \lim_{t\to\infty}\frac{\int_0^t e^{\gamma s}\eta(s)ds}{e^{\gamma t}} = \frac{1}{\gamma}\lim_{t\to\infty}\eta(t) = 0$.

Theorem 4.2. *Assume that there exist a functional $V \in \mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R}_+)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$, $\eta(t) \ge 0$ and some nonnegative constants $c_2c_3 \ne 0$, c_4, $\alpha > \beta$, $\lambda > 0$, such that condition (II), and the following conditions hold,*

(I″) $V(\varphi, t) \le c_2|\varphi(0)|^p + c_3\int_{-\tau}^{0} |\varphi(s)|^p d\mu_4(s)$;
(III″) $0 \le U_2(x) \le c_4|x|^p$, $U_2(x) - U_1(x) \le -\lambda|x|^p$, *for all* $x \in \mathbb{R}^n$,

where $\{\mu_i\}_{i=1}^4$ are probability measures defined on $[-\tau, 0]$. Then the unique global solution to SFDE (1) satisfies $\mathbb{E}|y(t)|^p \le Ce^{-\gamma t}$ for certain $C > 0$ if $\int_0^\infty e^{\gamma t}\eta(t)dt < \infty$, where $\gamma \in (0, \gamma^*)$, $\gamma^* = \gamma_0 \wedge \gamma_1$, $\gamma_1 = \frac{1}{\tau}\log(\frac{\alpha}{\beta})$, γ_0 is the root of $\gamma(c_2+c_3e^{\gamma\tau})-\lambda+c_4(e^{\gamma\tau}-1) = 0$.

Proof. Note that $\int_0^\infty e^{\gamma t}\eta(t)dt < \infty$ implies $\int_0^\infty \eta(t)dt < \infty$, and condition (III″) implies condition (III). Then Theorem 3.2 enables us to conclude that the solution is regular.

By $\alpha > \beta$, we have $\beta e^{\gamma\tau} - \alpha < 0$ for any $\gamma < \gamma_1 := \frac{1}{\tau}\log(\frac{\alpha}{\beta})$. Note also that condition (III″) implies $U_1(y(s)) - e^{\gamma\tau}U_2(y(s)) \ge \lambda|x|^p - c_4(e^{\gamma\tau} - 1)|x|^p$. Then we get from (13) that for $\gamma < \gamma_1$,

$$\mathbb{E}Z^k(t) \le C_7(\gamma) + \int_0^t e^{\gamma s}\eta(s)ds + h(\gamma)\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|y(s)|^p ds, \tag{18}$$

where

$$h(\gamma) = \gamma(c_2 + c_3e^{\gamma\tau}) - \lambda + c_4(e^{\gamma\tau} - 1). \tag{19}$$

Note that $h(0) = -\lambda$ and $h(\gamma) > 0$ for γ satisfying $\gamma(c_2 + c_3e^{\gamma\tau}) - \lambda > 0$. In addition, $h'(\gamma) > 0$ for any $\gamma > 0$. Then, there exists a unique root γ_2 such that $h(\gamma_2) = 0$. Hence, $h(\gamma) < 0$ for any $\gamma \in (0, \gamma_1 \wedge \gamma_2)$. Therefore, we have from (18) that $-h(\gamma)\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|y(s)|^p ds \le C_7(\gamma) + \int_0^t e^{\gamma s}\eta(s)ds$. By Fatou's lemma, letting $k \to \infty$, for any $t > 0$, $-h(\gamma)\int_0^t e^{\gamma s}\mathbb{E}|y(s)|^p ds \le C_5(\gamma) + \int_0^\infty e^{\gamma s}\eta(s)ds < \infty$, which implies the desired assertion. □

Remark 4.2. Theorem 2.1 in Kolmanovskii and Myshkis (1999, p. 389) asserts that if the coefficients f, g satisfy the global Lipschitz condition and there exist $C_1, C_2, C_3 > 0$ such that

(i) $C_1\mathbb{E}|y(t)|^2 \le \mathbb{E}V(y_t, t) \le C_2\sup_{-\tau\le s\le 0}\mathbb{E}|y(t + s)|^2$,
(ii) $\mathbb{E}(V(y_t, t) - V(y_s, s)) \le -C_3\int_s^t \mathbb{E}|y(u)|^2du$ for any $t \ge s \ge 0$,

then the solution is mean square asymptotically stable. Our stability results in Theorems 4.1 and 4.2 improve Theorem 2.1 in Kolmanovskii and Myshkis (1999) in the following four points: (1) Theorems 4.1 and 4.2 do not require the coefficients to be globally Lipschitz. (2) There is no requirement on the positive definiteness of $V(y_t, t)$. (3) Our results can produce the mean square exponential stability. (4) Condition (II) is broader than condition (ii) above since condition (II) can deal with the case with additive noises.

Theorems 4.1 and 4.2(c) present a pth moment exponential stability criterion. It is known in Mao (1997) that pth moment exponential stability implies almost sure exponential stability of SFDEs with multiplicative noises under the assumption that f, g satisfy a linear growth condition, that is, there exists a constant $c_5 > 0$ such that for any $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$,

$$|f(\varphi, t)|^2 \vee |g_i(\varphi, t)|^2 \le c_5\int_{-\tau}^{0} |\varphi(s)|^2 d\mu_5(s), \tag{20}$$

where $\mu_5(s)$ is a probability measure on $[-\tau, 0]$. Without the linear growth assumption, we can use the semi-martingale convergence theorem (Mao, 1997, p. 14) to examine the almost sure exponential stability of SFDEs with additive noise. To proceed, we give the almost sure stability analysis.

Theorem 4.3. *Assume that there exist a functional $V \in \mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R}^m)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$ and some nonnegative constants $c_2c_3 \ne 0$, c_4, $\alpha > \beta$, $\lambda > 0$ such that conditions (I″), (II), and (III″) hold. If $\int_0^\infty e^{\gamma t}\eta(t)dt < \infty$, then SFDE (1) admits a unique global solution and the solution satisfies that for any initial data $y_0 \in C([-\tau, 0]; \mathbb{R}^n)$, $\limsup \frac{1}{t}\log|y(t)| \le \frac{\gamma}{p}$, a.s., where $\gamma \in (0, \gamma^*)$ is defined in Theorem 4.2.*

Proof. For $\gamma > 0$, let $Z(t) = e^{\gamma t}V(y_t, t)$. Applying the functional Itô formula to $e^{\gamma t}V(y_t, t)$ and using (II) yield

$$\begin{aligned} Z(t) \le\ & V(y_0, 0) + \int_0^t e^{\gamma s}\eta(s)ds - \alpha\int_0^t e^{\gamma s}U_3(y(s))ds \\ & + c_2\gamma\int_0^t e^{\gamma s}|y(s)|^p ds - \int_0^t e^{\gamma s}U_1(y(s))ds + M(t) \\ & + c_3\gamma\int_0^t e^{\gamma s}\int_{-\tau}^{0} |y(u + s)|^p d\mu_4(u)ds \\ & + \beta\int_0^t e^{\gamma s}\int_{-\tau}^{0} U_3(y(u + s))d\mu_3(u)ds \\ & + \int_0^t e^{\gamma s}\int_{-\tau}^{0} U_2(y(u + s))d\mu_2(u)ds, \end{aligned} \tag{21}$$

where $M(t) = \int_0^t e^{-\gamma s}\Delta_x V(y_s, s)g(y_s, s)dw(s)$. Substituting (12) into (21), we get that

$$Z(t) \le C_9(\gamma) + (\beta e^{\gamma\tau} - \alpha)\int_0^t e^{\gamma s}U_3(y(s))ds + M(t) + h(\gamma)\int_0^t e^{\gamma s}|y(s)|^p ds + \int_0^t e^{\gamma s}\eta(s)ds, \tag{22}$$

where $C_9(\gamma) = V(y_0, 0) + c_3\gamma e^{\gamma\tau}\int_{-\tau}^0 |y(s)|^p ds + e^{\gamma\tau}\int_{-\tau}^0 U_2(y(s))ds + \beta e^{\gamma\tau}\int_{-\tau}^0 U_3(y(s))ds$, and $h(\cdot)$ is defined by (19). Let $\gamma \in (0, \gamma^*)$, then $h(\gamma) < 0$. Moreover, we have from (22) that

$$-h(\gamma)\int_0^t e^{\gamma s}|y(s)|^p ds \le Y(t), \tag{23}$$

where $Y(t) = C_7(\gamma) + (\beta e^{\gamma\tau} - \alpha)\int_0^t e^{\gamma s}U_3(y(s))ds + M(t) + \int_0^t e^{\gamma s}\eta(s)ds \ge 0$. Note that $\beta e^{\gamma\tau} - \alpha < 0$, $A_1(t) := \int_0^t e^{\gamma s}\eta(s)ds$ and $A_2(t) := (\alpha - \beta e^{\gamma\tau})\int_0^t e^{\gamma s}U_3(y(s))ds$ are increasing. Then, $\int_0^\infty e^{\gamma s}\eta(s)ds < \infty$ and the semi-martingale convergence theorem (Mao, 1997, p. 14) yield $\lim_{t\to\infty}Y(t) < \infty$, *a.s.* This together with (23) implies $\int_0^\infty e^{\gamma s}|y(s)|^p ds < \infty$, *a.s.* Therefore, the desired assertion follows. □

Note that the conditions of almost sure exponential stability in Theorem 4.3 are the same as that of pth moment exponential stability in Theorem 4.1. A natural question is: Can we obtain the almost sure asymptotic stability under the conditions for the pth moment asymptotic stability in Theorem 4.1? The following simple equation without delay shows that this may be impossible.

$$dy(t) = -y(t)dt + \sigma(t)dw(t). \tag{24}$$

It is easy to see that $\lim_{t\to\infty}\mathbb{E}|y(t)|^2 = 0$ if and only if

$$\lim_{t\to\infty}\int_0^t e^{-(t-s)}\sigma(s)^2 ds = 0. \tag{25}$$

From (Appleby, Cheng, & Rodkina, 2011), we know that $\lim_{t\to\infty}|y(t)| = 0$ a.s. if $\lim_{t\to\infty}\sigma(t)^2\log t = 0$, and only if

$$\liminf_{t\to\infty}\sigma(t)^2\log t = 0. \tag{26}$$

Hence, if we choose $\sigma(t) = \frac{1}{\sqrt{\log t}}$ for $t > 1$, then (25) holds (L'Hôpital's rule), and $\lim_{t\to\infty}\mathbb{E}|y(t)|^2 = 0$. But (26) is defied, and then $\mathbb{P}\{\lim_{t\to\infty}|y(t)| = 0\} < 1$.

Remark 4.3. In the above, we developed degenerate Lyapunov functionals to investigate the asymptotic behaviors of SFDEs. The conditions (I″) and (II) is required to hold for all $\varphi \in \mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R}_+)$. Actually, in many complex networked systems, we only require these conditions to be true on the solution path. Moreover, based on the solution paths, the structure properties of the complex systems might be helpful for the asymptotic analysis (see the multi-agent application in the subsequent section).

4.2. Stochastic stability of semi-linear SFDEs

Consider the following semi-linear SFDE,

$$dy(t) = [\sum_{i=0}^{d} A_i y(t-\tau_i) + f(y_t, t)]dt + \sum_{i=1}^{\bar d} g_i(y_t, t)dw_i(t), \tag{27}$$

where $y_t = \{y(t+\theta) : \theta \in [-\tau, 0]\}$, $A_i \in \mathbb{R}^{n\times n}$, $\tau > 0$, $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_d$ are delays, f and g_i satisfy the following linear growth condition.

Assumption 4.1. For each positive definite matrix P there exist matrices $D_{1P} \ge 0$ and $D_{2P} \ge 0$ such that for all $t \ge 0$ and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$,

$$f^T(\varphi, t)Pf(\varphi, t) = q^2\int_{-\tau}^0 \varphi^T(s)D_{1P}\varphi(s)d\overline{\mu}(s),$$

$$\sum_{i=1}^{\bar d} g_i^T(\varphi, t)Pg_i(\varphi, t) = \int_{-\tau}^0 \varphi^T(s)D_{2P}\varphi(s)d\bar\mu(s),$$

where $q \ge 0$, $\bar\mu(s)$ is a probability measure on $[-\tau, 0]$.

The delay-independent and delay-dependent stability has been discussed in the important works (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997, 1998; Kovalev, Kolmanovskii, & Shaikhet, 1998) for linear systems. In this section, we consider the delay-dependent stability for semi-linear systems, where without loss of generality, we assume that $\sum_{i=0}^{l}A_i$ is Hurwitz if and only if $l = d$, which is consistent with Theorem 5.3 in Kolmanovskii and Myshkis (1999) with $m_0 = m$.

Theorem 4.4. *Suppose that Assumption 4.1 holds. Let $\bar A = \sum_{i=0}^{d}A_i$. Then stochastic system (27) is mean square and almost surely exponentially stable if there exists a matrix $P > 0$ such that*

$$S := \bar A^T P + P\bar A + \sum_{i=1}^{d}\tau_i\bar A^T P\bar A + (1+q)\sum_{i=1}^{d}A_i^T PA_i\tau_i + qP + q(1 + \sum_{i=1}^{d}\tau_i)D_{1P} + D_{2P} < 0. \tag{28}$$

Proof. Let P be fixed. Define

$$V(\varphi, t) = V_1(\varphi) + V_2(\varphi), \varphi \in C([-\tau_d, 0]; \mathbb{R}^n), \tag{29}$$

with $V_1(\varphi) = (\varphi(0) + \sum_{i=1}^d\int_{-\tau_i}^0 A_i\varphi(s)ds)^T P(\varphi(0) + \sum_{i=1}^d\int_{-\tau_i}^0 A_i\varphi(s)ds)$ and $V_2(\varphi) = (1+q)\sum_{i=1}^d\int_{-\tau_i}^0\int_s^0 \varphi^T(\theta)A_i^T PA_i\varphi(\theta)d\theta ds$. It is easy to see that condition (I″) holds. It can be deduced that $\mathfrak{L}V_1(\varphi, t)$ has the following form

$$\mathfrak{L}V_1(\varphi, t) = \varphi^T(0)[\bar A^T P + P\bar A]\varphi(0) + 2\varphi^T(0)Pf(\varphi, t) + \sum_{i=1}^{\bar d} g_i^T(\varphi, t)Pg_i(\varphi, t) + \sum_{i=1}^{d} 2\varphi^T(0)\bar A^T PA_i\int_{-\tau_i}^0\varphi(s)ds + 2\sum_{i=1}^{d}\Big(\int_{-\tau_i}^0 A_i\varphi(s)ds\Big)^T Pf(\varphi, t).$$

By the elementary inequality $x^T y + y^T x \le x^T Ox + y^T O^{-1}y$, for any positive definite matrix $O \in \mathbb{R}^{n\times n}, \varepsilon > 0$, and $x, y \in \mathbb{R}^m$, we get $2\varphi^T(0)\bar A^T PA_i\int_{-\tau_i}^0\varphi(s)ds \le \tau_i\varphi^T(0)\bar A^T P\bar A\varphi(0) + \int_{-\tau_i}^0\varphi^T(s)A_i^T PA_i\varphi(s)ds$, $2\varphi^T(0)\ Pf(\varphi, t) \le q\varphi^T(0)P\varphi(0) + q\int_{-\tau}^0\varphi^T(s)D_{1P}\varphi(s)d\bar\mu(s)$ and $2\Big(\int_{-\tau_i}^0 A_i\varphi(s)ds\Big)^T Pf(\varphi, t) \le q\int_{-\tau_i}^0\varphi^T(s)A_i^T PA_i\varphi(s)ds + q\tau_i\int_{-\tau}^0\varphi(s)^T D_{1P}\varphi(s)d\bar\mu(s)$. Assume $q > 0$. Otherwise, the last two inequalities above vanish and the following estimations hold still. Then we get

$$\mathfrak{L}V_1(\varphi, t) \le \varphi^T(0)S_1\varphi(0) + \int_{-\tau}^0\varphi^T(s)D_{2P}\varphi(s)d\bar\mu(s) + q(1 + \sum_{i=1}^{d}\tau_i)\int_{-\tau}^0\varphi^T(s)D_{1P}\varphi(s)d\bar\mu(s) + (1+q)\sum_{i=1}^{d}\int_{-\tau_i}^0\varphi^T(s)A_i^T PA_i\varphi(s)ds,$$

where $S_1 = \bar{A}^T P + P\bar{A} + \sum_{i=1}^d \tau_i \bar{A}^T P\bar{A} + qP$. By the definition of $V_2(\varphi)$, we have $\mathfrak{L}V_2(\varphi, t) = (1+q)\sum_{i=1}^d \tau_i \varphi^T(0)A_i^T PA_i\varphi(0) - (1+q)\sum_{i=1}^d \int_{-\tau_i}^0 \varphi^T(\theta)A_i^T PA_i\varphi(\theta)d\theta$. This together with the estimations of $\mathfrak{L}V_1(\varphi, t)$ gives

$$\mathfrak{L}V(\varphi, t) \le \varphi^T(0)S_2\varphi(0) + \int_{-\tau}^0 \varphi^T(s)H_P\varphi(s)d\bar{\mu}(s), \tag{30}$$

where $S_2 = \bar{A}^T P + P\bar{A} + \sum_{i=1}^d \tau_i \bar{A}^T P\bar{A} + (1+q)\sum_{i=1}^d A_i^T PA_i\tau_i + qP$, $H_P = q(1+\sum_{i=1}^d \tau_i)D_{1P} + D_{2P}$. Note that condition (28) implies $S = S_2 + H_P < 0$. Then conditions (II) and (III″) hold with $c_4 = |H_P|$, $p = 2$ and $\lambda = -\lambda_{\max}(S)$. By Theorems 4.2 and 4.3, we know that the trivial solution to (27) is mean square and almost surely exponentially stable. □

If the nonlinear drift term $f(\varphi, t) \equiv 0$ $(q = 0)$, we have the following simple stability criterion.

Corollary 4.5. *Suppose that Assumption 4.1 hold with $q = 0$. Then stochastic system (27) is mean square and almost surely exponentially stable if there exists a positive definite matrix P such that*

$$\bar{A}^T P + P\bar{A} + \sum_{i=1}^d \tau_i \bar{A}^T P\bar{A} + \sum_{i=1}^d A_i^T PA_i\tau_i + D_{2P} < 0. \tag{31}$$

Remark 4.4. If the delays vanish ($\tau_i = 0$, $i = 1, \dots, d$), then Corollary 4.5 is consistent with stability criterion of the delay-free linear SDE; see (Rami & Zhou, 2000). Note that conditions (28) (or (31)) implies that matrix $\bar{A}$ is Hurwitz. When $\bar{A}^T + \bar{A}$ is still Hurwitz, we can choose $P = I_n$ in condition (28) (or (31)) to obtain the exponential stability criterion. That is, if $\bar{A}^T + \bar{A} + \sum_{i=1}^d \tau_i \bar{A}^T\bar{A} + (1+q)\sum_{i=1}^d A_i^T A_i\tau_i + qI_n + q(1+\sum_{i=1}^d \tau_i)D_{1I_n} + D_{2I_n} < 0$, (or $\bar{A}^T + \bar{A} + \sum_{i=1}^d \tau_i \bar{A}^T\bar{A} + \sum_{i=1}^d A_i^T A_i\tau_i + D_{2I_n} < 0$), then the trivial solution to (27) is mean square and almost surely exponentially stable.

Theorem 4.4 and Corollary 4.5 improve the linear mean square asymptotic stability in the previous works (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997, 1998; Kovalev et al., 1998) not only in considering the nonlinear terms, but also in that condition like $\sum_{i=1}^d \|A_i\|\tau_i < 1$ in Kolmanovskii and Myshkis (1999), Kolmanovskii and Shaikhet (1997, 1998) and Kovalev et al. (1998) is not required and the mean square and almost sure exponential stabilities are obtained. Here, we take the linear case

$$dy(t) = [A_0y(t) + A_1y(t-\tau_1)]dt + Cy(t)dw(t) \tag{32}$$

for example and show that the condition $\|A_1\|\tau_1 < 1$ required in Kolmanovskii and Myshkis (1999) and Kolmanovskii and Shaikhet (1997) is unnecessary. Consider $\tau_1 = 0.01$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 \\ -110 & -3.3 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

It is easy to see that $\|A_1\|\tau_1 > 1$ for any norm $\|\cdot\|$. But for $P = \begin{pmatrix} 120.12 & 2.3 \\ 2.3 & 1 \end{pmatrix}$, one obtains $\bar{A}^T P + P\bar{A} + (\bar{A}^T P\bar{A} + A_1^T PA_1)\tau_1 + C^T PC < 0$. Hence, Corollary 4.5 yields that the linear system (32) is still mean square and a.s. exponentially stable even if $\|A_1\|\tau_1 > 1$.

4.3. Non-global Lipschitz case

We consider the following scalar SDDE with non-global Lipschitz coefficients,

$$\begin{aligned} dy(t) = {} & [-\bar{\alpha}y(t-\tau) - \bar{\beta}y^3(t)]dt \\ & + \sigma_1 y^2(t-\bar{\tau})dw_1(t) + \sigma_2 y(t-\bar{\tau})dw_2(t), \end{aligned} \tag{33}$$

where $\tau, \bar{\tau} \ge 0$, $2\bar{\alpha} > \sigma_2^2 \ge 0$, $2\bar{\beta} > \sigma_1^2 \ge 0$. The following theorem can be considered as delay-dominated stability criterion.

Theorem 4.6. *For any given $\bar{\tau} \ge 0$, the trivial solution to (33) is mean square and almost surely exponentially stable if*

$$2\bar{\alpha}^2\tau < 2\bar{\alpha} - \sigma_2^2,\ \bar{\alpha}\bar{\beta}\tau^2 \le 2\bar{\beta} - \sigma_1^2. \tag{34}$$

Proof. Choose the degenerate Lyapunov functional

$$V(\varphi) = |\varphi(0) - \bar{\alpha}\int_{-\tau}^0 \varphi(s)ds|^2. \tag{35}$$

We obtain $\mathfrak{L}V(\varphi, t) = -2\bar{\alpha}|\varphi(0)|^2 - 2\bar{\beta}|\varphi(0)|^4 + 2\bar{\alpha}^2\varphi(0)\int_{-\tau}^0 \varphi(s)ds + \sigma_2^2|\varphi(-\bar{\tau})|^2 + 2\bar{\alpha}\bar{\beta}\varphi(0)^3\int_{-\tau}^0 \varphi(s)ds + \sigma_1^2|\varphi(-\bar{\tau})|^4$. By the elementary inequality $a^m b^n \le \frac{m}{m+n}a^{m+n} + \frac{n}{m+n}b^{m+n}$, $a, b \ge 0$, and the Hölder inequality, we have $2\bar{\alpha}^2\varphi(0)\int_{-\tau}^0 \varphi(s)ds \le \bar{\alpha}^2\tau|\varphi(0)|^2 + \bar{\alpha}^2\int_{-\tau}^0 |\varphi(s)|^2 ds$ and

$$\varphi(0)^3\int_{-\tau}^0 \varphi(s)ds \le \frac{3}{4}\tau^2|\varphi(0)|^4 + \frac{1}{4}\tau\int_{-\tau}^0 |\varphi(s)|^4 ds.$$

Hence, we get

$$\begin{aligned} \mathfrak{L}V(\varphi, t) \le {} & -(2\bar{\alpha} - \bar{\alpha}^2\tau)|\varphi(0)|^2 + \sigma_2^2|\varphi(-\bar{\tau})|^2 \\ & + \bar{\alpha}^2\int_{-\tau}^0 |\varphi(s)|^2 ds - (2\bar{\beta} - \frac{3}{2}\tau^2\bar{\alpha}\bar{\beta})|\varphi(0)|^4 \\ & + \frac{1}{2}\bar{\alpha}\bar{\beta}\tau\int_{-\tau}^0 |\varphi(s)|^4 ds + \sigma_1^2|\varphi(-\bar{\tau})|^4. \end{aligned} \tag{36}$$

Let $d\mu_2(\theta) = \frac{\sigma_2^2 d\delta_{-\bar{\tau}}(\theta) + \bar{\alpha}^2\tau d\bar{\mu}(\theta)}{\sigma_2^2 + \bar{\alpha}^2\tau}$, $d\bar{\mu}(\theta) = \frac{1}{\tau}1_{[-\tau,0]}(\theta)d\theta$, and $d\mu_3(\theta) = \frac{\sigma_1^2 d\delta_{-\bar{\tau}}(\theta) + \bar{\alpha}\bar{\beta}\tau^2 d\bar{\mu}(\theta)/2}{\sigma_1^2 + \bar{\alpha}\bar{\beta}\tau^2/2}$, where $\delta_{-\bar{\tau}}(\theta)$ is Dirac measure on $[-(\bar{\tau} \vee \tau), 0]$. Then, μ_2, μ_3 are the probability measures on $[-(\bar{\tau} \vee \tau), 0]$. Therefore, we get

$$\begin{aligned} \mathfrak{L}V(\varphi, t) \le {} & -r_1|\varphi(0)|^2 + r_2\int_{-(\bar{\tau}\vee\tau)}^0 |\varphi(\theta)|^2 d\mu_2(\theta) \\ & - r_3|\varphi(0)|^4 + r_4\int_{-(\bar{\tau}\vee\tau)}^0 |\varphi(s)|^4 d\mu_3(s), \end{aligned} \tag{37}$$

where $r_1 = 2\bar{\alpha} - \bar{\alpha}^2\tau$, $r_2 = \sigma_2^2 + \bar{\alpha}^2\tau$, $r_3 = 2\bar{\beta} - \frac{3}{2}\tau^2\bar{\alpha}\bar{\beta}$, $r_4 = \sigma_1^2 + \frac{1}{2}\bar{\alpha}\bar{\beta}\tau^2$. Note that condition (34) implies $r_1 > r_2$, $r_3 > r_4$. Then, by Theorems 4.1 and 4.3, we obtain the desired assertions. □

Remark 4.5. The degenerate Lyapunov functionals (29) and (35) also satisfy condition (I). We take (35) as example. In fact, by the inequality $|x-y|^2 \le (1+\varepsilon)(|x|^2 + \frac{1}{\varepsilon}|y|^2)$, $\varepsilon > 0$, we have

$$\begin{aligned} |\varphi(0)|^2 & \le (1+\varepsilon)\Big(|\varphi(0) - \bar{\alpha}\int_{-\tau}^0 \varphi(s)ds|^2 + \frac{1}{\varepsilon}|\bar{\alpha}\int_{-\tau}^0 \varphi(s)ds|^2\Big) \\ & \le (1+\varepsilon)V(\varphi) + \frac{1+\varepsilon}{\varepsilon}\bar{\alpha}^2\tau^2\int_{-\tau}^0 |\varphi(s)|^2\frac{ds}{\tau}. \end{aligned} \tag{38}$$

Let $c_0 = \frac{1}{1+\varepsilon}$, $c_1 = \frac{1}{\varepsilon}\bar{\alpha}^2\tau^2$, and $d\mu_1(s) = \frac{ds}{\tau}$ is a probability measure on $[-\tau, 0]$. Note that $\bar{\alpha}^2\tau^2 < 1$ (condition (34)), then there must exist a $\varepsilon > 0$ such that $c_0 > c_1$. That is, condition (I) holds. Similarly, we can prove (29) to satisfy condition (I).

Remark 4.6. Basin and Rodkina (2008) applied a degenerate Lyapunov functional to study the almost sure asymptotic stability of a scalar SDDEs with linear growth coefficients. Using our methods, we can show that not only the scalar SDDEs in Basin and Rodkina (2008) is almost surely asymptotically stable, but also mean square asymptotically stable.

We have investigated the asymptotic stability and exponential stability of SFDEs. This is different the L^2-stability of the random differential equations in Diblk, Dzhalladova, and Ružičková (2012). In fact, L^2-stability cannot yield the asymptotic stability and exponential stability. But our exponential stability can yield L^2 stability.

5. Applications to multi-agent systems with nonlinear dynamics and noises

In this section, we apply our main findings to multi-agent systems with distributed random delays and establish convergence and consentability of such systems that are difficult to treat when multiple or distributed delays are encountered. Wireless communication systems are essential part of networked systems and introduce significant impact on control design and system performance. In Xu, Wang, Yin, and Zhang (2014), coordinated control and communication design was studied in which TCP-based (Transmission Control Protocol) communication protocols were employed. The main consequence of the TCP channel uncertainties is random signal transmission delays.

Inter-vehicle communications use wireless networks and are subject to signal fading, delays in each hub's queues, re-transmission, scheduling policies in interference avoidance strategies, resulting in combined latency which is random, time varying, unpredictable, and distributed (Neely & Modiano, 2005). In our recent paper (Wang, Syed, Yin, Pandya, & Zhang, 2014), a weighted and constrained consensus control method was introduced to achieve platoon formation and robustness against communication channel uncertainties. However, distributed random delays could not be treated. Applying the new results of this paper, convergence to a desired platoon formation can now be achieved under more realistic communication latency conditions.

To proceed, we study consentability of nonlinear multi-agent systems consisting of N agents. Each agent is modeled by a dynamic system

$$\dot{y}_i(t) = f(y_i(t), y_i(t-\tau), t) + u_i(t), i = 1, \dots, N \tag{39}$$

where $y_i(t) \in \mathbb{R}^n$ is the state of the ith agent, τ is a delay, $u = [u_1, \dots, u_N]^T$ is the control to be designed, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a nonlinear function satisfying the following assumption.

Assumption 5.1. $f(0, 0, t) = 0$ and there exists a positive constant q (Lipschitz constant) such that for all $x, y, \bar{x}, \bar{y}$, $|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \le q(|x - \bar{x}| + |y - \bar{y}|)$.

The information flow structures among different agents are modeled as a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ denotes the set of directed edges and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly. Also, N_i denotes the set of the node i's neighbors, that is, for $j \in N_i$, $a_{ij} = 1$, and $\deg_i = \sum_{j=1}^N a_{ij}$ is called the degree of i. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}(\deg_1, \dots, \deg_N)$. It is obvious that $\mathcal{L}$ admits a zero eigenvalue, denoted by λ_1. If the digraph $\mathcal{G}$ is balanced, then $\widehat{\mathcal{L}} = \frac{\mathcal{L}^T+\mathcal{L}}{2}$ is the Laplacian matrix of the mirror digraph $\widehat{\mathcal{G}}$ of $\mathcal{G}$ (Olfati-Saber & Murray, 2004).

Consider the control

$$u_i(t) = K \sum_{j\in N_i} z_{ji}(t), \tag{40}$$

where the control gain $K \in \mathbb{R}^{n\times n}$ is symmetric, $z_{ji}(t) = y_j(t-\tau_1) - y_i(t-\tau_1) + g_{ji}(y_j(t-\tau_2) - y_i(t-\tau_2))\xi_{ji}(t)$ denotes the measurement of the agent i from its neighbor agent j, τ_1 and τ_2 are delays, $\xi_{ji}(t)$ is a scalar independent Gaussian white noise, and $g_{ji} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the noise intensity function. Here, we make the following assumptions.

Assumption 5.2. The noise processes $\{\xi_{ji}(t) \in \mathbb{R}, i, j = 1, 2, \dots, N\}$ satisfy $\int_0^t \xi_{ji}(s)ds = w_{ji}(t)$ for $t \ge 0$, where $\{w_{ji}(t), i, j = 1, 2, \dots, N\}$ are independent Brownian motions.

Assumption 5.3. For each (j, i), $g_{ji}(0) = 0$ and there exists a positive constant σ such that $|g_{ji}(x) - g_{ji}(y)| \le \sigma|x - y|$, for all $x, y \in \mathbb{R}^n$.

For the motivation of the measurement $z_{ji}(t)$ above, one can refer to (Zong, Li, Yin et al. 2017, Zong, Li, & Zhang 2017). We denote the collection of all admissible distributed protocols by $\mathcal{U} = \{u(t)|u_i(t)$ given by (40) for all symmetric $K, t \ge 0$, and $i = 1, \dots, N\}$.

Definition 5.1. The multi-agent systems (39) are said to be mean square (or almost surely) consentable with respect to (w.r.t.) $\mathcal{U}$, if there exists a protocol $u \in \mathcal{U}$ solving the mean square (or almost sure) consensus, that is, for any initial data $\varphi \in C([-(\tau \vee \tau_1 \vee \tau_2), 0]; \mathbb{R}^{Nn})$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty}\mathbb{E}|y_i(t) - y_j(t)|^2 = 0$ (or $\lim_{t\to\infty}|y_i(t) - y_j(t)| = 0$, a.s.).

Lemma 5.1. *For the Laplacian matrix $\mathcal{L}$ related to a balanced graph $\mathcal{G}$, there exists a matrix $\widetilde{Q} \in \mathbb{R}^{N\times(N-1)}$ such that the matrix $Q = (\frac{1}{\sqrt{N}}\mathbf{1}_N, \widetilde{Q}) \in \mathbb{R}^{N\times N}$ is orthogonal and*

$$Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^T \\ \widetilde{Q}^T \end{bmatrix}, \ Q^{-1}\mathcal{L}Q = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\mathcal{L}} \end{bmatrix}. \tag{41}$$

The proof of lemma can be found in Zong, Li, and Zhang (2018). We proceed to obtain the following result.

Theorem 5.2. *Suppose that Assumptions 5.1, 5.2, and 5.3 hold and the graph $\mathcal{G}$ is balanced. If*

$$4q[\lambda_2(\widehat{\mathcal{L}})\frac{N-1}{N}\sigma^2 + (1+q)|\mathcal{L}|^2\tau_1](2+\tau_1) < \lambda_2^2(\widehat{\mathcal{L}}), \tag{42}$$

then multi-agent systems are mean square and almost sure consentable w.r.t. $\mathcal{U}$. Moreover, the mean square and almost sure consensus can be solved by the protocol (40) with $K = kI_n$ satisfying $k \in (\underline{k}, \bar{k})$, $\underline{k} := \frac{\lambda_2(\widehat{\mathcal{L}}) - \sqrt{\lambda_2^2(\widehat{\mathcal{L}}) - 4qa(2+\tau_1)}}{2a}$, $\bar{k} := \frac{\lambda_2(\widehat{\mathcal{L}}) + \sqrt{\lambda_2^2(\widehat{\mathcal{L}}) - 4qa(2+\tau_1)}}{2a}$, where $a = \lambda_2(\widehat{\mathcal{L}})\frac{N-1}{N}\sigma^2 + (1+q)|\mathcal{L}|^2\tau_1$.

Proof. It is enough to prove that the protocol with k satisfying $k \in (\underline{k}, \bar{k})$ can solve the mean square and almost sure consensus. Note that $\underline{k}$ and $\bar{k}$ are well-defined since condition (42) holds. Let $y(t) = [y_1^T(t), \dots, y_N^T(t)]^T$, $F(t) = [f^T(y_1(t), y_1(t-\tau), t), \dots, f^T(y_N(t), y_N(t-\tau), t)]^T$, $\delta(t) = [(I_N - J_N) \otimes I_n]y(t)$, and $\delta(t) = [\delta_1^T(t), \dots, \delta_N^T(t)]^T$, where $J_N = \frac{1}{\sqrt{N}}\mathbf{1}_N\mathbf{1}_N^T$. Substituting the protocol (40) into the system (39) and using Assumption 5.2 yields $dy(t) = -[k(\mathcal{L} \otimes I_n)y(t-\tau_1) + F(t)]dt + dM_1(t)$, where $M_1(t) = k\sum_{i,j=1}^N a_{ij}\int_0^t [\eta_{N,i} \otimes (g_{ji}(\delta_j(s-\tau_2) - \delta_i(s-\tau_2)))]dw_{ji}(s)$. This together with the definition of $\delta(t)$ gives $d\delta(t) = -k(\mathcal{L} \otimes I_n)\delta(t-\tau_1)dt + \bar{F}(t)dt + dM_2(t)$, where $\bar{F}(t) = [(I_N - J_N) \otimes I_n]F(t) = [\bar{f}_1^T(t), \dots, \bar{f}_N^T(t)]^T$, $\bar{f}_i(t) = f(y_i(t), y_i(t-\tau), t) - \frac{1}{N}\sum_{j=1}^N f(y_j(t), y_j(t-\tau), t)$, $M_2(t) = k\sum_{i,j=1}^N a_{ij}\int_0^t\{[(I_N - J_N)\eta_{N,i}] \otimes \bar{g}_{ji}(s-\tau_2)\}dw_{ji}(s)$, $\bar{g}_{ji}(s) = g_{ji}(\delta_j(s) - \delta_i(s))$. Let $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^N y_j(t)$, then $\delta_i(t) = y_i(t) - \bar{y}(t)$. Note that $\bar{f}_i(t) = f(y_i(t), y_i(t-\tau), t) - f(y_i(t), \bar{y}(t-\tau), t) + f(y_i(t), \bar{y}(t-\tau), t) - f(\bar{y}(t), \bar{y}(t-\tau), t) + \frac{1}{N}\sum_{j=1}^N\Big((\bar{y}(t), \bar{y}(t-\tau), t) - (y_j(t), y_j(t-\tau), t)\Big)$. Write the above as $\bar{f}_i(t) = p_{i1}(t) + p_{i2}(t) + p_{03}(t)$. Define $\widetilde{\delta}(t) = (Q^{-1} \otimes I_n)\delta(t) = [\widetilde{\delta}_1^T(t), \dots, \widetilde{\delta}_N^T(t)]^T$, $\bar{\delta}(t) = [\widetilde{\delta}_2^T(t), \dots, \widetilde{\delta}_N^T(t)]^T$, $\widetilde{\delta}_i(t) \in \mathbb{R}^n$. Then, by the definition of Q^{-1} given in Lemma 5.1, we have $\widetilde{\delta}_1(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N^T \otimes I_n)\delta(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N^T(I_N - J_N) \otimes I_n)y(t) = 0$ and

$$d\bar{\delta}(t) = (\widetilde{Q}^T \otimes I_n)\bar{F}(t)dt - k(\widetilde{\mathcal{L}} \otimes I_n)\bar{\delta}(t-\tau_1)dt + dM_3(t), \tag{43}$$

where $M_3(t) = k\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}\int_0^t(\widetilde{Q}^T(I_N - J_N)\eta_{N,i}\otimes\bar{g}_{ji}(s-\tau_2))dw_{1ji}(s)$. Note that $\widetilde{Q}\widetilde{Q}^T = I_N - J_N$ and $(I_N - J_N)^2 = (I_N - J_N)$, $\eta_{N,i}^T(I_N - J_N)\eta_{N,i} = \frac{N-1}{N}$. It is deduced that $\sum_{i,j=1}^{N} a_{ij}|\bar{g}_{ji}(t)|^2 \le \frac{N-1}{N}\sigma^2\sum_{i,j=1}^{N} a_{ij}|\delta_j(t) - \delta_i(t)|^2 = \frac{N-1}{N}\sigma^2\Phi(t)$, where we used the definitions of $\widetilde{\delta}(t)$ and $\delta^T(t)[(\mathcal{L}^T + \mathcal{L})\otimes I_n]\delta(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N} a_{ij}|\delta_j(t) - \delta_i(t)|^2$, $\Phi(t) = \bar{\delta}^T(t)[(\widetilde{\mathcal{L}}^T + \widetilde{\mathcal{L}})\otimes I_n]\bar{\delta}(t)$. Firstly, we choose the degenerate Lyapunov functional $V_1(\varphi) = |\varphi(0) - k(\widetilde{\mathcal{L}}\otimes I_n)\int_{-\tau_1}^{0}\varphi(s)ds|^2$. We can compute $\mathcal{L}V_1(\cdot, t)$ on the solution path $\bar{\delta}(t)$ and obtain

$$
\begin{aligned}
\mathfrak{L}V_1(\bar{\delta}_t, t) \le{}& -k\Phi(t) + k^2\frac{N-1}{N}\sigma^2\Phi(t-\tau_2)\\
&- 2k\Big((\widetilde{\mathcal{L}}\otimes I_n)\int_{t-\tau_1}^{t}\bar{\delta}(s)ds\Big)^T(\widetilde{Q}^T\otimes I_n)\bar{F}(t)\\
&+ 2k^2\bar{\delta}^T(t)(\widetilde{\mathcal{L}}^T\widetilde{\mathcal{L}}\otimes I_n)\int_{t-\tau_1}^{t}\bar{\delta}(s)ds\\
&+ 2\bar{\delta}^T(t)(\widetilde{Q}^T\otimes I_n)\bar{F}(t).
\end{aligned}
$$

By the inequality $2x^Ty \le \frac{1}{\varepsilon}|x|^2 + \varepsilon|y|^2$, we have

$$
\begin{aligned}
&2\bar{\delta}^T(t)(\widetilde{\mathcal{L}}^T\widetilde{\mathcal{L}}\otimes I_n)\int_{t-\tau_1}^{t}\bar{\delta}(s)ds\\
&\quad\le |\widetilde{\mathcal{L}}|^2\Big(\tau_1|\bar{\delta}(t)|^2 + \int_{t-\tau_1}^{t}|\bar{\delta}(s)|^2ds\Big). \qquad (44)
\end{aligned}
$$

By Assumption 5.1, we obtain $2\delta_i^T(t)p_{i1}(t) \le q(|\delta_i(t)|^2 + |\delta_i(t-\tau)|^2)$, $2\delta_i^T(t)p_{i2}(t) \le 2q|\delta_i(t)|^2$. From the definitions of $\delta(t)$ and $\bar{\delta}(t)$ and the orthogonality of Q, we know that $|\delta(t)|^2 = |\bar{\delta}(t)|^2$ and $\sum_{i=1}^{N}\delta_i^T(t)p_{03}(t) = 0$. Hence,

$$
\begin{aligned}
&2\bar{\delta}^T(t)(\widetilde{Q}^T\otimes I_n)\bar{F}(t)\\
&\quad= 2\delta^T(t)(\widetilde{Q}\widetilde{Q}^T\otimes I_n)[(I_N - J_N)\otimes I_n]F(t)\\
&\quad\le 3q|\bar{\delta}(t)|^2 + q|\bar{\delta}(t-\tau)|^2. \qquad (45)
\end{aligned}
$$

Note that $\widetilde{\mathcal{L}} = \widetilde{Q}^T\mathcal{L}\widetilde{Q}$, $\widetilde{Q}\widetilde{Q}^T = I_N - J_N$, $(I_N - J_N)^2 = (I_N - J_N)$, $(I_N - J_N)\mathcal{L} = \mathcal{L} = \mathcal{L}(I_N - J_N)$, $\bar{\delta}(t) = (\widetilde{Q}^T\otimes I_n)\delta(t)$ and $\mathcal{L}^T\mathbf{1}_N = 0$. Then,

$$
\begin{aligned}
&2k\Big((\widetilde{\mathcal{L}}\otimes I_n)\int_{t-\tau_1}^{t}\bar{\delta}(s)ds\Big)^T(\widetilde{Q}^T\otimes I_n)\bar{F}(t)\\
&\quad\le 2k^2q|\widetilde{\mathcal{L}}|^2\int_{t-\tau_1}^{t}|\bar{\delta}(s)|^2ds + q\tau_1(|\bar{\delta}(t)|^2 + |\bar{\delta}(t-\tau_1)|^2). \qquad (46)
\end{aligned}
$$

Combining (44)–(46) yields

$$
\begin{aligned}
\mathfrak{L}V_1(\bar{\delta}_t, t) \le{}& -k\Phi(t) + (k^2|\widetilde{\mathcal{L}}|^2\tau_1 + 3q + q\tau_1)|\bar{\delta}(t)|^2\\
&+ (1+2q)k^2|\widetilde{\mathcal{L}}|^2\int_{t-\tau_1}^{t}|\bar{\delta}(s)|^2ds + q|\bar{\delta}(t-\tau)|^2\\
&+ q\tau_1|\bar{\delta}(t-\tau_1)|^2 + k^2\frac{N-1}{N}\sigma^2\Phi(t-\tau_2). \qquad (47)
\end{aligned}
$$

Define a new degenerate Lyapunov functional

$$
\begin{aligned}
V(\bar{\delta}_t, t) ={}& V_1(\bar{\delta}_t) + k^2(2q+1)|\widetilde{\mathcal{L}}|^2\int_{-\tau_1}^{0}\Big[\int_{s}^{0}|\bar{\delta}_t(\theta)|^2d\theta\Big]ds\\
&+ q\int_{t-\tau}^{t}|\bar{\delta}(s)|^2ds + q\tau_1\int_{t-\tau_1}^{t}|\bar{\delta}(s)|^2ds\\
&+ k^2\frac{N-1}{N}\sigma^2\int_{t-\tau_2}^{t}\Phi(s)ds. \qquad (48)
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
\mathfrak{L}V(\bar{\delta}_t, t) \le{}& -(k - k^2\frac{N-1}{N}\sigma^2)\Phi(t)\\
&+ 2[(1+q)k^2|\widetilde{\mathcal{L}}|^2\tau_1 + 2q + q\tau_1]|\bar{\delta}(t)|^2. \qquad (49)
\end{aligned}
$$

Moreover, due to the balanced graph, we have $\Phi(t) = \delta^T(t)[(\mathcal{L}^T + \mathcal{L})\otimes I_n]\delta(t) \ge 2\lambda_2(\widehat{\mathcal{L}})|\bar{\delta}(t)|^2$. From (49), we get $\mathfrak{L}V(\bar{\delta}_t, t) \le 2h(k)|\bar{\delta}(t)|^2$, where $h(k) = ak^2 - \lambda_2(\widehat{\mathcal{L}})k + q(2+\tau_1)$, $a = \frac{N-1}{N}\sigma^2\lambda_2(\widehat{\mathcal{L}}) + (1+q)|\widetilde{\mathcal{L}}|^2\tau_1$. Moreover, it is easy to see that $h(k) < 0$ for $k \in (\underline{k}, \bar{k})$. Therefore, Theorem 4.2 and the linear growth condition of $f(\cdot,\cdot,t)$ and $g_{ji}(\cdot)$ imply the mean square and almost sure exponential stability of the solution to (43), that is, $\mathbb{E}|\bar{\delta}(t)|^2 \le C_0e^{-\gamma_0 t}$, $\limsup_{t\to\infty}\frac{1}{t}\log|\bar{\delta}(t)| < -\frac{\gamma_0}{2}$, a.s. for certain $C_0, \gamma > 0$. Note that $\delta_i(t) = y_i - \frac{1}{N}\sum_{j=1}^{N}y_j(t)$, $i = 1,\ldots,N$. Then for $i \ne j$, $|y_j(t) - y_i(t)| \le |\delta_j(t)| + |\delta_i(t)|$. The desired assertion follows. □

Theorem 5.3. *Suppose that Assumptions 5.1–5.3 hold, that the graph $\mathcal{G}$ is balanced, that $\tau_1 = 0$, and that*

$$
8q\frac{N-1}{N}\sigma^2 < \lambda_2(\widehat{\mathcal{L}}). \qquad (50)
$$

Then multi-agent systems are mean square and almost surely consentable w.r.t. $\mathcal{U}$. Moreover, the mean square and a.s. consensus can be solved by protocol (40) with $K = kI_n$ satisfying $k \in (\underline{k}, \bar{k})$, $\underline{k} = \frac{\lambda_2(\widehat{\mathcal{L}}) - \sqrt{\lambda_2^2(\widehat{\mathcal{L}}) - 8qa}}{2a}$, $\bar{k} = \frac{\lambda_2(\widehat{\mathcal{L}}) + \sqrt{\lambda_2^2(\widehat{\mathcal{L}}) - 8qa}}{2a}$, $a = \lambda_2(\widehat{\mathcal{L}})\frac{N-1}{N}\sigma^2$.

Remark 5.1. Theorems 5.2 and 5.3 show that if condition (42) or condition (50) holds, then for any delay τ in the nonlinear dynamics and delay τ_2 in noise terms, the mean square and almost sure consensus can be achieved.

6. Concluding remarks

This work established the stochastic stability of SFDEs by using degenerate Lyapunov functionals under local Lipschitz conditions. The analysis of delay-dependent stability shows that pure delay terms in SFDEs may dominate the asymptotic behavior of SFDEs. Most importantly, these stability criteria enable us to design delayed state-feedback controllers for networked systems. Note that the degenerate Lyapunov functionals may also be used to examine the ergodicity of SFDEs whose coefficients may not satisfy the classical one-sided Lipschitz condition in Bao, Yin, and Yuan (2014), and pure delay terms may dominate the existence of stationary distributions. For another extension, the degenerate Lyapunov functionals can also be applied to examine SFDEs with Poisson jumps, neutral SFDEs, and their stabilization problems. In addition, this paper established the consentability for first-order multi-agent systems. We hope that the stability theorems can be applied to the heterogeneous multi-agent systems, and their leader-following versions.

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Appendix. Proof of Theorem 2.1

With the motivation of Cont and Fournié (2013), an outline of the proof is given below. Let the interval $[-\tau, t]$ be fixed. Without loss of generality, we use the uniform partition of $[-\tau, 0]$: $\{t_{-m}, \ldots, t_{-1}\}$ with $t_k = k\Delta$, $k = -m, \ldots, -1$, $\Delta > 0$. We assume that there exists an N such that $t = N\Delta$. Then we have the uniform partition of $[0, t]$: $\{t_0, \ldots, t_N\}$. For any continuous semi-martingale $y(t)$, $y(t)$ is uniformly continuous on $[-\tau, t]$. Then $h_\Delta \to 0$ as Δ tends to zero, where $h_\Delta = \sup\{|y(u) - y(t_k)|, u \in [t_k, t_{k+1}), k = -m, \ldots, N\}$. Define $\bar{y}_{t_k}(\theta) = \sum_{j=-m}^{-1} y(t_{k+j})\mathbf{1}_{[j,j+1)}(\theta) + y(t_k)\mathbf{1}_{\{0\}}(\theta)$, $k = 0, \ldots, N$, which is an approximation for function y_{t_k}. Consider the decomposition

$$\begin{aligned} &V(\bar{y}_{t_{k+1}}, t_{k+1}) - V(\bar{y}_{t_k}, t_k) \\ &= V(\bar{y}_{t_{k+1}}, t_{k+1}) - V(\bar{y}_{t_{k+1}^-}, t_{k+1}) \\ &\quad + V(\bar{y}_{t_{k+1}^-}, t_{k+1}) - V(\bar{y}_{t_k}, t_k). \end{aligned} \tag{A.1}$$

Let $h_k = y(t_{k+1}) - y(t_k)$ and $P(h) = V(\bar{y}^h_{t_{k+1}^-}, t_{k+1})$. Then, $P(h_k) - P(0) = V(\bar{y}^{h_k}_{t_{k+1}^-}, t_{k+1}) - V(\bar{y}_{t_{k+1}^-}, t_{k+1})$. Since $V \in \mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R})$, P is a $\mathcal{C}^2$ function with $P'(h) = \nabla_x V(\bar{y}^h_{t_{k+1}^-}, t_{k+1})$ and $P''(h) = \nabla_{xx} V(\bar{y}^h_{t_{k+1}^-}, t_{k+1})$. Note that $\bar{y}_{t_{k+1}^-}$ is $\mathfrak{F}_{t_k}$ measurable. Applying the Itô formula yields that for $\delta < \Delta$,

$$\begin{aligned} &P(y(t_k + \delta) - y(t_k)) \\ &= P(0) + \int_{t_k}^{t_k+\delta} \nabla_x V(\bar{y}^{y(u)-y(t_k)}_{t_{k+1}^-}, t_{k+1}) dy(u) \\ &\quad + \int_{t_k}^{t_k+\delta} \frac{1}{2}\mathrm{Tr}(\nabla_{xx} V(\bar{y}^{y(u)-y(t_k)}_{t_{k+1}^-}, t_{k+1}) d[y](u)). \end{aligned} \tag{A.2}$$

Then $\bar{y}_{t_{k+1}^-} = \Delta\bar{y}_{t_k}$, which is a horizontal extension. So the second term on the right side of (A.1) has the form $Q(\delta) - Q(0) = V(\delta\bar{y}_{t_k}, t_k+\delta) - V(\bar{y}_{t_k}, t_k)$, $0 < \delta \le \Delta$, where $Q(\delta) = V(\delta\bar{y}_{t_k}, t_k+\delta)$. Note that $V \in \mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R}^d)$. Then, $V(\bar{y}_{t_{k+1}^-}, t_{k+1}) - V(\bar{y}_{t_k}, t_k) = \int_{t_k}^{t_{k+1}} \mathcal{D}V(\bar{y}_{t_k}, s)ds$. Hence, taking sum form $i = 0$ to $i = N - 1$ in (A.1) yields

$$\begin{aligned} &V(\bar{y}_t, t) - V(y_0, 0) \\ &= \int_0^t \mathcal{D}V(\bar{y}_{\check{s}}, s)ds + \int_0^t \nabla_x V(\bar{y}^{y(s)-y(\check{s})}_{\tilde{s}}, \hat{s}) dy(s) \\ &\quad + \int_0^t \frac{1}{2}\mathrm{Tr}(\nabla_{xx} V(\bar{y}^{y(s)-y(\check{s})}_{\tilde{s}}, \hat{s}) d[y](s)), \end{aligned} \tag{A.3}$$

where $\check{s} = t_k \mathbf{1}_{[t_k, t_{k+1})}(s)$, $\tilde{s} = t_{k+1}^- \mathbf{1}_{[t_k, t_{k+1})}(s)$ and $\hat{s} = t_{k+1}\mathbf{1}_{[t_k, t_{k+1})}(s)$, $k = 1, \ldots, N - 1$. It is easy to see that for any given $t \ge 0$, $\bar{y}_t \to y_t$ uniformly on $[-\tau, 0]$ as $\Delta \to 0$. This together with the continuity of y_t yields $d_D(\bar{y}_t, y_t) \to 0$ as $\Delta \to 0$ (Kushner, 1984 p. 31). The approach in Cont and Fournié (2013) leads to the desired assertion as $\delta \to 0$.

References

Appleby, J. A. D., Cheng, J., & Rodkina, A. (2011). Characterisation of the asymptotic behaviour of scalar linear differential equations with respect to a fading stochastic perturbation. *Discrete and Continuous Dynamical Systems*, 79–90 Suppl.

Bao, J., Yin, G., & Yuan, C. (2014). Ergodicity for functional stochastic differential equations and applications. *Nonlinear Analysis: Theory, Methods and Applications*, *98*, 66–82.

Basin, M., & Rodkina, A. (2008). On delay-dependent stability for a class of nonlinear stochastic systems with multiple state delays. *Nonlinear Analysis: Theory, Methods and Applications*, *68*(8), 2147–2157.

Bejarano, F. J., & Zheng, G. (2014). Observability of linear systems with commensurate delays and unknown inputs. *Automatica*, *50*(8), 2077–2083.

Bresch-Pietri, D., & Krstic, M. (2014). Delay-adaptive control for nonlinear systems. *IEEE Transactions on Automatic Control*, *59*(5), 1203–1218.

Caraballo, T., Real, J., & Taniguchi, T. (2007). The exponential stability of neutral stochastic delay partial differential equations. *Discrete and Continuous Dynamical Systems*, *18*(2/3), 295–313.

Cheng, L., Hou, Z.-G., & Tan, M. (2014). A mean square consensus protocol for linear multi-agent systems with communication noises and fixed topologies. *IEEE Transactions on Automatic Control*, *59*(1), 261–267.

Cont, R., & Fournié, D.-A. (2013). Functional Itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, *41*(1), 109–133.

Diblk, J., Dzhalladova, I., & Ružičková, M. (2012). The stability of nonlinear differential systems with random parameters. *Abstract and Applied Analysis*, *2012*, 509–512.

Dupire, B. (2009). Functional Itô's Calculus. Bloomberg Portfolio Research Paper No. 2009-04-FRONTIERS Available at SSRN: http://ssrn.com/abstract=1435551 or http://dx.doi.org/10.2139/ssrn.1435551.

Gershon, E., Shaked, U., & Berman, N. (2007). H_∞ control and estimation of retarded state-multiplicative stochastic systems. *IEEE Transactions on Automatic Control*, *52*(9), 1773–1779.

Hale, J. K., & Lunel, S. M. V. (1993). *Introduction to functional differential equations*. New York: Springer.

Huang, L., & Mao, X. (2009a). Delay-dependent exponential stability of neutral stochastic delay systems. *IEEE Transactions on Automatic Control*, *54*(1), 147–152.

Huang, L., & Mao, X. (2009b). Robust delayed-state-feedback stabilization of uncertain stochastic systems. *Automatica*, *45*(5), 1332–1339.

Janković, S., Randjelović, J., & Jovanović, M. (2009). Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations. *Journal of Mathematical Analysis and Applications*, *355*(2), 811–820.

Karimi, H. R. (2011). Robust delay-dependent control of uncertain time-delay systems with mixed neutral, discrete, and distributed time-delays and Markovian switching parameters. *IEEE Transactions on Circuits and Systems I*, *58*(8), 1910–1923.

Kolmanovskii, V., & Myshkis, A. (1999). *Introduction to the theory and applications of functional differential equations*. Dordrecht: Kluwer.

Kolmanovskii, V. B., & Nosov, V. R. (1986). *Stability of functional differential equations*. New York: Academic Press.

Kolmanovskii, V. B., & Shaikhet, L. E. (1997). Matrix Riccati equations and stability of stochastic linear systems with nonincreasing delays. *Functional Differential Equations*, *4*(3–4), 279–293.

Kolmanovskii, V. B., & Shaikhet, L. E. (1998). Riccati equations and stability of stochastic linear systems with distributed delay. *Advances in Systems, Signals, Control and Computers*, *1*, 97–100.

Kovalev, A. A., Kolmanovskii, V. B., & Shaikhet, L. E. (1998). Riccati equations in the stability of retarded stochastic linear systems. *Avtomatika i Telemekhanika*,(10), 35–54 (in Russian). Translated in Automatic Remote Control 59 (1998), N.10, part 1, 1379-1394.

Kushner, H. J. (1984). *Approximation and weak convergence methods for random processes, with applications to stochastic systems theory*. MIT press.

Li, T., Fu, M., Xie, L., & Zhang, J.-F. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control*, *56*(2), 279–292.

Luo, Q., Gong, Y., & Jia, C. (2017). Stability of gene regulatory networks with Lévy noise. *Science China: Information Sciences*, *60*(7), 072204.

Luo, Q., Mao, X., & Shen, Y. (2006). New criteria on exponential stability of neutral stochastic differential delay equations. *Systems and Control Letters*, *55*(10), 826–834.

Mao, X. (1996). Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stochastic Processes and their Applications*, *65*(2), 233–250.

Mao, X. (1997). *Stochastic differential equations and their applications*. Chichester: Horwood.

Mao, X. (2002). Exponential stability of stochastic delay interval systems with Markovian switching. *IEEE Transactions on Automatic Control*, *47*(10), 1604–1612.

Mohammed, S. E. A. (1986). *Stochastic functional differential equations*. New York: Longman.

Neely, M. J., & Modiano, E. (2005). Capacity and delay tradeoffs for Ad-Hoc mobile networks. *IEEE Transactions on Information Theory*, *51*(6), 1917–1936.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Rakkiyappan, R., Balasubramaniam, P., & Lakshmanan, S. (2008). Robust stability results for uncertain stochastic neural networks with discrete interval and distributed time-varying delays. *Physics Letters A*, *372*(32), 5290–5298.

Rami, M. A., & Zhou, X. Y. (2000). Linear matrix inequalities, riccati equations, and indefinite stochastic linear quadratic controls. *IEEE Transactions on Automatic Control*, *45*(6), 1131–1143.

Shaikhet, L. (2013). *Lyapunov functionals and stability of stochastic functional differential equations*. Springer.

Song, Y., & Shen, Y. (2013). New criteria on asymptotic behavior of neutral stochastic functional differential equations. *Automatica*, *49*(2), 626–632.

Strogatz, S. H. (2001). Exploring complex networks. *Nature*, *410*(6825), 268–276.

Teel, A. R. (1998). Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, *43*(7), 960–964.

Verriest, E. I., & Florchinger, P. (1995). Stability of stochastic systems with uncertain time delays. *Systems and Control Letters*, *24*(1), 41–47.

Wang, L. Y., Syed, A., Yin, G., Pandya, A., & Zhang, H. (2014). Control of vehicle platoons for highway safety and efficient utility: Consensus with communications and vehicle dynamics. *Journal of Systems Science and Complexity*, *27*(4), 605–631.

Wu, F., & Hu, S. (2012). Razumikhin-type theorems on general decay stability and robustness for stochastic functional differential equations. *International Journal of Robust and Nonlinear Control*, *22*(7), 763–777.

Wu, F., Yin, G., & Wang, L. Y. (2015). Razumikhin-type theorems on moment exponential stability of functional differential equations involving two-time-scale Markovian switching. *Mathematical Control and Related Fields*, *5*(3), 697–719.

Xu, L., Wang, L. Y., Yin, G., & Zhang, H. (2014). Communication information structures and contents for enhanced safety of highway vehicle platoons. *IEEE Transactions on Vehicular Technology*, *63*(9), 4206–4220.

Yu, W., Chen, G., Cao, M., & Ren, W. (2013). Delay-induced consensus and quasi-consensus in multi-agent dynamical systems. *IEEE Transactions on Circuits and Systems I*, *60*(10), 2679–2687.

Zhu, B., Xie, L., Han, D., Meng, X., & Teo, R. (2017). A survey on recent progress in control of swarm systems. *Science China Information Sciences*, *60*(7), 070201.

Zong, X., Li, T., Yin, G., Wang, L. Y., & Zhang, J.-F. (2017). Stochastic consentability of linear systems with time delays and multiplicative noises. *IEEE Transactions on Automatic Control*. http://dx.doi.org/10.1109/TAC.2017.2732823.

Zong, X., Li, T., & Zhang, J.-F. (2018). Consensus conditions for continuous-time multi-agent systems with additive and multiplicative measurement noises. *SIAM Journal on Control and Optimization*, *56*(1), 19–52.

Zong, X., Li, T., & Zhang, J.-F. (2017). Consensus conditions for continuous-time multi-agent systems with time-delays and measurement noises, arXiv:1602.00069v3.

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