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CONSENSUS CONDITIONS OF CONTINUOUS-TIME MULTI-AGENT SYSTEMS WITH ADDITIVE AND MULTIPLICATIVE MEASUREMENT NOISES*

XIAOFENG ZONG†, TAO LI‡, AND JI-FENG ZHANG§

Abstract. This work is concerned with the consensus problem of multi-agent systems with additive and multiplicative measurement noises. By developing general stochastic stability lemmas for nonautonomous stochastic differential equations, stochastic weak and strong consensus conditions are investigated under fixed and time-varying topologies, respectively. For the case with fixed topologies and additive noises, the necessary and sufficient conditions for almost sure strong consensus are given. It is revealed that almost sure strong consensus and mean square strong consensus are equivalent under general digraphs, and almost sure weak consensus implies mean square weak consensus under undirected graphs; if multiplicative noises appear, then small noise intensities do not affect the control gain to guarantee stochastic strong consensus. For the case with time-varying topologies, sufficient consensus conditions are given under the periodically connected condition of the topology flow.

Key words. multi-agent system, additive noise, multiplicative noise, mean square consensus, almost sure consensus

AMS subject classifications. 93E03, 93E15, 60H10, 94C15

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1. Introduction.

Consensus is a typical collective behavior and has a wide range of applications that involve coordination of multiple entities with only limited neighborhood information to reach a global goal for the entire team, such as multi-agent systems in robotics [15], flocking behavior and swarms [26, 29], and sensor networks [1, 17, 33]. Related algorithms and theoretical developments were reported in [6, 36, 37, 39, 40, 46]. Since real networks are often in uncertain environments, and each agent cannot measure its neighbors' states accurately, multi-agent systems subject to the phenomenon of stochasticity have been a topic of increasing investigation in recent years (see [2, 7, 18, 19, 25, 32, 47] and the references therein). The stochasticity is often modeled to be additive or multiplicative noises, and then the overall system becomes a stochastic system. For such stochastic systems, the convergence analysis shows more complexity and involves various convergence concepts

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†School of Automation, China University of Geosciences, Wuhan 430074, China, and Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, Wuhan 430074, China (xfzong87816@gmail.com).

‡Corresponding author. Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, Department of Mathematics, East China Normal University, Shanghai 200241, China (tli@math.ecnu.edu.cn).

§Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (jif@iss.ac.cn).

(see [28, 48]). The mean square convergence and the almost sure convergence are two very important convergence concepts and refer to mean square consensus and almost sure consensus, respectively, in the stochastic consensus problem of multi-agent systems. To date, much literature has contributed to the stochastic consensus problem with additive measurement noises, where the intensity of noise is independent of agents' states. For discrete-time models, the distributed stochastic approximation method was introduced to attenuate the impact of communication/measurement noises, and conditions were given for stochastic consensus. For the case with independent channel noises, Huang and Manton [13] investigated the decreasing control gains for mean square and almost sure consensus under fixed topologies. For the case with martingale difference–type noises, Li and Zhang [22] gave the necessary conditions and sufficient conditions for mean square and almost sure consensus under both fixed and time-varying balanced topologies, respectively. This was extended to the case with more general observation noises in Fang, Chen, and Wen [10]. Huang and colleagues [12, 14] considered almost sure and mean square consensus under randomly switching digraphs. Xu, Zhang, and Xie [45] examined the almost sure convergence rates for stochastic approximation methods under fixed topologies. Aysal and Barner [5] considered the general discrete-time consensus models and gave sufficient conditions for almost sure consensus. For continuous-time models, Li and Zhang [21] gave the necessary and sufficient conditions on the control gain to ensure mean square strong consensus under balanced digraphs. For the case with linear dynamics with absolute state feedback, Cheng, Hou, and Tan [9] studied mean square strong consensus conditions. Wang and Zhang [42] investigated the sufficient conditions for almost sure strong consensus. Tang and Li [41] gave the relationship between mean square and almost sure convergence rates of the consensus error and a representative class of consensus gains. Recently, some researchers have also paid attention to the case with multiplicative noises, where the intensity of noise depends on agents' states. Ni and Li [31] investigated the consensus problems of the continuous-time systems with multiplicative noises, where the noise intensities are proportional to the absolute value of the relative states of agents and their neighbors. Then this work was extended to the discrete-time version in [27]. Li, Wu, and Zhang [19, 20] studied the distributed consensus with the general multiplicative noises and developed some small consensus gain theorems to give sufficient conditions for mean square and almost sure consensus under undirected topologies.

Most of the above works deal with the consensus problem with additive and multiplicative measurement noises separately. When the two types of noises coexist, the continuous-time consensus problem has not been considered before. In fact, even for the case with only additive noises, there is no unified result under general digraphs, and some basic problems still remain open, for example, the necessary and sufficient conditions for almost sure strong consensus, the relationship between mean square and almost sure strong consensus, and the necessary and sufficient conditions for mean square and almost sure weak consensus. Moreover, for the case with time-varying topologies, little is known about the consensus conditions if the digraph is not strongly connected all the time.

In reference to the existing literature, this paper furthers our recent quest [19, 21, 41] on continuous-time stochastic consensus by developing unified tools under directed networks and different types of noises. Based on the matrix theory and the algebraic graph theory, by utilizing the variable transformation, the closed-loop system is transformed into a nonautonomous stochastic differential equation (SDE) driven by additive or compound noises. There is no existing result to deal with the stochastic stability of such SDEs. To this end, we first develop some useful stability criteria, which involve sufficient conditions and necessary conditions, for the stability of nonautonomous SDEs with additive noises. The criteria show a powerful ability in examining the consensus conditions. Then similar analysis tools are developed for the cases with compound noises and time-varying topologies. The contribution and findings of this work are summarized as follows.

(a) Networks with fixed topologies.

 (i) Additive noises case: (i-1) Stochastic stability is established for the nonautonomous SDEs with additive noises. By the matrix theorem and the semidecoupled methods, we develop the necessary conditions and sufficient conditions for the mean square asymptotic stability. By the semimartingale convergence theorem, the law of the iterated logarithm for martingales, and the variation of constants formula, we establish the necessary conditions and sufficient conditions for the almost sure asymptotic stability. (i-2) By the conditions for the mean square asymptotic stability, we prove that the sufficient conditions on control gain $c(t)$ to guarantee mean square weak consensus are $\int_0^\infty c(t)dt = \infty$ and $\lim_{t\to\infty} \int_0^t e^{-2\underline{\lambda} \int_s^t c(u)du} c^2(s)ds = 0$, and that the necessary conditions are $\int_0^\infty c(t)dt = \infty$ and $\lim_{t\to\infty} \int_0^t e^{-2\bar{\lambda} \int_s^t c(u)du} c^2(s)ds = 0$, where $\underline{\lambda}$ and $\bar{\lambda}$ are the minimum and maximum real parts of the eigenvalues of the Laplacian matrix, respectively. (i-3) By the conditions for the almost sure asymptotic stability, we get that $\int_0^\infty c(t)dt = \infty$ and $\int_0^\infty c^2(s)ds < \infty$ are the necessary and sufficient conditions for almost sure strong consensus, and we show that almost sure weak consensus can be achieved if $\int_0^\infty c(t)ds = \infty$ and $\lim_{t\to\infty} c(t) \log \int_0^t c(s)\ ds = 0$, and only if $\int_0^\infty c(t)ds = \infty$ and $\liminf_{t\to\infty} c(t) \log \int_0^t c(s)ds = 0$ for the case with undirected graphs. We also reveal that almost sure consensus and mean square strong consensus are equivalent, and for the case with undirected graphs, almost sure weak consensus implies mean square weak consensus.

 (ii) Compound noises case: We develop necessary conditions and sufficient conditions for mean square weak and strong consensus, and sufficient conditions for almost sure strong consensus. It is revealed that multiplicative noises with small noise intensities do not affect mean square and almost sure strong consensus.

(b) Networks with time-varying balanced topologies and the frequent connectivity condition.

 (i) Additive noises case: We introduce some sufficient conditions on control gain $c(t)$ for guaranteeing mean square weak consensus, and we show that mean square and almost sure strong consensus can be achieved if $\sum_{j=0}^\infty \int_{\tau_{i_j}}^{\tau_{i_j+1}} c(u)du = \infty$ and $\int_0^\infty c^2(s)ds < \infty$, where $\{\tau_i\}_{i=0}^\infty$ is the sequence of switching time instants of the graph flow, and $\{\tau_{i_k}\}_{k=0}^\infty \subseteq \{\tau_i\}_{i=0}^\infty$ are the time instants when the digraphs are strongly connected.

 (ii) Compound noises case: We obtain sufficient conditions on the control gain for mean square weak consensus and show that $\sum_{j=0}^\infty \int_{\tau_{i_j}}^{\tau_{i_j+1}} c(u)du = \infty$ and $\int_0^\infty c^2(s)ds < \infty$ can guarantee mean square and almost sure strong consensus if the upper bound $\bar{\sigma}$ of multiplicative noise intensities is so small that $\bar{\sigma}^2 < \frac{N}{(N-1)\sup_{t\geq 0} c(t)}$, where N is the number of agents.

The rest of the paper is organized as follows. First, the networked system and consensus problem are introduced in section 2. Section 3 presents a stochastic consensus theorem for multi-agent systems with fixed topologies, containing two subsections for the cases with additive noises and compound noises, respectively. Section 4 extends our study to the case with time-varying topologies. Section 5 introduces the simulations to demonstrate the theoretical analysis. Section 6 concludes the paper.

Notation. Throughout this paper, unless otherwise specified, we use the following notation. For any complex number λ, $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote its real part and imaginary part, respectively. $\mathbf{1}_n$ denotes the n-dimensional column vector with all ones. $\eta_{N,i}$ denotes the N-dimensional column vector with the ith element being 1 and others being zero. I_N denotes the N-dimensional identity matrix. For a given matrix or vector A, its transpose is denoted by A^T, and its Euclidean norm is denoted by $\|A\|$. The eigenvalues of the matrix $A \in \mathbb{R}^{N\times N}$ are denoted by $\{\lambda_1(A), \dots, \lambda_N(A)\}$. If $A \in \mathbb{R}^{N\times N}$ is a real symmetric matrix, its eigenvalues are arranged as $\lambda_{\min}(A) = \lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_N(A) = \lambda_{\max}(A)$. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. Let $(\mathbf{\Omega}, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions; namely, it is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For a given random variable or vector X, its mathematical expectation is denoted by $\mathbb{E}X$. For a continuous martingale $M(t)$, its quadratic variation is denoted by $\langle M\rangle(t)$ (see [38]).

2. Problem formulation. We consider the consensus control for a network of agents with the dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, N,\ t \ge 0, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ denote the state and the control input of the ith agent, respectively. Here, each agent has n control channels, and each component of $x_i(t)$ is controlled by a control channel. Denote $x(t) = [x_1^T(t), \dots, x_N^T(t)]^T$ and $u(t) = [u_1^T(t), \dots, u_N^T(t)]^T$. The information flow structures among different agents are modeled as a directed graph (digraph) $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)\}$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}(t)$ denotes the set of directed edges, and $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}(t)$ with element $a_{ij}(t) = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly at time t. Also, $N_i(\mathcal{G}(t))$ denotes the set of the node i's neighbors; that is, for $j \in N_i(\mathcal{G}(t))$, $a_{ij}(t) = 1$. And $\deg_i(t) = \sum_{j=1}^N a_{ij}(t)$ is called the degree of i at time t. The Laplacian matrix of $\mathcal{G}(t)$ is defined as $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where $\mathcal{D}(t) = \mathrm{diag}(\deg_1(t), \dots, \deg_N(t))$. If $\mathcal{G}(t)$ is balanced, then $\widehat{\mathcal{L}}(t) = \frac{\mathcal{L}^T(t)+\mathcal{L}(t)}{2}$ is the Laplacian matrix of the mirror graph $\widehat{\mathcal{G}}(t)$ of $\mathcal{G}(t)$ [35].

In real multi-agent networks, for each agent, the information from its neighbors may have different types of communication/measurement noises. Hence, we consider that the measurements of relative states by agent i have the following form:

$$z_{ji}(t) = x_j(t) - x_i(t) + \mathbf{1}_n\sigma_{ji}\xi_{1ji}(t) + f_{ji}(x_j(t) - x_i(t))\xi_{2ji}(t), \tag{2.2}$$

where $j \in N_i(\mathcal{G}(t))$, $\xi_{lji}(t) \in \mathbb{R}$, $l = 1, 2$, denote the measurement noises, $\sigma_{ji} > 0$, and $f_{ji}(\cdot)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. We assume that the measurement noises are independent Gaussian white noises.

ASSUMPTION 2.1. *The noise process $\xi_{lji}(t)$ satisfies $\int_0^t \xi_{lji}(s)ds = w_{lji}(t)$, $t \ge 0$, $i, j = 1, 2, \dots, N$, $l = 1, 2$, where $\{w_{lji}(t),\ i, j = 1, 2, \dots, N,\ l = 1, 2\}$ are scalar independent Brownian motions.*

Also assume that the noise intensity $f_{ji}(\cdot)$ is Lipschitz continuous.

ASSUMPTION 2.2. *$f_{ji}(0) = 0$, $i, j = 1, 2, \ldots, N$, and there exists a constant $\bar{\sigma} \geq 0$ such that for any $x, y \in \mathbb{R}^n$,*

$$\|f_{ji}(x) - f_{ji}(y)\| \leq \bar{\sigma}\|x - y\|, \quad i, j = 1, 2, \ldots, N. \tag{2.3}$$

Remark 2.3. Distributed consensus problems with additive and multiplicative noises for continuous-time models were studied, respectively, in [21] and [19], where the measurements of $x_j(t) - x_i(t)$ have the forms $z_{ji}(t) = x_j(t) - x_i(t) + \mathbf{1}_n \sigma_{ji} \xi_{1ji}(t)$ and $z_{ji}(t) = x_j(t) - x_i(t) + f_{ji}(x_j(t) - x_i(t))\xi_{2ji}(t)$. The case with compound noises is motivated by the demand in applications. In many real models, the measurement by multiple sensors is often disturbed by both additive and multiplicative noises in multisensor multirate systems [11, Chapter 16]. The measurement model with compound noises was also proposed in [43, 44]. In those papers, the consensus algorithm was proposed with the constant control gain, and mean square weak consensus was achieved by assuming zero additive noises. Different from [43, 44], here time-varying control gains are used to attenuate the additive measurement noises, and sufficient conditions and necessary conditions for the stochastic (mean square and almost sure) weak and strong consensus will be established for continuous-time algorithms. Our methods and results can also be applied to the discrete-time models.

We introduce the following definitions to describe the stochastic consensus on the protocol $u(t)$ for the system (2.1).

DEFINITION 2.4. *A distributed protocol u is called a mean square weak consensus protocol if it results in the systems* (2.1) *and* (2.2) *having the following property: for any given $x(0) \in \mathbb{R}^{Nn}$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x_j(t)\|^2 = 0$. If, in addition, there is a random vector $x^* \in \mathbb{R}^n$, such that $\mathbb{E}\|x^*\|^2 < \infty$ and $\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x^*\|^2 = 0$, $i = 1, 2, \ldots, N$, then u is called a mean square strong consensus protocol. Particularly, if $\mathbb{E}x^* = \frac{1}{N}\sum_{j=1}^N x_j(0)$, then u is called an asymptotically unbiased mean square average consensus (AUMSAC) protocol, and $\mathbb{E}\|x^* - \frac{1}{N}\sum_{j=1}^N x_j(0)\|^2$ is called the mean square steady-state error.*

DEFINITION 2.5. *A distributed protocol u is called an almost sure weak consensus protocol if it results in the systems* (2.1) *and* (2.2) *having the following property: for any given $x(0) \in \mathbb{R}^{Nn}$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, a.s. If, in addition, there is a random vector $x^* \in \mathbb{R}^n$, such that $\mathbb{P}\{\|x^*\| < \infty\} = 1$ and $\lim_{t\to\infty} \|x_i(t) - x^*\| = 0$, a.s., $i = 1, 2, \ldots, N$, then u is called an almost sure strong consensus protocol. Particularly, if $\mathbb{E}x^* = \frac{1}{N}\sum_{j=1}^N x_j(0)$, then u is called an asymptotically unbiased almost sure average consensus (AUASAC) protocol.*

It is obvious that mean square (or almost sure) strong consensus implies mean square (or almost sure) weak consensus. The weak consensus aims to describe the generalized asymptotic behavior of the agents, and implies that all agents will get together but may not converge to a finite value (or random variable). For the strong consensus, all the states must converge to a common value, which may depend on the initial values, the noises, and the consensus algorithm. The average consensus is the special case of the strong consensus and means that the state of each agent converges to the average of initial states, which is usually investigated under balanced digraphs [34].

Note that the additive noises are included in the measurement (2.2), and then the fixed control gain fails to solve the consensus problems. In order to attenuate the effects of additive noises as $t \to \infty$, we use the following stochastic consensus protocol:

$$u_i(t) = c(t) \sum_{j=1}^{N} a_{ij}(t) z_{ji}(t), \quad i = 1, 2, \ldots, N, \tag{2.4}$$

where the time-varying control gain $c(t) : [0, \infty) \to [0, \infty)$ is a continuous function. The central issue of stochastic consensus lies in how to choose the control gain $c(t)$ for guaranteeing mean square or almost sure consensus. There are many works on the choice of control gain $c(t)$ for multi-agent systems with additive noises, such as [21, 22] and [24], where the following conditions on the control gain $c(t)$ were addressed for stochastic strong consensus:

(C1) $\int_0^\infty c(t)dt = \infty$.
(C2) $\int_0^\infty c^2(t)dt < \infty$.
(C3) $\lim_{t\to\infty} c(t) = 0$.

It will be revealed that (C2) is not a necessary condition for the stochastic weak consensus, and some new conditions on the control gain $c(t)$ will be proposed.

3. Networks with fixed topologies. In this section, we consider the consensus problem with additive noises and the fixed topology, i.e., $\mathcal{G}(t) \equiv \mathcal{G}$. The other notation related to the topology will also be given in the simplified form free of the time t and the topology $\mathcal{G}(t)$; for example, $a_{ij}(t)$ will be denoted by a_{ji} for short.

It is well known that the existence of a spanning tree is a minimum requirement for the deterministic consensus and the mean square strong consensus under the fixed topology [21, 42]. Hence, the following assumption will be examined.

ASSUMPTION 3.1. *The digraph $\mathcal{G}$ contains a spanning tree.*

We first develop an auxiliary lemma, which generalizes Lemma 4 in [14]. The proof is given in Appendix A.

LEMMA 3.2. *For the Laplacian matrix $\mathcal{L}$, we have the following assertions:*
1. *There exists a probability measure π such that $\pi^T \mathcal{L} = 0$.*
2. *There exists a matrix $\widetilde{Q} \in \mathbb{R}^{N\times(N-1)}$ such that the matrix $Q = (\frac{1}{\sqrt{N}}\mathbf{1}_N, \widetilde{Q}) \in \mathbb{R}^{N\times N}$ is nonsingular and*

$$Q^{-1} = \begin{pmatrix} \nu^T \\ \overline{Q} \end{pmatrix}, \qquad Q^{-1}\mathcal{L}Q = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathcal{L}} \end{pmatrix}, \tag{3.1}$$

where $\overline{Q} \in \mathbb{R}^{(N-1)\times N}$, $\widetilde{\mathcal{L}} \in \mathbb{R}^{(N-1)\times(N-1)}$, and ν is the left eigenvector of $\mathcal{L}$ with $\nu^T \mathcal{L} = 0$ and $\frac{1}{\sqrt{N}}\nu^T \mathbf{1}_N = 1$.
3. *Assumption 3.1 holds if and only if all the eigenvalues of $\widetilde{\mathcal{L}}$ have positive real parts. Moreover, if Assumption 3.1 holds, then the probability measure π is unique and $\nu = \sqrt{N}\pi$.*

Especially, if the digraph is balanced, then $\pi = \frac{1}{N}\mathbf{1}_N$, and Q can be constructed as an orthogonal matrix with the form $Q = (\frac{1}{\sqrt{N}}\mathbf{1}_N, \widetilde{Q})$ and the inverse of Q is represented in the form

$$Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^T \\ \widetilde{Q}^T \end{bmatrix}.$$

3.1. Stochastic consensus under additive noises.

3.1.1. Stochastic stability. By Lemma 3.2, the consensus problem will come down to the mean square and the almost sure asymptotic stability analysis of the SDE with the following form:

$$dX(t) = -c_1(t)DX(t)dt + \sum_{i=1}^{d} \sigma_i c_2(t) dw_i(t), \quad X(0) \in \mathbb{R}^m, \tag{3.2}$$

where $D \in \mathbb{R}^{m\times m}$, $\sigma_i = [\sigma_{i1}, \ldots, \sigma_{im}]^T$, m and d are integers, $\sigma_i \neq 0$ for some i, $c_1(t), c_2(t) : [0, \infty) \to [0, \infty)$ are continuous functions, and $\{w_i(t)\}_{i=1}^d$ are scalar independent Brownian motions. To facilitate the consensus analysis, here we develop the stability criteria of the solution to (3.2) with the proofs given in Appendix A.

Let $\underline{\lambda} = \min_{1\le i\le m} \mathrm{Re}(\lambda_i(D))$ and $\bar{\lambda} = \max_{1\le i\le m} \mathrm{Re}(\lambda_i(D))$. We first give the mean square asymptotic stability of the solution to (3.2).

LEMMA 3.3. *The SDE* (3.2) *is mean square asymptotically stable, that is,* $\lim_{t\to\infty} \mathbb{E}\|X(t)\|^2 = 0$ *for any initial value* $X(0)$*, if every eigenvalue of* D *has strictly positive real part,* $\int_0^\infty c_1(s)ds = \infty$*, and* $\lim_{t\to\infty} \int_0^t e^{-2\underline{\lambda} \int_s^t c_1(u)du} c_2^2(s)ds = 0$*, and only if every eigenvalue of* D *has strictly positive real part,* $\int_0^\infty c_1(s)ds = \infty$*, and*

$$\lim_{t\to\infty} \int_0^t e^{-2\bar{\lambda} \int_s^t c_1(u)du} c_2^2(s)ds = 0.$$

For the almost sure stability, we have the following lemma.

LEMMA 3.4. *The SDE* (3.2) *is almost surely asymptotically stable, that is,* $\lim_{t\to\infty} X(t) = 0$*, a.s., for any initial value* $X(0)$*, if every eigenvalue of* D *has strictly positive real part,* $\int_0^\infty c_1(s)ds = \infty$*, and* $\int_0^\infty c_2^2(s)ds < \infty$ *(or* $\int_0^\infty c_1(s)ds = \infty$ *and* $\lim_{t\to\infty} c_2^2(t) \log \int_0^t c_1(s)ds / c_1(t) = 0$*), and only if* $\int_0^\infty c_1(s)ds = \infty$ *and every eigenvalue of* D *has strictly positive real part. Especially, if all the eigenvalues of the matrix* D *are real, then the SDE* (3.2) *is almost surely asymptotically stable only if all the eigenvalues of the matrix* D *are positive,* $\int_0^\infty c_1(s)ds = \infty$*, and* $\liminf_{t\to\infty} c_2^2(t) \log \int_0^t c_1(s)ds / c_1(t) = 0$.

Remark 3.5. In the previous literature, there are many results on the mean square and almost sure asymptotic stability of SDEs (see [3, 4, 8, 28, 48] and the references therein). For the mean square asymptotical stability, almost all existing stability results are based on the multiplicative noises, and the necessary and sufficient conditions for the linear multidimensional SDEs with additive noises have not been established. For the almost sure asymptotic stability, the works [3, 4, 8] introduced the necessary conditions and sufficient conditions of almost sure asymptotic stability for SDEs with additive noises. But, they focused on the scalar SDEs with time-invariant drift terms and cannot be used to examine (3.2). Motivated by the above works, in Lemmas 3.3 and 3.4, we develop mean square and almost sure asymptotic stability conditions for n-dimensional and nonautonomous SDEs that can be used to solve mean square and almost sure consensus problems of multi-agent systems with additive noises and time-varying noise intensities.

3.1.2. Mean square weak consensus. The necessary and sufficient conditions of mean square strong consensus are now clear; see [21, 42]. Here, we concentrate on mean square weak consensus and introduce two new conditions on the control gain $c(t)$:

(C4) $\lim_{t\to\infty}\int_0^t e^{-2\underline{\lambda}\int_s^t c(u)du}c^2(s)ds = 0$, where $\underline{\lambda} = \min_{2\le i\le N}\mathrm{Re}(\lambda_i(\mathcal{L}))$.

(C4′) $\lim_{t\to\infty}\int_0^t e^{-2\bar{\lambda}\int_s^t c(u)du}c^2(s)ds = 0$, where $\bar{\lambda} = \max_{2\le i\le N}\mathrm{Re}(\lambda_i(\mathcal{L}))$.

THEOREM 3.6. *Suppose that Assumption* 2.1 *holds and* $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. *Then the protocol* (2.4) *is a mean square weak consensus protocol if Assumption* 3.1, (C1), *and* (C4) *hold, and only if Assumption* 3.1, (C1), *and* (C4′) *hold.*

Proof. Substituting the protocol (2.4) into the system (2.1) and using Assumption 2.1 produce

$$dx(t) = -c(t)(\mathcal{L}\otimes I_n)x(t)dt + c(t)\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}(\eta_{N,i}\otimes \mathbf{1}_n)dw_{1ji}(t). \tag{3.3}$$

Let ν be defined as in Lemma 3.2, and let $J_N = \frac{1}{\sqrt{N}}\mathbf{1}_N\nu^T$. Noting that $\mathcal{L}\mathbf{1}_N = 0$ and $\nu^T\mathcal{L} = 0$, then $(I_N - J_N)\mathcal{L} = \mathcal{L} = \mathcal{L}(I_N - J_N)$. Let $\delta(t) = [(I_N - J_N)\otimes I_n]x(t)$; then we have

$$d\delta(t) = -c(t)(\mathcal{L}\otimes I_n)\delta(t)dt + c(t)\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}((I_N - J_N)\eta_{N,i}\otimes \mathbf{1}_n)dw_{1ji}(t).$$

Define $\widetilde{\delta}(t) = (Q^{-1}\otimes I_n)\delta(t) = [\widetilde{\delta}_1^T(t),\dots,\widetilde{\delta}_N^T(t)]^T$ and $\overline{\delta}(t) = [\widetilde{\delta}_2^T(t),\dots,\widetilde{\delta}_N^T(t)]^T$. By the definition of Q^{-1} given in Lemma 3.2, we have $\widetilde{\delta}_1(t) = (\nu^T\otimes I_n)\delta(t) = (\nu^T(I_N - J_N)\otimes I_n)x(t) = 0$ and

$$d\overline{\delta}(t) = -c(t)(\widetilde{\mathcal{L}}\otimes I_n)\overline{\delta}(t)dt + c(t)\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}(\overline{Q}(I_N - J_N)\eta_{N,i}\otimes \mathbf{1}_n)dw_{1ji}(t), \tag{3.4}$$

where $\overline{Q}$ is defined in Lemma 3.2. Note that $\delta_i(t) = x_i - \frac{1}{\sqrt{N}}\sum_{k=1}^N \nu_k x_k(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \nu_k(x_i - x_k)$. It can be seen that if the protocol (2.4) is a mean square weak consensus protocol, then the solution to SDE (3.4) must be mean square asymptotically stable. Note that Assumption 3.1 holds if and only if all the eigenvalues of $\widetilde{\mathcal{L}}$ have positive real parts (Lemma 3.2). Letting $c_1(t) = c_2(t)$ and applying Lemma 3.3 to the SDE (3.4) produce the necessity of Assumption 3.1, (C1), and (C4′) immediately.

Assume now that Assumption 3.1 and conditions (C1) and (C4) hold and let $\nu = \sqrt{N}\pi$, where π is the unique probability measure satisfying $\pi^T\mathcal{L} = 0$. Hence, applying Lemma 3.3 yields the mean square asymptotic stability of the SDE (3.4), which also produces $\lim_{t\to\infty}\mathbb{E}\|\delta(t)\|^2 = 0$ for any initial value $x(0)$. Note that $\delta_i(t) = x_i - \sum_{k=1}^N \pi_k x_k(t)$, $i = 1,\dots,N$; then for $i \ne j$, $\lim_{t\to\infty}\mathbb{E}\|x_j(t) - x_i(t)\|^2 \le 2\lim_{t\to\infty}\mathbb{E}\|x_j(t) - \sum_{k=1}^N \pi_k x_k(t)\|^2 + 2\lim_{t\to\infty}\mathbb{E}\|x_i(t) - \sum_{k=1}^N \pi_k x_k(t)\|^2 = 0$. That is, mean square weak consensus follows, and the sufficiency is obtained. ∎

The following corollary argues the relationship between conditions (C1)–(C4) and mean square weak consensus.

COROLLARY 3.7. *Suppose that Assumption* 2.1 *holds and* $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. *Then the protocol* (2.4) *is a mean square weak consensus protocol if Assumption* 3.1, (C1), *and* (C3) *hold. Especially, if* $c(t)$ *is a decreasing function, then the protocol* (2.4) *is a mean square weak consensus protocol if and only if Assumption* 3.1, (C1), *and* (C3) *hold.*

Proof. By (C1), (C3), and l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t \exp\Big\{-\lambda\int_s^t c(u)du\Big\}c^2(s)ds
&= \lim_{t\to\infty}\frac{\int_0^t \exp\left\{\lambda\int_0^s c(u)du\right\}c^2(s)ds}{\exp\left\{\lambda\int_0^t c(u)du\right\}}\\
&= \frac{1}{\lambda}\lim_{t\to\infty} c(t) = 0 \quad \forall\,\lambda>0.
\end{aligned}$$

Then (C3) under (C1) implies (C4). By Theorem 3.6, the desired mean square weak consensus follows.

Assume that $c(t)$ is a decreasing function. We need only prove the "only if" part. If the protocol (2.4) is a mean square weak consensus protocol, by Theorem 3.6, we obtain that Assumption 3.1 and conditions (C1) and (C4′) hold. Note that $c(t)$ is a decreasing function; then for any $\lambda>0$,

$$\begin{aligned}
\int_0^t e^{-2\lambda\int_s^t c(u)du}c^2(s)ds &\ge c(t)e^{-2\lambda\int_0^t c(u)du}\int_0^t e^{2\lambda\int_0^s c(u)du}c(s)ds\\
&= \frac{c(t)}{2\lambda}\big(1-e^{-2\lambda\int_0^t c(u)du}\big),
\end{aligned}$$

which implies $c(t)\le 2\lambda\int_0^t e^{-2\lambda\int_s^t c(u)du}c^2(s)ds(1-e^{-2\lambda\int_0^t c(u)du})^{-1}$ and tends to zero as $t\to\infty$. Hence, (C4′) under (C1) also implies (C3) if $c(t)$ is a decreasing function, and then the necessity follows. ∎

Remark 3.8. It was shown in [21] that the necessary and sufficient conditions of mean square strong consensus are (C1) and (C2). Since mean square strong consensus implies mean square weak consensus and Theorem 3.6 shows that mean square weak consensus implies (C4′), one may wonder whether the condition (C4′) conflicts with (C2). In fact, they are not contradictory, and (C2) under (C1) implies (C4′) (see [21]).

3.1.3. Almost sure strong and weak consensus. Compared with the analysis of mean square consensus, almost sure consensus is more difficult. This difficulty stems from the almost sure asymptotic stability theory of SDEs. Only a few efforts have been made in the analysis of almost sure consensus; see [13, 22] for discrete-time systems and [25, 42] for continuous-time models, which showed that (C1) and (C2) are the sufficient conditions for almost sure strong consensus. We first prove that (C1) and (C2) are also necessary for almost sure strong consensus.

THEOREM 3.9. *Suppose that Assumption* 2.1 *holds and* $f_{ji}(x)\equiv 0$, $i,j\in\mathcal{V}$. *Then the protocol* (2.4) *is an almost sure strong consensus protocol if and only if Assumption* 3.1*,* (C1)*, and* (C2) *hold.*

Proof. The sufficiency follows from [42], and the necessity is proved as follows. If the protocol (2.4) is an almost sure strong consensus protocol, then we have $\lim_{t\to\infty}\|\overline{\delta}(t)\|=0$, a.s., where $\overline{\delta}(t)$ is defined by (3.4). Note that Assumption 3.1 is equivalent to assuming that all the eigenvalues of $\widetilde{\mathcal{L}}$ have positive real parts (Lemma 3.2). Then by Lemma 3.4, we have that Assumption 3.1 and condition (C1) hold. Then we argue the necessity of (C2). By the property of the matrix $\mathcal{L}$ and (3.3), we have

$$(\pi^T\otimes I_n)x(t) = (\pi^T\otimes I_n)x(0)+\bar{M}(t), \tag{3.5}$$

where $\bar{M}(t) = \sum_{i,j=1}^{N} a_{ij}\sigma_{ji}\int_0^t c(s)[\pi^T\eta_{N,i}\otimes\mathbf{1}_n]dw_{1ji}(s)$. It is evident that the almost sure strong consensus implies that $(\pi^T\otimes I_n)x(t)$ converges almost surely to a random variable. Note that $(\pi^T\otimes I_n)x(t)$ converges almost surely if and only if the limit of the continuous local martingale $\bar{M}(t)$ exists almost surely, denoted by $\bar{M}(\infty)$. But this is also equivalent to $\lim_{t\to\infty}\langle\bar{M}\rangle(t) < \infty$, a.s. (see [38, Proposition 1.8]). Note also that $\langle\bar{M}\rangle(t) = n\sum_{i,j=1}^{N} a_{ij}^2\sigma_{ji}^2\pi_i^2\int_0^t c^2(s)ds$ (see [38, Theorem 1.8]). Hence, $(\pi^T\otimes I_n)x(t)$ converges almost surely to a random variable if and only if (C2) holds. ∎

Remark 3.10. Combining Theorem 3.9 and Remark 3.8 gives that mean square strong consensus and almost sure strong consensus are equivalent. Here, (C2) aims at guaranteeing that all agents converge to a common value in the sense of both mean square and probability one.

Note that almost sure strong consensus implies almost sure weak consensus. That is, (C1) and (C2) can guarantee almost sure weak consensus. But it is natural to pursue the weaker conditions to guarantee almost sure weak consensus. Thanks to Lemma 3.4, we can examine almost sure weak consensus without condition (C2). Moreover, we can obtain a finer necessary condition under the following assumption. In fact, the case with undirected graphs falls in this assumption.

ASSUMPTION 3.11. *All the eigenvalues of the Laplacian matrix $\mathcal{L}$ are real.*

Here, we introduce the following two conditions for the almost sure weak consensus, which will be proved to be sufficient and necessary, respectively:

(C5) $\lim_{t\to\infty} c(t)\log\int_0^t c(s)ds = 0$.

(C5′) $\liminf_{t\to\infty} c(t)\log\int_0^t c(s)ds = 0$.

Note that the almost sure weak consensus problem of the multi-agent system (2.1) with additive noises is actually the almost sure asymptotic stability problem of the SDE (3.4). By Lemmas 3.2 and 3.4, we can easily obtain the following consensus theorem.

THEOREM 3.12. *Suppose that Assumption* 2.1 *holds and $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. Then the protocol* (2.4) *is an almost sure weak consensus protocol if Assumption* 3.1*,* (C1)*, and* (C5) *hold, and only if Assumption* 3.1 *and* (C1) *hold. Especially, if Assumption* 3.11 *holds, then the protocol* (2.4) *is an almost sure weak consensus protocol only if Assumption* 3.1*,* (C1)*, and* (C5′) *hold.*

Remark 3.13. Note that almost sure strong consensus implies almost sure weak consensus. Further, if Assumption 3.1, (C1), and (C2) hold, then almost sure weak consensus follows. Theorem 3.12 showed that condition (C5′) is necessary for almost sure weak consensus. One may hope that (C2) can produce (C5′). In fact, that is true under (C1). To see this, if (C5′) fails, then there exists a constant $\underline{c} > 0$, such that $\liminf_{t\to\infty} c(t)\log\int_0^t c(s)ds > \underline{c}$. This implies that for some $T > 0$, $c(t)\log\int_0^t c(s)ds > \underline{c}/2$, $t > T$. Hence, from (C1), $\int_0^t c^2(s)ds > \frac{\underline{c}}{2}\int_T^t \frac{d(\int_0^s c(u)du)}{\log\int_0^s c(u)du} \to \infty$ as t tends to ∞, which is in conflict with (C2).

Remark 3.14. Note that almost sure weak consensus implies (C1) and (C5′), which lead to (C3). Then mean square weak consensus is obtained from Corollary 3.7. Hence, almost sure weak consensus implies mean square weak consensus under Assumption 3.11.

Remark 3.15. Here, we remark that weak consensus may not be strong consensus. For example, consider $c(t) = (1+t)^{-1/3}$, which satisfies (C1), (C3), and (C5), but defies (C2). In view of the consensus results above, mean square and almost sure weak consensus both hold, but neither mean square nor almost sure strong consensus holds. This is also demonstrated in the numerical examples of section 5.

Remark 3.16. The necessary and sufficient conditions for mean square strong consensus were established in [21, 42]. However, the necessary and sufficient conditions for almost sure strong consensus, and conditions for stochastic weak consensus have not been taken into account. This paper fills in the gap and reveals the relationship between mean square and almost sure consensus. For examining stochastic weak consensus, stochastic stability criteria are developed by the semidecoupled technique and the law of the iterated logarithm for martingales (see the proofs of Lemmas 3.3 and 3.4 in Appendix A). These criteria differ from those in [21, 42] on stochastic strong consensus.

3.2. Stochastic consensus under compound noises. We continue to investigate the more complex models, where multiplicative and additive noises coexist in the information communication. We first examine the two-agent case ($N = 2$) with $f_{ji}(x) = \bar{\sigma}_{ji}x$.

3.2.1. Two-agent case. It is easy to understand that the existence of a spanning tree is necessary for stochastic consensus for the two-agent case and implies that there exists at least one pair (i, j) such that $a_{ij} \neq 0$. Here, we let $\bar{a} = a_{12} + a_{21}$, $\underline{a} = (a_{12}\bar{\sigma}_{21}^2 + a_{21}\bar{\sigma}_{12}^2)/2$, and $\widetilde{\mu}(t) = \bar{a}c(t) - \underline{a}c^2(t),\ t \geq 0$, and give the corresponding convergence conditions.

THEOREM 3.17. *Suppose Assumption* 2.1 *holds, and* $f_{ji}(x) = \bar{\sigma}_{ji}x$ *with* $\bar{\sigma}_{ji} > 0$, $i, j = 1, 2, N = 2$. *Then the protocol* (2.4) *is a mean square weak consensus protocol if and only if* $\int_0^\infty \widetilde{\mu}(s)ds = \infty$ *and* $\lim_{t\to\infty} \int_0^t e^{-2\int_s^t \widetilde{\mu}(u)du} c^2(s)ds = 0$. *Moreover, the protocol* (2.4) *is a mean square strong consensus protocol if and only if* (C1) *and* (C2) *hold.*

Proof. Let $\widetilde{x}(t) = x_1(t) - x_2(t)$; then by the definition of $x_i(t)$, $i = 1, 2$, we have

$$\begin{aligned} d\widetilde{x}(t) = &-\bar{a}c(t)\widetilde{x}(t)dt - a_{12}\bar{\sigma}_{21}c(t)\widetilde{x}(t)dw_{221}(t) \\ &- a_{21}\bar{\sigma}_{12}c(t)\widetilde{x}(t)dw_{212}(t) + dM_1(t), \end{aligned} \tag{3.6}$$

where $M_1(t) = \mathbf{1}_n(a_{12}\sigma_{21}\int_0^t c(s)dw_{121}(s) - a_{21}\sigma_{12}\int_0^t c(s)dw_{212}(s))$ is a martingale vanishing at zero. By the Itô formula,

$$\begin{aligned} d\|\widetilde{x}(t)\|^2 = &-2\widetilde{\mu}(t)\|\widetilde{x}(t)\|^2dt + np_0c^2(t)dt - 2a_{12}\bar{\sigma}_{21}c(t)\|\widetilde{x}(t)\|^2dw_{221}(t) \\ &- 2a_{21}\bar{\sigma}_{12}c(t)\|\widetilde{x}(t)\|^2dw_{212}(t) + 2\widetilde{x}(t)^TdM_1(t), \end{aligned} \tag{3.7}$$

where $p_0 = \sum_{i,j=1}^2 a_{ij}\sigma_{ij}^2 > 0$. Taking expectations on both sides gives

$$\mathbb{E}\|\widetilde{x}(t)\|^2 = \|\widetilde{x}(0)\|^2 - 2\int_0^t \widetilde{\mu}(s)\|\widetilde{x}(s)\|^2ds + np_0\int_0^t c^2(s)ds, \tag{3.8}$$

which implies

$$\mathbb{E}\|\widetilde{x}(t)\|^2 = \|\widetilde{x}(0)\|^2e^{-2\int_0^t \widetilde{\mu}(s)ds} + np_0\int_0^t e^{-2\int_s^t \widetilde{\mu}(u)du}c^2(s)ds. \tag{3.9}$$

Hence, the necessary and sufficient conditions of mean square weak consensus are obtained.

We now prove the second assertion. Assume (C1) and (C2) hold. Let $\pi_1 = \frac{a_{12}}{\bar{a}}$, $\pi_2 = \frac{a_{21}}{\bar{a}}$, and $\bar{x}(t) = \pi_1x_1(t) + \pi_2x_2(t)$; then $\bar{x}(t) = \bar{x}(0) + M_2(t)$, where $M_2(t) =$

$\pi_1 a_{12}\bar{\sigma}_{21}\int_0^t c(s)\widetilde{x}(s)dw_{221}(s) - \pi_2 a_{21}\bar{\sigma}_{12}\int_0^t c(s)\widetilde{x}(s)dw_{212}(t) + \mathbf{1}_n\pi_1 a_{12}\sigma_{21}\int_0^t c(s)\,dw_{121}(s) - \mathbf{1}_n\pi_2 a_{21}\sigma_{12}\int_0^t c(s)dw_{212}(s)$ satisfies

$$\begin{aligned}\mathbb{E}\|M_2(t)\|^2 &= (\pi_1^2 a_{12}\bar{\sigma}_{21}^2 + \pi_2^2 a_{21}\bar{\sigma}_{12}^2)\int_0^t c^2(s)\mathbb{E}\|\widetilde{x}(s)\|^2 ds\\ &\quad + n(\pi_1^2 a_{12}\sigma_{21}^2 + \pi_2^2 a_{21}\sigma_{12}^2)\int_0^t c^2(s)ds. \qquad (3.10)\end{aligned}$$

If (C1) and (C2) hold, then $\int_0^\infty \widetilde{\mu}(s)ds = \bar{a}\int_0^\infty c(t)dt - \underline{a}\int_0^\infty c^2(t)dt = \infty$. Moreover,

$$\begin{aligned}\lim_{t\to\infty}\int_0^t e^{-2\int_s^t \widetilde{\mu}(u)du}c^2(s)ds &= \lim_{t\to\infty}\int_0^t e^{2\underline{a}\int_s^t c^2(u)du}e^{-2\bar{a}\int_s^t c(u)du}c^2(s)ds\\ &\le e^{2\underline{a}\int_0^\infty c^2(u)du}\lim_{t\to\infty}\int_0^t e^{-2\bar{a}\int_s^t c(u)du}c^2(s)ds = 0. \qquad (3.11)\end{aligned}$$

Hence, mean square weak consensus follows, and then $\mathbb{E}\|\widetilde{x}(t)\|^2$ is bounded, which together with (3.10) also implies $\lim_{t\to\infty}\mathbb{E}\|M_2(t)\|^2 < \infty$. The martingale convergence theorem [16, Corollary 7.22] admits that $M_2(t)$ converges in the sense of mean square to a random variable, denoted by $M_2(\infty)$. Then $\bar{x}(t)$ converges to $x^* := \bar{x}(0) + M_2(\infty)$ in the sense of mean square, and we have $\lim_{t\to\infty}\mathbb{E}\|x_1(t) - x^*\|^2 \le 2\lim_{t\to\infty}\mathbb{E}\|x_1(t) - \bar{x}(t)\|^2 + 2\lim_{t\to\infty}\mathbb{E}\|\bar{x}(t) - x^*\|^2 = 0$. This together with mean square weak consensus also yields $\lim_{t\to\infty}\mathbb{E}\|x_2(t) - x^*\|^2 = 0$, and mean square strong consensus follows.

It remains to show the necessity of (C1) and (C2) for mean square strong consensus. Note that mean square strong consensus implies mean square weak consensus, which together with the first assertion also gives $\int_0^\infty \widetilde{\mu}(s)ds = \infty$. Also note that mean square strong consensus implies the mean square convergence of $M_2(t)$, which together with (3.10) produces (C2). Hence, $\bar{a}\int_0^\infty c(t)dt = \int_0^\infty \widetilde{\mu}(s)ds + \underline{a}\int_0^\infty c^2(t)dt = \infty$. The proof is completed. ∎

THEOREM 3.18. *Suppose that Assumption* 2.1 *holds and* $f_{ji}(x) = \bar{\sigma}_{ji}x$ *with* $\bar{\sigma}_{ji} > 0$, $i,j = 1,2, N = 2$. *Then the protocol* (2.4) *is an almost sure strong consensus protocol if* (C1), (C2) *hold and there exists a constant* $t_0 > 0$ *such that* $\widetilde{\mu}(t) = \bar{a}c(t) - \underline{a}c^2(t) > 0$ *for all* $t \ge t_0$, *and only if* (C1) *and* (C2) *hold.*

Proof. Note that (3.6) admits the explicit solution

$$\widetilde{x}(t) = \widetilde{x}(0)y(t,0) + \int_0^t y(t,s)dM_1(s), \qquad (3.12)$$

where $y(t,s) = \exp\{-\bar{a}\int_s^t c(u)du\}\mathcal{E}(t,s)$ with $\mathcal{E}(t,s) = \exp\{-\underline{a}\int_s^t c^2(u)du - a_{12}\bar{\sigma}_{21}\int_s^t c(u)dw_{221}(u) - a_{21}\bar{\sigma}_{12}\int_s^t c(u)dw_{212}(u)\}$. It is easy to see that $\mathcal{E}(t,0)$ is an exponential martingale.

First we show the "only if" part. Note that the almost sure strong consensus gives that $\bar{x}(t)$ converges almost surely, which is equivalent to saying that the limit of the continuous local martingale $M_2(t)$ exists almost surely, denoted by $M_2(\infty)$. This is also equivalent to $\lim_{t\to\infty}\langle M_2\rangle(t) < \infty$, a.s. (see [38, Proposition 1.8, p. 183]). Note that

$$\begin{aligned}\langle M_2\rangle(t) &= (\pi_1^2 a_{12}\bar{\sigma}_{21}^2 + \pi_2^2 a_{21}\bar{\sigma}_{12}^2)\int_0^t c^2(s)\|\widetilde{x}(s)\|^2 ds\\ &\quad + n(\pi_1^2 a_{12}\sigma_{21}^2 + \pi_2^2 a_{21}\sigma_{12}^2)\int_0^t c^2(s)ds. \qquad (3.13)\end{aligned}$$

That is, almost sure strong consensus gives (C2). Moreover, almost sure strong consensus implies that $\lim_{t\to\infty}\|\widetilde{x}(t)\|=0$, a.s., for any initial value $\widetilde{x}(0)$. This together with (3.12) also implies $\lim_{t\to\infty}\|y(t,0)\|=0$, a.s., and $\lim_{t\to\infty}\int_0^t y(t,s)dM_1(s)=0$, a.s. Condition (C2) together with the martingale convergence theorem implies that the stochastic integrals $\int_0^t c(u)dw_{221}(u)$ and $\int_0^t c(u)dw_{212}(u)$ converge almost surely, and then gives that the exponential martingale $\mathcal{E}(t,0)$ converges almost surely to some positive random variable e_0. By the definition of $y(t,0)$, we have $\lim_{t\to\infty} y(t,0)=e_0e^{-\bar{a}\int_0^\infty c(s)ds}$, a.s. Hence, (C1) holds. Therefore, the "only if" part follows.

Now we need to prove the "if" part. By (3.7), we get that for any $t\geq t_0$,

$$\|\widetilde{x}(t)\|^2=\|\widetilde{x}(t_0)\|^2-A_1(t,t_0)+A_2(t,t_0)+M_3(t,t_0), \tag{3.14}$$

where $A_1(t,t_0)=2\int_{t_0}^t\widetilde{\mu}(s)\|\widetilde{x}(s)\|^2ds$, $A_2(t,t_0)=np_0\int_{t_0}^t c^2(s)ds$, and $M_3(t,t_0)=-2a_{12}\bar{\sigma}_{21}\int_{t_0}^t c(s)\|\widetilde{x}(s)\|^2dw_{221}(s)-2a_{21}\bar{\sigma}_{12}\int_{t_0}^t c(s)\|\widetilde{x}(s)\|^2dw_{212}(s)+2\int_{t_0}^t\widetilde{x}(s)^TdM_1(s)$. Note that $\widetilde{\mu}(t)>0$ for $t\geq t_0$; then $A_1(t,t_0)$ is increasing. It is observed from (C2) that $A_2(\infty,t_0)<\infty$. Hence, by the semimartingale convergence theorem (Lemma A.1), we can see that $\lim_{t\to\infty}\|\widetilde{x}(t)\|<\infty$, a.s. Theorem 3.17 shows that conditions (C1) and (C2) produce $\lim_{t\to\infty}\mathbb{E}\|\widetilde{x}(t)\|^2=0$, and then there exists a subsequence converging to zero almost surely. The uniqueness of the limit admits $\lim_{t\to\infty}\|\widetilde{x}(t)\|=0$, a.s. Then almost sure weak consensus follows. Note that $\lim_{t\to\infty}\|\widetilde{x}(t)\|=0$, a.s. implies $\|\widetilde{x}(t)\|$ is bounded almost surely. Combining (3.13), (C2), and Proposition 1.8 of [38] gives $x^*:=\lim_{t\to\infty}\bar{x}(t)<\infty$, a.s., which together with almost sure weak consensus implies almost sure strong consensus. ∎

3.2.2. Multi-agent case. The appearance of multiplicative noises makes the transformed stochastic systems become SDEs with compound noises, which cannot be semidecoupled as those in the proofs of Lemmas 3.3 and 3.4 with only additive noises. In this subsection, we aim to seek other strategies to examine the consensus problem with compound noises. Note that the matrix $-\widetilde{\mathcal{L}}$ defined in Lemma 3.2 is Hurwitz if the digraph $\mathcal{G}$ contains a spanning tree. Then we have the following lemma.

LEMMA 3.19. *If Assumption* 3.1 *holds, then there exists a unique positive definite matrix P such that*

$$-P\widetilde{\mathcal{L}}-\widetilde{\mathcal{L}}^TP=-I_{N-1}.$$

First, we concentrate on mean square consensus. Let q_i be the ith row of $\widetilde{Q}$, $\overline{Q}(i)=\overline{Q}(I_N-J_N)\eta_{N,i}$. For the matrix $P>0$ determined by Lemma 3.19, define $\widetilde{B}=\sum_{i=1}^N[\overline{Q}^T(i)\ P\overline{Q}(i)]\sum_{j=1}^N a_{ij}[(q_j-q_i)^T(q_j-q_i)]$ and $\mu(t)=c(t)\lambda_{\min}(I_{N-1}-c(t)\bar{\sigma}^2\widetilde{B})$. We need the following assumption.

(C6) There exists a constant $t_0>0$ such that $\mu(t)>0$ for all $t\geq t_0$.

THEOREM 3.20. *Suppose that Assumptions* 2.1 *and* 2.2 *hold. Then the protocol* (2.4) *is a mean square weak consensus protocol if Assumption* 3.1 *and* (C6) *hold,* $\int_0^\infty\mu(s)ds=\infty$, *and* $\lim_{t\to\infty}\int_0^t e^{-\lambda_{\max}^{-1}(P)\int_s^t\mu(u)du}c^2(s)ds=0$, *and only if Assumption* 3.1, (C1), *and* (C4′) *hold. Moreover, the protocol* (2.4) *is a mean square strong consensus protocol if Assumption* 3.1, (C1), (C2), *and* (C6) *hold, and only if Assumption* 3.1, (C1), *and* (C2) *hold.*

Proof. Substituting the protocol (2.4) into the system (2.1) and using Assumption

2.1 yield

$$dx(t) = -c(t)(\mathcal{L} \otimes I_n)x(t)dt + dM_4(t), \tag{3.15}$$

where $M_4(t) = \sum_{i,j=1}^N a_{ij}\sigma_{ji}(\eta_{N,i} \otimes \mathbf{1}_n)\int_0^t c(s)dw_{1ji}(s) + \sum_{i,j=1}^N a_{ij}\int_0^t c(s)[\eta_{N,i} \otimes f_{ji}(x_j(s) - x_i(s))]dw_{2ji}(s)$. Continuing to use the definitions of $\delta(t)$, $\widetilde{\delta}(t)$, and $\overline{\delta}(t)$, similarly to the proof of Theorem 3.6, we obtain

$$d\overline{\delta}(t) = -c(t)(\widetilde{\mathcal{L}} \otimes I_n)\overline{\delta}(t)dt + dM_5(t), \tag{3.16}$$

where $M_5(t) = \sum_{i,j=1}^N a_{ij}\sigma_{ji}(\overline{Q}(i) \otimes \mathbf{1}_n)\int_0^t c(s)dw_{1ji}(s) + \sum_{i,j=1}^N a_{ij}\int_0^t c(s)[\overline{Q}(i) \otimes f_{ji}(\delta_j(s) - \delta_i(s))]dw_{2ji}(s)$. Note that Assumption 3.1 holds. Let $V(t) = \overline{\delta}(t)^T(P \otimes I_n)\overline{\delta}(t)$, where P is the positive definite matrix defined in Lemma 3.19. Using Itô's formula and Lemma 3.19, we have

$$dV(t) = -c(t)\|\overline{\delta}(t)\|^2dt + c^2(t)F(t)dt + C_2c^2(t)dt + dM_6(t), \tag{3.17}$$

where $F(t) = \sum_{i,j=1}^N a_{ij}\overline{Q}(i)^TP\overline{Q}(i)\|f_{ji}(\delta_j(t) - \delta_i(t))\|^2$, $C_2 = n\sum_{i,j=1}^N a_{ij}\sigma_{ji}^2[\overline{Q}(i)^TP\,\overline{Q}(i)]$, and $M_6(t) = 2\sum_{i,j=1}^N a_{ij}\sigma_{ji}\int_0^t c(s)\overline{\delta}(s)^T(P\overline{Q}(i) \otimes \mathbf{1}_n)dw_{1ji}(s) + 2\sum_{i,j=1}^N a_{ij}\int_0^t \overline{\delta}(s)^T[P\overline{Q}(i) \otimes f_{ji}(\delta_j(s) - \delta_i(s))]c(s)dw_{2ji}(s)$. Note that

$$\delta(t) = (Q \otimes I_n)\widetilde{\delta}(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N \otimes I_n)\widetilde{\delta}_1(t) + (\widetilde{Q} \otimes I_n)\overline{\delta}(t). \tag{3.18}$$

Then $\delta_i(t) = \frac{1}{\sqrt{N}}\widetilde{\delta}_1(t) + (q_i \otimes I_n)\overline{\delta}(t)$ and $\delta_j(t) - \delta_i(t) = [(q_j - q_i) \otimes I_n]\overline{\delta}(t)$. By Assumption 2.2 and the definition of $\widetilde{B}$, we have

$$\begin{aligned} F(t) &\le \bar{\sigma}^2\sum_{i=1}^N[\overline{Q}(i)^TP\overline{Q}(i)]\sum_{j=1}^N a_{ij}\|\delta_j(t) - \delta_i(t)\|^2 \\ &= \bar{\sigma}^2\overline{\delta}(t)^T\left(\sum_{i=1}^N[\overline{Q}(i)^TP\overline{Q}(i)]\sum_{j=1}^N a_{ij}[(q_j - q_i)^T(q_j - q_i) \otimes I_n]\right)\overline{\delta}(t) \\ &= \bar{\sigma}^2\overline{\delta}(t)^T(\widetilde{B} \otimes I_n)\overline{\delta}(t). \end{aligned} \tag{3.19}$$

Let $U(t) = e^{\lambda_{\max}^{-1}(P)\int_0^t \mu(s)ds}V(t)$. Note that (C6) holds, that is, $\mu(t) > 0$ for $t \ge t_0$. Substituting (3.19) into (3.17) and applying integration by parts to $U(t)$ give

$$\begin{aligned} dU(t) &= \lambda_{\max}^{-1}(P)\mu(t)U(t)dt + e^{\lambda_{\max}^{-1}(P)\int_0^t \mu(s)ds}dV(t) \\ &\le C_2e^{\lambda_{\max}^{-1}(P)\int_0^t \mu(s)ds}c^2(t)dt + e^{\lambda_{\max}^{-1}(P)\int_0^t \mu(s)ds}dM_6(t), \quad t \ge t_0. \end{aligned}$$

Noting that $M_6(t)$ is a martingale vanishing at 0, we obtain

$$\mathbb{E}V(t) \le e^{-\lambda_{\max}^{-1}(P)\int_{t_0}^t \mu(s)ds}\mathbb{E}V(t_0) + C_2\int_{t_0}^t e^{\lambda_{\max}^{-1}(P)\int_s^t \mu(u)du}c(s)^2ds. \tag{3.20}$$

Therefore, from $\int_0^\infty \mu(s)ds = \infty$ and $\lim_{t\to\infty}\int_0^t e^{-\lambda_{\max}^{-1}(P)\int_s^t \mu(u)du}c^2(s)ds = 0$, we have $\lim_{t\to\infty}\mathbb{E}V(t) = 0$, which implies $\lim_{t\to\infty}\mathbb{E}\|\overline{\delta}(t)\|^2 = 0$, and then mean square weak consensus follows from $x_j(t) - x_i(t) = [(q_j - q_i) \otimes I_n]\overline{\delta}(t)$.

The necessity is proved as follows. By the Jordan matrix decomposition, there exists an invertible matrix P_1 such that $P_1^{-1}\widetilde{\mathcal{L}}P_1 = J_{\widetilde{\mathcal{L}}}$. Here, $J_{\widetilde{\mathcal{L}}}$ is the Jordan normal form of $\widetilde{\mathcal{L}}$, i.e.,

$$J_{\widetilde{\mathcal{L}}} = \mathrm{diag}(J_{\lambda_1,n_1}, J_{\lambda_2,n_2}, \dots, J_{\lambda_l,n_l}), \quad \sum_{k=1}^{l} n_k = N-1,$$

where

$$J_{\lambda_k,n_k} = \begin{pmatrix} \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & \lambda_k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix}$$

is the corresponding Jordan block of size n_k with eigenvalue λ_k. Here, $\lambda_1, \lambda_2, \dots, \lambda_l$ are all the eigenvalues of $\widetilde{\mathcal{L}}$. Let $\zeta(t) = (P_1^{-1} \otimes I_n)\overline{\delta} = [\zeta_1(t)^T, \dots, \zeta_{N-1}(t)^T]^T$, $\zeta_i(t) \in \mathbb{R}^n$, $i = 1, \dots, N-1$. We have from (3.16) that

$$d\zeta(t) = -c(t)(J_{\widetilde{\mathcal{L}}} \otimes I_n)\zeta(t)dt + (P_1^{-1} \otimes I_n)dM_5(t).$$

Considering the kth Jordan block and its corresponding parts $\zeta_k = [\zeta_{k,1}^T, \dots, \zeta_{k,n_k}^T]^T$ and $P_1^{-1}(k) = [p_{k,1}^T, \dots, p_{k,n_k}^T]^T$, where $\zeta_{k,j} = \zeta_{k_j}(t)$ with $k_j = (\sum_{i=1}^{k-1} n_i + j)$, and $p_{k,j} = p_{k_j}$ with p_{k_j} being the k_jth row of P_1^{-1}, we have

$$d\zeta_k(t) = -c(t)(J_{\lambda_k,n_k} \otimes I_n)\zeta_k(t)dt + (P_1^{-1}(k) \otimes I_n)dM_5(t).$$

Then by the variation of constants formula, we obtain

$$\zeta_{k,n_k}(t) = e^{-\lambda_k \int_0^t c(u)du}\zeta_{k,n_k}(0) + \int_0^t e^{-\lambda_k \int_s^t c(u)du}(p_{k,n_k} \otimes I_n)dM_5(s). \tag{3.21}$$

Hence,

$$\begin{aligned} \mathbb{E}\|\zeta_{k,n_k}(t)\|^2 &= e^{-2\mathrm{Re}(\lambda_k)\int_0^t c(u)du}\|\zeta_{k,n_k}(0)\|^2 + S(t) \\ &\quad + n\sum_{i,j=1}^{N} a_{ij}\sigma_{ji}^2\|p_{k,n_k}\overline{Q}(i)\|^2 \int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c(u)du}c^2(s)ds, \end{aligned} \tag{3.22}$$

where $S(t) = \sum_{i,j=1}^N a_{ij}\|p_{k,n_k}\overline{Q}(i)\|^2 \int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c(u)du}c^2(s)\|f_{ji}(\delta_j(s) - \delta_i(s))\|^2 ds$ ≥ 0. Note that mean square strong consensus implies that for $\zeta_{k,n_k}(0) \neq 0$, $\lim_{t\to\infty} \mathbb{E}\|\zeta_{k,n_k}(t)\|^2 = 0$, which means that the three terms on the right-hand side of (3.22) all tend to zero. It gives $\mathrm{Re}(\lambda_k) > 0$ and condition (C1) that the first term tends to zero. By Lemma 3.2, we know that Assumption 3.1 holds. It gives (C4′) that the third term tends to zero.

Now, we aim to examine mean square strong consensus under (C1), (C2), and (C6). First, we see that $\mu(t) \geq c(t) - c^2(t)\bar{\sigma}^2\lambda_{\max}(\widetilde{B})$, which together with (C1) and (C2) implies $\int_0^\infty \mu(s)ds = \infty$. Moreover, by the methods used in proving (3.11), we have $\lim_{t\to\infty}\int_0^t e^{-\lambda_{\max}^{-1}(P)\int_s^t \mu(u)du}c^2(s)ds = 0$. Hence, the first assertion tells us that if (C1), (C2), and (C6) hold, then mean square weak consensus follows, which together with the continuity of $\mathbb{E}\|\bar{\delta}(t)\|^2$ gives that $\mathbb{E}\|\bar{\delta}(t)\|^2$ is bounded by some $C_3 > 0$. Let $\bar{M}_4(t) = (\pi^T \otimes I_n)M_4(t)$. Note that (3.15) implies

$$(\pi^T \otimes I_n)x(t) = (\pi^T \otimes I_n)x(0) + \bar{M}_4(t). \tag{3.23}$$

Then we have

$$\mathbb{E}\|\bar{M}_4(t)\|^2 = C_4 \int_0^t c^2(s)ds + \sum_{i,j=1}^{N} a_{ij}\pi_i^2 \int_0^t c^2(s)\mathbb{E}\|f_{ji}(\delta_j(s) - \delta_i(s))\|^2 ds, \tag{3.24}$$

where $C_4 = n\sum_{i,j=1}^N a_{ij}\sigma_{ji}^2\pi_i^2$. By Assumption 2.2 and $\mathbb{E}\|\bar{\delta}(t)\|^2 \le C_3$, we get

$$\mathbb{E}\|\bar{M}_4(t)\|^2 \le (C_4 + C_5)\int_0^t c^2(s)ds, \tag{3.25}$$

where $C_5 = C_3\bar{\sigma}^2\sum_{i,j=1}^N a_{ij}\pi_i^2\|q_j - q_i\|^2$. Hence, under (C2), $\mathbb{E}\|\bar{M}_4(t)\|^2$ is bounded by $C_6 = (C_4 + C_5)\int_0^\infty c^2(s)ds > 0$. The martingale convergence theorem [16, Corollary 7.22] admits that $\bar{M}_4(t)$ converges in mean square to a random variable, denoted by $\bar{M}_4(\infty)$. Let $x^* = (\pi^T \otimes I_n)x(0) + \bar{M}_4(\infty)$. Note that $x_i(t) - (\pi^T \otimes I_n)x(t) = x_i(t) - \sum_{j=1}^N \pi_j x_j(t) = \sum_{j=1}^N \pi_j(x_i(t) - x_j(t))$. Then the mean square weak consensus implies that $\lim_{t\to\infty}\mathbb{E}\|x_i(t) - (\pi^T \otimes I_n)x(t)\|^2 = 0$, and then

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\|x_i(t) - x^*\|^2 \le{}& 2\lim_{t\to\infty}\mathbb{E}\|x_i(t) - (\pi^T \otimes I_n)x(t)\|^2\\ &+ 2\lim_{t\to\infty}\mathbb{E}\|x^* - (\pi^T \otimes I_n)x(t)\|^2 = 0, \quad i \in \mathcal{V}.\end{aligned}$$

Therefore, mean square strong consensus follows. The necessities of (C1) and Assumption 3.1 are proved in the assertion above related to mean square weak consensus. Note that mean square strong consensus implies that $(\pi^T \otimes I_n)x(t)$ converges in the sense of mean square, which means that $\bar{M}_4(t)$ converges in mean square. This together with (3.24) produces condition (C2). Hence, the proof is completed. ∎

Similar to Corollary 3.7, we have the following corollary, which gives another sufficient condition for mean square weak consensus.

COROLLARY 3.21. *Suppose that Assumptions* 2.1, 2.2, *and* 3.1 *hold. If* $\int_0^\infty \mu(s)ds = \infty$ *and* (C3) *holds, then the protocol* (2.4) *is a mean square weak consensus protocol.*

Proof. It is obvious that (C3) implies (C6). Moreover, by $\int_0^\infty \mu(s)ds = \infty$, (C3), and l'Hôpital's rule, we have

$$\begin{aligned}\lim_{t\to\infty}\int_0^t e^{-\lambda_{\max}^{-1}(P)\int_s^t \mu(u)du}c^2(s)ds &= \lim_{t\to\infty}\frac{\int_0^t \exp\left\{\lambda_{\max}^{-1}(P)\int_0^s \mu(u)du\right\}c^2(s)ds}{\exp\left\{\lambda_{\max}^{-1}(P)\int_0^t \mu(u)du\right\}}\\ &= \lambda_{\max}(P)\lim_{t\to\infty}\frac{c^2(t)}{\mu(t)} = 0.\end{aligned}$$

That is, (C3) under $\int_0^\infty \mu(s)ds = \infty$ implies $\lim_{t\to\infty}\int_0^t e^{-\lambda_{\max}^{-1}(P)\int_s^t \mu(u)du}c^2(s)ds = 0$. By Theorem 3.20, the desired mean square weak consensus follows. ∎

Based on the semimartingale convergence theorem, we have the following almost sure strong consensus criterion.

THEOREM 3.22. *Suppose that Assumptions* 2.1, 2.2, *and* 3.1 *hold. Then the protocol* (2.4) *is an almost sure strong consensus protocol if* (C1), (C2), *and* (C6) *hold, and only if* (C2) *holds.*

Proof. For any $t \geq t_0$, we have from (C6) that $c(t)\|\bar{\delta}(t)\|^2 - c^2(t)F(t) > 0$. Let $A_1(t) = C_2 \int_{t_0}^t c^2(s)ds$ and $A_2(t) = \int_{t_0}^t (c(s)\|\bar{\delta}(s)\|^2 - c^2(s)F(s))ds$, $t \geq t_0$. Then by (C6), we know that $A_1(t)$ and $A_2(t)$ are two increasing processes for any $t \geq t_0$. By (3.17), we have

$$V(t) = V(t_0) + A_1(t) - A_2(t) + M_6(t) - M_6(t_0). \tag{3.26}$$

Similar to the proof of almost sure strong consensus in Theorem 3.18, we can use the semimartingale convergence theorem and Theorem 3.20 to obtain almost sure strong consensus.

Note that almost sure strong consensus gives that the martingale $\bar{M}_4(t)$ converges almost surely. This together with Proposition 1.8 in [38, p. 183]) also implies $\lim_{t\to\infty}\langle \bar{M}_4\rangle(t) < \infty$, a.s. It can be seen that

$$\langle \bar{M}_4\rangle(t) = C_4 \int_0^t c^2(s)ds + \sum_{i,j=1}^N a_{ij}\pi_i^2 \int_0^t c^2(s)\|f_{ji}(\delta_j(s) - \delta_i(s))\|^2 ds. \tag{3.27}$$

Hence, (C2) holds. ∎

Remark 3.23. Note that (C3) implies (C6). Then if Assumption 3.1 and (C1)–(C3) hold (for example, $c(t) = \frac{1}{1+t}$), we can obtain mean square and almost sure strong consensus.

Remark 3.24. Remark 3.8 and Theorem 3.9 show that if (C1)–(C2) hold, we can obtain mean square and almost sure strong consensus of multi-agent systems with only additive noises. When multiplicative noises also appear, we can get from Theorems 3.20 and 3.22 that the multiplicative noises do not affect the control gain to assure mean square and almost sure strong consensus if the corresponding noise intensities are so small that $\bar{\sigma}^2 < (\sup_{t\geq 0} c(t)\lambda_{\max}(\widetilde{B}))^{-1}$.

4. Networks with time-varying topologies. Note that the agent number N is finite and $a_{ij}(t)$ takes two values for $t \geq 0$; then the set of possible digraphs $\mathcal{G}(t)$ is also finite, i.e., $\mathcal{G}(t) \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{m_0}\}$, $m_0 > 0$, is an integer, where $\mathcal{G}_i = \{\mathcal{V}, \mathcal{E}_i, \mathcal{A}_i\}$, $i = 1, \ldots, m_0$. In this section, we assume that the time-varying topology $\mathcal{G}(t)$ satisfies the following assumption.

ASSUMPTION 4.1. *All the possible digraphs of* $\mathcal{G}(t)$ *are balanced.*

4.1. Stochastic consensus under additive noises. For the case with balanced topologies, we define $\lambda_2(t) = \lambda_2(\widehat{\mathcal{L}}(t)) \geq 0$. Let us introduce the following conditions for stochastic consensus:

(C1′) $\int_0^\infty \lambda_2(s)c(s)ds = \infty$.

(C4″) $\lim_{t\to\infty} \int_0^t e^{-2\int_s^t \lambda_2(u)c(u)du} c^2(s)ds = 0$.

THEOREM 4.2. *Suppose Assumptions* 2.1 *and* 4.1 *hold and* $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. *Then the protocol* (2.4) *is a mean square weak consensus protocol if conditions* (C1′) *and* (C4″) *hold. Moreover, the protocol* (2.4) *is an AUMSAC protocol if conditions* (C1′) *and* (C2) *hold.*

Proof. Similar to (3.3), we have

$$dx(t) = -c(t)(\mathcal{L}(t) \otimes I_n)x(t)dt + c(t) \sum_{i,j=1}^{N} a_{ij}(t)\sigma_{ji}(\eta_{N,i} \otimes \mathbf{1}_n)dw_{1ji}(t). \tag{4.1}$$

Let $J_N = \frac{1}{\sqrt{N}}\mathbf{1}_N\mathbf{1}_N^T$ and $\delta(t) = [(I_N - J_N) \otimes I_n]x(t)$. Under Assumption 4.1 and by the property of the balanced digraph, we get $d\delta(t) = -c(t)(\mathcal{L}(t) \otimes I_n)\delta(t)dt + c(t)\sum_{i,j=1}^N a_{ij}(t)\sigma_{ji}\ ((I_N - J_N)\eta_{N,i} \otimes \mathbf{1}_n)dw_{1ji}(t)$. Using the Itô formula, we have that for all $t \geq t_1 \geq 0$,

$$\begin{aligned} \|\delta(t)\|^2 = \|\delta(t_1)\|^2 &- 2\int_{t_1}^t c(s)\delta^T(s)(\widehat{\mathcal{L}}(s) \otimes I_n)\delta(s)ds \\ &+ n\frac{N-1}{N}\int_{t_1}^t p(s)c^2(s)ds + M_7(t) - M_7(t_1), \end{aligned} \tag{4.2}$$

where $p(t) = \sum_{i,j=1}^N a_{ij}(t)\sigma_{ji}^2 \leq \sum_{i,j=1}^N \sigma_{ji}^2 =: \bar{p}$ and $M_7(t) = 2\sum_{i,j=1}^N \int_0^t \delta^T(t)\ [(I_N - J_N)\eta_{N,i} \otimes \mathbf{1}_n]a_{ij}(s)\sigma_{ji}c(s)dw_{1ji}(s)$. Note that Theorem 3 in [34] gives

$$\delta^T(\widehat{\mathcal{L}}(t) \otimes I_n)\delta \geq \lambda_2(t)\|\delta\|^2 \quad \forall\ 1^T\delta = 0. \tag{4.3}$$

Using this property and taking expectations on the both sides of (4.2), we obtain that for any $t \geq t_1 \geq 0$,

$$\mathbb{E}\|\delta(t)\|^2 \leq \mathbb{E}\|\delta(t_1)\|^2 - 2\int_{t_1}^t \lambda_2(s)c(s)\mathbb{E}\|\delta(s)\|^2ds + n\frac{N-1}{N}\bar{p}\int_{t_1}^t c^2(s)ds. \tag{4.4}$$

By the comparison theorem, we also get that for any $t \geq 0$,

$$\mathbb{E}\|\delta(t)\|^2 \leq \|\delta(0)\|^2 e^{-2\int_0^t \lambda_2(s)c(s)ds} + n\frac{N-1}{N}\bar{p}\int_0^t e^{-2\int_s^t \lambda_2(u)c(u)du}c^2(s)ds. \tag{4.5}$$

Hence, by (C1′) and (C4″), $\lim_{t\to\infty} \mathbb{E}\|\delta(t)\|^2 = 0$, which together with the definition of $\delta(t)$ implies mean square weak consensus.

We now prove that (C1′) and (C2) guarantee mean square strong consensus. We first claim that (C1′) and (C2) imply (C4″). By (C2), for any $\varepsilon > 0$, there exists a positive constant $t_2 = t_2(\varepsilon)$, such that $\int_{t_2}^\infty c^2(s)ds < \varepsilon/2$, and then by (C1′) and (C2), there exists a positive constant $t_3 = t_3(\varepsilon) > t_2$, such that for any $t > t_3$,

$$e^{-\int_{t_2}^t \lambda_2(u)c(u)du} < \frac{\varepsilon}{2\int_0^\infty c^2(s)ds}.$$

Hence, for any $\varepsilon > 0$, there exists $T = T(\varepsilon) \geq t_3(\varepsilon)$, such that for any $t > T$,

$$\begin{aligned} \int_0^t e^{-\int_s^t \lambda_2(u)c(u)du}c^2(s)ds &= \int_0^{t_2} e^{-\int_s^t \lambda_2(u)c(u)du}c^2(s)ds + \int_{t_2}^t e^{-\int_s^t \lambda_2(u)c(u)du}c^2(s)ds \\ &\leq e^{-\int_{t_2}^t \lambda_2(u)c(u)du}\int_0^{t_2} c^2(s)ds + \int_{t_2}^\infty c^2(s)ds < \varepsilon. \end{aligned}$$

That is, condition (C4″) holds. This together with (C1′) implies mean square weak consensus.

Note that $\mathcal{G}(t)$ is balanced and the probability measure $\pi = \mathbf{1}_N/N$ satisfies the first assertion in Lemma 3.2. Similar to (3.5), we have from (4.1) that

$$\bar{x}(t) = \bar{x}(0) + \bar{M}(t), \tag{4.6}$$

where $\bar{x}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$ and $\bar{M}(t) = \frac{1}{N}\mathbf{1}_n \sum_{i,j=1}^N \int_0^t a_{ij}(s)\sigma_{ji}c(s)dw_{1ji}(s)$. It is easy to see from (C2) that

$$\mathbb{E}\|\bar{M}(t)\|^2 = \frac{n}{N^2}\int_0^t p(s)c^2(s)ds \le \frac{n\bar{p}}{N^2}\int_0^\infty c^2(s)ds < \infty. \tag{4.7}$$

Then by methods similar to those in the proof of Theorem 3.20, we can obtain the desired assertion. ∎

Remark 4.3. In Theorem 4.2, we give the sufficient conditions for mean square weak and strong consensus. In fact, we can also obtain some necessary conditions for mean square weak and strong consensus if $\mathcal{E}_i \neq \emptyset$, $i = 1, \ldots, m_0$. To show this, let $\lambda_N(t) = \lambda_N(\widehat{\mathcal{L}}(t))$; then the condition $\mathcal{E}_i \neq \emptyset$, $i = 1, \ldots, m_0$, gives $0 < \lambda_N(t) \le \bar{\lambda}_N := \max_{1\le i\le m_0} \lambda_N(\widehat{\mathcal{L}}_i)$. This together with $\delta^T(t)(\widehat{\mathcal{L}}(t)\otimes I_n)\delta(t) \le \lambda_N(t)\|\delta(t)\|^2$ and (4.2) can produce $\mathbb{E}\|\delta(t)\|^2 \ge \|\delta(0)\|^2 e^{-2\bar{\lambda}_N\int_0^t c(s)ds} + \int_0^t e^{-2\bar{\lambda}_N\int_s^t c(u)du}p(t)c^2(s)ds$. Note that $p(t) \ge \min_{i,j\in\mathcal{V}} \sigma_{ji}^2$. Hence, (C1) and $\lim_{t\to\infty}\int_0^t e^{-2\bar{\lambda}_N\int_s^t c(u)du}c^2(s)ds = 0$ are necessary for mean square weak consensus. Moreover, it can be observed from (4.7) that conditions (C1) and (C2) are necessary for mean square strong consensus under the condition $\mathcal{E}_i \neq \emptyset$, $i = 1, \ldots, m_0$.

Note that Theorem 4.2 does not tell us directly whether it can be relaxed that $\mathcal{G}(t)$ contains a spanning tree for all $t \ge 0$. In fact, Theorem 4.2 covers the case where the graph flow $\{\mathcal{G}(t),\, t \ge 0\}$ is switching and does not contain a spanning tree all the time. For the switching graph flow $\{\mathcal{G}(t),\, t \ge 0\}$, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \infty$ be a sequence of switching time instants with $\cup_{i=0}^\infty[\tau_i, \tau_{i+1}) = [0, \infty)$, such that $\mathcal{G}(t)$ is fixed over $[\tau_i, \tau_{i+1})$ and $G(\tau_{i+1}) \neq G(\tau_i)$, $i = 0, 1, 2, \ldots$. We have the following frequent connectivity assumption.

ASSUMPTION 4.4. *There exists a strictly increasing subsequence* $\{\tau_{i_k}\}_{k=0}^\infty \subseteq \{\tau_i\}_{i=0}^\infty$, *with* $\lim_{k\to\infty}\tau_{i_k} = \infty$ *such that* $\mathcal{G}(\tau_{i_k})$ *contains a spanning tree,* $k = 0, 1, 2, \ldots$.

Note that Assumption 4.4 includes the periodic connectivity [15, 30] as the special case. Bear in mind that the number of possible digraphs is finite. Then, under Assumption 4.4, we know that $\underline{\lambda}_2 := \min_{k\ge 0}\lambda_2(\tau_{i_k}) > 0$ and

$$\int_0^\infty \lambda_2(s)c(s)ds = \sum_{i=0}^\infty \int_{\tau_i}^{\tau_{i+1}} \lambda_2(\tau_i)c(s)ds \ge \underline{\lambda}_2 \sum_{j=0}^\infty \int_{\tau_{i_j}}^{\tau_{i_j+1}} c(s)ds.$$

Hence (C1′) is implied by the condition below.

(C1″) $\sum_{j=0}^\infty \int_{\tau_{i_j}}^{\tau_{i_j+1}} c(s)ds = \infty$.

Then Theorem 4.2 leads to the following corollary.

COROLLARY 4.5. *Suppose Assumptions* 2.1, 4.1, *and* 4.4 *hold and* $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. *Then the protocol* (2.4) *is a mean square weak consensus protocol if* (C1″) *and* (C4″) *hold. Moreover, the protocol* (2.4) *is an AUMSAC protocol if* (C1″) *and* (C2) *hold.*

Remark 4.6. Define the dwell time of $\{\mathcal{G}(t),\, t \ge 0\}$ as the infimum of the lengths of maximum time intervals over which $\mathcal{G}(t)$ remains fixed, i.e., $\inf_{i\ge 0}(\tau_{i+1} - \tau_i)$. If

$\{\mathcal{G}(t),\, t \geq 0\}$ has a positive dwell time and Assumption 4.4 holds with $\sup_{k\geq 0}(\tau_{i_{k+1}} - \tau_{i_k}) < \infty$, then (C1$''$) and (C2) are easily satisfied. For example, let $\tau_i = i\tau$, $i = 0, 1, \ldots,$ and $\tau_{i_j} = n_0\tau_j$, $j = 0, 1, \ldots,$ for some positive constant τ and integer n_0. Choose $c(t) = \frac{1}{(1+t)^{3/4}}$. Then (C2) holds. In addition, $\sum_{j=0}^{\infty} \int_{\tau_{i_j}}^{\tau_{i_j}+1} c(s)ds = \sum_{j=0}^{\infty} \int_{n_0 j\tau}^{n_0 j\tau+\tau} \frac{1}{(1+t)^{3/4}} dt \geq \sum_{j=0}^{\infty} \frac{\tau}{(1+n_0 j\tau+\tau)^{3/4}} = \infty$, which implies (C1$''$).

Applying the semimartingale convergence theorem, we have the following almost sure strong consensus.

THEOREM 4.7. *Suppose Assumptions* 2.1 *and* 4.1 *hold and* $f_{ji}(x) \equiv 0$, $i, j \in \mathcal{V}$. *Then the protocol* (2.4) *is an AUASAC protocol if conditions* (C1$'$) *and* (C2) *hold. Moreover, if Assumption* 4.4 *holds, then the protocol* (2.4) *is an AUASAC protocol if conditions* (C1$''$) *and* (C2) *hold.*

Proof. Applying the semimartingale convergence theorem to (4.2), we obtain that $\lim_{t\to\infty} \|\delta(t)\| < \infty$, a.s. By Theorem 4.2, we know that (C1$'$) (or (C1$''$)) and (C2) can guarantee the mean square strong consensus. Hence, $\lim_{t\to\infty} \mathbb{E}\|\delta(t)\|^2 = 0$, which implies that there exists a subsequence converging to zero almost surely. The uniqueness of limit admits $\lim_{t\to\infty} \|\delta(t)\| = 0$, a.s. Therefore, the almost sure weak consensus follows. Note that under (C2),

$$\lim_{t\to\infty} \langle \bar{M}\rangle(t) = \frac{n}{N^2} \int_0^{\infty} p(s)c^2(s)ds \leq \frac{n\bar{p}}{N^2} \int_0^{\infty} c^2(s)ds < \infty, \tag{4.8}$$

where $\bar{M}(t)$ is defined in (4.6). Hence, by Proposition 1.8 in [38, p. 183], we have that $\bar{M}(t)$ converges almost surely and the protocol (2.4) is an AUASAC protocol. ☐

4.2. Stochastic consensus under compound noises. In this subsection, we define $\kappa(t) = c(t)(1 - \bar{\sigma}^2 \frac{N-1}{N} c(t))$ and use the following condition.

(C6$'$) There exists a constant $t_0 > 0$ such that $\kappa(t) > 0$ for all $t \geq t_0$.

We have the following theorem.

THEOREM 4.8. *Suppose that Assumptions* 2.1, 2.2, *and* 4.1 *hold. Then it follows that the protocol* (2.4) *is a mean square weak consensus protocol if* $\int_0^{\infty} \lambda_2(s)\kappa(s)ds = \infty$, $\lim_{t\to\infty} \int_0^t e^{-2\int_s^t \lambda_2(u)\kappa(u)du} c^2(s)ds = 0$, *and* (C6$'$) *holds. Moreover, the protocol* (2.4) *is an AUMSAC and AUASAC protocol if* $\int_0^{\infty} \lambda_2(s)\kappa(s)ds = \infty$, (C2), *and* (C6$'$) *hold.*

Proof. By techniques similar to those used in obtaining (4.2), we can get

$$d\|\delta(t)\|^2 = -\widetilde{V}(t)dt + dM_8(t) + n\frac{N-1}{N} \sum_{i,j=1}^{N} a_{ij}(t)\sigma_{ji}^2 c^2(t)dt, \tag{4.9}$$

where $\widetilde{V}(t) = 2c(t)\delta^T(t)(\widehat{\mathcal{L}}(t) \otimes I_n)\delta(t) - c^2(t)\frac{N-1}{N} \sum_{i,j=1}^{N} a_{ij}(t)\|f_{ji}(\delta_j(t) - \delta_i(t))\|^2$ and $M_8(t) = 2\sum_{i,j=1}^{N} \int_0^t c(s)a_{ij}(s)\sigma_{ji}\delta^T(s)[(I_N - J_N)\eta_{N,i} \otimes \mathbf{1}_n]\, dw_{1ji}(s) + 2\sum_{i,j=1}^{N} \int_0^t c(s) a_{ij}(s)\delta^T(s)[(I_N - J_N)\eta_{N,i} \otimes f_{ji}(\delta_j(s) - \delta_i(s))]dw_{2ji}(s)$. Noting that the digraph $\mathcal{G}(t)$ is balanced for each $t \geq 0$ (Assumption 4.1), then the Laplacian matrix $\mathcal{L}(t)$ satisfies the following sum-of-squares property:

$$2\delta^T(t)(\widehat{\mathcal{L}}(t) \otimes I_n)\delta(t) = \sum_{i,j=1}^{N} a_{ij}(t)\|\delta_j(t) - \delta_i(t)\|^2. \tag{4.10}$$

By Assumption 2.2, we have

$$\sum_{i,j=1}^{N} a_{ij}(t)\|f_{ji}(\delta_j(t)-\delta_i(t))\|^2 \le 2\bar{\sigma}^2\delta^T(t)(\widehat{\mathcal{L}}(t)\otimes I_n)\delta(t). \tag{4.11}$$

Then $\widetilde{V}(t) \ge 2c(t)(1-\frac{N-1}{N}c(t)\bar{\sigma}^2)\delta^T(t)(\widehat{\mathcal{L}}(t)\otimes I_n)\delta(t)$. Substituting this inequality into (4.9) yields

$$d\|\delta(t)\|^2 \le -2\kappa(t)\delta^T(t)(\widehat{\mathcal{L}}(t)\otimes I_n)\delta(t)dt + C_6c^2(t)dt + dM_8(t), \tag{4.12}$$

where $C_6 = n\frac{N-1}{N}\sum_{i,j=1}^{N}\sigma_{ji}^2$. This together with (4.3) produces

$$d\|\delta(t)\|^2 \le -2\lambda_2(t)\kappa(t)\|\delta(t)\|^2dt + C_6c^2(t)dt + dM_4(t). \tag{4.13}$$

Note that (C6$'$) holds; then $\kappa(t) > 0$ for $t \ge t_0$. By the comparison theorem, we get that for any $t \ge t_0 \ge 0$,

$$\mathbb{E}\|\delta(t)\|^2 \le \mathbb{E}\|\delta(t_0)\|^2 e^{-2\int_{t_0}^{t}\lambda_2(s)\kappa(s)ds} + C_6\int_{t_0}^{t} e^{-2\int_s^t \lambda_2(u)\kappa(u)du}c^2(s)ds. \tag{4.14}$$

Hence, $\int_0^\infty \lambda_2(s)\kappa(s)ds = \infty$, and $\lim_{t\to\infty}\int_0^t e^{-2\int_s^t\lambda_2(u)\kappa(u)du}c^2(s)ds = 0$ implies $\lim_{t\to\infty}\mathbb{E}\|\delta(t)\|^2 = 0$, which together with the definition of $\delta(t)$ implies the mean square weak consensus. Then it remains only to repeat the corresponding part in the proofs of Theorems 3.20 and 3.22. ∎

COROLLARY 4.9. *Suppose that Assumptions* 2.1*,* 2.2*,* 4.1*, and* 4.4 *hold. If* (C1$''$)*,* (C2)*, and* (C6$'$) *hold, then the protocol* (2.4) *is an AUMSAC and AUASAC protocol.*

Proof. Note that $\underline{\lambda}_2 = \min_{k\ge 0}\lambda_2(\tau_{i_k}) > 0$. By the definition of $\kappa(t)$, we have

$$\begin{aligned}\int_0^\infty \lambda_2(s)\kappa(s)ds &= \int_0^\infty \lambda_2(s)c(s)ds - \frac{N-1}{N}\bar{\sigma}^2\int_0^\infty \lambda_2(s)c^2(s)ds\\ &\ge \underline{\lambda}_2\sum_{j=0}^{\infty}\int_{\tau_{i_j}}^{\tau_{i_j}+\tau} c(s)ds - \frac{N-1}{N}\bar{\sigma}^2\bar{\lambda}_N\int_0^\infty c^2(s)ds.\end{aligned}$$

Hence, (C1$''$) and (C2) imply $\int_0^\infty \lambda_2(s)\kappa(s)ds = \infty$, and then the desired assertions follow from Theorem 4.8. ∎

Remark 4.10. Combining Corollary 4.5, Theorem 4.7, and Corollary 4.9, we can see that if (C1$''$) and (C2) hold, then the multiplicative noises with small intensities satisfying $\bar{\sigma}^2 < \frac{N}{(N-1)\sup_{t\ge 0}c(t)}$ do not affect the control gain to guarantee mean square and almost sure strong consensus.

5. Simulation examples. We consider the stochastic consensus for a four-agent example. Consider a dynamic network of four scalar agents with the topology graph $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{A}\}$, where $\mathcal{V} = \{1,2,3,4\}$, $\mathcal{E} = \{(1,2),(2,3),(4,3),(3,2),\ (2,1)\}$, and $\mathcal{A} = [a_{ij}]_{4\times 4}$ with $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = 1$ and other values being zero. It is easy to see that the graph $\mathcal{G}$ contains a spanning tree. The initial value is $x(0) = [-7,4,3,-8]^T$.

Additive noise case. Assume that $\sigma_{ji} = 0.8$, $i, j = 1, 2, 3, 4$. We first choose the control gain $c(t)$ as $c(t) = (1+t)^{-1/3}$. Then it can be seen that conditions (C1) and (C5) hold. Hence, Theorem 3.12 tells us that the four agents achieve almost sure weak consensus, and the necessary condition (C2) in Theorem 3.9 implies that the four agents never achieve almost sure strong consensus under $c(t) = (1+t)^{-1/3}$. That is, all agents' states will converge in the future, but cannot converge to a common value, which is depicted in Figure 1. However, if we choose the control gain $c(t) = (1+t)^{-1}$, then conditions (C1)–(C2) hold. Theorem 3.9 gives that almost sure strong consensus is achieved. That is, all agents' states will tend to a common value, which is depicted in Figure 2.

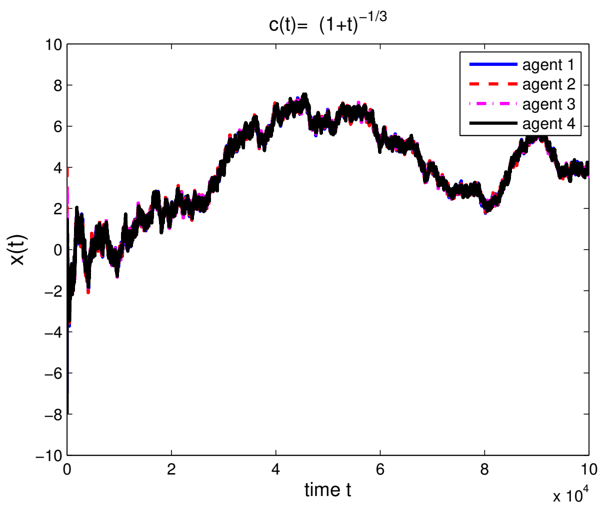

Fig. 1. *States of the four agents with additive noises:* $c(t) = (1+t)^{-1/3}$.

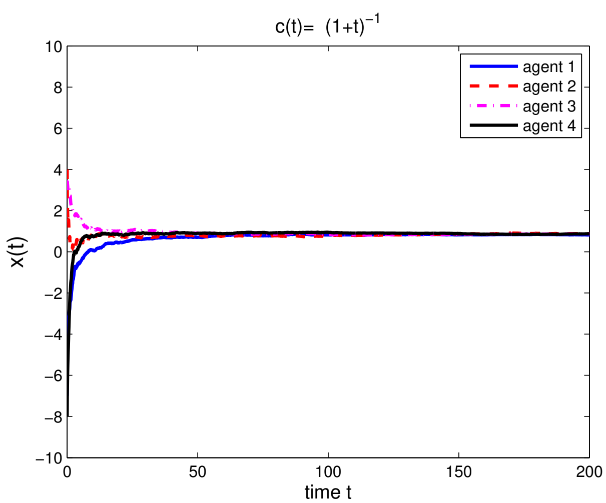

Fig. 2. *States of the four agents with additive noises:* $c(t) = (1+t)^{-1}$.

Under $c(t) = (1+t)^{-1/2}$, Corollary 3.7 gives mean square weak consensus. To simulate such behavior, we consider the relative state mean square errors $\{\mathbb{E}|x_i(t) - x_1(t)|^2\}_{i=2,3,4}$. We generate 10^3 sample paths. Then, taking the mean square average, we obtain Figure 3, which shows that the agents achieve mean square weak consensus.

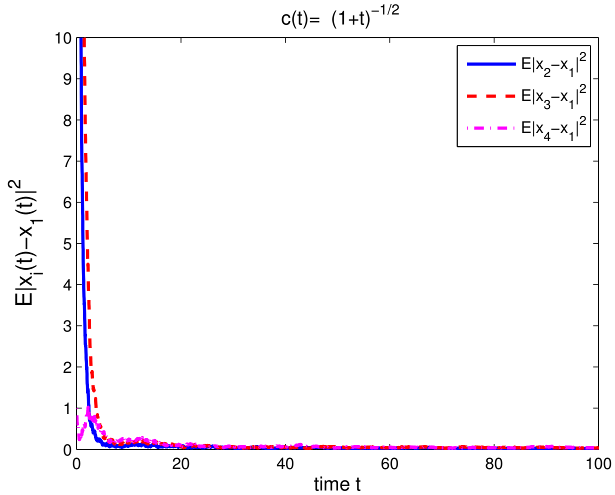

FIG. 3. *Mean square errors with additive noises:* $c(t) = (1+t)^{-1/2}$.

Compounding noise case. Assume additionally that $a_{43} = 1$; then the graph $\mathcal{G}$ is balanced. Let $f_{ji}(x) = 0.2x$, $i,j = 1,2,3,4$. We first see from Corollary 3.21 that the choice $c(t) = (1+t)^{-1/2}$ can guarantee the mean square weak consensus of the four agents with compound noises, which is revealed in Figure 4. If we choose $c(t) = (1+t)^{-1}$, then Theorem 3.22 gives that the four agents can achieve almost sure strong consensus, which is shown in Figure 5.

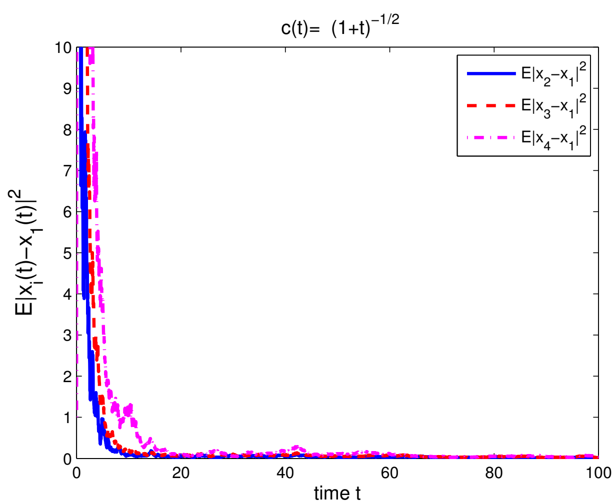

FIG. 4. *Mean square errors with compound noises:* $c(t) = (1+t)^{-1/2}$.

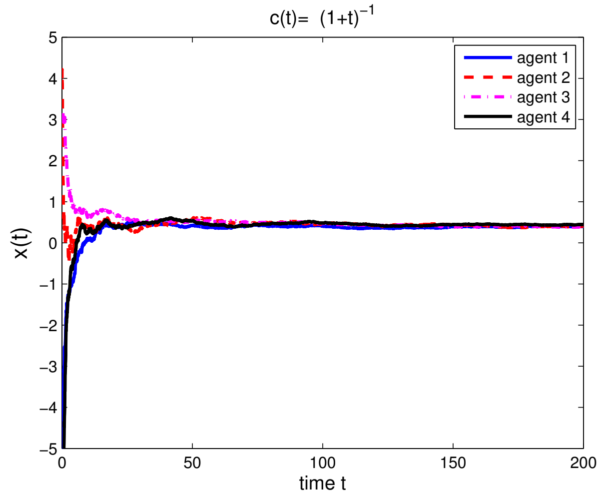

FIG. 5. *States of the four agents with compound noises:* $c(t) = (1+t)^{-1}$.

6. Concluding remarks. In this work, consensus conditions have been examined for the multi-agent systems with additive and multiplicative measurement noises. Based on the matrix theory and the algebraic graph theory, we utilize the variable transformation to transform the closed-loop system into a nonautonomous stochastic differential equation (SDE) driven by the additive or the compound noises. By establishing the stochastic stability of SDEs with additive noises, some necessary conditions and sufficient conditions were obtained for mean square weak consensus and almost sure weak and strong consensus under fixed topologies and additive noises. When the multiplicative noises appear, some necessary conditions and sufficient conditions for mean square and almost sure consensus were obtained. The efforts have also been devoted to the networks with time-varying topologies, and we have shown that mean square and almost sure consensus can be guaranteed under the periodical connectivity of the topology flow.

When the additive noise vanishes, the current results actually show that the time-varying control gain can be used to guarantee stochastic consensus of multi-agent systems with multiplicative noises. In the future, it will be of interest to consider the second-order consensus and containment control, and take the time-delay into consideration.

Appendix A. Proofs of lemmas in section 3.

LEMMA A.1 (semimartingale convergence theorem [23]). *Let $A_1(t)$ and $A_2(t)$ be the two $\mathfrak{F}_t$-adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$, a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$, a.s., and let ζ be a nonnegative $\mathfrak{F}_0$-measurable random variable. Assume that $X(t)$ is nonnegative and*

$$X(t) = \zeta + A_1(t) - A_2(t) + M(t), \quad t \geq 0.$$

If $\lim_{t\to\infty} A_1(t) < \infty$, a.s., then for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} X(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} A_2(t) < \infty.$$

Proof of Lemma 3.2. If the digraph $\mathcal{G}$ contains a spanning tree, then from Lemma 4 in [14], there exists a unique probability measure π such that $\pi^T\mathcal{L} = 0$. Hence, for assertion 1, we need only consider the case that the digraph $\mathcal{G}$ does not contain a spanning tree. In this case, there exist at least two separate subgroups or at least two agents in the group who do not receive any information. Then there exists an elementary translational transformation S such that

$$S^T\mathcal{L}S = \begin{pmatrix} \mathcal{L}_{11} & 0 & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ 0 & 0 & \mathcal{L}_{33} \end{pmatrix}, \tag{A.1}$$

where $\mathcal{L}_{11}$ is a Laplacian matrix related to a nonempty communicating class $C = \{i_1, \dots, i_s\}$, $s < N$, and $\mathcal{L}_{21}, \mathcal{L}_{22}, \mathcal{L}_{23}, \mathcal{L}_{33}$ are matrices with the appropriate dimensions. Therefore, there exists a probability measure $\widetilde{\pi}$ such that $\widetilde{\pi}^T\mathcal{L}_{11} = 0$. Let $\widehat{\pi} = (\widetilde{\pi}^T, 0, \dots, 0)^T$; then $\widehat{\pi}^T S^T \mathcal{L} S = 0$. Let $\pi = S\widehat{\pi}$; then $\pi^T\mathcal{L} = 0$ due to the reversibility of S.

For assertion 2, we introduce the following class of $N \times (N-k)$ matrices:

$$\mathcal{C} = \Big\{\phi \in \mathbb{R}^{N\times(N-k)} | \mathrm{span}\{\phi\} = \mathrm{span}\{\mathcal{L}\}\Big\}, \tag{A.2}$$

where $k \geq 1$ denotes the number of zero eigenvalues and $\mathrm{span}\{\mathcal{L}\}$ denotes the linear space spanned by the columns of $\mathcal{L}$. Then $\mathrm{rank}(\mathcal{L}) = N - k$, and each $\phi \in \mathcal{C}$ has $\mathrm{rank}(\phi) = N - k$. Denote $\mathcal{S} = \mathrm{span}\{\phi\} = \mathrm{span}\{\mathcal{L}\}$. We claim that $\mathbf{1}_N \notin \mathcal{S}$. Otherwise, $\mathbf{1}_N \in \mathcal{S}$, then there exists $\xi \in \mathbb{R}^N$ such that $\mathbf{1}_N = \mathcal{L}\xi$, and then $0 < \pi^T\mathbf{1}_N = \pi^T\mathcal{L}\xi = 0$, which is a contradiction. Hence, we have $\mathrm{rank}((\frac{1}{\sqrt{N}}\mathbf{1}_N, \phi)) = N - k + 1$. We can choose $\varphi \in \mathbb{R}^{N\times(k-1)}$ such that $\mathrm{rank}((\frac{1}{\sqrt{N}}\mathbf{1}_N, \phi, \varphi)) = N$. Let $\widetilde{Q} = (\phi, \varphi)$; then $Q = (\frac{1}{\sqrt{N}}\mathbf{1}_N, \widetilde{Q})$ is nonsingular. Let $Q^{-1} = [\begin{smallmatrix} \nu^T \\ Q \end{smallmatrix}]$, where ν^T is the first row of Q^{-1}. Hence, $\frac{1}{\sqrt{N}}\nu^T\mathbf{1}_N = 1$ and $\nu^T\phi = 0$. Note that $\mathcal{L} \in \mathrm{span}\{\phi\}$. Then there exists an $(N-k) \times N$ matrix Γ such that $\mathcal{L} = \phi\Gamma$, which implies $\nu^T\mathcal{L} = 0$. Therefore, the second matrix equality in (3.1) holds. Assertion 3 and the special case are from [14] and [12]. $\square$

Proof of Lemma 3.3. By the Jordan matrix decomposition, there exists an invertible matrix P_1 such that $P_1^{-1}DP_1 = J_D$. Here, J_D is the Jordan normal form of D, i.e.,

$$J_D = \mathrm{diag}(J_{\lambda_1,n_1}, J_{\lambda_2,n_2}, \dots, J_{\lambda_l,n_l}), \quad \sum_{k=1}^{l} n_k = m,$$

where $\lambda_1, \lambda_2, \dots, \lambda_l$ are all the eigenvalues of D and

$$J_{\lambda_k,n_k} = \begin{pmatrix} \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & \lambda_k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix}$$

is the corresponding Jordan block of size n_k with eigenvalue λ_k. Letting $\zeta(t) = P_1^{-1}X(t)$, we have

$$d\zeta(t) = -c_1(t)J_D\zeta(t)dt + \sum_{i=1}^{d} P_{1i}c_2(t)dw_i(t), \tag{A.3}$$

where $P_{1i} = P_1^{-1}\sigma_i$. Considering the kth Jordan block and the corresponding components $\zeta_k = [\zeta_{k,1}, \dots, \zeta_{k,n_k}]^T$ and $P_{1i}(k) = [p^i_{k,1}, \dots, p^i_{k,n_k}]^T$, where $\zeta_{k,j}$ and $p^i_{k,j}$ are the $(\sum_{i=1}^{k-1} n_i + j)$th elements of ζ and P_{1i}, respectively, we have

$$d\zeta_k(t) = -c_1(t)(J_{\lambda_k,n_k})\zeta_k(t)dt + \sum_{i=1}^{d} P_{1i}(k)c_2(t)dw_i(t).$$

This implies that

$$d\zeta_{k,n_k}(t) = -c_1(t)\lambda_k\zeta_{k,n_k}(t)dt + \sum_{i=1}^{d} p^i_{k,n_k}c_2(t)dw_i(t) \tag{A.4}$$

and for $j = 1, \dots, n_k - 1$,

$$d\zeta_{k,j}(t) = -c_1(t)\lambda_k\zeta_{k,j}(t)dt - c_1(t)\zeta_{k,j+1}(t)dt + \sum_{i=1}^{d} p^i_{k,j}c_2(t)dw_i(t), \tag{A.5}$$

which are semidecoupled equations. By means of a variation of constants formula for (A.4) and (A.5), we obtain

$$\zeta_{k,n_k}(t) = e^{-\lambda_k\int_0^t c_1(u)du}\zeta_{k,n_k}(0) + Z_{k,n_k}(t) \tag{A.6}$$

and

$$\zeta_{k,j}(t) = e^{-\lambda_k\int_0^t c_1(u)du}\zeta_{k,j}(0) + Z_{k,j}(t) - \int_0^t e^{-\lambda_k\int_s^t c_1(u)du}c_1(s)\zeta_{k,j+1}(s)ds, \tag{A.7}$$

where $Z_{k,j}(t) = \sum_{i=1}^d p^i_{k,j}\int_0^t e^{-\lambda_k\int_s^t c_1(u)du}c_2(s)dw_i(s)$, $j = 1, \dots, n_k$. Then the mean square asymptotic stability is equivalent to $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$, $k = 1, \dots, l$, $j = 1, 2, \dots, n_k$ for any initial value $X(0)$.

Sufficiency. We can see from (A.6) that

$$\begin{aligned}\mathbb{E}\|\zeta_{k,n_k}(t)\|^2 &= e^{-2\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\|\zeta_{k,n_k}(0)\|^2\\ &\quad + C_{k,n_k}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du}c_2^2(s)ds,\end{aligned} \tag{A.8}$$

where $C_{k,n_k} = \sum_{i=1}^d \|p^i_{k,n_k}\|^2$. Note that $\mathrm{Re}(\lambda_k) > 0$, $\int_0^\infty c_1(s)ds = \infty$, and $\lim_{t\to\infty}\int_0^t e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds = 0$. Hence, $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,n_k}(t)\|^2 = 0$. We now use the backstepping method to prove $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$, $k = 1, \dots, l$, $j = 1, 2, \dots, n_k$. That is, assuming that $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j+1}(t)\|^2 = 0$ for some fixed $j < n_k$, we will show $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$. It is easy to see from (A.7) that

$$\begin{aligned}\mathbb{E}\|\zeta_{k,j}(t)\|^2 &\le 2e^{-2\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\|\zeta_{k,j}(0)\|^2 + C_{k,j}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du}c_2^2(s)ds\\ &\quad + 2\mathbb{E}\Big(\int_0^t e^{-\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du}c_1(s)\|\zeta_{k,j+1}(s)\|ds\Big)^2,\end{aligned}$$

where $C_{k,j} = \sum_{i=1}^d \|p^i_{k,j}\|^2$. Note that the first two terms tend to zero; then we need only prove that the last term vanishes at infinite time. Let k, j be fixed, and write $S_{k,j}(t) = \int_0^t e^{-\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_1(s)\|\zeta_{k,j+1}(s)\|ds$. By Minkowski's inequality for integrals, we have

$$\sqrt{\mathbb{E}(S_{k,j}(t))^2} \le \int_0^t e^{-\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_1(s)\sqrt{\mathbb{E}\|\zeta_{k,j+1}(s)\|^2}ds.$$

Let $Q(t) = \int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du} c_1(s)\sqrt{\mathbb{E}\|\zeta_{k,j+1}(s)\|^2}ds$. Then if $\lim_{t\to\infty} Q(t) < \infty$, we get from $\int_0^\infty c_1(s)ds = \infty$ that

$$\lim_{t\to\infty}\sqrt{\mathbb{E}(S_{k,j}(t))^2} \le \lim_{t\to\infty} e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du} Q(t) = 0.$$

Note that $\lim_{t\to\infty}\mathbb{E}|\zeta_{k,j+1}(s)|^2 = 0$; then if $\lim_{t\to\infty} Q(t) = \infty$, l'Hôpital's rule gives

$$\lim_{t\to\infty}\sqrt{\mathbb{E}(S_{k.j}(t))^2} \le \lim_{t\to\infty}\frac{Q(t)}{e^{\mathrm{Re}(\lambda_k)\int_0^t c(u)du}} = \lim_{t\to\infty}\frac{\sqrt{\mathbb{E}\|\zeta_{k,j+1}(t)\|^2}}{\mathrm{Re}(\lambda_k)} = 0.$$

Hence, we have $\lim_{t\to\infty}\mathbb{E}\|S_{k,j}(t)\|^2 = 0$. Above all, we have shown that

$$\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0 \quad \forall j < n_k.$$

Repeating the induction above gives $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$ for all $j = 1,\dots,n_k$, and therefore, $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$ for all $k = 1,\dots,l$ and $j = 1,\dots,n_k$. That is, the mean square asymptotic stability follows, and the sufficiency is proved.

Necessity. Assume that $\|\zeta(0)\|^2 \ne 0$, $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,j}(t)\|^2 = 0$, $k = 1,\dots,l$, $j = 1,\dots,n_k$. From (A.8), we can see that for any $\|\zeta_{k,n_k}(0)\|^2 \ne 0$, $\lim_{t\to\infty}\mathbb{E}\|\zeta_{k,n_k}(t)\|^2 = 0$ implies $\int_0^\infty c_1(s)ds = \infty$, $\mathrm{Re}(\lambda_k) > 0$, and

$$C_{k,n_k}\lim_{t\to\infty}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_2^2(s)ds = 0, \quad k = 1,\dots,l. \tag{A.9}$$

Hence, $\int_0^\infty c_1(s)ds = \infty$ and $\mathrm{Re}(\lambda_k) > 0$, $k = 1,\dots,l$. It remains to show that $\lim_{t\to\infty}\int_0^t e^{-2\bar\lambda\int_s^t c_1(u)du} c_2^2(s)ds = 0$.

Note that $\sigma_i \ne 0$ for certain i and P_1 is invertible. Hence, there must exist k, j such that $C_{k,j} > 0$, $0 < j \le n_k$. Let k be fixed. If $C_{k,n_k} > 0$, then (A.9) gives $\lim_{t\to\infty}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_2^2(s)ds = 0$, which also produces $\lim_{t\to\infty}\int_0^t e^{-2\bar\lambda\int_s^t c_1(u)du} c_2^2(s)ds = 0$. If $C_{k,n_k} = 0$, we define $l_0 = \max\{n_k > j > 1, C_{k,j} = \sum_{i=1}^d \|p_{k,j}^i\|^2 > 0\}$. Then we have from (A.7) that

$$\mathbb{E}\|Z_{k,l_0}(t)\|^2 \le 3\mathbb{E}\|\zeta_{k,l_0}(t)\|^2 + 3e^{-2\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\|\zeta_{k,l_0}(0)\|^2 + 3\mathbb{E}\|S_{k,l_0}(t)\|^2. \tag{A.10}$$

Employing methods similar to those used in proving the sufficiency, we can obtain $\lim_{t\to\infty}\mathbb{E}\|S_{k.l_0}(t)\|^2 = 0$, and then $\lim_{t\to\infty}\mathbb{E}\|Z_{k,l_0}(t)\|^2 = 0$. Note that

$$\mathbb{E}\|Z_{k,l_0}(t)\|^2 = C_{k,l_0}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_2^2(s)ds,$$

where $C_{k,l_0} > 0$. Therefore, the necessity is proved. □

Proof of Lemma 3.4. The proof is split into the following four parts:

(a) We prove that $\int_0^\infty c_1(s)ds = \infty$ and $\int_0^\infty c_2^2(s)ds < \infty$ imply the almost sure asymptotic stability. We first show that $\int_0^\infty c_1(s)ds = \infty$ and $\int_0^\infty c_2^2(s)ds < \infty$

imply $\lim_{t\to\infty}\int_0^t e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds = 0$. For any $\varepsilon > 0$, by $\int_0^\infty c_2^2(s)ds < \infty$, there exists a positive constant $t_0 = t_0(\varepsilon)$, such that $\int_{t_0}^\infty c_2^2(s)ds < \varepsilon/2$, and then by $\int_0^\infty c_1(s)ds = \infty$, there exists a positive constant $t_1 = t_1(\varepsilon) > t_0$, such that $e^{-2\underline{\lambda}\int_{t_0}^t c_1(u)du} < \varepsilon/(2\int_0^\infty c_2^2(s)ds)$. Hence, for any $\varepsilon > 0$, there exists $T = T(\varepsilon) \geq t_1(\varepsilon)$, such that for any $t > T$,

$$\begin{aligned}\int_0^t e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds &\leq \int_0^{t_0} e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds + \int_{t_0}^t e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds \\ &\leq e^{-2\underline{\lambda}\int_{t_0}^t c_1(u)du}\int_0^{t_0} c_2^2(s)ds + \int_{t_0}^\infty c_2^2(s)ds \leq \varepsilon.\end{aligned}$$

That is, $\lim_{t\to\infty}\int_0^t e^{-2\underline{\lambda}\int_s^t c_1(u)du}c_2^2(s)ds = 0$. Then, by Lemma 3.3, $\lim_{t\to\infty}\mathbb{E}\|X(t)\|^2 = 0$ for any $X(0)\in\mathbb{R}^m$. Note that every eigenvalue of D has strictly positive real part; then there exists a positive definite matrix P such that $D^TP + PD = I_m$. Letting $V(t) = X(t)^TPX(t)$ and applying the Itô formula yields

(A.11)

$$V(t) = V(0) - \int_0^t c_1(s)\|X(s)\|^2ds + C\int_0^t c_2^2(s)ds + 2\int_0^t \sum_{i=1}^d X(s)^TP\sigma_i c_2(s)dw_i(s),$$

where $C = \sum_{i=1}^d \sigma_i^TP\sigma_i > 0$. Then the semimartingale convergence theorem gives that $\lim_{t\to\infty} V(t) < \infty$, a.s., which together with the definiteness of P implies $\lim_{t\to\infty}\|X(t)\| < \infty$, a.s. Notice that $\lim_{t\to\infty}\mathbb{E}\|X(t)\|^2 = 0$ implies that there exists a subsequence converging to zero almost surely. The uniqueness of limit admits $\lim_{t\to\infty}\|X(t)\| = 0$, a.s. Therefore, the almost sure asymptotic stability follows.

(b) We prove that if $\int_0^\infty c_1(s)ds = \infty$ and $\lim_{t\to\infty} c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t) = 0$, then the solution to (3.2) is almost sure asymptotically stable. Note that the almost sure asymptotic stability is equivalent to $\lim_{t\to\infty}\|\zeta_{k,j}(t)\| = 0$, a.s., for all $k = 1,\dots,l$, $j = 1,2,\dots,n_k$, and any initial value $X(0)$, where $\zeta_{k,j}(t)$ is defined by (A.4) and (A.5).

We first fix k and show $\lim_{t\to\infty}\|\zeta_{k,n_k}(t)\| = 0$, a.s., under $\int_0^\infty c_1(s)ds = \infty$, $\lim_{t\to\infty} c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t) = 0$, and $\mathrm{Re}(\lambda_k) > 0$. It can be seen that $\lim_{t\to\infty} e^{-\lambda_1\int_0^t c_1(s)ds} = 0$. Then in order for $\lim_{t\to\infty}\|\zeta_{k,n_k}(t)\| = 0$, a.s., we need to show that $\lim_{t\to\infty} Z_{k,n_k}(t) = 0$, a.s. Let

$$M_{k,n_k}(t) = \sum_{l=1}^d p_{k,n_k}^l\int_0^t c_2(s)dw_i(s),$$

$$M_k^1(t) = \sum_{l=1}^d \mathrm{Re}(p_{k,n_k}^l)\int_0^t c_2(s)dw_i(s), \qquad M_k^2(t) = \sum_{l=1}^d \mathrm{Im}(p_{k,n_k}^l)\int_0^t c_2(s)dw_i(s);$$

then $M_{k,n_k}(t) = M_k^1(t) + iM_k^2(t)$, where $i^2 = -1$, and $M_k^1(t)$ and $M_k^2(t)$ are real-valued martingales. Letting $\bar{M}_k(t) = \int_0^t e^{\lambda_k\int_0^s c_1(u)du}dM_{k,n_k}(s)$, we get

$$\begin{aligned}\bar{M}_k(t) \;=\; &\int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\cos(p_k(s))dM_k^1(s) \\ &- \int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\sin(p_k(s))dM_k^2(s) \\ &+ i\int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\sin(p_k(s))dM_k^1(s)\end{aligned}$$

$$
\begin{aligned}
&+i\int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\cos(p_k(s))dM_k^2(s)\\
&=: \bar{m}_{k1}(t)-\bar{m}_{k2}(t)+i\bar{m}_{k3}(t)+i\bar{m}_{k4}(t),
\end{aligned}
\tag{A.12}
$$

where $p_k(s)=\mathrm{Im}(\lambda_k)\int_0^s c_1(u)du$. Note that

$$
\begin{aligned}
Z_{k,n_k}(t)&=e^{-\lambda_k\int_0^t c_1(u)du}\bar{M}_k(t)\\
&=e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\big(\cos(p_k(t))+i\sin(p_k(t))\big)\bar{M}_k(t).
\end{aligned}
\tag{A.13}
$$

Hence, for $\lim_{t\to\infty} Z_{k,n_k}(t)=0$, a.s., it is enough to show that

$$
z_{kj}(t)=e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\cos(p_k(t))\bar{m}_{kj}(t)
$$

and

$$
\bar{z}_{kj}(t)=e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\sin(p_k(t))\bar{m}_{kj}(t),\quad j=1,2,3,4,
$$

tend to zero almost surely as $t\to\infty$. Here, we show only $\lim_{t\to\infty} z_{k1}(t)=0$, a.s., and the others can be obtained similarly. Note that $\bar{m}_{k1}(t)$ is a continuous martingale, vanishing at zero. Then we have that either

$$
\lim_{t\to\infty}\langle\bar{m}_{k1}\rangle(t)=C_{k,n_k}\int_0^\infty e^{2\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\cos^2(p_k(s))c_2^2(s)ds<\infty
\tag{A.14}
$$

or

$$
\lim_{t\to\infty}\langle\bar{m}_{k1}\rangle(t)=\infty,
\tag{A.15}
$$

where $C_{k,n_k}=\sum_{i=1}^d|\mathrm{Re}(p_{k,n_k}^i)|^2$. In case (A.14), the martingale $M_1(t)$ converges almost surely to a finite limit by the martingale convergence theorem [38, Proposition 1.8, p. 183]. Noting $\int_0^\infty c_1(s)ds=\infty$, $\mathrm{Re}(\lambda_k)>0$, and

$$
z_{k1}(t)=e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\cos(p_k(t))\bar{m}_{k1}(t),
$$

we immediately have the desired result $\lim_{t\to\infty} z_{k1}(t)=0$, a.s. In case (A.15), we use the law of the iterated logarithm for martingales (see [38, p. 186]),

$$
\limsup_{t\to\infty}\frac{|\bar{m}_{k1}(t)|}{\sqrt{2\langle\bar{m}_{k1}\rangle(t)\log\log(\langle\bar{m}_{k1}\rangle(t))}}=1,\quad \text{a.s.}
\tag{A.16}
$$

Thus, for all ω in an a.s. event, there is a finite $T(\omega)>0$ such that for all $t>T(\omega)$, $\log\log(\langle\bar{m}_{k1}\rangle(t))>0$ and

$$
|z_{kj}(t)|\le 2\sqrt{e^{-2\mathrm{Re}(\lambda_k)\int_0^t c_1(s)ds}\langle\bar{m}_{k1}\rangle(t)\log\log(\langle\bar{m}_{k1}\rangle(t))},\quad \text{a.s.}
\tag{A.17}
$$

Let $C(t)=\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du}\cos^2(p_k(s))c_2^2(s)ds\log\int_0^t c_1(s)ds$. By l'Hôpital's rule and $\lim_{t\to\infty} c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$, we have

$$
\begin{aligned}
\lim_{t\to\infty}C(t)&=\lim_{t\to\infty}\frac{\int_0^t e^{2\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du}\cos^2(p_k(s))c_2^2(s)ds}{e^{2\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\log^{-1}\int_0^t c_1(s)ds}\\
&=\lim_{t\to\infty}\frac{c_2^2(t)\log\int_0^t c_1(u)du\cos^2(p_k(t))/c_1(t)}{2\mathrm{Re}(\lambda_k)-\frac{1}{\int_0^t c_1(u)du\log\int_0^t c_1(s)ds}}=0.
\end{aligned}
\tag{A.18}
$$

Hence, for any $\epsilon \in (0,1)$ satisfying $C_{k,n_k}\epsilon < 1$, there exists $T(\epsilon) > 0$ such that $\int_0^t c_1(s)ds > e$ and $\int_0^t e^{2\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du} c_2^2(s)ds \le \epsilon e^{2\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}$ for all $t > T(\epsilon)$. This also implies

$$\log\log\left(C_{k,n_k}\int_0^t e^{2\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du} c^2(s)ds\right) \le \log(2\mathrm{Re}(\lambda_k)) + \log\int_0^t c_1(u)du.$$

Thus, for $t > \max\{T(\omega), T(\epsilon)\}$, we have from (A.17)

$$|z_{k1}(t)|^2 \le 4C_{k,n_k}\int_0^t e^{-2\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_2^2(s)ds\left[\log(2\mathrm{Re}(\lambda_k)) + \log\int_0^t c_1(s)ds\right].$$

This together with (A.18) gives $\lim_{t\to\infty}\|z_{k1}(t)\| = 0$, a.s. Similarly, we have $\lim_{t\to\infty}\|z_{kj}(t)\| = 0$, a.s., and $\lim_{t\to\infty}\|\bar{z}_{kj}(t)\| = 0$, a.s., $j = 1,2,3,4$. Hence, $\lim_{t\to\infty} Z_{k,n_k}(t) = 0$, a.s. Therefore, $\lim_{t\to\infty}\|\zeta_{k,n_k}(t)\| = 0$, a.s.

We now use the the backstepping method to prove $\lim_{t\to\infty}\|\zeta_{k,j}(t)\| = 0$, a.s., for all $j = 1,\dots,n_k$. Assume that $\lim_{t\to\infty}\|\zeta_{k,j+1}(t)\| = 0$, a.s., for certain $j < n_k$. We will show that $\lim_{t\to\infty}\|\zeta_{k,j}(t)\| = 0$, a.s. Note that $\zeta_{k,j}(t)$ satisfies (A.5). Then we have

$$|\zeta_{k,j}(t)| \le e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\|\zeta_{k,j}(0)\| + \|Z_{k,j}(t)\| + S_{k,j}(t),$$

where $S_{k,j}(t) = \int_0^t e^{-\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_1(s)\|\zeta_{k,j+1}(s)\|ds$. Similarly to the estimation of $\zeta_{k,n_k}(t)$ above, we can obtain

$$\lim_{t\to\infty} e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\|\zeta_{k,j}(0)\| = 0 \quad \text{and} \quad \lim_{t\to\infty}\|Z_{k,j}(t)\| = 0, \quad \text{a.s.}$$

Hence, we need only show

$$\lim_{t\to\infty} S_{k.j}(t) = \lim_{t\to\infty}\int_0^t e^{-\mathrm{Re}(\lambda_k)\int_s^t c_1(u)du} c_1(s)\|\zeta_{k,j+1}(s)\|ds = 0, \quad \text{a.s.}$$

Let k, j be fixed, and let $\bar{Q}(t) = \int_0^t e^{\mathrm{Re}(\lambda_k)\int_0^s c_1(u)du} c_1(s)\|\zeta_{k,j+1}(s)\|ds$. If $\lim_{t\to\infty}\bar{Q}(t) < \infty$, then from $\int_0^\infty c_1(s)\ ds = \infty$, we get

$$\lim_{t\to\infty} S_{k,j}(t) \le \lim_{t\to\infty} e^{-\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}\bar{Q}(t) = 0.$$

Otherwise, $\lim_{t\to\infty}\bar{Q}(t) = \infty$; then, by l'Hôpital's rule, we still have

$$\lim_{t\to\infty} S_{k,j}(t) = \lim_{t\to\infty}\frac{\bar{Q}(t)}{e^{\mathrm{Re}(\lambda_k)\int_0^t c_1(u)du}} = \frac{1}{\mathrm{Re}(\lambda_k)}\lim_{t\to\infty}\|\zeta_{k,j+1}(s)\| = 0, \quad \text{a.s.}$$

Hence, $\lim_{t\to\infty}\|\zeta_{k,j}(t)\| = 0$, a.s. By repeating the induction above for $j = 1,\dots,n_k$, we have $\lim_{t\to\infty}\|\zeta_{k,j}(t)\| = 0$, a.s., for all $j = 1,\dots,n_k$. Repeating the process above for $k = 1,2,\dots,l$ gives $\lim_{t\to\infty}\|X(t)\| = 0$, a.s., and then the almost sure asymptotic stability follows.

(c) We prove that almost sure asymptotical stability implies $\mathrm{Re}(\lambda_k) > 0$, $k = 1,\dots,l$, and $\int_0^\infty c_1(s)ds = \infty$.

We now assume that the solution to (3.2) is almost sure asymptotically stable. By methods similar to those used in the proof of Lemma 3.3, we can obtain (A.6). Note that the first term on the right-hand side of (A.6) is deterministic and convergent for each $\zeta_{k,n_k}(0)$. Hence, $\lim_{t\to\infty}\|\zeta_{k,n_k}(t)\|=0$, a.s., gives $\lim_{t\to\infty}e^{-\lambda_k\int_0^t c_1(u)du}=0$, which also implies $\mathrm{Re}(\lambda_k)>0$, $k=1,\dots,l$, and $\int_0^\infty c_1(s)ds=\infty$.

(d) We prove that if all the eigenvalues of the matrix D are real, then almost sure asymptotical stability implies $\mathrm{Re}(\lambda_k)>0$, $k=1,\dots,l$, $\int_0^\infty c_1(s)ds=\infty$, and $\liminf_{t\to\infty}c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$.

Note that the necessity of every eigenvalue of D having strictly positive real part and $\int_0^\infty c_1(s)ds=\infty$ is proved in (c). Hence, it remains to prove necessity of $\liminf_{t\to\infty}c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$ for almost sure stability under $\int_0^\infty c_1(s)ds=\infty$ and $\lambda_k>0$, $k=1,\dots,l$.

Due to the fact that all the eigenvalues of the matrix D are real, the Jordan matrix decomposition in the proof of Lemma 3.3 can be obtained under a real invertible matrix P_1 such that $P_1^{-1}DP_1=J_D$. In view of this, the coefficients in (A.4) and (A.5) are real. Note that the almost sure asymptotic stability gives that for any initial value $\zeta(0)$, $\lim_{t\to\infty}\|\zeta_{k,n_k}(t)\|=0$, a.s., $k=1,2,\dots,l$. This together with (A.6) implies that

$$\lim_{t\to\infty}\|Z_{k,n_k}(t)\|=0,\quad \text{a.s.} \tag{A.19}$$

Let $\hat{M}_{k,j}(t)=\sum_{i=1}^d p_{k,j}^i\int_0^t e^{\lambda_k\int_0^s c_1(u)du}c_2(s)dw_i(s)$, $j=1,\dots,n_k$. Then $Z_{k,j}(t)=e^{-\lambda_k\int_0^t c_1(u)du}M_{k,j}$, and

$$\lim_{t\to\infty}\langle\hat{M}_{k,j}\rangle(t)=C_{k,j}\int_0^t e^{2\lambda_k\int_0^s c_1(u)du}c_2^2(s)ds,\quad j=1,2,\dots,n_k, \tag{A.20}$$

where $C_{k,j}=\sum_{i=1}^d\|p_{k,j}^i\|^2$. Note that $\sigma_i\neq 0$ for certain i and P_1 is invertible. Hence there must exist k,j such that $C_{k,j}>0$, $0<j\le n_k$. Let k be fixed. We first assume that $C_{k,n_k}>0$ and claim that (A.19) implies $\liminf_{t\to\infty}c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$ holds. We suppose now, contrary to the desired result, that

$$\liminf_{t\to\infty}c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)\neq 0;$$

then there is $\underline{c}>0$ such that

$$\liminf_{t\to\infty}\frac{c_2^2(t)\log\int_0^t c_1(u)du}{c_1(t)}>\underline{c}, \tag{A.21}$$

which implies that for some $T>0$,

$$\frac{c_2^2(t)\log\int_0^t c_1(u)du}{c_1(t)}>\frac{\underline{c}}{2},\quad t>T. \tag{A.22}$$

Then we have that for $t>T$,

$$\int_0^t e^{2\lambda_k\int_0^s c_1(u)du}c_2^2(s)ds\ge\frac{\underline{c}}{2}\frac{\int_T^t e^{2\lambda_k\int_0^s c_1(u)du}c_1(s)ds}{\log\int_0^t c_1(u)du}, \tag{A.23}$$

which together with (A.20) gives $\lim_{t\to\infty}\langle\hat{M}_{k,n_k}\rangle(t)=\infty$. By the law of the iterated logarithm for martingales, we have $\limsup_{t\to\infty}\frac{\|Z_{k,n_k}(t)\|^2}{Z_1(t)}=1$, a.s., where

$$Z_1(t)=2e^{-2\lambda_k\int_0^t c_1(s)ds}\langle\hat{M}_{k,j}\rangle(t)\log\log\langle\hat{M}_{k,j}\rangle(t).$$

Therefore, if $\lim_{t\to\infty}\|Z_{k,n_k}(t)\|=0$, a.s., then we must have $\liminf_{t\to\infty} Z_1(t)=0$. Note that (A.23) implies that

$$\liminf_{t\to\infty}\frac{\log\log\int_0^t e^{2\lambda_k\int_0^s c_1(u)du}c_2^2(s)ds}{\log\int_0^t c_1(u)du}\geq 1 \tag{A.24}$$

and

$$\begin{aligned} &e^{-2\lambda_k\int_0^t c_1(s)ds}\int_0^t e^{2\lambda_k\int_0^s c_1(u)du}c_2^2(s)ds\log\int_0^t c_1(u)du\\ &\geq \frac{\underline{c}}{2}\frac{\int_T^t e^{2\lambda_k\int_0^s c_1(u)du}c_1(s)ds}{e^{2\lambda_k\int_0^t c_1(s)ds}},\quad t>T. \end{aligned} \tag{A.25}$$

Note that l'Hôpital's rule and $\int_0^\infty c_1(s)\ ds=\infty$ give

$$\lim_{t\to\infty}\frac{\int_T^t e^{2\lambda_k\int_0^s c_1(u)du}c_1(s)ds}{e^{2\lambda_k\int_0^t c_1(s)ds}}=\frac{1}{2\lambda_k}. \tag{A.26}$$

By the definition of $Z_1(t)$ and (A.24), (A.25), (A.26), we get

$$\begin{aligned}\liminf_{t\to\infty} Z_1(t)\geq C_{k,n_k}\underline{c}\liminf_{t\to\infty}&\frac{\log\log\int_0^t e^{2\lambda_k\int_0^s c_1(u)du}c_2^2(s)ds}{\log\int_0^t c_1(u)du}\\ &\times\frac{\int_T^t e^{2\lambda\int_0^s c_1(u)du}c_1(s)ds}{e^{2\lambda\int_0^t c_1(s)ds}}>\frac{C_{k,n_k}\underline{c}}{2\lambda_k}>0,\end{aligned}$$

which is a contradiction, and then $\liminf_{t\to\infty} c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$. If $C_{k,n_k}=0$, we define $l_0=\max\{n_k>j>1, C_{k,j}=\sum_{i=1}^d\|p_{k,j}^i\|^2>0\}$. Then we have from (A.7) that

$$\|Z_{k,l_0}(t)\|\leq\|\zeta_{k,l_0}(t)\|+e^{-2\lambda_k\int_0^t c_1(u)du}\|\zeta_{k,l_0}(0)\|+\|S_{k,l_0}(t)\|. \tag{A.27}$$

Note that $\lim_{t\to\infty}\|\zeta_{k,l_0}(t)\|=0$, $\lambda_k>0$, and $\lim_{t\to\infty}\|S_{k,l_0}(t)\|=0$. Hence, $\lim_{t\to\infty}\|Z_{k,l_0}(t)\|^2=0$. By methods similar to those used for $C_{k,n_k}>0$ above, we can obtain that $\liminf_{t\to\infty} c_2^2(t)\log\int_0^t c_1(s)ds/c_1(t)=0$ under $C_{k,l_0}>0$. Therefore, the proof is complete. ∎

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