Stability of Stochastic Functional Differential Systems Using Degenerate Lyapunov Functionals and Applications

Xiaofeng Zong, George Yin, Le Yi Wang, Tao Li, Ji-Feng Zhang

a School of Automation, China University of Geosciences, Wuhan 430074, China.
b Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.
c Department of Electrical and Computer Engineering, Wayne State University, Detroit, MI 48202, USA.
d Department of Mathematics, East China Normal University, Shanghai 200241, China.
e Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
f School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Abstract

In view of the seminal work of Dupire [9], this work investigates asymptotic behaviors of stochastic functional differential systems represented by stochastic functional differential equations (SFDEs). Stability of general delay-dependent SFDEs is investigated by using degenerate Lyapunov functionals, which are only positive semi-definite rather than positive definite as used in the classical work. This paper first establishes boundedness and regularity of SFDEs by using degenerate Lyapunov functionals. Then moment and almost sure exponential stabilities are obtained based on degenerate Lyapunov functionals and the semi-martingale convergence theorem. As applications of the stability criteria, consentability of stochastic multi-agent systems with nonlinear dynamics are studied.

Key words: Degenerate Lyapunov functional, stochastic functional differential equation, asymptotic behavior, stabilization, consentability.

1 Introduction

In a seminal work [9], Dupire introduced a method to extend the Itô formula to a functional setting using a path-wise functional derivative that quantifies the sensitivity of a functional variation in the endpoint of a path. Prior to this work, although there were many excellent works on stochastic delay equations, because of the solution processes being non-Markovian due to delay, there were virtually no bona fide operators and functional Itô formulas except for some general setup in a Banach space. The general setup in a Banach space, however, cannot be used in analysis of various systems involving stochastic functional differential equations. The newly developed Itô formula has substantially eased the difficulties and enabled development with a wide range of applications. The current paper aims to use the newly developed functional Itô formula to examine stability and related issues of stochastic functional differential systems. Many real systems inevitably involve delays, in which the future states depend not only on the present inputs, but also on the past history. Such systems have drawn much attention owing to their wide ranges of applications in population biology, epidemiology, economics, neural networks, and control of mechanical and electrical systems. Delay differential equations (DDEs) and more general functional differential equations (FDEs) were used to model delay systems in [11,17,24]. To date, various properties of FDEs have been investigated extensively. One of the most important properties is stability, which is the key in control theory [4,5]. It has been well observed that small delays may be of help for stability in special systems [39], but typically they are a source of instability and poor performance, and can greatly increase difficulty in stability analysis and control design. For some simple

Preprint submitted to Automatica 6 September 2017
linear systems, certain system structures (such as eigenvalues of the system matrices) can be used to deduce stability [11]. For more complex functional equations, one often resorts to Lyapunov functional and Razumikhin methods to derive sufficient stability conditions [11,16]. More often than not, delay systems are subject to random disturbances. Thus naturally they can be modeled by stochastic delay differential equations (SDDEs) and stochastic functional differential equations (SFDEs); see [25,28]. To treat random disturbances, various notions of stability such as moment stability, almost sure stability, and stability in probability, are commonly used. Different from deterministic linear FDEs, SFDEs usually do not have special structures that enable us to obtain necessary and sufficient conditions of mean square and almost sure stabilities for linear multi-dimensional SDDEs. Hence, Razumikhin methods and Lyapunov functionals are two of the main machineries for stability analysis in the literature. The stochastic versions of Razumikhin methods were developed in [14,26,37,38] to study moment asymptotic stability. However, unlike its deterministic counterpart, the stochastic Razumikhin methods have limited success [11] and [35]. Choosing an appropriate Lyapunov function or functional is the key for stability analysis and control design in applications. While sufficient conditions on delay-independent stability of SFDEs are often obtained by using Lyapunov functions, studies on delay-dependent stability are often based on Lyapunov functionals and linear matrix inequalities (LMIs). Mao [25] gave the delay-independent pth-moment stability by using Lyapunov functions and obtained the almost sure stability from the moment exponential stability and a linear growth condition. Huang and Mao [12] studied the mean square exponential stability of neutral SDDEs by using a Lyapunov functional, and certain stability conditions were given in terms of LMIs. Caraballo, Real, and Taniguchi [6] studied the almost sure exponential stability and ultimate boundedness of solutions for a class of neutral stochastic semi-linear partial delay differential equations. Radiolarian, Parachromatism, and Leishmania [30] used Lyapunov functionals to derive LMI-type stability conditions for uncertain stochastic neural networks with discrete intervals and distributed time-varying delays. Grison, Shaken, and Berman [10] investigated $H_{\infty}$ state-feedback control of stochastic delay systems by Lyapunov functions and LMIs. Shaikhet [32] introduced different Lyapunov functionals to examine stochastic stabilities of different SFDEs. Note that these results were given under the specific Lyapunov functionals, and a fundamental stability theory related to Lyapunov functionals has not been developed. Hence, one does not know under which conditions on general functionals, the corresponding SFDE is moment and almost surely stable. There were many results on delayed feedback stabilization problems of stochastic systems. Verriest and Florchinger [36] gave delay-independent conditions on delayed feedback stabilization of linear SDDEs by Riccati-type equations in which multi-matrices must be determined. Mao [27] and Huang and Mao [13] obtained delay-dependent exponential stability conditions for linear SDDEs in terms of LMIs. Karimi [15] investigated robust mode-dependent delayed state feedback control for a class of uncertain delay systems in terms of LMIs. These results are based on the positive definitive Lyapunov functionals.

In this paper, we develop several fundamental stability theorems using degenerate Lyapunov functionals. These results include moment asymptotic stability, exponential stability, and almost sure exponential stability. Degenerate Lyapunov functionals being positive semi-definite were well-investigated in [17] for deterministic neutral FDEs. Some specific degenerate Lyapunov functionals were also applied to examine the moment stability of stochastic functional differential equations with global Lipschitz conditions in the references (for example, [17,18,32]). However, a fundamental stability theorem related to a degenerate Lyapunov functional has not been established yet for SFDEs due to the lack of a general functional Itô formula, to the best of our knowledge. Due to the lack of bona fide operators and Itô formulas, whether degenerate Lyapunov functionals can be used to examine stability of SFDEs with non-linear growth and non-Lipschitz coefficients has not been settled until now. This paper fills in the gap and reveals that an appropriate degenerate Lyapunov functional can simplify stability analysis and is beneficial for control design.

Using our fundamental stability theorems, we treat an application-motivated multi-agent consentability and consensus problem, see some recent work in multi-agent systems [7,22,41,42]. The problem captures the common backbone platform of many communication systems in networked mobile systems, such as highway autonomous vehicles, unmanned aerial vehicles, team robots, among many other applications. As pointed out in [34] that due to dynamical complexity, the nodes could be nonlinear dynamical systems. In fact, nonlinear multi-agent systems have attracted much attention in control theory; see the survey [40]. However, there are no results available for stochastic multi-agent consentability and consensus with nonlinear dynamics, even for the delay-free case. Due to the noises and nonlinear dynamics, delayed measurements should work positively for stability. Note that the LMI-type conditions can not reveal the explicit conditions for consentability. However, our constructed fundamental stability theorems can be used to resolve the problem. Assuming that the nonlinear terms satisfy a Lipschitz condition with uncertainty, we give sufficient stochastic consentability conditions, and reveal explicitly that the multi-agent consentability depends on the delay in feedback, noise intensities, balanced graphs, and the Lipschitz constant of the nonlinear dynamics.

The rest of the paper is structured as follows. Section 2 introduces notation and develops the functional Itô formula for SFDEs. Section 3 establishes the fundamental degenerate Lyapunov functional theorem to produce the
regularity and moment boundedness of SFDEs. Section 4 presents the fundamental degenerate Lyapunov functional theorem for the asymptotic moment estimates, moment stability, and almost sure stability based on the semi-martingale convergence theorem. In particular, mean square and almost sure stabilities are established for stochastic systems represented by semi-linear SFDEs. Section 5 applies the theorems using degenerate Lyapunov functionals to examine stochastic consentability of multi-agent systems with nonlinear dynamics. Section 6 concludes the paper with some remarks. Finally, an appendix containing the proof of a technical result is included at the end of the paper.

2 Functional Itô Formula

We work with the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $\cdot$. For a vector or a matrix $A$, its transpose is denoted by $A^T$. For a matrix $A$, denote its trace norm by $|A| = \sqrt{\text{trace}(A^T A)}$. For a symmetric matrix $A$, denote its transpose is denoted by $A^T$. The metric $l_2$-dimensional Euclidean space $\mathbb{R}^n$ is equipped with the Euclidean norm $| \cdot |$.

Assumption 2.1

Before proceeding further, we need to develop the functional Itô formula given by a stochastic functional differential equation (SFDE) where $\psi_0(\theta) = \phi(\theta), \theta \in [-\delta,\delta]$;

and $\delta \phi(\theta) = \phi(\theta), \theta \in [-\delta,\delta]$;

and $\nabla \phi(\theta) = \phi(\theta), \theta \in [-\delta,\delta]$;

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.

and $\nabla \phi(\theta)$.
sup \mathbf{d}_\tau \mathbf{d}_s = \lambda |x|^p, \text{ for all } x \in \mathbb{R}^n.
}

Hence, the lower bound condition (I) is bounded in the sense of pth order, that is, sup_{s \geq 0} E|y(s)|^p < \infty.

**Remark 3.1** The conditions in Theorem 3.1 can be considered as extension of the classical Lyapunov theorem for stochastic systems without delay. In fact, if $\tau = 0$, $\{\mu_3\}_{1=1}^n$ become the Dirac measure $d_0$ and the functional $V(x, t)$ is certain function $V(x, \tau)$. Then condition (I) reduces to $(c_0 - c_1)|x|^p \leq V(x, \tau)$, which is frequently used in moment stability analysis, and which justifies the requirement $c_0 > c_1$. In the non-delay case, conditions (II) and (III) become $\mathbf{E}V(x, t) \leq \eta(t) - |x|^p$, which is a classical condition in the Lyapunov asymptotic analysis of stochastic systems [25]. Intuitively, conditions (II)-(III) imply that the functional can grow since $\lambda$ can be negative. But condition (I) is added to avoid exploding in finite time. In fact, if $\eta(t) = 0$, the growth rate is no more than the exponential rate, which is implied in (9) below. However, if $\lambda > 0$, then conditions (II)-(III) imply that the functional cannot grow forever. In fact, if $\eta(t) = 0$, the overall solution tends to be bounded. Such conditions are often required for Lyapunov function stability of SDEEs (see [23, 33]). Hence, the lower bound condition (I) is not required if $\lambda > 0$ and $\int_0^\infty \eta(t)dt < \infty$. More details will be given in Theorem 3.2.

Note also that we require $V(t, \tau) \geq 0$, but do not always require $c_0|\varphi(0)|^p - c_1 \int_{-\tau}^0 |\varphi(s)|^p \mathbf{d}_s \geq 0$. In fact, for the case $\mathbf{d}_s \mathbf{d}_t = ds, \tau = 1, \varphi(s) = (2 + s)^{-1/2}, c_0 = 0.11$ and $c_1 = 0.1$, then $c_0|\varphi(0)|^p - c_1 \int_{-\tau}^0 |\varphi(s)|^p \mathbf{d}_s < 0$. This is different from the classical Lyapunov functionals used in previous works; see [13] for example, where $c_0|\varphi(0)|^p \leq V(\varphi, t)$ is required. Hence, the functional in this paper is known as degenerate Lyapunov functional. Note also that we require $V(t, \tau) \geq 0$, but do not always require $c_0|\varphi(0)|^p - c_1 \int_{-\tau}^0 |\varphi(s)|^p \mathbf{d}_s \geq 0$. In fact, for the case $\mathbf{d}_s \mathbf{d}_t = ds, \tau = 1, \varphi(s) = (2 + s)^{-1/2}, c_0 = 0.11$ and $c_1 = 0.1$, then $c_0|\varphi(0)|^p - c_1 \int_{-\tau}^0 |\varphi(s)|^p \mathbf{d}_s < 0$. This is different from the classical Lyapunov functionals used in previous works; see [13] for example, where $c_0|\varphi(0)|^p \leq V(\varphi, t)$ is required. Hence, the functional in this paper is known as degenerate Lyapunov functional. Sections 4.2 and 4.3 give some specific examples of Lyapunov functionals for stochastic stability analysis.

**Proof** For any given initial data $y_0 \in C([-\tau, 0], \mathbb{R}^n)$, by Assumption 2.1, there exists a unique maximal local strong solution $y(t)$ to (1) on $t \in [-\tau, \rho_k)$, where $\rho_k$ is the explosion time [25]. To prove that this solution is global, we need only show that $\rho_k = \infty$ a.s. Define the stopping time $\rho_k = \inf\{t \in [0, \rho_k) : |y(t)| \geq k\}$. Clearly, $\rho_k$ is increasing as $k \to \infty$ and $\rho_k \to \rho = \infty$ a.s. If we can show $\rho_\infty = \infty$ a.s., then $\tau_e = \infty$, which implies that the solution $y(t)$ is actually global. Let $V^k(t) = \mathbf{E} y(t \wedge \rho_k, t \wedge \rho_k)$. By the generalized Itô formula (3) and condition (II), we obtain that for any $k > 0$ and $t \geq 0$,

$$V^k(t) = \mathbf{E} y_0 + \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} V(y_s, s) ds \leq \mathbf{E} y_0 + \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_2(y_s + \theta) \mathbf{d}_s + \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_3(y_s) ds - \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_1(y_s) ds + \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_4(y_s) ds,$$

By the Fubini theorem, we get that for $i = 2, 3$,

$$\mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_i(y_s + \theta) \mathbf{d}_s = \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_i(y_s + \theta) \mathbf{d} \mu_\iota \mathbf{d}_s \leq \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_i(y_s + \theta) \mathbf{d}_s ds \leq \mathbf{E} \int_0^{t \wedge \rho_k} \mathbf{E} U_i(y_s) ds$$

Note that $\alpha \geq \beta$ and $U_2(x) - U_1(x) \leq -\lambda |x|^p$. Hence, substituting the inequalities above into (4) yields

$$V^k(t) \leq \mathbf{E} y_0 + \mathbf{E} \int_0^t \mathbf{E} U_2(y(s) + \theta) ds + \mathbf{E} \int_0^t \mathbf{E} U_3(y(s)) ds - \mathbf{E} \int_0^t \mathbf{E} U_1(y(s)) ds + \mathbf{E} \int_0^t \mathbf{E} U_4(y(s)) ds,$$

where $C_1 = 1 + \mathbf{E} y_0, \mathbf{E} \int_0^t \mathbf{E} U_2(y(s) + \theta) ds + \beta \mathbf{E} \int_0^t \mathbf{E} U_3(y(s)) ds$. By condition (I), we have that for any $s > 0$,

$$c_0 \mathbf{E} |y(s + \rho_k)|^p \leq \mathbf{E} y(s) + \mathbf{E} |y(s + \rho_k)|^p \leq \mathbf{E} y(s) + \mathbf{E} |y(s + \rho_k)|^p.$$

Note that $\rho_k \geq 0$. Then we get

$$\sup_{s \in [-\tau, 0]} \mathbf{E} |y(s + \rho_k)|^p \leq \mathbf{E} y_0 + \mathbf{E} |y(s + \rho_k)|^p \leq \mathbf{E} y_0 + \mathbf{E} |y(s + \rho_k)|^p,$$

which together with $c_1 < c_0$ gives $\mathbf{E} |y(s + \rho_k)|^p \leq \sup_{s \in [-\tau, 0]} \mathbf{E} |y(s + \rho_k)|^p$. Substituting the inequality above into (6) yields $\mathbf{E} |y(t + \rho_k)|^p \leq (1 + \mathbf{E} |y(s)|^p) C_2 +$
\( C\int_0^T \mathbb{E}[\eta(s \wedge \rho_k)]^p ds \), where \( C_2 = \frac{1}{\epsilon_{\sigma-s}}(c_0C(y_0) + C_1) \), \( C_3 = \frac{1}{\epsilon_{\sigma-s}} \). Applying the Gronwall inequality produces

\[ \mathbb{E}[y(t \wedge \rho_k)]^p \leq (1 + \int_0^t \eta(s) ds) C_2 + C_3 \int_0^t e^{C_3(t-s)} (1 + \int_0^s \eta(u) du) ds \leq (1 + \int_0^t \eta(s) ds) C_2 e^{C_3 t} =: C_4(t), \] (9)

Note that \( \mathbb{E}[y(t \wedge \rho_k)]^p \geq \mathbb{E}[[y(t \wedge \rho_k)]^p 1_{\{\rho_k < \ell\}}] \geq k^p \mathbb{P}\{\rho_k \leq t\} \). Hence, \( k^p \mathbb{P}\{\rho_k \leq t\} \leq C_4(t) \). Then, for any \( t > 0 \), \( \lim_{t \to \infty} \mathbb{P}\{\rho_k \leq t\} = 0 \), which together with the arbitrariness of \( t \) implies \( \rho_k = \infty \) a.s. Therefore, the solution \( y(t) \) is global. Especially, if \( \int_0^\infty \eta(s)ds \leq \infty \) and \( \lambda \geq 0 \), then we can see that \( C_3 \) is still true with \( C_3 \). Hence, we get from (9) that for any \( t \geq 0 \), \( \mathbb{E}[y(t)]^p = \mathbb{E}[\lim_{t \to \infty} (y(t \wedge \rho_k))]^p \leq \lim_{t \to \infty} \mathbb{E}[y(t \wedge \rho_k)]^p \leq (1 + \int_0^t \eta(s) ds) C_2 \), where we used Fatou’s lemma and \( \rho_k = \infty \). □

**Theorem 3.2** Assume that there exist a functional \( V \in C^{2,1}(D[\tau/2]; \mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \), three functions \( U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+ \), and two nonnegative constants \( \alpha \geq \beta \geq 0 \) such that conditions (II)-(III) hold with \( \int_0^\tau \eta(t) dt < \infty \) and \( \lambda > 0 \). Then SFDE (1) admits a unique global solution for each initial data \( y_0 \in C([\tau, 0]; \mathbb{R}^n) \). Moreover, the solution to SFDE (1) is bounded in the sense of \( p \)th moment.

**Proof** Note that \( \alpha \geq \beta \) and \( U_2(x) - U_1(x) \leq -\lambda |x|^p \), \( \lambda > 0 \). Then, substituting (5) in (4) yields

\[ \mathbb{E}[V_k(t)] \leq \mathbb{E}[V(0)] + \int_0^t \mathbb{E}[U_2(y(\theta))] d\theta + \int_0^t \eta(s) ds \] (10)

Therefore, for any \( t \geq 0 \), \( \lambda \int_0^t \mathbb{E}[y(s \wedge \rho_k)]^p ds < C_5 + \int_0^\tau \mathbb{E}[\eta(s) ds \), where \( C_5 = \mathbb{E}[V(0)] + E \int_0^\tau U_2(y(\theta)) d\theta \) and \( \beta \mathbb{E} \int_0^\tau U_3(y(\theta)) d\theta \). This also implies \( \mathbb{E}[y(s \wedge \rho_k)]^p \leq C_6 \) for certain \( C_6 > 0 \). Then, we use similar methods as above to prove that \( \rho_k = \infty \) a.s. Therefore, the solution \( y(t) \) is global. Similarly to the proof of Theorem 3.1, we obtain the moment boundedness of the solution by using Fatou’s lemma. □

4 Asymptotic Analysis and Stability

4.1 Asymptotic analysis of nonlinear SFDEs

**Theorem 4.1** Assume that there exist a functional \( V \in C^{2,1}(D[\tau/2]; \mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \), three functions \( U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+ \), \( \eta(t) \geq 0 \) and some nonnegative constants \( c_0 > c_1, c_2 \neq 0, \alpha > \beta, \) such that

\[ \mathbb{E}[|\phi(0)|^p] - c_1 \int_0^\tau |\phi(s)|^p d\mu_1(s) \leq V(\varphi, t) \leq c_2 |\phi(0)|^p + c_3 \int_0^\tau |\phi(s)|^p d\mu_4(s); \]

(II) \( \mathbb{E}[V(\varphi, t)] \leq \mathbb{E}[-U_2(\varphi(s))ds + \alpha U_3(\varphi(s))] \leq \mathbb{E}[\int_0^\tau U_3(\varphi(s)) d\mu_3(s); \]

(III) \( U_2(x) - U_1(x) \leq 0 \) and there exists a \( \gamma_0 > 0 \) such that \( \gamma_0(c_2 \epsilon_\tau)^{\gamma_0} |x|^p - U_1(x) + \epsilon^{\gamma_0} U_2(x) \leq 0 \) for all \( x \in \mathbb{R}^n \), where \( \{\mu_i\}_{i=1}^\tau \) are probability measures defined on \([\tau, 0] \).

Then the unique global solution to SFDE (1) satisfies

(a) \( \lim_{t \to \infty} \mathbb{E}[y(t)]^p \leq \frac{\eta_0}{\gamma_0(\epsilon_0 \epsilon_\tau)^{\gamma_0}} \) if \( \eta(t) \leq \eta_0 \) for any \( t > 0; \)

(b) \( \lim_{t \to \infty} \mathbb{E}[y(t)]^p = 0 \) if \( \lim_{t \to \infty} \int_0^t e^{-\gamma(t-s)} \eta(s) ds = 0; \)

(c) \( \mathbb{E}[y(t)]^p \leq C e^{-t} \) for some \( C > 0 \) if \( \int_0^\infty e^{\gamma t} \eta(t) dt < \infty, \)

where \( \gamma \in (0, \gamma^*), \gamma^* = \gamma_0 \gamma_1 \gamma_2, \gamma_1 = \frac{1}{\tau} \log(\frac{3}{2}), \gamma_2 = \frac{1}{2} \log(\frac{c_0}{c_1}). \)

**Proof** Note that \( U_2(x) - U_1(x) \leq 0 \) implies \( U_2(x) - U_1(x) \leq |\lambda||x|^p \). Then, condition (II) holds. Based on Theorem 3.1, the solution is regular. Letting \( Z^k(t) = e^{(\gamma \rho_k) t} V(y_t \rho_k, t \wedge \rho_k) \) and applying Itô’s formula (2), we obtain that

\[ Z^k(t) = V(y_0, 0) + \int_0^{t \wedge \rho_k} e^{-\gamma \lambda \Delta z} V(y, s) g(y, s) daw(s) \] + \[ \int_0^{t \wedge \rho_k} e^{\gamma \lambda \Delta z} V(y, s) d\mathbb{E}(\Delta z) ds \]
\[ \leq V(y_0, 0) + \int_0^{t \wedge \rho_k} e^{-\gamma \lambda \Delta z} V(y, s) d\mathbb{E}(\Delta z) ds \] + \[ \int_0^{t \wedge \rho_k} \gamma \lambda \Delta z V(y, s) d\mathbb{E}(\Delta z) ds \] (11)

Note that for \( i = 2, 3, 4, \) and \( h : \mathbb{R}^n \to \mathbb{R}_+ \),

\[ \int_0^t e^{\gamma \lambda \Delta z} V(y, s) ds \]
\[ = \int_0^t e^{\gamma \lambda \Delta z} V(y, s) ds d\mu_1(u) \]
\[ \leq \epsilon^{\gamma_0} \int_0^t e^{\gamma \lambda \Delta z} V(y, s) ds + \epsilon^{\gamma_0} \int_0^t e^{\gamma \lambda \Delta z} V(y, s) ds. \] (12)

Substituting (12) into (11), we get

\[ \mathbb{E}Z^k(t) \leq C_7(\gamma) + E \int_0^{t \wedge \rho_k} e^{\gamma \lambda \Delta z} V(y, s) ds \]
\[ \gamma < \gamma \]
\[ C \]

and satisfies \( \lim_{t \to \infty} \eta(t) = 0 \).

From \( \alpha > \beta \), we know that \( \beta e^{\gamma t} - \alpha < 0 \) for any \( \gamma < \gamma_1 := \frac{1}{\beta} \log(\frac{\alpha}{\beta}) \). Then we get from (13) that for \( \gamma < \gamma_1 \),

\[ E\mathbb{Z}^k(t) \leq C_7(\gamma) + \int_0^t e^\gamma \eta(s)ds + \mathbb{E}\int_0^t e^\gamma \eta h(y, g, s)ds, \tag{14} \]

where \( h(x, y) = \gamma (c_2 + c_3 e^{\gamma t}) |x|^p - U_1(x) + e^{\gamma} U_2(x) \).

Hence, from condition (III), \( h(x, y) \leq 0 \) for any \( \gamma \in (0, \gamma_0 \wedge \gamma_1) \). Letting \( k \to \infty \) in (14), we have

\[ \mathbb{E} e^{\gamma} [V(y, t)] \leq C_7(\gamma) + \int_0^t e^\gamma \eta(s)ds. \tag{15} \]

Let \( Y(s) = \sup_{u \in [s-t, s]} e^{\gamma u} \mathbb{E}|y(u)|^p \). By condition (I), we have that for any \( s > 0 \),

\[ c_0 e^{\gamma s} \mathbb{E}|y(s)|^p \leq e^{\gamma s} \mathbb{E}(y(s), s) + c_1 e^{\gamma s} \sup_{u \in [s-t, s]} e^{\gamma u} \mathbb{E}|y(u)|^p \]

\[ \leq e^{\gamma s} \mathbb{E}(y(s), s) + c_2 e^{\gamma Y(t)}. \]

Then for any \( \gamma > 0 \),

\[ c_0 Y(t) \leq c_0 Y(0) + c_0 Y(t) \]

\[ \leq c_0 Y(0) + \sup_{s \in [0, t]} e^{\gamma s} \mathbb{E}(y(s), s) + c_1 e^{\gamma Y(t)}. \tag{16} \]

From \( c_1 < c_0 \), we know that \( c_0 - c_1 e^{\gamma t} > 0 \) for any \( \gamma < \gamma_3 := \frac{1}{\beta} \log(\frac{\alpha}{\beta}) \). This together with (15) and (16) implies that for any \( \gamma \in (0, \gamma_1 \wedge \gamma_2 \wedge \gamma_3) \),

\[ Y(t) = \frac{c_0 Y(0) + c_1 e^{\gamma t} \sup_{s \in [0, t]} e^{\gamma s} \mathbb{E}(y(s, s))}{c_0 - c_1 e^{\gamma t}} \]

\[ \leq C_8(\gamma) + \frac{1}{c_0 - c_1 e^{\gamma t}} \int_0^t e^\gamma \eta(s)ds, \tag{17} \]

where \( C_8(\gamma) = \frac{1}{c_0 - c_1 e^{\gamma t}}(c_0 Y(0) + C_7(\gamma)) \). Therefore, for any \( t > 0 \), \( e^{\gamma t} \mathbb{E}(y(t)|^p \leq C_8(\gamma) + \frac{1}{c_0 - c_1 e^{\gamma t}} \int_0^t e^\gamma \eta(s)ds \).

That is, \( \mathbb{E}|y(t)|^p \leq C_8(\gamma) e^{\gamma t} + \frac{1}{c_0 - c_1 e^{\gamma t}} \int_0^t e^\gamma \eta(s)ds \).

Then the desired assertions can be easily obtained. \( \square \)

**Remark 4.1** Note that \( \lim_{t \to \infty} \int_0^t e^{-\gamma(t-s)} \eta(s)ds = 0 \) is guaranteed by \( \eta \in L^1((0, \infty); \mathbb{R}^s) \) or \( \eta \) is continuous and satisfies \( \lim_{t \to \infty} \eta(t) = 0 \). In fact, if \( \eta \) is continuous and satisfies \( \lim_{t \to \infty} \eta(t) = 0 \), we use L'Hôpital's rule to deduce \( \lim_{t \to \infty} \int_0^t e^{-\gamma(t-s)} \eta(s)ds = \lim_{t \to \infty} \int_0^t e^{\gamma u} \eta(s)du = 0 \).

**Theorem 4.2** Assume that there exist a functional \( V \in C^2(D([-\tau, 0]; \mathbb{R}^s)) \times \mathbb{R}_+ \), three functions \( U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+ \), \( \eta(t) \geq 0 \) and some nonnegative constants \( c_2, c_3 \neq 0, c_4, \alpha > \beta, \lambda > 0 \), such that condition (II), and the following conditions hold,

\[ \forall (t^*) \quad V(\varphi, t) \leq c_2 |\varphi(0)|^p + c_3 \int_0^t |\varphi(s)|^p ds \]

\[ (III) \quad 0 \leq U_2(x) \leq c_4 |x|^p, U_3(x) \leq -\lambda |x|^p, \text{ for all } x \in \mathbb{R}^n, \]

where \( \{\mu_i\}_{i=1}^N \) are probability measures defined on \([-\tau, 0)\). Then the unique global solution to SFDE (1) satisfies \( \mathbb{E}|y(t)|^p \leq C e^{-\gamma t} \) for certain \( C > 0 \) if \( \int_0^\infty e^{\gamma t} \eta(t)dt < \infty \), where \( \gamma \in (0, \gamma^*), \gamma^* = \gamma_0 \wedge \gamma_1 = \frac{1}{\beta} \log(\frac{\alpha}{\beta}) \), if \( c_2 + c_3 e^{\gamma t} < -\lambda + c_4 (e^{\gamma t} - 1) \).

**Proof** Note that \( \int_0^\infty e^{\gamma t} \eta(t)dt < \infty \) implies \( \int_0^\infty \eta(t)dt < \infty \), and condition (III) implies condition (III'). Then Theorem 3.2 enables us to conclude that the solution is regular.

By \( \alpha > \beta \), we have \( \beta e^{\gamma t} - \alpha < 0 \) for any \( \gamma < \gamma_1 := \frac{1}{\beta} \log(\frac{\alpha}{\beta}) \). Note also that condition (III') implies \( U_1(y(s)) - e^{\gamma} U_2(y(s)) \geq \lambda |x|^p - c_4 (e^{\gamma t} - 1)|x|^p \) then we get from (13) that for \( \gamma < \gamma_1 \),

\[ E\mathbb{Z}^k(t) \leq C_7(\gamma) + \int_0^t e^\gamma \eta(s)ds + h(\gamma)\mathbb{E}\int_0^t e^\gamma \eta(y(s))ds \tag{18} \]

where \( h(\gamma) = \gamma (c_2 + c_3 e^{\gamma t}) - \lambda + c_4 (e^{\gamma t} - 1) \).

Note that \( h(0) = -\lambda \) and \( h(\gamma) > 0 \) for \( \gamma \) satisfying \( \gamma (c_2 + c_3 e^{\gamma t}) - \lambda > 0 \). In addition, \( h'(\gamma) > 0 \) for any \( \gamma > 0 \). Then, there exists a unique root \( \gamma_2 \) such that \( h(\gamma_2) = 0 \). Hence, \( h(\gamma) < 0 \) for any \( \gamma \in (0, \gamma_1 \wedge \gamma_2) \). Therefore, we have from (18) that \( -h(\gamma) \mathbb{E}\int_0^t e^{\gamma t} \eta(y(s))ds \leq C_7(\gamma) + \int_0^t e^\gamma \eta(s)ds \).

By Fatou's lemma, letting \( k \to \infty \), for any \( t > 0 \), \( -h(\gamma) \mathbb{E}\int_0^t e^{\gamma t} \eta(y(s))ds \leq C_7(\gamma) + \int_0^\infty e^{\gamma t} \eta(s)ds \), which implies the desired assertion. \( \square \)

**Remark 4.2** Theorem 2.1 in [17, p.389] asserts that if the coefficients \( f, g \) satisfy the global Lipschitz condition and there exist \( C_1, C_2, C_3 > 0 \) such that

(i) \( C_1 \mathbb{E}|y(t)|^2 \leq \mathbb{E} V(y, t) \leq C_2 \sup_{-\tau \leq s \leq 0} \mathbb{E} |y(t + s)|^2 \),

(ii) \( \mathbb{E} (V(y, t) - V(y, s)) \leq -C_3 \int_0^s \mathbb{E}|y(u)|^2du \) for any \( t \geq s \geq 0 \),

then the solution is mean square asymptotically stable. Our stability results in Theorems 4.1 and 4.2 improve Theorem 2.1 in [17] in the following four points: (1) Theorems 4.1 and 4.2 do not require the coefficients to be globally Lipschitz. (2) There is no requirement on the positive definiteness of \( V(y, t) \). (3) Our results can produce the mean square exponential stability. (4) Condition (II) is broader than condition (ii) above since condition (II) can deal with the case with additive noises.
Theorem 4.2 and assertion (c) in Theorem 4.1 present a pth moment exponential stability criterion. It is known in [25] that pth moment exponential stability implies almost sure exponential stability of SFDEs with multiplicative noises under the assumption that \( f, g \) satisfy a linear growth condition, that is, there exists a constant \( c_5 > 0 \) such that for any \( \varphi \in C([-\tau, 0]; \mathbb{R}^n) \),

\[
|f(\varphi, t)|^2 + |g(\varphi, t)|^2 \leq c_5 \int_{-\tau}^0 |\varphi(s)|^2 d\mu_5(s),
\]

where \( \mu_5(s) \) is a probability measure on \([-\tau, 0] \). Without the linear growth assumption, we can resort to the semi-martingale convergence theorem [25, p. 14] to examine the almost sure exponential stability of SFDEs, where the noise can be additive. To proceed, we give the almost sure stability analysis.

**Theorem 4.3** Assume that there exist a functional \( V \in C^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^n) \), three functions \( U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+ \), and some nonnegative constants \( c_2, c_3, c > \beta, \lambda > 0 \) such that conditions (I'), (II'), and (III') hold. If \( \{t_0 e^{\gamma \tau} \Omega(t) dt < \infty \), then SFDE (1) admits a unique global solution and the solution satisfies that for any initial data \( y_0 \in C([-\tau, 0]; \mathbb{R}^n) \),

\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(t)| \leq \gamma \text{ a.s.},
\]

where \( \gamma \in (0, \gamma^*) \) is defined in Theorem 4.2.

**Proof** For \( \gamma > 0 \), let \( Z(t) = e^{t\gamma}V(y(t), t) \). Applying the functional Itô formula to \( e^{t\gamma}V \) and using (II) yield

\[
Z(t) \leq V(y_0, 0) + \int_0^t e^{t\gamma} \Omega(y(s), s)ds - \alpha \int_0^t e^{t\gamma} U_3(y(s))ds
+ c_2 \gamma \int_0^t e^{t\gamma} |y(s)|^2 ds - \int_0^t e^{t\gamma} U_1(y(s))ds + M(t)
+ c_3 \gamma \int_0^t e^{t\gamma} \int_{-\tau}^0 |y(u + s)|^p d\mu_4(u)ds
+ \beta \int_0^t e^{t\gamma} \int_{-\tau}^0 U_3(y(u + s))d\mu_3(u)ds
+ \int_0^t e^{t\gamma} \int_{-\tau}^0 U_2(y(u + s))d\mu_2(u)ds,
\]

where \( M(t) = \int_0^t e^{-\gamma \tau} \Delta_V V(y(s), s)g(y(s), s)dw(s) \). Substituting (12) into (22), we get that

\[
Z(t) \leq C_0(\gamma) + (\beta e^{\gamma \tau} - \alpha) \int_0^t e^{t\gamma} U_3(y(s))ds + M(t)
+ h(\gamma) \int_0^t e^{t\gamma} |y(s)|^p ds + \int_0^t e^{t\gamma} \eta(s),
\]

where \( C_0(\gamma) = V(y_0, 0) + c_2 \gamma e^{\gamma \tau} \int_{-\tau}^0 |y(s)|^p ds + e^{\gamma \tau} \int_{-\tau}^0 U_2(y(s))ds + \beta e^{\gamma \tau} \int_{-\tau}^0 U_3(y(s))ds \), and \( h(\gamma) \) is defined by (19). Let \( \gamma \in (0, \gamma^*) \), then \( h(\gamma) < 0 \). Moreover, we have from (23) that

\[
-h(\gamma) \int_0^t e^{t\gamma} |y(s)|^p ds \leq Y(t),
\]

where \( Y(t) = C_0(\gamma) + (\beta e^{\gamma \tau} - \alpha) \int_0^t e^{t\gamma} U_3(y(s))ds + M(t)
+ \int_0^t e^{t\gamma} \eta(s)ds \geq 0 \). Note that \( \beta e^{\gamma \tau} - \alpha < 0 \), \( A_1(t) := \int_0^t e^{t\gamma} \eta(s)ds \) and \( A_2(t) := (\alpha - \beta e^{t\gamma}) \int_0^t e^{t\gamma} U_3(y(s))ds \) are increasing. Then, \( \int_0^\infty e^{t\gamma} \eta(s)ds < \infty \) and the semi-martingale convergence theorem [25, p. 14] yield \( \lim_{t \to \infty} Y(t) < \infty \), a.s. Therefore, together with (24) implies \( \int_0^\infty e^{t\gamma} |y(s)|^p ds < \infty \), a.s. This desired assertion (21) follows. □

Note that the conditions of almost sure exponential stability in Theorem 4.3 are the same as that of pth moment exponential stability in Theorem 4.1. Now, one may have a question whether we can obtain the almost sure asymptotic stability under the conditions for the pth moment asymptotic stability in Theorem 4.1. In fact, this may be impossible. To see it, we consider the following simple equation without delay,

\[
dy(t) = -y(t)dt + \sigma(t)dw(t).
\]

It is easy to see that \( \lim_{t \to \infty} \mathbb{E}|y(t)|^2 = 0 \) if and only if

\[
\lim_{t \to \infty} \int_0^t e^{-t\gamma} \sigma(s)^2 ds = 0.
\]

From [1], we know that \( \lim_{t \to \infty} |y(t)| = 0 \) a.s. if \( \lim_{t \to \infty} \sigma(t)^2 \log t = 0 \), and only if

\[
\lim_{t \to \infty} \sigma(t)^2 \log t = 0.
\]

Hence, if we choose \( \sigma(t) = \frac{1}{\sqrt{\log t}} \) for \( t > 1 \), then (26) holds (L'Hôpital's rule), and \( \lim_{t \to \infty} \mathbb{E}|y(t)|^2 = 0 \). But (27) is defied, and then \( \mathbb{P}\{\lim_{t \to \infty} |y(t)| = 0\} < 1 \).

**Remark 4.3** In the above, we developed degenerate Lyapunov functionals to investigate the asymptotic behaviors of SFDEs. The conditions (I') and (II) is required to hold for all \( \varphi \in C^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathbb{R}_+) \). Actually, in many complex networked systems, we only require these conditions to be true on the solution path. Moreover, based on the solution paths, the structure properties of the complex systems might be helpful for the asymptotic analysis (see the multi-agent application in the subsequent section).

### 4.2 Stochastic stability of semi-linear SFDEs

Consider the following semi-linear SFDE,

\[
dy(t) = \sum_{i=0}^d A_i y(t-\tau_i) + f(y(t), t)dt \sum_{i=1}^d g_i(y, t)dw_i(t),
\]

for \( i = 1, 2, \ldots, d \).

...
where $y_t = \{y(t+\theta) : \theta \in [-\tau, 0]\}, A_t \in \mathbb{R}^{n \times n}, \tau > 0, 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_d$ are delays, $f$ and $g_i$ satisfy the following linear growth condition.

**Assumption 4.1** For each positive definite matrix $P$ there exist matrices $D_{1P} \geq 0$ and $D_{2P} \geq 0$ such that for all $t \geq 0$ and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$,

$$f^T(\varphi, t)Pf(\varphi, t) = q^2 \int_{-\tau}^{0} \varphi^T(s)D_{1P}\varphi(s)d\mu(s),$$

$$\sum_{i=1}^{d} g_i^T(\varphi, t)P g_i(\varphi, t) = \int_{-\tau}^{0} \varphi^T(s)D_{2P}\varphi(s)d\mu(s),$$

where $q \geq 0, \mu(s)$ is a probability measure on $[-\tau, 0]$. The delay-independent and delay-dependent stability has been discussed in the important works [17–20] for linear systems. In this section, we consider the delay-dependent stability for semi-linear systems, where without loss of generality, we assume that $\sum_{i=0}^{d} A_i$ is Hurwitz if and only if $l = d$, which is consistent with Theorem 5.3 in [17] with $m_0 = m$.

**Theorem 4.4** Suppose that Assumption 4.1 holds. Let $\bar{A} = \sum_{i=0}^{d} A_i$. Then stochastic system (28) is mean square and almost surely exponentially stable if there exists a matrix $P > 0$ such that

$$\begin{align*}
S := \bar{A}^T P + P \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T P \bar{A} + (1 + q) \sum_{i=1}^{d} A_i^T P A_i \tau_i + q P + q(1 + \sum_{i=1}^{d} \tau_i) D_{1P} + D_{2P} & < 0. (29)
\end{align*}$$

**Proof** Let $P$ be fixed. Define

$$V(\varphi, t) = V_1(\varphi) + V_2(\varphi), \varphi \in C([-\tau_d, 0]; \mathbb{R}^n),$$

with $V_1(\varphi) = (\varphi(0) + \sum_{i=1}^{d} \int_{-\tau_i}^{0} A_i \varphi(s)ds)\|\bar{A}\|_{\infty}$ and $V_2(\varphi) = (1 + q) \sum_{i=1}^{d} \int_{-\tau_i}^{0} \varphi^T(s)A_i^T P A_i \varphi(s)ds$. It is easy to see that condition (14) holds. It can be deduced that $2V_1(\varphi, t)$ has the following form

$$2V_1(\varphi, t) = \varphi^T(0) [\bar{A}^T P + P \bar{A}] \varphi(0)$$

$$+ 2p^T(0) Pf(\varphi, t) + \sum_{i=1}^{d} g_i^T(\varphi, t) P g_i(\varphi, t)$$

$$+ 2 \sum_{i=1}^{d} \int_{-\tau_i}^{0} A_i \varphi(s)ds$$

$$+ 2 \sum_{i=1}^{d} \left( \int_{-\tau_i}^{0} A_i \varphi(s)ds \right)^T P f(\varphi, t).$$

By the elementary inequality $x^T y + y^T x \leq x^T Q x + y^T O^{-1} y$, for any positive definite matrix $O \in \mathbb{R}^{n \times n}, \varepsilon > 0$, and $x, y \in \mathbb{R}^n$, we get $2p^T(0) \bar{A}^T P A_i \int_{-\tau_i}^{0} \varphi(s)ds \leq \tau_i p^T(0) \bar{A}^T P A_i \varphi(0) + \int_{-\tau_i}^{0} p^T(\varphi(s)) A_i^T P A_i \varphi(ds) + 2p^T(0) P f(\varphi, t) \leq \tau_i p^T(0) P A_i \varphi(0) + q \int_{-\tau_i}^{0} p^T(\varphi(s)) A_i^T P A_i \varphi(ds) + 2q \int_{-\tau_i}^{0} p^T(\varphi(s)) A_i^T P A_i \varphi(ds) + q \tau_i \int_{-\tau_i}^{0} \varphi^T(s) D_{1P} \varphi(ds)$. Assume $q > 0$. Otherwise, the last two inequalities above vanish and the following estimations hold still. Then we get

$$2V_1(\varphi, t) \leq \varphi^T(0) S_1 \varphi(0) + \int_{-\tau}^{0} \varphi^T(s) D_{2P} \varphi(ds)$$

$$+ q(1 + \sum_{i=1}^{d} \tau_i) \int_{-\tau}^{0} \varphi^T(s) D_{1P} \varphi(ds)$$

$$+ (1 + q) \sum_{i=1}^{d} \int_{-\tau_i}^{0} \varphi^T(s) A_i^T P A_i \varphi(ds)$$

where $S_1 = \bar{A}^T P + P \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T P \bar{A} + q P$. By the definition of $V_2(\varphi)$, we have $2V_2(\varphi, t) = (1 + q) \sum_{i=1}^{d} \int_{-\tau_i}^{0} \varphi^T(\varphi(s)) A_i^T P A_i \varphi(\varphi(s))ds$. This together with the estimations of $2V_1(\varphi, t)$ gives

$$2V(\varphi, t) \leq \varphi^T(0) S_2 \varphi(0) + \int_{-\tau}^{0} \varphi^T(s) H P \varphi(ds),$$

where $S_2 = \bar{A}^T P + P \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T P \bar{A} + (1 + q) \sum_{i=1}^{d} \int_{-\tau_i}^{0} \varphi^T(\varphi(s)) A_i^T P A_i \varphi(\varphi(s))ds + q \int_{-\tau}^{0} \varphi^T(s) D_{1P} \varphi(ds) + D_{2P}$. Note that condition (29) implies $S = S_2 + H P < 0$. Then conditions (II) and (III’) hold with $c_4 = |H P|$, $p = 2$ and $\lambda = -\lambda_{\max}(S)$. By Theorems 4.2 and 4.3, we know that the trivial solution to (28) is mean square and almost surely exponentially stable. □

If the nonlinear drift term $f(\varphi, t) \neq 0 (q = 0)$, we have the following simple stability criterion.

**Corollary 4.5** Suppose that Assumption 4.1 hold with $q = 0$. Then stochastic system (28) is mean square and almost surely exponentially stable if there exists a positive definite matrix $P > 0$ such that

$$\bar{A}^T P + P \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T P A_i + \sum_{i=1}^{d} A_i^T P A_i \tau_i + D_{2P} < 0.$$  

(32)

**Remark 4.4** If the delays vanish ($\tau_i = 0, i = 1, \ldots, d$), then Corollary 4.5 is consistent with stability criterion of the delay-free linear SDE; see [31]. Note that conditions (29) or (32)) implies that matrix $\bar{A}$ is Hurwitz. When $\bar{A}^T + \bar{A}$ is still Hurwitz, we can choose $P = I_n$ in condition (29) or (32)) to obtain the exponential stability criterion. That is, if $\bar{A}^T + \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T A_i + (1 + q) \sum_{i=1}^{d} A_i^T A_i \tau_i + q I_d + q(1 + \sum_{i=1}^{d} \tau_i) D_{1P} + D_{2P} < 0$, or ($\bar{A}^T + \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^T A_i + \sum_{i=1}^{d} A_i^T A_i \tau_i + D_{2P} < 0$...
Theorem 4.4 and Corollary 4.5 improve the linear mean square asymptotic stability in the previous works [17–20] not only in considering the nonlinear terms, but also in that condition like \( \sum_{i=1}^{d} \| A_i \| \tau_i < 1 \) in [17–20] is not required and the mean square and almost sure exponential stabilities are obtained. Here, we take the linear case

\[
   dy(t) = [A_0 y(t) + A_1 y(t - \tau_1)] dt + C y(t) dw(t) \tag{33}
\]

for example and show that the condition \( \| A_1 \| \tau_1 < 1 \) required in [18,17] is unnecessary. Consider \( \tau_1 = 0.01, \)

\[
   A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -110 & -3.3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}.
\]

It is easy to see that \( \| A_1 \| \tau_1 > 1 \) for any norm \( \| \cdot \| \). But for \( P = \begin{pmatrix} 102.12 & 2.3 \\ 2.3 & 1 \end{pmatrix} \), one obtains \( \bar{A}^T P + P \bar{A} + \bar{A}^T P \bar{A} + C^T P C < 0 \). Hence, Corollary 4.5 tells us that the linear system (33) is still mean square and almost surely exponentially stable even if \( \| A_1 \| \tau_1 > 1 \).

### 4.3 Non-global Lipschitz case

We consider the following scalar SDDE with non-global Lipschitz coefficients,

\[
   dy(t) = [-\alpha y(t - \tau) - \beta y^3(t)] dt + \sigma_1 y^2(t - \bar{\tau}) dw_1(t) + \sigma_2 y(t - \bar{\tau}) dw_2(t), \tag{34}
\]

where \( \tau, \bar{\tau} \geq 0, 2\alpha > \sigma_2^2 \geq 0, 2 \beta > \sigma_2^2 \geq 0 \). The following theorem can be considered as delay-dominated stability criterion.

**Theorem 4.6** For any given \( \bar{\tau} > 0 \), the trivial solution to (34) is mean square and almost surely exponentially stable if

\[
   2\alpha^2 \bar{\tau} < 2\alpha^2 - \sigma_2^2, \quad 3\beta \bar{\tau}^2 \leq 2\beta - \sigma_2^2. \tag{35}
\]

**Proof** Choose the degenerate Lyapunov functional

\[
   V(\varphi) = |\varphi(0) - \bar{\alpha} \int_{-\bar{\tau}}^{0} \varphi(s) ds|^2. \tag{36}
\]

We obtain \( \mathcal{L} V(\varphi, t) = -2\bar{\alpha} |\varphi(0)|^2 - 2\beta |\varphi(0)|^4 + 2\sigma_2^2 \int_{-\bar{\tau}}^{0} \varphi(s) ds + 2\sigma_2^2 |\varphi(-\bar{\tau})|^2 + 2\bar{\alpha} \beta |\varphi(0)|^3 \int_{-\bar{\tau}}^{0} \varphi(s) ds + \sigma_2^2 |\varphi(-\bar{\tau})|^2 \). By the elementary inequality \( a^m \leq \frac{m^m}{m^n} a^{m-n} + \frac{n^m}{m^n} b^{m-n}, \ a, b \geq 0 \), and the Hölder inequality, we have \( 2\bar{\alpha} \beta |\varphi(0)|^3 \int_{-\bar{\tau}}^{0} \varphi(s) ds \leq \bar{\alpha}^2 \beta |\varphi(0)|^2 + \bar{\alpha}^2 \int_{-\bar{\tau}}^{0} |\varphi(s)|^2 ds \) and

\[
   \varphi(0)^3 \int_{-\bar{\tau}}^{0} \varphi(s) ds \leq \frac{3}{4} \bar{\alpha}^2 |\varphi(0)|^4 + \frac{1}{4\bar{\alpha}^2} \int_{-\bar{\tau}}^{0} \varphi(s) ds |\varphi(0)|^4.
\]

Hence, we get

\[
   \mathcal{L} V(\varphi, t) \leq -(2\bar{\alpha} - \bar{\alpha}^2) |\varphi(0)|^2 + \sigma_2^2 |\varphi(-\bar{\tau})|^2 + \bar{\alpha}^2 \int_{-\bar{\tau}}^{0} |\varphi(s)|^2 ds - (2\beta - \frac{3}{2} \sigma_2^2 \bar{\alpha}^2) |\varphi(0)|^4 + \frac{1}{2} \bar{\alpha} \beta \bar{\tau} \int_{-\bar{\tau}}^{0} |\varphi(s)|^4 ds + \sigma_2^4 |\varphi(-\bar{\tau})|^4. \tag{37}
\]

Let \( \mu_{\bar{\tau}}(\theta) = \frac{\sigma_2^2 d\bar{\tau} \delta_\theta}{2\sigma_2^2 + \sigma_2^2 + \sigma_2^2} d\mu(\theta) = \frac{1}{2} \delta_{1-\tau}(\theta) d\theta \), and \( \delta_{-\bar{\tau}}(\theta) = \frac{\sigma_2^2 d\bar{\tau} \delta_\theta}{2\sigma_2^2 + \sigma_2^2 + \sigma_2^2} d\mu(\theta) \), where \( \delta_\theta = \delta_{\bar{\tau}}(\theta) \) is Dirac measure on \([-\bar{\tau}, 0] \). Then, \( \mu_{\bar{\tau}}, \mu_{\theta} \) are the probability measures on \([-\bar{\tau}, 0] \). Therefore, we get

\[
   \mathcal{L} V(\varphi, t) \leq -r_1 |\varphi(0)|^2 + r_2 \int_{-\bar{\tau}}^{0} |\varphi(s)|^2 d\mu_{\bar{\tau}}(\theta) - r_3 |\varphi(0)|^4 + r_4 \int_{-\bar{\tau}}^{0} |\varphi(s)|^4 d\mu_{\bar{\tau}}(\theta). \tag{38}
\]

We take \( r_1 = 2\bar{\alpha} - \sigma_2^2 \bar{\tau}r_2 = \sigma_2^2 + \sigma_2^2 \bar{\tau}, r_3 = 2\bar{\alpha} - \sigma_2^2, r_4 = \sigma_2^2 + \sigma_2^2 \bar{\tau} \). Note that condition (35) implies \( r_1 > r_2, r_3 > r_4 \). Then, by Theorems 4.1 and 4.3, we obtain the desired assertions.

**Remark 4.5** The degenerate Lyapunov functional (30) and (36) also satisfy condition (1). We take (36) as example. In fact, by the inequality \(|x - y|^2 \leq (1 + \varepsilon) |x|^2 + \frac{1}{\varepsilon} |y|^2 \), \( \varepsilon > 0 \), we have

\[
   |\varphi(0)|^2 \leq (1 + \varepsilon) \left( |\varphi(0) - \bar{\alpha} \int_{-\bar{\tau}}^{0} \varphi(s) ds|^2 + \frac{1}{\varepsilon} |\bar{\alpha} \int_{-\bar{\tau}}^{0} \varphi(s) ds|^2 \right) \leq (1 + \varepsilon) |\varphi(0)|^2 + \frac{1}{\varepsilon} |\bar{\alpha} \int_{-\bar{\tau}}^{0} \varphi(s) ds|^2 \leq (1 + \varepsilon) V(\varphi) + \frac{1}{\varepsilon} (\bar{\alpha} |\varphi|^2)^2 \int_{-\bar{\tau}}^{0} |\varphi(s)|^2 ds. \tag{39}
\]

Let \( c_0 = \frac{1}{1 + \varepsilon}, c_1 = \frac{1}{\varepsilon} \bar{\alpha} |\varphi|^2, \) and \( d\mu_1(\theta) = \frac{d\mu}{\bar{\tau}} \) is a probability measure on \([-\bar{\tau}, 0] \). Note that \( \bar{\alpha}^2 \bar{\tau}^2 \) is a condition (35)), then there must exist a \( \varepsilon > 0 \) such that \( \mu_0 > c_1 \). That is, condition (1) holds. Similarly, we can prove (30) to satisfy condition (1).

**Remark 4.6** Basin and Rodkina [3] applied a degenerate Lyapunov functional to study the almost sure asymptotic stability of a scalar SDDEs with linear growth coefficients. Using our methods, we can show that not only the scalar SDDEs in [3] is almost surely asymptotically stable, but also mean square asymptotically stable.

### 5 Applications to Multi-agent Systems with Nonlinear Dynamics and Noises

In this subsection, we apply the main findings of this paper to practical multi-agent systems with distribut
ed random delays and establish convergence and consentability of such systems that are difficult to treat when systems contain multiple or distributed delays.

Wireless communication systems are part of networked systems and introduce significant impact on control design and system performance. In [44], coordinated control and communication design was studied in which TCP-based (Transmission Control Protocol) communication protocols were employed. The main consequence of the TCP channel uncertainties is random signal transmission delays which have profound, often detrimental, effects on team mobile systems, such as highway autonomous platoons, team UAVs (unmanned aerial vehicles), surveillance robots, among many others. Inter-vehicle communications use wireless networks and are subject to terrain obstructions, signal fading, signal interference from other vehicles, etc. Communication latency is further caused by delays in each hub’s queues, retransmission, scheduling policies in interference avoidance strategies, resulting in combined latency of several hundreds of milliseconds even several seconds. When the TCP protocol is used in data communications, packet-carrying capacity of a path between two vehicles will be limited by its bandwidth-delay product, leading to random, time varying, unpredictable, and distributed delays in different channels. Such scenarios have been well studied under the IEEE 802.11 standards with experimental latency data [45].

In our recent paper [46], a weighted and constrained consensus control method was introduced to achieve platoon formation and robustness against communication channel uncertainties. However, distributed random delays could not be treated due to lack of rigorous analysis and design methods for such uncertainties. Applying the new results of this paper, convergence to a desired platoon formation can now be achieved under more realistic communication latency conditions.

To proceed, we study consentability of nonlinear multi-agent systems consisting of $N$ agents by using the developed stability theory. Each agent is modeled by a dynamic system

$$\dot{y}_i(t) = f(y_i(t), y_i(t - \tau), t) + u_i(t), \quad (40)$$

where $y_i(t) \in \mathbb{R}^n$ is the state of the $i$th agent, $\tau$ is a delay, $u = [u_1, \ldots, u_N]^T$ is the control input, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a nonlinear function, which may have certain uncertainty satisfying the following assumption.

**Assumption 5.1** $f(0, 0, t) = 0$ and there exists a positive constant $q$ (Lipschitz constant) such that for all $x, y, \bar{x}, \bar{y}, |f(x, y, t) - f(\bar{x}, \bar{y}, t)| \leq q(|x - \bar{x}| + |y - \bar{y}|)$.

The information flow structures among different agents are modeled as a directed graph $\mathcal{G} = [\mathcal{V}, \mathcal{E}, \mathcal{A}]$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes with $i$ representing the $i$th agent, $\mathcal{E}$ denotes the set of directed edges and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent $j$ to agent $i$ directly. Also, $N_i$ denotes the set of the node $i$’s neighbors, that is, for $j \in N_i$, $a_{ij} = 1$, and $\deg_i = \sum_{j=1}^{N} a_{ij}$ is called the degree of $i$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}(\deg_1, \ldots, \deg_N)$. It is obvious that $\mathcal{L}$ admits a zero eigenvalue, denoted by $\lambda_1$. If the digraph $\mathcal{G}$ is balanced, then $\tilde{\mathcal{L}} = \frac{1}{\sqrt{2}} \mathcal{L}$ denotes the Laplacian matrix of the mirror digraph $\tilde{\mathcal{G}}$ of $\mathcal{G}$ [29]. Consider the control

$$u_i(t) = K \sum_{j \in N_i} z_{ji}(t), \quad (41)$$

where the control gain $K \in \mathbb{R}^{n \times n}$ is to be designed to be symmetric, $z_{ji}(t) = y_j(t - \tau_1) - y_i(t - \tau_1) + g_{ji}(y_j(t - \tau_2) - y_i(t - \tau_2))$, $\xi_{ji}(t)$ denotes the measurement of the agent $i$ from its neighbor agent $j$, $\tau_1$ and $\tau_2$ are delays, $\xi_{ji}(t)$ is the scalar independent Gaussian white noise, and $g_{ji} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the noise intensity function. Here, we make the following assumptions.

**Assumption 5.2** The noise processes $\{\xi_{ji}(t) \in \mathbb{R}, i, j = 1, 2, \ldots, N\}$ satisfy $\int_0^t \xi_{ji}(s)ds = w_{ji}(t)$ for $t \geq 0$, where $\{w_{ji}(t), i, j = 1, 2, \ldots, N\}$ are independent Brownian motions.

**Assumption 5.3** For each $(j, i)$, $g_{ji}(0) = 0$ and there exists a positive constant $g$ such that $g_{ji}(x) - g_{ji}(y) \leq g|x - y|$, for all $x, y \in \mathbb{R}^n$.

For the motivation of the measurement $z_{ji}(t)$ above, one can refer to [41,42]. We denote the collection of all admissible distributed protocols by

$$\mathcal{U} = \{u(t)|u_i(t) = K \sum_{j \in N_i} z_{ji}(t), t \geq 0, \quad \text{for all symmetric } K, \ i = 1, \ldots, N\}. \quad (42)$$

**Definition 5.1** We say that the multi-agent systems (40) are mean square (or almost surely) consensurable with respect to (w.r.t.) $\mathcal{U}$, if there exists a protocol $u \in \mathcal{U}$ solving the mean square (or almost sure) consensus, that is, for any initial data $\varphi \in C(\{-\tau \vee \tau_1 \vee \tau_2\}, 0; \mathbb{R}^N)$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t \rightarrow \infty} \mathbb{E}[y_i(t) - y_j(t)]^2 = 0$ or $\lim_{t \rightarrow \infty} \mathbb{E}[y_i(t) - y_j(t)] = 0$, a.s.

**Lemma 5.1** For the Laplacian matrix $\mathcal{L}$ related to a balanced graph $\mathcal{G}$, there exists a matrix $\mathcal{Q} \in \mathbb{R}^{N \times (N - 1)}$ such that the matrix $\mathcal{Q} = \left(\frac{1}{\sqrt{N}} \mathcal{L}\right)^T$ is orthogonal and

$$\mathcal{Q}^{-1} = \left[\frac{1}{\sqrt{N}} \mathcal{L}\right]^T, \quad \mathcal{Q}^{-1} \mathcal{L} \mathcal{Q} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{E} \end{bmatrix}. \quad (43)$$

The proof of lemma can be found in [43]. We proceed to obtain the following result.

**Theorem 5.2** Suppose that Assumptions 5.1, 5.2, and 5.3 hold and the graph $\mathcal{G}$ is balanced. If

$$4q[\lambda_2(\mathcal{L})N - N^{-1} \sigma^2 + (1+q)|\mathcal{L}|^2(t_0 + \tau_1)(2+\tau_2)] < \lambda_2^2(\mathcal{L}), \quad (44)$$

then multi-agent systems are mean square and almost
sure consistent w.r.t. \( U \). Moreover, the mean square and almost sure consensus can be solved by the protocol (41) with \( K = kI_n \) satisfying \( k \in (k_\ast, \bar{k}) \), \( \bar{k} = \frac{\lambda_2(\hat{\mathcal{L}}) - \sqrt{\lambda_2(\hat{\mathcal{L}})^2 - 4n(2n+1)}}{2n} \), where \( a = \lambda_2(\hat{\mathcal{L}})N_n\frac{1}{2} + (1 + q)|\mathcal{L}|^2 \gamma_1 \).

**Proof** It is enough to prove that the protocol with \( k \) satisfying \( k \in (k_\ast, \bar{k}) \) can solve the mean square and almost sure consensus. Note that \( k \) and \( \bar{k} \) are well-defined since condition (44) holds. Let \( y(t) = [y_1^T(t), \ldots, y_n^T(t)]^T \), \( F(t) = [f_1^T(y(t), y(t - \tau), t), \ldots, f_n^T(y_n(t), y_n(t - \tau), t)]^T \), \( \delta(t) = [(I_N - J_N) \otimes I_n]y(t) \), and \( \delta(t) = [\delta_1^T(t), \ldots, \delta_n^T(t)]^T \), where \( J_N = \frac{1}{\sqrt{n}}1_N1_N^T \). Substituting the protocol into the system (40) and using Assumption 5.2 yields

\[
dy(t) = [k(\mathcal{L} \otimes I_n)y(t - \tau_1) + F(t)] dt + dM(t),
\]

where \( M(t) = k \sum_{i,j=1}^N a_{ij} f_0^j_0 [\eta_Nn_i \otimes (g_{ji}(\delta(s - \tau) - \delta(s - \tau_2))] \tau \eta_{ji}(s) \tau \eta_{ji}(s) \] \( y(t) = \frac{1}{N} \sum_{j=1}^N y_j(t) \), then \( \delta(t) = y(t) - \tilde{y}(t) \). Note that

\[
f_j(t) = f_j(y(t), y(t - \tau), t) - f_j(y(t), \tilde{y}(t), t) + f_j(y(t), \tilde{y}(t), t) - f_j(y(t), \tilde{y}(t), t) + \frac{1}{N} \sum_{j=1}^N \left( (\tilde{y}(t), \tilde{y}(t) - t) - (y(t), y(t - \tau), t) \right)
= p_{11}(t) + p_{22}(t) + p_{03}(t).
\]

Define \( \delta(t) = (Q^{-1} \otimes I_n)\delta(t) = [\tilde{\delta}_1^T(t), \ldots, \tilde{\delta}_n^T(t)]^T \), \( \tilde{\delta}(t) = [\tilde{\delta}_1^T(t), \ldots, \tilde{\delta}_n^T(t)]^T \), \( \tilde{\delta}(t) \in \mathbb{R}^n \). Then, by the definition of \( Q^{-1} \) given in Lemma 5.1, we have \( \tilde{\delta}_1(t) = \frac{1}{\sqrt{n}}1_N(1_N \otimes I_n)\delta(t) = \frac{1}{\sqrt{n}}1_N(1_N - J_N) \otimes I_n)\delta(t) = 0 \) and

\[
d\tilde{\delta}(t) = (Q^{T} \otimes I_n)\tilde{F}(t) dt - k(\tilde{\mathcal{L}} \otimes I_n)\tilde{\delta}(t) dt + dM(t) \}
\]

where \( M_3(t) = k \sum_{i,j=1}^N a_{ij} g_{ji}(t) (Q^{T}(I_N - J_N) \otimes I_n)\tilde{\delta}(t) dt \) \( g_{ji}(s) \) \( \tilde{\delta}(t) = (Q^{T} \otimes I_n)\tilde{F}(t) dt - k(\tilde{\mathcal{L}} \otimes I_n)\tilde{\delta}(t) dt + dM(t) \}
\]

where we use the definitions of \( \tilde{\delta}(t) \) and \( \delta^T(t)[(\mathcal{L}^T + \mathcal{L}) \otimes I_n] \delta(t) = \frac{k}{N} \sum_{i,j=1}^N a_{ij} (\tilde{\delta}_j(t) - \delta(t))^2 \), \( \Phi(t) = \tilde{\delta}^T(t)[(\tilde{\mathcal{L}}^T + \tilde{\mathcal{L}}) \otimes I_n] \tilde{\delta}(t) \). Firstly, we choose the degenerate Lyapunov functional \( V_1(\varphi) = \|\varphi - k(\tilde{\mathcal{L}} \otimes I_n)\int_{t-\tau_1}^t \varphi(s) ds \|^2 \). We can compute \( \mathcal{L}V_1(\cdot, t) \) on the solution path \( \tilde{\delta}(t) \) and obtain

\[
2V_1(\tilde{\delta}, t) = 2\left( \tilde{\mathcal{L}}(t) - k(\tilde{\mathcal{L}} \otimes I_n) \right) \int_{t-\tau_1}^t \tilde{\delta}(s) ds \right)^T \left[ -k(\tilde{\mathcal{L}} \otimes I_n) \tilde{\delta}(t) + (\tilde{\mathcal{L}} \otimes I_n) \tilde{\delta}(t) \right]
+ k^2 \sum_{i,j=1}^N a_{ij} \|\tilde{\delta}_j(t) - \delta(t)\|^2

\leq -k\Phi(t) + k^2 \sum_{i,j=1}^N a_{ij} \|\tilde{\delta}_j(t) - \delta(t)\|^2
-2k \tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \int_{t-\tau_1}^t \tilde{\delta}(s) ds \right)^T (\tilde{\mathcal{L}}^T \otimes I_n) F(t)
+ 2k \tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \int_{t-\tau_1}^t \tilde{\delta}(s) ds
+ 2\tilde{\mathcal{L}} F(t)
+ 2\tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n F(t) \right.
\]

By the inequality \( 2tx \leq \frac{1}{2}x^2 + 2x^2 \), we have

\[
2\tilde{\mathcal{L}} F(t) \leq \frac{1}{2}\|\tilde{\delta}(t)\|^2 + 2\|\tilde{\delta}(t)\|^2 \right)
\leq \|\tilde{\mathcal{L}}\|^2 \|\tilde{\delta}(t)\|^2 + \int_{t-\tau_1}^t \|\tilde{\delta}(s)\|^2 ds \right)
\]

By Assumption 5.1, we obtain \( 2\tilde{\mathcal{L}} F(t) \leq q(\|\tilde{\delta}(t)\|^2 + \|\tilde{\delta}_j(t) - \delta(t)\|^2) \). From the definitions of \( \delta(t) \) and \( \tilde{\delta}(t) \) and the orthogonality of \( Q \), we know that \( \|\tilde{\delta}(t)\|^2 = \|\tilde{\delta}(t)\|^2 \) and \( \sum_{i=1}^N \|\tilde{\delta}_i(t)\|^2 \leq 0 \). Hence,

\[
2\tilde{\mathcal{L}} F(t) \leq 2\tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \tilde{F}(t)
= 2\tilde{\mathcal{L}} \tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \tilde{F}(t)
= 2\sum_{i=1}^N \left( \delta_i^T(t) p_{ii}(t) + \tilde{\delta}_i^T(t) p_{ii}(t) \right)
\leq 3q\|\tilde{\delta}(t)\|^2 + q\|\tilde{\delta}(t)\|^2 \right)
\]

Note that \( \tilde{\mathcal{L}} = \tilde{\mathcal{L}} \tilde{\mathcal{L}} \tilde{\mathcal{L}} \tilde{\mathcal{L}} = \mathcal{L} = \mathcal{L} \otimes I_n \), \( \tilde{\mathcal{L}} = \mathcal{L} \otimes I_n \), \( \tilde{\mathcal{L}} \otimes I_n \mathcal{L} \otimes I_n = \mathcal{L} = \mathcal{L} \otimes I_n \), \( \mathcal{L} \) \( \mathcal{L} \otimes I_n \), \( \tilde{\mathcal{L}} \otimes I_n \) \( \mathcal{L} \otimes I_n \) \( \mathcal{L} \otimes I_n \) \( \mathcal{L} \otimes I_n \). Then,

\[
2k \left( \mathcal{L} \mathcal{L} \otimes I_n \right) \int_{t-\tau_1}^t \tilde{\delta}(s) ds \right)^T \left( \tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \tilde{F}(t)
= 2k \left( \mathcal{L} \mathcal{L} \otimes I_n \right) \int_{t-\tau_1}^t \tilde{\delta}(s) ds \right)^T \left( \mathcal{L} \mathcal{L} \tilde{\mathcal{L}} \tilde{\mathcal{L}} \otimes I_n \tilde{F}(t)
= 2k \left( \mathcal{L} \mathcal{L} \otimes I_n \right) \int_{t-\tau_1}^t \tilde{\delta}(s) ds \right)^T \left( \mathcal{L} \mathcal{L} \otimes I_n \tilde{F}(t)
\]
Suppose that Assumptions 5.1-5.3 hold, that the graph $\mathcal{G}$ is balanced, that $t_1 = 0$, and that
\[ 8q \frac{N - 1}{N} \sigma^2 < \lambda_2(\tilde{L}). \tag{55} \]
Then multi-agent systems are mean square and almost surely converge w.r.t. $U$. Moreover, the mean square and almost sure consensus can be solved by the protocol (41) with $K = kI_n$ satisfying $k \in [k, K]$.

**Remark 5.1** Theorems 5.2 and 5.3 show that if condition (44) or condition (55) holds, then for any delay $\tau$ in the nonlinear dynamics and delay $\tau_2$ in noise terms, the mean square and almost sure consensus can be achieved.

### 6 Conclusion Remarks

This work established the stochastic stability of SFDEs by using degenerate Lyapunov functionals under local Lipschitz conditions. The analysis of delay-dependent stability shows that pure delay terms in SFDEs may dominate the asymptotic behavior of SFDEs. Most importantly, these stability criteria enable us to design delayed state-feedback controllers for networked systems. Note that the degenerate Lyapunov functionals may also be used to examine the ergodicity of SFDEs whose coefficients may not satisfy the classical one-sided Lipschitz condition in $[2]$, and thus, pure delay terms may dominate the existence of stationary distributions. For another extension, the degenerate Lyapunov functionals can also be applied to examine SFDEs with Poisson jumps, neutral SFDEs, and their stabilization problems. This work has studied the consensusability of first-order multi-agent systems. We hope that the stability theorems can be applied to the heterogeneous multi-agent systems, and their leader-following versions.

**Acknowledgements.** This work was supported in part by the National Natural Science Foundation of China under grants 61522310 and 61703378, Shanghai Rising-Star Program under grant 15QA1402000, the National Key Basic Research Program of China (973 Program) under Grant No. 2014CB845301, the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) (No. CUG170610), and the U.S. Army Research Office under W911NF-15-1-0218.

### A Appendix: Proof of Theorem 2.1

To prove Theorem 2.1, we need to introduce some notation. Let the interval $[-\tau, t]$ be fixed. Without loss of generality, we use the uniform partition of $[-\tau, 0]$: $\{t_{m-1}, \ldots, t_0\}$ with $t_m = \tau$, $k = m, \ldots, -1$, $\Delta > 0$. Moreover, we assume that there exists an $N$ such that $t = N\Delta$. Then we have the uniform partition of $[0, t]$: $\{t_0, \ldots, t_N\}$. Motivated by [8], we provide the outline of the proof. For any continuous semi-martingale $y(t)$, $y(t)$ is uniformly continuous on $[-\tau, t]$. Then $\lim_{\Delta \to 0}$ as $\Delta$ tends to zero, where $\lim_{\Delta \to 0} = \sup_{m} \{||y(u) - y(t_k)||, u \in [t_k, t_{k+1}], k = m, \ldots, N\}$. Define $\gamma_k(\theta) = \sum_{j=m}^{k} y(t_{j+1})(\theta) + y(t_k)(\theta)$, $k = 0, \ldots, N$, which is an approximation for function $y(t_k)$. Consider the decomposition
\[
V(y_{k+1}, t_{k+1}) - V(y_k, t_k) = V(y_{k+1}^{\tilde{y}_{k+1}}, t_{k+1}) - V(y_{k+1}, t_{k+1}) + V(y_{k+1}, t_{k+1}) - V(y_k, t_k).
\tag{A.1}
\]
Let $h_k = y(t_{k+1}) - y(t_k)$ and $P(h) = V\left(\dot{y}_{k+1}^h, t_{k+1}\right)$. Then, $P(h_k) - P(0) = V\left(\dot{y}_{k+1}^h, t_{k+1}\right) - V\left(\dot{y}_{k}^0, t_{k+1}\right)$. Since $V \in C^2(\mathbb{D}([-\tau, 0] ; \mathbb{R}^n) \times \mathbb{R}_+^n; \mathbb{R})$, $P$ is a $C^2$ function with $P(h) = \nabla_x V\left(\dot{y}_{k+1}^h, t_{k+1}\right)$ and $P''(h) = \nabla_{xx} V\left(\dot{y}_{k+1}^h, t_{k+1}\right)$. Note that $\dot{y}_{k+1}^h$ is measurable. Applying the Itô formula yields that for $\delta < \Delta$,

\[
P(y(t_k + \delta) - y(t_k)) = P(0) + \int_{t_k}^{t_k + \delta} \nabla_x V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(u) + \frac{1}{2} \int_{t_k}^{t_k + \delta} \text{Tr}\left(\nabla_{xx} V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(u)\right). \quad (A.2)
\]

Then $\dot{y}_{k+1} = \Delta \dot{y}_k$, which is a horizontal extension. So the second term in the right side of (A.1) has the form

\[
Q(\delta) - Q(0) = V(\delta \dot{y}_k, t_k + \delta) - V(\dot{y}_k, t_k), \quad 0 < \delta \leq \Delta,
\]

where $Q(\delta) = V(\delta \dot{y}_k, t_k + \delta) + \Delta$. Note that $V \in C^2(\mathbb{D}([-\tau, 0] ; \mathbb{R}^n) \times \mathbb{R}_+^n; \mathbb{R})$. Then, $V(\dot{y}_{k+1} - \dot{y}_k, t_k) = \int_{t_k}^{t_{k+1}} DV(\dot{y}_k, s)ds$. Hence, taking sum form $i = 0$ to $i = N - 1$ in (A.1) yields

\[
V(\dot{y}_i, t) - V(\dot{y}_0, 0) = \sum_{k=0}^{N-1} DV(\dot{y}_k, s)ds + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \nabla_x V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(s)
\]

\[
+ \frac{1}{2} \int_{0}^{\Delta} \text{Tr}\left(\nabla_{xx} V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(s)\right)ds + \int_{0}^{t_{k+1}} \frac{1}{2} \text{Tr}\left(\nabla_{xx} V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(s)\right)ds + \int_{0}^{t_{k+1}} \frac{1}{2} \text{Tr}\left(\nabla_{xx} V\left(\dot{y}_{k+1}^h - y(t_k), t_{k+1}\right)dy(s)\right)ds,
\]

where $\delta = t_{k+1} \dot{y}_k(t_{k+1})\left[\dot{y}_k(t_{k+1})\right]$, $\delta = t_{k+1} \dot{y}_k(t_{k+1})\left(\dot{y}_k(t_{k+1})\right)$, $k = 1, \ldots, N - 1$. It is easy to see that for any given $t > 0$, $\dot{y}_k \to y$ uniformly on $[\tau, 0]$ as $\Delta \to 0$. This together with the continuity of $\dot{y}_k$ yields $dP(\dot{y}_k, y) \to 0$ as $\Delta \to 0$ [21, p.31]. The approach in [8] leads to the desired assertion as $\delta \to 0$.

References


