Time-Inconsistent Mean-Field Stochastic LQ Problem: Open-Loop Time-Consistent Control

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***Abstract*—This paper is concerned with the open-loop time-consistent solution of time-inconsistent mean-field stochastic linear-quadratic (LQ) optimal control. Different from standard stochastic linear-quadratic problems, both the system matrices and the weighting matrices are depending on the initial times, and the conditional expectations of the control and state enter quadratically into the cost functional. Such features will ruin Bellman's principle of optimality and result in the time inconsistency of optimal control. Based on the dynamical nature of the systems involved, a kind of open-loop time-consistent equilibrium control is investigated in this paper. It is shown that the existence of open-loop equilibrium control for a fixed initial pair is equivalent to the solvability of a set of forward–backward stochastic difference equations with stationary condition and convexity condition. By decoupling the forward–backward stochastic difference equations, necessary and sufficient conditions in terms of linear difference equations and generalized difference Riccati equations are given for the existence of open-loop equilibrium control for a fixed initial pair. Moreover, the existence of open-loop time-consistent equilibrium controls for all the initial pairs is shown to be equivalent to the solvability of a set of coupled constrained generalized difference Riccati equations and two sets of constrained linear difference equations.**

***Index Terms*—Forward–backward stochastic difference equation, mean-field theory, stochastic linear-quadratic optimal control, time inconsistency.**

I. Introduction

A. Time Consistency versus Time Inconsistency

THOUGH not mentioned frequently, time consistency is indeed an essential notion in optimal control theory, which relates to Bellman's principle of optimality. To see this, recall a standard discrete-time stochastic optimal control problem, whose system dynamics and cost functional are given, respectively, by

$$\begin{cases} X_{k+1} = f(k, X_k, u_k, w_k) \\ X_t = x \in \mathbb{R}^n, \ k \in \mathbb{T}_t, \ t \in \mathbb{T} \end{cases} \tag{1}$$

and

$$J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}\left[e^{-\delta(k-t)} L(k, X_k, u_k)\right] + \mathbb{E}\left[e^{-\delta(N-t)} h(X_N)\right]. \tag{2}$$

Here, $\mathbb{T}_t = \{t, \ldots, N-1\}$, $\mathbb{T} = \{0, 1, \ldots, N-1\}$, and N is a positive integer; $\{X_k, k \in \widetilde{\mathbb{T}}_t\}$ and $\{u_k, k \in \mathbb{T}_t\}$ with $\widetilde{\mathbb{T}}_t = \{t, \ldots, N\}$ are the state process and the control process, respectively; $\{w_k, k \in \mathbb{T}\}$ is a stochastic disturbance process; $\mathbb{E}$ is the operator of mathematical expectation. Without loss of generality, the functions f, L, and h are assumed bounded. Let $\mathcal{U}[t, N-1]$ be a set of admissible controls. Then, we have the following optimal control problem.

Problem (C): Letting $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, find a $\bar{u} \in \mathcal{U}[t, N-1]$ such that

$$J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}[t, N-1]} J(t, x; u). \tag{3}$$

Above Problem will be called Problem (C) for the initial pair (t, x), and Problem (C) for other initial pairs can be similarly formulated. Any $\bar{u} \in \mathcal{U}[t, N-1]$ satisfying (3) is called an optimal control for the initial pair (t, x), and $\bar{X} = \{\bar{X}_k = \bar{X}(k; t, x, \bar{u}), k \in \widetilde{\mathbb{T}}_t\}$ is the corresponding optimal trajectory. Furthermore, $(\bar{X}, \bar{u})$ is referred to as an optimal pair for the initial pair (t, x).

Let $(\bar{X}, \bar{u})$ be an optimal pair for the initial pair (t, x); as the dynamics evolves, we indeed face a family of optimal control problems, namely, Problem (C) for the initial pairs $\{(k, \bar{X}_k), k \in \mathbb{T}_t\}$. Bellman's principle of optimality tells us that the optimal controls of this family of problems are interrelated, namely, for any $\tau \in \mathbb{T}_{t+1} = \{t+1, \ldots, N-1\}$, $\bar{u}|_{\mathbb{T}_\tau} = \{\bar{u}_\tau, \ldots, \bar{u}_{N-1}\}$ (the restriction of $\bar{u}$ on $\mathbb{T}_\tau = \{\tau, \ldots, N-1\}$) is an optimal control of Problem (C) for the initial pair $(\tau, \bar{X}_\tau)$. This property is the cornerstone of Bellman's dynamic programming and is referred to as the time consistency of optimal control, which is essential to handle optimal control problems like Problem (C) and its continuous-time counterpart. In such situation, we call that Problem (C) is time consistent.

However, the time-consistency fails quite often in many situations. For instance, when the exponential discounting function $e^{-\delta(k-t)}$ in (2) is replaced by other discounting functions, the

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corresponding problem is not time consistent, i.e., time inconsistent; see examples in [5] and [19] about the hyperbolic discounting and quasi-geometric discounting. In addition, when the conditional expectations of the state and/or control enters nonlinearly into the cost functional, the considered optimal control problems are time inconsistent too; a notable example is the mean–variance utility [2], [5], [9], [11], [22], [24]. In such case, the smoothing property of conditional expectation will not be sufficient to ensure the time consistency of optimal control.

B. Literature Review

Problems with nonlinear terms of conditional expectation (in the cost functional) are classified into the mean-field stochastic optimal control [37]. In [22], recognizing the time inconsistency (called nonseparability there), Li and Ng derived the optimal policy of multiperiod mean–variance portfolio selection by using an embedding scheme. Note that the optimal policy of [22] is with respect to the initial pair, i.e., it makes sense to be optimal only when viewed at the initial time. This derivation is called the precommitment optimal solution now.

Precommitment optimal solution is a static notion, which maps the considered initial pair into an admissible control set. By applying a precommitment optimal control (for an initial pair), its restriction to the tail time horizon is not an optimal control for the intertemporal initial pair. This static trait conflicts with the dynamic nature of (time-inconsistent) optimal control, as the time is involved in the problem setting. Though the static solution is of some practical and theoretical values, it neglects and has not really addressed the time inconsistency. Differently, another approach handles the time inconsistency in a dynamic manner; instead of seeking a precommitment optimal control, some kinds of equilibrium solutions are dealt with. This is mainly motivated by practical applications in economics and finance, and has recently attracted considerable interest and efforts.

The explicit formulation of time inconsistency was initiated by Strotz [29] in 1955, whereas its qualitative analysis can be traced back to the work of Smith [28]. Strotz studied the general discounting problem, and in the discrete-time case, his idea is to tackle the time inconsistency by a lead-follower game with hierarchical structure. Specifically, controls at different time points were viewed as different selves (players), and every self-integrated the policies of his successor into his own decision. By a backward procedure, the equilibrium policy (*if it exists*) was obtained. Inspired by Strotz and intending to tackling practical problems in economics and finance, hundreds of works were concerned with time inconsistency of dynamic systems described by ordinary difference or differential equations; see, for example, [12], [13], [15], [19], [20], [26] and references therein. Unfortunately, as pointed out by Ekeland [12], [13], it is hard to prove the existence of Strotz's equilibrium policy. Therefore, it is necessary and of great importance to develop a general theory on time inconsistent optimal control. This, on the one hand, can enrich the optimal control theory, and on the other hand, can provide instructive methodology to push the solvability of practical problems. Recently, this topic has attracted considerable attention from the theoretic control community; see, for example, [5], [17], [18], [30], [32], [34], [37] and references therein.

For the time-inconsistent LQ problems, two kinds of time-consistent equilibrium solutions are studied, which are the open-loop equilibrium control and the closed-loop equilibrium strategy [17], [18], [32], [34], [37]. The separate investigations of such two formulations are due to the fact that in the dynamic game theory, open-loop control distinguishes significantly from closed-loop strategy [3], [36]. To compare, open-loop formulation is to find an open-loop equilibrium "control," whereas the "strategy" is the object of closed-loop formulation. By a strategy, we mean a decision rule that a controller uses to select a control action based on the available information set. Mathematically, a strategy is a mapping or operator on the information set. When substituting the available information into a strategy, the open-loop value or open-loop realization of this strategy is obtained. Strotz's equilibrium solution [29] is essentially a closed-loop equilibrium strategy, which is further elaborately developed by Yong to the LQ optimal control [32], [37] as well as the nonlinear optimal control [33], [34]. In contrast, open-loop equilibrium control is extensively studied in [17], [18], and [37]. In particular, the closed-loop formulation can be viewed as the extension of Bellman's dynamic programming, and the corresponding equilibrium strategy (*if it exists*) is derived by a backward procedure [32]–[34], [37]. Differently, the open-loop equilibrium control is characterized via the maximum-principle-like methodology [17], [18].

Portfolio selection is to seek a best allocation of wealth among a basket of securities. The (single-period) mean–variance formulation is pioneered by Markowitz [24] in 1952, which is the cornerstone of modern portfolio theory and is widely used in both academia and industry. The multiperiod mean–variance portfolio selection is the natural extension of [24], which has been extensively studied. Until 2000 and for the first time, Li and Ng [22] and Zhou and Li [38] reported the analytical precommitment optimal policies for the discrete-time case and the continuous-time case, respectively. Noted above, multiperiod mean–variance portfolio selection is a particular example of time-inconsistent optimal control; the recent developments in time-inconsistent optimal control and the revisits of multiperiod mean–variance portfolio selection [2], [6], [9], [10], [17], [18] are mutually stimulated.

It is noted that some nondegenerate assumptions are posed in [2], [6], [9], [10], [17], and [18]. Specifically, the volatilities of the stocks in [2], [6], [17], and [18] and the return rates of the risky securities in [9] and [10] are assumed to be nondegenerate. To make the formulation more practical, it is natural to consider, at least in theory, how to generalize these results to the case where degeneracy is allowed. In fact, mean–variance portfolio selection problems with degenerate covariance matrices may date back to 1970s. In [7] or the "corrected" version [27], Buser *et al.* propose the single-period version with possibly singular covariance matrix. Clearly, such class of problems are more general than the classical ones [24], and more consistent with the reality.

To address the case with possible degenerate return rates, it is better to put multiperiod mean–variance portfolio selection within the framework of time-inconsistent mean-field stochastic LQ optimal control (with indefinite weighting matrices), which has not been established yet. Note that the running weighting matrices in [17], [18], [32], [34], and [37] are assumed to be nonnegative definite and positive definite. For standard time-consistent indefinite stochastic LQ optimal control, readers are referred to, for example, [1], [8], [31] and reference therein.

C. Contents of This Paper

In this paper, we shall investigate a time-inconsistent indefinite mean-field stochastic LQ optimal control problem. The matrices in system dynamics and cost functional are also dependent on the initial times; this is an extension of the general discounting functions that are in cost functionals. The contents of this paper are as follows.

The notion of open-loop equilibrium control is introduced in Section II, which is a discrete-time counterpart of that for the continuous-time problem [17], [18]. Different from the precommitment optimal control, the equilibrium control is only locally optimal in an infinitesimal sense. Furthermore, the open-loop equilibrium control is defined for a fixed initial time-state pair; its existence is shown to be equivalent to some stationary condition and convexity condition, which are involved with a set of forward–backward stochastic difference equations (FBSΔEs). Furthermore, necessary and sufficient conditions are obtained, respectively, for the stationary condition and the convexity condition; and by combining them, the existence of open-loop equilibrium control is further characterized.

The convexity condition is equivalent to the nonnegative definiteness of some matrices relating to a set of linear difference equations (LDEs), which is called the solvability of those constrained LDEs. The stationary condition is characterized via a property about the ranges of some matrices that are involved with another set of LDEs and a set of generalized difference Riccati equations (GDREs). If we further let the initial pair vary, some neater result about the existence of open-loop equilibrium control will be obtained. Specifically, for any initial pair problem (LQ) admitting an open-loop equilibrium control is shown to be equivalent to that two sets of constrained LDEs (39), (41), and a set of constrained GDREs (40) are solvable. It is worth pointing out that (if it is solvable) the set of GDREs (40) does not have symmetric structure, i.e., its solution is not symmetric. Furthermore, all the open-loop equilibrium controls are obtained.

As application of the derived theory, Section V investigates the multiperiod mean–variance portfolio selection. Necessary and sufficient condition is given on the existence of open-loop equilibrium portfolio control, which is completely characterized by the returns of the risky and riskless assets. If the return rates of the risky securities are nondegenerate, the equilibrium portfolio control will exist.

From our derived results, we have the following remarks.

1) Most existing results about time-inconsistent LQ problems are for the continuous-time case [17], [18], [32], [34], [37], and the study of discrete-time case is lagging behind. Noted above, the discrete-time multiperiod mean–variance portfolio selection is a notable example of discrete-time time-inconsistent LQ problems, and its full investigation motivates and needs to develop general theory about discrete-time time-inconsistent LQ optimal control. This is the aim of this paper.

2) The novelties of this paper are as follows.

First, no definiteness constraint is posed on the weighting matrices of cost functional, namely, the considered problem is an indefinite LQ optimal control. On the one hand, the indefinite setting provides a maximal capacity to model and deal with LQ-type problems, whose study will generalize existing results to some extent. On the other hand and most importantly, general explicit answers have not been reported about whether or not the definite weighting matrices could ensure the existence of open-loop equilibrium control for a time-inconsistent LQ problem. Therefore, the essential and weakest conditions are much desired for ensuring the existence of open-loop equilibrium control; and it is not necessary to pose the definiteness constraint on the weighting matrices.

Second, necessary and sufficient conditions are obtained on the existence of open-loop equilibrium control of problem (LQ) for both the case with a fixed initial pair and the case with all the initial pairs. The conditions are in terms of discrete-time LDEs and GDREs, which are easy to be verified by iteratively solving the LDEs and GDREs.

Third, necessary and sufficient condition is derived on the existence of open-loop equilibrium portfolio control of multiperiod mean–variance portfolio selection [problem (MV)]. The obtained condition is completely characterized by the returns of the risky and riskless assets. If the return rates of the risky securities are nondegenerate (this is the common assumption in the literature), the equilibrium portfolio control will exist.

If the system dynamics and cost functional are both independent of the initial time, the corresponding LQ problem will be a dynamic version of that considered in [25], where the conditional expectation operators are replaced by the expectation operators. For more details on mean-field stochastic optimal control and related mean-field games, we refer to [4], [11], [14], [16], [21], [25], [35] and the references therein.

The rest of this paper is organized as follows. Section II introduces the notion of open-loop equilibrium control of problem (LQ). In Sections III and IV, necessary and sufficient conditions on the existence of open-loop equilibrium control are presented for both the case with a fixed initial pair and the case with all the initial pairs. Section V studies the multiperiod mean–variance portfolio selection, and some concluding remarks are given in Section VI.

II. Open-Loop Equilibrium Control

Consider the following controlled stochastic difference equation (SΔE)

$$\begin{cases} X_{k+1}^t = \big(A_{t,k}X_k^t + \bar{A}_{t,k}\mathbb{E}_t X_k^t \\ \qquad\quad + B_{t,k}u_k + \bar{B}_{t,k}\mathbb{E}_t u_k + f_{t,k}\big) \\ \qquad\quad + \sum_{i=1}^p \big(C_{t,k}^i X_k^t + \bar{C}_{t,k}^i \mathbb{E}_t X_k^t \\ \qquad\quad + D_{t,k}^i u_k + \bar{D}_{t,k}^i \mathbb{E}_t u_k + d_{t,k}^i\big) w_k^i \\ X_t^t = x,\ k \in \mathbb{T}_t,\ t \in \mathbb{T} \end{cases} \tag{4}$$

where $A_{t,k}, \bar{A}_{t,k}, C_{t,k}^i, \bar{C}_{t,k}^i \in \mathbb{R}^{n\times n}$, $B_{t,k}, \bar{B}_{t,k}, D_{t,k}^i, \bar{D}_{t,k}^i \in \mathbb{R}^{n\times m}$, and $f_{t,k}, d_{t,k}^i \in \mathbb{R}^n$ are deterministic matrices. In (4), the noise process $\{w_k = (w_k^1, \dots, w_k^p)^T, k \in \mathbb{T}\}$ is assumed to be a vector-valued martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with

$$\mathbb{E}_k[w_k] = 0,\ \mathbb{E}_k[w_k w_k^T] = \Gamma_k,\ k \geq 0. \tag{5}$$

$\mathbb{E}_t$ in (4) is the conditional mathematical expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$, where $\mathcal{F}_t = \sigma\{w_l, l = 0, 1, \dots, t-1\}$, and $\mathcal{F}_0$ is understood as $\{\emptyset, \Omega\}$. Furthermore, $\Gamma_k = (\gamma_k^{ij})_{p\times p}$ is assumed to be deterministic.

The cost functional associated with the system (4) is

$$J(t,x;u)=\sum_{k=t}^{N-1}\mathbb{E}_t\Big[(X_k^t)^TQ_{t,k}X_k^t+(\mathbb{E}_tX_k^t)^T\bar{Q}_{t,k}\mathbb{E}_tX_k^t+u_k^TR_{t,k}u_k+(\mathbb{E}_tu_k)^T\bar{R}_{t,k}\mathbb{E}_tu_k+2q_{t,k}^TX_k^t+2\rho_{t,k}^Tu_k\Big]+\mathbb{E}_t\big[(X_N^t)^TG_tX_N^t\big]+(\mathbb{E}_tX_N^t)^T\bar{G}_t\mathbb{E}_tX_N^t+2\mathbb{E}_t(g_t^TX_N^t) \tag{6}$$

where $Q_{t,k},\bar{Q}_{t,k},R_{t,k},\bar{R}_{t,k},k\in\mathbb{T}_t,\ G_t,\bar{G}_t$ are deterministic symmetric matrices of appropriate dimensions, and $q_{t,k},\rho_{t,k}$, $k\in\mathbb{T}_t, g_t$ are deterministic vectors.

In (4), the initial state x is in $l^2_{\mathcal{F}}(t;\mathbb{R}^n)$, which is defined as

$$l^2_{\mathcal{F}}(t;\mathbb{R}^n)=\Big\{\zeta\in\mathbb{R}^n\ \big|\ \zeta \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}|\zeta|^2<\infty\Big\}.$$

Similarly, we can define $l^2_{\mathcal{F}}(k;\mathbb{R}^n)$ and $l^2_{\mathcal{F}}(k;\mathbb{R}^m)$, $k\in\mathbb{T}$. Furthermore, let

$$l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)=\left\{\nu=\{\nu_k,k\in\mathbb{T}_t\}\ \Big|,\ \begin{array}{l}\nu_k \text{ is } \mathcal{F}_k\text{-measurable}\\ \mathbb{E}|\nu_k|^2<\infty,k\in\mathbb{T}_t\end{array}\right\}.$$

Then, we pose the following optimal control problem.

Problem (LQ): For the initial pair (t,x), find a $u^*\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ such that

$$J(t,x;u^*)=\inf_{u\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)}J(t,x;u) \tag{7}$$

holds.

Due to the time inconsistency, we in this paper intend finding an equilibrium control of the following type.

Definition II.1: Given $t\in\mathbb{T}$ and $x\in l^2_{\mathcal{F}}(t;\mathbb{R}^n)$, $u^{t,x,*}\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ is called an open-loop equilibrium control of problem (LQ) for the initial pair (t,x), if

$$J(k,X_k^{t,x,*};u^{t,x,*}|_{\mathbb{T}_k})\le J(k,X_k^{t,x,*};(u_k,u^{t,x,*}|_{\mathbb{T}_{k+1}})) \tag{8}$$

holds for any $k\in\mathbb{T}_t$ and any $u_k\in L^2_{\mathcal{F}}(k;\mathbb{R}^m)$. Here, $u^{t,x,*}|_{\mathbb{T}_k}$ and $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ (with $\mathbb{T}_k=\{k,...,N-1\},\mathbb{T}_{k+1}=\{k+1,...,N-1\}$) are the restrictions of $u^{t,x,*}$ on $\mathbb{T}_k$ and $\mathbb{T}_{k+1}$, respectively; and $X^{t,x,*}$ is given by

$$\begin{cases}X_{k+1}^{t,x,*}=\big[(A_{k,k}+\bar{A}_{k,k})X_k^{t,x,*}+(B_{k,k}+\bar{B}_{k,k})u_k^{t,x,*}+f_{k,k}\big]\\ \qquad+\sum_{i=1}^p\big[(C_{k,k}^i+\bar{C}_{k,k}^i)X_k^{t,x,*}+(D_{k,k}^i+\bar{D}_{k,k}^i)u_k^{t,x,*}+d_{k,k}^i\big]w_k^i\\ X_t^{t,x,*}=x,\ k\in\mathbb{T}_t\end{cases} \tag{9}$$

which is called the equilibrium state corresponding to $u^{t,x,*}$.

Noting that $u^{t,x,*}|_{\mathbb{T}_k}=(u_k^{t,x,*},u^{t,x,*}|_{\mathbb{T}_{k+1}})$, the control $(u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})$ on the right-hand side of (8) differs from $u^{t,x,*}|_{\mathbb{T}_k}$ only at time instant k. Intuitively, the cost functional will increase if one deviates from $u^{t,x,*}$. Hence, $\{u_t^{t,x,*},...,u_{N-1}^{t,x,*}\}$ can be viewed as an equilibrium of a multiperson game with hierarchical structure. By its definition, $u^{t,x,*}$ is time consistent in the sense that for any $k\in\mathbb{T}_t$, $u^{t,x,*}|_{\mathbb{T}_k}$ is an open-loop equilibrium control for the initial pair $(k,X_k^{t,x,*})$.

Throughout this paper, we adopt the following notations:

$$\begin{cases}\mathcal{A}_{k,\ell}=A_{k,\ell}+\bar{A}_{k,\ell},\ \ \mathcal{B}_{k,\ell}=B_{k,\ell}+\bar{B}_{k,\ell}\\ \mathcal{C}_{k,\ell}^i=C_{k,\ell}^i+\bar{C}_{k,\ell}^i,\ \ \mathcal{D}_{k,\ell}^i=D_{k,\ell}^i+\bar{D}_{k,\ell}^i\\ \mathcal{Q}_{k,\ell}=Q_{k,\ell}+\bar{Q}_{k,\ell},\ \ \mathcal{R}_{k,\ell}=R_{k,\ell}+\bar{R}_{k,\ell}\\ \mathcal{G}_k=G_k+\bar{G}_k,\ \ i=1,...,p,\ \ k\in\mathbb{T}_t,\ \ \ell\in\mathbb{T}_k.\end{cases} \tag{10}$$

Then, (9) is simply rewritten as

$$\begin{cases}X_{k+1}^{t,x,*}=\big[\mathcal{A}_{k,k}X_k^{t,x,*}+\mathcal{B}_{k,k}u_k^{t,x,*}+f_{k,k}\big]\\ \quad+\sum_{i=1}^p\big[\mathcal{C}_{k,k}^iX_k^{t,x,*}+\mathcal{D}_{k,k}^iu_k^{t,x,*}+d_{k,k}^i\big]w_k^i\\ X_t^{t,x,*}=x,\ k\in\mathbb{T}_t.\end{cases} \tag{11}$$

III. Problem (LQ) for a Fixed Initial Pair

A. First Characterization on the Existence of Open-Loop Equilibrium Control

Throughout Section III, we will study Problem (LQ) for the fixed initial pair (t,x), which will be simply denoted as Problem $(\text{LQ})_{tx}$. First, a difference formula of cost functionals is given.

Lemma III.1: Let $\zeta\in l^2_{\mathcal{F}}(k;\mathbb{R}^n)$, $u=\{u_\ell,k\in\mathbb{T}_k\}\in l^2_{\mathcal{F}}(\mathbb{T}_k;\mathbb{R}^m)$, $\bar{u}_k\in l^2_{\mathcal{F}}(k;\mathbb{R}^m)$ and $\lambda\in\mathbb{R}$. Then, we have

$$J(k,\zeta;(u_k+\lambda\bar{u}_k,u|_{\mathbb{T}_{k+1}}))-J(k,\zeta;u)=\lambda^2\widehat{J}(k,0;\bar{u}_k)+2\lambda\Big[\mathcal{R}_{k,k}u_k+\mathcal{B}_{k,k}^T\mathbb{E}_kZ_{k+1}^k+\sum_{i=1}^p(\mathcal{D}_{k,k}^i)^T\mathbb{E}_k(Z_{k+1}^kw_k^i)+\rho_{k,k}\Big]^T\bar{u}_k \tag{12}$$

where $u|_{\mathbb{T}_{k+1}}=\{u_{k+1},...,u_{N-1}\}$ and

$$\widehat{J}(k,0;\bar{u}_k)=\mathbb{E}_k\big[\bar{u}_k^T\mathcal{R}_{k,k}\bar{u}_k\big]+\sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(Y_\ell^{k,\bar{u}_k})^TQ_{k,\ell}Y_\ell^{k,\bar{u}_k}+(\mathbb{E}_kY_\ell^{k,\bar{u}_k})^T\bar{Q}_{k,\ell}\mathbb{E}_kY_\ell^{k,\bar{u}_k}\Big]+\mathbb{E}_k\big[(Y_N^{k,\bar{u}_k})^TG_kY_N^{k,\bar{u}_k}\big]+(\mathbb{E}_kY_N^{k,\bar{u}_k})^T\bar{G}_k\mathbb{E}_kY_N^{k,\bar{u}_k} \tag{13}$$

with

$$\begin{cases}Y_{\ell+1}^{k,\bar{u}_k}=A_{k,\ell}Y_\ell^{k,\bar{u}_k}+\bar{A}_{k,\ell}\mathbb{E}_kY_\ell^{k,\bar{u}_k}\\ \qquad+\sum_{i=1}^p\big(C_{k,\ell}^iY_\ell^{k,\bar{u}_k}+\bar{C}_{k,\ell}^i\mathbb{E}_kY_\ell^{k,\bar{u}_k}\big)w_\ell^i\\ Y_{k+1}^{k,\bar{u}_k}=\mathcal{B}_{k,k}\bar{u}_k+\sum_{i=1}^p\mathcal{D}_{k,k}^i\bar{u}_kw_k^i\\ Y_k^{k,\bar{u}_k}=0,\ \ \ell\in\mathbb{T}_{k+1}.\end{cases} \tag{14}$$

Furthermore, Z^k_{k+1} in (12) is computed via the following FBSΔE

$$\begin{cases} X^k_{\ell+1} = \big(A_{k,\ell}X^k_\ell + \bar{A}_{k,\ell}\mathbb{E}_k X^k_\ell \\ \qquad + B_{k,\ell}u_\ell + \bar{B}_{k,\ell}\mathbb{E}_k u_\ell + f_{k,\ell}\big) \\ \qquad + \sum_{i=1}^p \big(C^i_{k,\ell}X^k_\ell + \bar{C}^i_{k,\ell}\mathbb{E}_k X^k_\ell \\ \qquad + D^i_{k,\ell}u_\ell + \bar{D}^i_{k,\ell}\mathbb{E}_k u_\ell + d^i_{k,\ell}\big)w^i_\ell \\ Z^k_\ell = Q_{k,\ell}X^k_\ell + \bar{Q}_{k,\ell}\mathbb{E}_k X^k_\ell + q_{k,\ell} \\ \qquad + A^T_{k,\ell}\mathbb{E}_\ell Z^k_{\ell+1} + \bar{A}^T_{k,\ell}\mathbb{E}_k Z^k_{\ell+1} \\ \qquad + \sum_{i=1}^p \big[(C^i_{k,\ell})^T\mathbb{E}_\ell(Z^k_{\ell+1}w^i_\ell) \\ \qquad + (\bar{C}^i_{k,\ell})^T\mathbb{E}_k(Z^k_{\ell+1}w^i_\ell)\big] \\ X^k_k = \zeta,\ Z^k_N = G_k X^k_N + \bar{G}_k\mathbb{E}_k X^k_N + g_k \\ \ell \in \mathbb{T}_k. \end{cases} \tag{15}$$

Proof: See Appendix A. ■

From Lemma III.1, we have the following result.

Theorem III.1: The following statements are equivalent.

i) Problem $(\mathrm{LQ})_{tx}$ admits an open-loop equilibrium control.

ii) The following assertions hold.

a) The convexity condition

$$\inf_{\bar{u}_k\in l^2_{\mathcal{F}}(k;\mathbb{R}^m)} \widehat{J}(k,0;\bar{u}_k) \geq 0,\quad k\in\mathbb{T}_t \tag{16}$$

is satisfied, where $\widehat{J}(k,0;\bar{u}_k)$ is given in (13).

b) There exists a $u^{t,x,*}\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ such that the stationary condition

$$0 = \mathcal{R}_{k,k}u^{t,x,*}_k + \mathcal{B}^T_{k,k}\mathbb{E}_k Z^{k,t,x}_{k+1} + \sum_{i=1}^p (\mathcal{D}^i_{k,k})^T\mathbb{E}_k(Z^{k,t,x}_{k+1}w^i_k) + \rho_{k,k},\ k\in\mathbb{T}_t \tag{17}$$

is satisfied. Here, $Z^{k,t,x}_{k+1}$ is computed via the FBSΔE

$$\begin{cases} X^{k,t,x}_{\ell+1} = \big(A_{k,\ell}X^{k,t,x}_\ell + \bar{A}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \quad + B_{k,\ell}u^{t,x,*}_\ell + \bar{B}_{k,\ell}\mathbb{E}_k u^{t,x,*}_\ell + f_{k,\ell}\big) \\ \quad + \sum_{i=1}^p \big(C^i_{k,\ell}X^{k,t,x}_\ell + \bar{C}^i_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \quad + D^i_{k,\ell}u^{t,x,*}_\ell + \bar{D}^i_{k,\ell}\mathbb{E}_k u^{t,x,*}_\ell \\ \quad + d^i_{k,\ell}\big)w^i_\ell \\ Z^{k,t,x}_\ell = Q_{k,\ell}X^{k,t,x}_\ell + \bar{Q}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \quad + q_{k,\ell} + A^T_{k,\ell}\mathbb{E}_\ell Z^{k,t,x}_{\ell+1} + \bar{A}^T_{k,\ell}\mathbb{E}_k Z^{k,t,x}_{\ell+1} \\ \quad + \sum_{i=1}^p \big[(C^i_{k,\ell})^T\mathbb{E}_\ell(Z^{k,t,x}_{\ell+1}w^i_\ell) \\ \quad + (\bar{C}^i_{k,\ell})^T\mathbb{E}_k(Z^{k,t,x}_{\ell+1}w^i_\ell)\big] \\ X^{k,t,x}_k = X^{t,x,*}_k \\ Z^{k,t,x}_N = G_k X^{k,t,x}_N + \bar{G}_k\mathbb{E}_k X^{k,t,x}_N + g_k \\ \ell\in\mathbb{T}_k \end{cases} \tag{18}$$

and the initial state $X^{t,x,*}_k$ of the forward SΔE of (18) is computed via

$$\begin{cases} X^{t,x,*}_{k+1} = \big[\mathcal{A}_{k,k}X^{t,x,*}_k + \mathcal{B}_{k,k}u^{t,x,*}_k + f_{k,k}\big] \\ \qquad + \sum_{i=1}^p \big[\mathcal{C}^i_{k,k}X^{t,x,*}_k + \mathcal{D}^i_{k,k}u^{t,x,*}_k \\ \qquad + d^i_{k,k}\big]w^i_k \\ X^{t,x,*}_t = x,\ k\in\mathbb{T}_t. \end{cases} \tag{19}$$

Under any of the above conditions, $u^{t,x,*}$ given in ii) is an open-loop equilibrium control for the initial pair (t,x).

Proof: See Appendix B. ■

Remark III.1: As the stationary condition (17) holds for $k\in\mathbb{T}_t$, we have a set of FBSΔEs, which are coupled with (19) via the initial states $X^{k,t,x}_k = X^{t,x,*}_k, k\in\mathbb{T}_t$.

B. Convexity Condition

This subsection studies the convexity condition (16). First, we give a compact form of $\widehat{J}(k,0;\bar{u}_k)$.

Lemma III.2: $\widehat{J}(k,0;\bar{u}_k)$ can be expressed as

$$\widehat{J}(k,0;\bar{u}_k) = \bar{u}^T_k\mathbb{W}_k\bar{u}_k \tag{20}$$

where

$$\mathbb{W}_k = \mathcal{R}_{k,k} + \mathcal{B}^T_{k,k}\mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \sum_{i,j=1}^p \gamma^{ij}_k(\mathcal{D}^i_{k,k})^T P_{k,k+1}\mathcal{D}^j_{k,k} \tag{21}$$

with $P_{k,k+1}$ and $\mathcal{P}_{k,k+1}$ computed via

$$\begin{cases} P_{k,\ell} = Q_{k,\ell} + A^T_{k,\ell}P_{k,\ell+1}A_{k,\ell} \\ \qquad + \sum_{i,j=1}^p \gamma^{ij}_\ell(C^i_{k,\ell})^T P_{k,\ell+1}C^j_{k,\ell} \\ \mathcal{P}_{k,\ell} = \mathcal{Q}_{k,\ell} + \mathcal{A}^T_{k,\ell}\mathcal{P}_{k,\ell+1}\mathcal{A}_{k,\ell} \\ \qquad + \sum_{i,j=1}^p \gamma^{ij}_\ell(\mathcal{C}^i_{k,\ell})^T P_{k,\ell+1}\mathcal{C}^j_{k,\ell} \\ P_{k,N} = G_k,\ \mathcal{P}_{k,N} = \mathcal{G}_k,\ \ell\in\mathbb{T}_k. \end{cases} \tag{22}$$

Proof: From (14), it follows that

$$\begin{cases} \mathbb{E}_k Y^{k,\bar{u}_k}_{\ell+1} = \mathcal{A}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell,\ \ \ell\in\mathbb{T}_{k+1} \\ \mathbb{E}_k Y^{k,\bar{u}_k}_{k+1} = \mathcal{B}_{k,k}\mathbb{E}_k\bar{u}_k \\ \mathbb{E}_k Y^{k,\bar{u}_k}_k = 0. \end{cases}$$

Let $\bar{P}_{k,\ell} = \mathcal{P}_{k,\ell} - P_{k,\ell}, \ell\in\mathbb{T}_k$. By adding to and subtracting

$$\sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(Y^{k,\bar{u}_k}_{\ell+1})^T P_{k,\ell+1}Y^{k,\bar{u}_k}_{\ell+1} - (Y^{k,\bar{u}_k}_\ell)^T P_{k,\ell}Y^{k,\bar{u}_k}_\ell + (\mathbb{E}_k Y^{k,\bar{u}_k}_{\ell+1})^T\bar{P}_{k,\ell+1}\mathbb{E}_k Y^{k,\bar{u}_k}_{\ell+1} - (\mathbb{E}_k Y^{k,\bar{u}_k}_\ell)^T\bar{P}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big]$$

from (13), we have

$$\widehat{J}(k,0;\bar{u}_k) = \sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(Y^{k,\bar{u}_k}_\ell)^T Q_{k,\ell}Y^{k,\bar{u}_k}_\ell + (\mathbb{E}_k Y^{k,\bar{u}_k}_\ell)^T\bar{Q}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell$$

$$
\begin{aligned}
&+ (Y_{\ell+1}^{k,\bar{u}_k})^T P_{k,\ell+1} Y_{\ell+1}^{k,\bar{u}_k} - (Y_{\ell}^{k,\bar{u}_k})^T P_{k,\ell} Y_{\ell}^{k,\bar{u}_k} \\
&+ (\mathbb{E}_k Y_{\ell+1}^{k,\bar{u}_k})^T \bar{P}_{k,\ell+1} \mathbb{E}_k Y_{\ell+1}^{k,\bar{u}_k} \\
&- (\mathbb{E}_k Y_{\ell}^{k,\bar{u}_k})^T \bar{P}_{k,\ell} \mathbb{E}_k Y_{\ell}^{k,\bar{u}_k} \Big] \\
&+ \bar{u}_k^T \mathcal{R}_{k,k} \bar{u}_k \\
= &\sum_{\ell=k+1}^{N-1} \mathbb{E}_k \Big[(\mathbb{E}_k Y_{\ell}^{k,\bar{u}_k})^T \Big(\mathcal{Q}_{k,\ell} + \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1} \mathcal{A}_{k,\ell} \\
&+ \sum_{i,j=1}^{p} \gamma_\ell^{ij} (\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1} \mathcal{C}_{k,\ell}^j - \mathcal{P}_{k,\ell} \Big) \mathbb{E}_k Y_{\ell}^{k,\bar{u}_k} \\
&+ (Y_{\ell}^{k,\bar{u}_k} - \mathbb{E}_k Y_{\ell}^{k,\bar{u}_k})^T \Big(Q_{k,\ell} + A_{k,\ell}^T P_{k,\ell+1} A_{k,\ell} \\
&+ \sum_{i,j=1}^{p} \gamma_\ell^{ij} (C_{k,\ell}^i)^T P_{k,\ell+1} C_{k,\ell}^j - P_{k,\ell} \Big) (Y_{\ell}^{k,\bar{u}_k} - \mathbb{E}_k Y_{\ell}^{k,\bar{u}_k}) \Big] \\
&+ \bar{u}_k^T \Big[\mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \mathcal{P}_{k,k+1} \mathcal{B}_{k,k} \\
&+ \sum_{i,j=1}^{p} \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T P_{k,k+1} \mathcal{D}_{k,k}^j \Big] \bar{u}_k \\
= &\,\bar{u}_k^T \Big[\mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \mathcal{P}_{k,k+1} \mathcal{B}_{k,k} \\
&+ \sum_{i,j=1}^{p} \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T P_{k,k+1} \mathcal{D}_{k,k}^j \Big] \bar{u}_k. \qquad (23)
\end{aligned}
$$

■

By Lemma III.2 and Theorem III.1, the following result is straightforward.

Theorem III.2: The following statements are equivalent.

1) The convexity condition (16) is satisfied.
2) The following inequalities
$$\mathbb{W}_k \succeq 0, \quad k \in \mathbb{T}_t \qquad (24)$$
hold, i.e., $\mathbb{W}_k, k \in \mathbb{T}_t$ are nonnegative definite, where $\mathbb{W}_k$ is given in (21).

C. Stationary Condition

We now switch to the stationary condition (17). The following lemma gives an expression of the backward state $Z^{k,t,x}$ of the FBSΔE (18), provided that $u^{t,x,*}$ is a linear function of $X^{t,x,*}$.

Lemma III.3: Letting $k \in \mathbb{T}, \tau \in \mathbb{T}_k$, suppose that $\{u_\ell^{t,x,*}, \ell \in \mathbb{T}_\tau\}$ in (18) has the form $u_\ell^{t,x,*} = \Psi_\ell X_\ell^{t,x,*} + \alpha_\ell,\ \ell \in \mathbb{T}_\tau$ with $\mathbb{T}_\tau = \{\tau, ..., N-1\}$ and $\Psi_\ell, \alpha_\ell, \ell \in \mathbb{T}_\tau$ being deterministic matrices. Then, the backward state $\{Z_\ell^{k,t,x}, \ell \in \mathbb{T}_\tau\}$ has the following expression:

$$
\begin{aligned}
Z_\ell^{k,t,x} &= P_{k,\ell} X_\ell^{k,t,x} + \bar{P}_{k,\ell} \mathbb{E}_k X_\ell^{k,t,x} + T_{k,\ell} X_\ell^{t,x,*} \\
&\quad + \bar{T}_{k,\ell} \mathbb{E}_k X_\ell^{t,x,*} + \pi_{k,\ell}, \quad \ell \in \mathbb{T}_\tau. \qquad (25)
\end{aligned}
$$

Here, $\bar{P}_{k,\ell} = \mathcal{P}_{k,\ell} - P_{k,\ell}$ with $P_{k,\ell}, \mathcal{P}_{k,\ell}$ computed via (22); and $T_{k,\ell}, \bar{T}_{k,\ell}, \pi_{k,\ell}$ are given by

$$
\begin{cases}
T_{k,\ell} = A_{k,\ell}^T T_{k,\ell+1} \mathcal{A}_{\ell,\ell} \\
\quad + \sum_{i,j=1}^p \gamma_\ell^{ij} (C_{k,\ell}^i)^T T_{k,\ell+1} \mathcal{C}_{\ell,\ell}^j \\
\quad + \Big\{ A_{k,\ell}^T P_{k,\ell+1} B_{k,\ell} + A_{k,\ell}^T T_{k,\ell+1} \mathcal{B}_{\ell,\ell} \\
\quad + \sum_{i,j=1}^p \Big[(C_{k,\ell}^i)^T P_{k,\ell+1} D_{k,\ell}^j \\
\quad + (C_{k,\ell}^i)^T T_{k,\ell+1} \mathcal{D}_{\ell,\ell}^j \Big] \Big\} \Psi_\ell \\
\bar{T}_{k,\ell} = A_{k,\ell}^T \bar{T}_{k,\ell+1} \mathcal{A}_{\ell,\ell} + \bar{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1} \mathcal{A}_{\ell,\ell} \\
\quad + \sum_{i,j=1}^p \gamma_\ell^{ij} (\bar{C}_{k,\ell}^i)^T T_{k,\ell+1} \mathcal{C}_{\ell,\ell}^j \\
\quad + \Big\{ A_{k,\ell}^T P_{k,\ell+1} \bar{B}_{k,\ell} + A_{k,\ell}^T \bar{P}_{k,\ell+1} \mathcal{B}_{k,\ell} \\
\quad + A_{k,\ell}^T \bar{T}_{k,\ell+1} \mathcal{B}_{\ell,\ell} + \bar{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1} \mathcal{B}_{k,\ell} \\
\quad + \bar{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1} \mathcal{B}_{\ell,\ell} \\
\quad + \sum_{i,j=1}^p \gamma_\ell^{ij} \Big[(C_{k,\ell}^i)^T P_{k,\ell+1} \bar{D}_{k,\ell}^j \\
\quad + (\bar{C}_{k,\ell}^i)^T P_{k,\ell+1} \mathcal{D}_{k,\ell}^j \\
\quad + (\bar{C}_{k,\ell}^i)^T T_{k,\ell+1} \mathcal{D}_{\ell,\ell}^j \Big] \Big\} \Psi_\ell \\
T_{k,N} = 0,\ \bar{T}_{k,N} = 0 \\
\ell \in \mathbb{T}_\tau
\end{cases} \qquad (26)
$$

and

$$
\begin{cases}
\pi_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1} \big(\mathcal{B}_{k,\ell} \alpha_\ell + f_{k,\ell} \big) \\
\quad + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1} \big(\mathcal{B}_{\ell,\ell} \alpha_\ell + f_{\ell,\ell} \big) + \mathcal{A}_{k,\ell}^T \pi_{k,\ell+1} \\
\quad + \sum_{i,j=1}^p \gamma_\ell^{ij} \big[(\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1} \big(\mathcal{D}_{k,\ell}^j \alpha_\ell + d_{k,\ell}^j \big) \\
\quad + (\mathcal{C}_{k,\ell}^i)^T T_{k,\ell+1} \big(\mathcal{D}_{\ell,\ell}^j \alpha_\ell + d_{\ell,\ell}^j \big) \big] + q_{k,\ell} \\
\pi_{k,N} = g_k \\
\ell \in \mathbb{T}_\tau
\end{cases} \qquad (27)
$$

with $\mathcal{T}_{k,\ell} = T_{k,\ell} + \bar{T}_{k,\ell}, \ell \in \mathbb{T}_\tau$.

Proof: See Appendix C. ■

To prove above lemma, we have used a backward deduction method, namely, starting from $k = N$ and $k = N-1$, the expression (25) can be deductively obtained.

For a given matrix $M \in \mathbb{R}^{n \times m}$, its Moore–Penrose inverse is denoted as $M^\dagger$, which is in $\mathbb{R}^{m \times n}$. The following lemma is from [1].

Lemma III.4: Let matrices L, M, and N be given with appropriate size. Then, $LXM = N$ has a solution X if and only if $LL^\dagger NMM^\dagger = N$. Moreover, the solution of $LXM = N$ can be expressed as $X = L^\dagger NM^\dagger + Y - L^\dagger LYMM^\dagger$, where Y is a matrix with appropriate size.

Based on above results, we have the following theorem.

Theorem III.3: The following statements are equivalent.

i) The stationary condition (17) is satisfied.
ii) The condition
$$\mathcal{H}_k X_k^{t,x,*} + \beta_k \in \mathrm{Ran}(\mathcal{W}_k), \ k \in \mathbb{T}_t \qquad (28)$$

holds. Here, $\mathcal{H}_k, \mathcal{W}_k, \beta_k, k \in \mathbb{T}_t$ are given by

$$\begin{cases} \mathcal{W}_k = \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{B}_{k,k} \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \\ \qquad \times \big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{D}_{k,k}^j \\ \mathcal{H}_k = \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{A}_{k,k} \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \\ \qquad \times \big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{C}_{k,k}^j \\ \beta_k = \mathcal{B}_{k,k}^T\big[\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)f_{k,k} + \pi_{k,k+1}\big] \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \\ \qquad \times \big(P_{k,k+1} + T_{k,k+1}\big)d_{k,k}^j + \rho_{k,k} \\ k \in \mathbb{T}_t \end{cases} \tag{29}$$

with

$$\begin{cases} \begin{cases} T_{k,\ell} = A_{k,\ell}^T T_{k,\ell+1}\mathcal{A}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij} (C_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell}^j \\ \qquad - \Big\{A_{k,\ell}^T P_{k,\ell+1}B_{k,\ell} + A_{k,\ell}^T T_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij}\Big[(C_{k,\ell}^i)^T P_{k,\ell+1}D_{k,\ell}^j \\ \qquad + (C_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}^j\Big]\Big\}\mathcal{W}_\ell^\dagger\mathcal{H}_\ell \\ \mathcal{T}_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\mathcal{A}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij} (\mathcal{C}_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell}^j \\ \qquad - \Big\{\mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\mathcal{B}_{k,\ell} + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^p (\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1}\mathcal{D}_{k,\ell}^j \\ \qquad + \mathcal{C}_{k,\ell}^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\Big)\mathcal{W}_\ell^\dagger\mathcal{H}_\ell \\ T_{k,N} = 0,\ \mathcal{T}_{k,N} = 0 \\ \ell \in \mathbb{T}_k \end{cases} \\ k \in \mathbb{T}_t \end{cases} \tag{30}$$

and

$$\begin{cases} \begin{cases} \pi_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\big(f_{k,\ell} - \mathcal{B}_{k,\ell}\mathcal{W}_\ell^\dagger\beta_\ell\big) \\ \qquad + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\big(f_{\ell,\ell} - \mathcal{B}_{\ell,\ell}\mathcal{W}_\ell^\dagger\beta_\ell\big) \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij}\big[(\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1} \\ \qquad \times \big(d_{k,\ell}^j - \mathcal{D}_{k,\ell}^j\mathcal{W}_\ell^\dagger\beta_\ell\big) \\ \qquad + (\mathcal{C}_{k,\ell}^i)^T T_{k,\ell+1}\big(d_{\ell,\ell}^j - \mathcal{D}_{\ell,\ell}^j\mathcal{W}_\ell^\dagger\beta_\ell\big)\big] \\ \qquad + \mathcal{A}_{k,\ell}^T \pi_{k,\ell+1} + q_{k,\ell} \\ \pi_{k,N} = g_k \end{cases} \\ k \in \mathbb{T}_t. \end{cases} \tag{31}$$

Furthermore, in (28) $\mathrm{Ran}(\mathcal{W}_k)$ is the range of $\mathcal{W}_k$, and $X_k^{t,x,*}$ is computed via

$$\begin{cases} X_{k+1}^{t,x,*} = \big[\big(\mathcal{A}_{k,k} - \mathcal{B}_{k,k}\mathcal{W}_k^\dagger\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{B}_{k,k}\mathcal{W}_k^\dagger\beta_k + f_{k,k}\big] \\ \qquad + \sum_{i=1}^p \big[\big(\mathcal{C}_{k,k}^i - \mathcal{D}_{k,k}^i\mathcal{W}_k^\dagger\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{D}_{k,k}^i\mathcal{W}_k^\dagger\beta_k + d_{k,k}^i\big]w_k^i \\ X_t^{t,x,*} = x,\ k \in \mathbb{T}_t. \end{cases} \tag{32}$$

Under any of the above conditions, $u^{t,x,*}$ in (17) is selected as

$$u_k^{t,x,*} = -\mathcal{W}_k^\dagger\mathcal{H}_k X_k^{t,x,*} - \mathcal{W}_k^\dagger\beta_k,\ k \in \mathbb{T}_t \tag{33}$$

with $X^{t,x,*}$ given in (32). Furthermore, we have

$$\begin{aligned} Z_\ell^{k,t,x} &= P_{k,\ell}\big(X_\ell^{k,t,x} - \mathbb{E}_k X_\ell^{k,t,x}\big) + \mathcal{P}_{k,\ell}\mathbb{E}_k X_\ell^{k,t,x} \\ &\quad + T_{k,\ell}\big(X_\ell^{t,x,*} - \mathbb{E}_k X_\ell^{t,x,*}\big) \\ &\quad + \mathcal{T}_{k,\ell}\mathbb{E}_k X_\ell^{t,x,*} + \pi_{k,\ell},\quad \ell \in \mathbb{T}_k. \end{aligned} \tag{34}$$

Proof: See Appendix D. ■

Remark III.2: Noting that $P_{k,\ell}$ and $\mathcal{P}_{k,\ell}$ are symmetric, $T_{k,\ell}$ and $\bar{T}_{k,\ell}$ are generally nonsymmetric as $\mathcal{A}_{\ell,\ell}, \mathcal{B}_{\ell,\ell}, \mathcal{C}_{\ell,\ell}$, and $\mathcal{D}_{\ell,\ell}$ appear in the expressions of $T_{k,\ell}$ and $\bar{T}_{k,\ell}$. Let the stationary condition (17) hold. From Lemma III.4 and "i)⇒ii)" of the proof of Theorem III.3, we indeed have that control of the following form

$$u_k^{t,x,*} = -\mathcal{W}_k^\dagger\mathcal{H}_k X_k^{t,x,*} - \mathcal{W}_k^\dagger\widetilde{\beta}_k + \widetilde{Y}_k,\ k \in \mathbb{T}_t \tag{35}$$

satisfies (17). Here, $\widetilde{Y}_k = (I - \mathcal{W}_k^\dagger\mathcal{W}_k)Y_k$ with $Y_k \in \mathbb{R}^m$, and $\{\widetilde{\beta}_k, k \in \mathbb{T}_t\}$ is given by

$$\begin{cases} \widetilde{\beta}_k = \mathcal{B}_{k,k}^T\big[\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)f_{k,k} + \widetilde{\pi}_{k,k+1}\big] \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T\big(P_{k,k+1} + T_{k,k+1}\big)d_{k,k}^j \\ \qquad + \rho_{k,k} \\ k \in \mathbb{T}_t \end{cases}$$

with

$$\begin{cases} \begin{cases} \widetilde{\pi}_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\big[f_{k,\ell} - \mathcal{B}_{k,\ell}(\mathcal{W}_\ell^\dagger\widetilde{\beta}_\ell - \widetilde{Y}_k)\big] \\ \qquad + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\big[f_{\ell,\ell} - \mathcal{B}_{\ell,\ell}(\mathcal{W}_\ell^\dagger\widetilde{\beta}_\ell - \widetilde{Y}_k)\big] \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij}\big[(\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1} \\ \qquad \times \big(d_{k,\ell}^j - \mathcal{D}_{k,\ell}^j(\mathcal{W}_\ell^\dagger\widetilde{\beta}_\ell - \widetilde{Y}_k)\big) \\ \qquad + (\mathcal{C}_{k,\ell}^i)^T T_{k,\ell+1}\big(d_{\ell,\ell}^j - \mathcal{D}_{\ell,\ell}^j(\mathcal{W}_\ell^\dagger\widetilde{\beta}_\ell - \widetilde{Y}_k)\big)\big] \\ \qquad + \mathcal{A}_{k,\ell}^T \widetilde{\pi}_{k,\ell+1} + q_{k,\ell} \\ \widetilde{\pi}_{k,N} = g_k \end{cases} \\ k \in \mathbb{T}_t. \end{cases}$$

Furthermore, $X^{t,x,*}$ in (35) is given by

$$\begin{cases} X_{k+1}^{t,x,*} = \big[\big(\mathcal{A}_{k,k} - \mathcal{B}_{k,k}\mathcal{W}_k^{\dagger}\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{B}_{k,k}(\mathcal{W}_k^{\dagger}\widetilde{\beta}_k - \widetilde{Y}_k) + f_{k,k}\big] \\ \qquad + \sum_{i=1}^{p}\big[\big(\mathcal{C}_{k,k}^i - \mathcal{D}_{k,k}^i\mathcal{W}_k^{\dagger}\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{D}_{k,k}^i(\mathcal{W}_k^{\dagger}\widetilde{\beta}_k - \widetilde{Y}_k) + d_{k,k}^i\big]w_k^i \\ X_t^{t,x,*} = x, \ k \in \mathbb{T}_t. \end{cases}$$

D. Second Characterization on the Existence of Open-Loop Equilibrium Control

By Theorem III.1, Theorem III.2, Theorem III.3, and Remark III.2, we have the following result, which gives conditions on the existence of open-loop equilibrium control.

Theorem III.4: The following statements are equivalent.

i) Problem $(\mathrm{LQ})_{tx}$ admits an open-loop equilibrium control.

ii) The conditions (24) and (28) hold.

Under any of the above conditions, control of the following form:

$$u_k^{t,x,*} = -\mathcal{W}_k^{\dagger}\mathcal{H}_k X_k^{t,x,*} - \mathcal{W}_k^{\dagger}\widetilde{\beta}_k + \widetilde{Y}_k, \ k \in \mathbb{T}_t \quad (36)$$

is an open-loop equilibrium control.

Above-mentioned theorem is concerned with the existence of open-loop equilibrium control. Another important issue is the uniqueness of open-loop equilibrium control, which is studied in the following theorem.

Theorem III.5: The following statements are eqivalent.

i) Problem $(\mathrm{LQ})_{tx}$ admits a unique open-loop equilibrium control.

ii) The following assertions hold.

a) The condition (24) is satisfied.

b) $\mathcal{W}_k, k \in \mathbb{T}_t$, are invertible, where $\mathcal{W}_k$ is given in (29).

Under any of the above conditions, the unique open-loop equilibrium control is given by

$$u_k^{t,x,*} = -\mathcal{W}_k^{-1}\mathcal{H}_k X_k^{t,x,*} - \mathcal{W}_k^{-1}\beta_k, \ k \in \mathbb{T}_t \quad (37)$$

with $X^{t,x,*}$ given by

$$\begin{cases} X_{k+1}^{t,x,*} = \big[\big(\mathcal{A}_{k,k} - \mathcal{B}_{k,k}\mathcal{W}_k^{-1}\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{B}_{k,k}\mathcal{W}_k^{-1}\beta_k + f_{k,k}\big] \\ \qquad + \sum_{i=1}^{p}\big[\big(\mathcal{C}_{k,k}^i - \mathcal{D}_{k,k}^i\mathcal{W}_k^{-1}\mathcal{H}_k\big)X_k^{t,x,*} \\ \qquad - \mathcal{D}_{k,k}^i\mathcal{W}_k^{-1}\beta_k + d_{k,k}^i\big]w_k^i \\ X_t^{t,x,*} = x, \ k \in \mathbb{T}_t. \end{cases}$$

Proof: i)⇒ii). The condition (24) naturally holds. We further have b). Otherwise, controls of form (35) are also open-loop equilibrium control.

ii)⇒i). According to Theorem III.1 and Theorem III.4, Problem $(\mathrm{LQ})_{tx}$ admits an open-loop equilibrium control. Due to the nonsingularity of $\mathcal{W}_k, k \in \mathbb{T}_t$ and the proof of Theorem III.3, the open-loop equilibrium control is unique, which is given by (37). ■

IV. Case with All the Initial Pairs

In this section, we will let the initial time t and initial state x range over $\mathbb{T}$ and $l_{\mathcal{F}}^2(t;\mathbb{R}^n)$, respectively; this is referred to as the case with all the initial pairs. Problem (LQ) for the initial pair (t,x) will be simply denoted as Problem $(\mathrm{LQ})_{tx}$, and similar meanings hold for other initial pairs.

First, we give an interesting result on the unique existence of open-loop equilibrium control, which follows from Theorem III.5.

Proposition IV.1: Let $t \in \mathbb{T}$ and $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$. Then, the following statements are equivalent.

i) Problem $(\mathrm{LQ})_{tx}$ admits a unique open-loop equilibrium control.

ii) For any $k \in \mathbb{T}_t$ and any $\xi \in l_{\mathcal{F}}^2(k;\mathbb{R}^n)$, Problem $(\mathrm{LQ})_{k\xi}$ admits a unique open-loop equilibrium control.

Unfortunately, result similar to Proposition IV.1 does not hold if we just consider the existence of open-loop equilibrium control. Alternatively, the following assertion holds.

Theorem IV.1: The following statements are equivalent.

i) For any $t \in \mathbb{T}$ and any $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$, Problem $(\mathrm{LQ})_{tx}$ admits an open-loop equilibrium control.

ii) The set of constrained LDEs

$$\begin{cases} \begin{cases} P_{k,\ell} = Q_{k,\ell} + A_{k,\ell}^T P_{k,\ell+1}A_{k,\ell} \\ \qquad + \sum_{i,j=1}^{p}\gamma_\ell^{ij}(C_{k,\ell}^i)^T P_{k,\ell+1}C_{k,\ell}^j \\ \mathcal{P}_{k,\ell} = \mathcal{Q}_{k,\ell} + \mathcal{A}_{k,\ell}^T\mathcal{P}_{k,\ell+1}\mathcal{A}_{k,\ell} \\ \qquad + \sum_{i,j=1}^{p}\gamma_\ell^{ij}(C_{k,\ell}^i)^T P_{k,\ell+1}\mathcal{C}_{k,\ell}^j \\ P_{k,N} = G_k, \ \mathcal{P}_{k,N} = \mathcal{G}_k, \ \ell \in \mathbb{T}_k \end{cases} \\ \mathbb{W}_k \succeq 0 \\ k \in \mathbb{T} \end{cases} \quad (38)$$

and the set of constrained GDREs

$$\begin{cases} \begin{cases} T_{k,\ell} = A_{k,\ell}^T T_{k,\ell+1}\mathcal{A}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^{p}\gamma_\ell^{ij}(C_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell}^j \\ \qquad - \Big\{A_{k,\ell}^T P_{k,\ell+1}B_{k,\ell} + A_{k,\ell}^T T_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^{p}\gamma_\ell^{ij}\Big[(C_{k,\ell}^i)^T P_{k,\ell+1}D_{k,\ell}^j \\ \qquad + (C_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}^j\Big]\Big\}\mathcal{W}_\ell^{\dagger}\mathcal{H}_\ell \\ \mathcal{T}_{k,\ell} = \mathcal{A}_{k,\ell}^T\mathcal{T}_{k,\ell+1}\mathcal{A}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^{p}\gamma_\ell^{ij}(\mathcal{C}_{k,\ell}^i)^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell}^j \\ \qquad - \Big\{\mathcal{A}_{k,\ell}^T\mathcal{P}_{k,\ell+1}\mathcal{B}_{k,\ell} + \mathcal{A}_{k,\ell}^T\mathcal{T}_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \qquad + \sum_{i,j=1}^{p}(\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1}\mathcal{D}_{k,\ell}^j \\ \qquad + \mathcal{C}_{k,\ell}^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\Big\}\mathcal{W}_\ell^{\dagger}\mathcal{H}_\ell \\ T_{k,N} = 0, \ \mathcal{T}_{k,N} = 0 \\ \ell \in \mathbb{T}_k \end{cases} \\ \mathcal{W}_k\mathcal{W}_k^{\dagger}\mathcal{H}_k - \mathcal{H}_k = 0 \\ k \in \mathbb{T} \end{cases} \quad (39)$$

and the set of constrained LDEs

$$\begin{cases} \begin{cases} \pi_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\big(f_{k,\ell} - \mathcal{B}_{k,\ell}\mathcal{W}_\ell^\dagger \beta_\ell\big) \\ \qquad + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\big(f_{\ell,\ell} - \mathcal{B}_{\ell,\ell}\mathcal{W}_\ell^\dagger \beta_\ell\big) \\ \qquad + \sum_{i,j=1}^p \gamma_\ell^{ij}\big[(\mathcal{C}_{k,\ell}^i)^T P_{k,\ell+1} \\ \qquad \times \big(d_{k,\ell}^j - \mathcal{D}_{k,\ell}^j \mathcal{W}_\ell^\dagger \beta_\ell\big) + (\mathcal{C}_{k,\ell}^i)^T \\ \qquad \times T_{k,\ell+1}\big(d_{\ell,\ell}^j - \mathcal{D}_{\ell,\ell}^j \mathcal{W}_\ell^\dagger \beta_\ell\big)\big] \\ \qquad + \mathcal{A}_{k,\ell}^T \pi_{k,\ell+1} + q_{k,\ell} \\ \pi_{k,N} = g_k \end{cases} \\ \mathcal{W}_k \mathcal{W}_k^\dagger \beta_k - \beta_k = 0 \\ k \in \mathbb{T} \end{cases} \tag{40}$$

are solvable in the sense that

$$\begin{cases} \mathbb{W}_k \succeq 0 \\ \mathcal{W}_k \mathcal{W}_k^\dagger \mathcal{H}_k - \mathcal{H}_k = 0 \\ \mathcal{W}_k \mathcal{W}_k^\dagger \beta_k - \beta_k = 0 \\ k \in \mathbb{T} \end{cases} \tag{41}$$

holds, i.e., the solutions of (38)–(40) satisfy (41). Here

$$\begin{cases} \mathbb{W}_k = \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \mathcal{P}_{k,k+1}\mathcal{B}_{k,k} \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T P_{k,k+1} \mathcal{D}_{k,k}^j \\ \mathcal{W}_k = \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{B}_{k,k} \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{D}_{k,k}^j \\ \mathcal{H}_k = \mathcal{B}_{k,k}^T \big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{A}_{k,k} \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{C}_{k,k}^j \\ \beta_k = \mathcal{B}_{k,k}^T \big[\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big) f_{k,k} + \pi_{k,k+1}\big] \\ \qquad + \sum_{i,j=1}^p \gamma_k^{ij} (\mathcal{D}_{k,k}^i)^T \big(P_{k,k+1} + T_{k,k+1}\big) d_{k,k}^j \\ \qquad + \rho_{k,k} \\ k \in \mathbb{T}. \end{cases}$$

Under any of the above conditions, control of the form (36) is an open-loop equilibrium control of Problem (LQ)$_{tx}$.

Proof: i)⇒ii). From Theorem III.4, the constrained LDEs (38) are solvable, and for any $t \in \mathbb{T}, x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$, the condition (28) holds, i.e.,

$$\mathcal{H}_k X_k^{t,x,*} + \beta_k \in \mathrm{Ran}(\mathcal{W}_k),\ k \in \mathbb{T}_t.$$

Especially, we have

$$\mathcal{H}_t x + \beta_t \in \mathcal{W}_t,\ t \in \mathbb{T}$$

equivalently

$$\mathcal{W}_t \mathcal{W}_t^\dagger (\mathcal{H}_t x + \beta_t) = \mathcal{H}_t x + \beta_t,\ t \in \mathbb{T}. \tag{42}$$

Let $x = 0$ in (42), we have

$$\mathcal{W}_t \mathcal{W}_t^\dagger \beta_t = \beta_t,\ t \in \mathbb{T}$$

which further implies

$$\mathcal{W}_t \mathcal{W}_t^\dagger \mathcal{H}_t x = \mathcal{H}_t x,\ t \in \mathbb{T}. \tag{43}$$

Noting that (43) holds for any $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$, we obtain

$$\mathcal{W}_k \mathcal{W}_k^\dagger \mathcal{H}_k - \mathcal{H}_k = 0,\ t \in \mathbb{T}.$$

Hence, (39) and (40) are solvable.

ii)⇒i). As (38)–(40) are solvable and by Theorem III.4, for any $t \in \mathbb{T}$ and any $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$ Problem (LQ)$_{tx}$ admits an open-loop equilibrium control. ■

Corollary IV.1: Let

$$Q_{k,\ell}, \bar{Q}_{k,\ell} \succeq 0,\ R_{k,\ell}, \bar{R}_{k,\ell} \succ 0,\ k \in \mathbb{T},\ \ell \in \mathbb{T}_k. \tag{44}$$

Then, the following statements are equivalent.

1) For any $t \in \mathbb{T}$ and any $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$, Problem (LQ)$_{tx}$ admits an open-loop equilibrium control.
2) Equations (39) and (40) are solvable.

Proof: In this situation, $\mathbb{W}_k, k \in \mathbb{T}$ are positive definite, i.e., $\mathbb{W}_k \succ 0, k \in \mathbb{T}$. Hence, the conclusion follows. ■

Let us make some rough observations under the condition (44). Assuming (44), consider Problem (LQ)$_{tx}$ for $t \in \mathbb{T}$ and $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$. Let us begin with $t = N-1$. Noting that $\mathcal{W}_{N-1} = \mathbb{W}_{N-1} \succ 0$, Problem (LQ)$_{N-1,x}$ admits a unique open-loop equilibrium control and $u_{N-1}^{N-1,x,*}$ is easily obtained

$$u_{N-1}^{N-1,x,*} = -\mathcal{W}_{N-1}^{-1}\mathcal{H}_{N-1}x - \mathcal{W}_{N-1}^{-1}\beta_{N-1}.$$

Now move to the case $t = N-2$. If we have selected $u_{N-1}^{N-2,x,*}$, from Lemmas III.1 and III.2 we have

$$\begin{aligned} &J(N-2,x;(u_{N-2},u_{N-1}^{N-2,x,*})) \\ &= u_{N-2}^T \mathbb{W}_{N-2} u_{N-2} \\ &\quad + 2\Big[\rho_{N-2,N-2} + \mathcal{B}_{N-2,N-2}^T \mathbb{E}_{N-2} Z_{N-1}^{N-2,0} \\ &\quad + \sum_{i=1}^p (\mathcal{D}_{N-2,N-2}^i)^T \mathbb{E}_{N-1}(Z_{N-1}^{N-2,0} w_{N-2}^i)\Big]^T u_{N-2} \\ &\quad + J(N-2,x;(0,u_{N-1}^{N-2,x,*})) \\ &\triangleq \langle \mathbb{W}_{N-2}u_{N-2},u_{N-2}\rangle + 2\langle M_{N-2}(Z_{N-1}^{N-2,0}),u_{N-2}\rangle \\ &\quad + J(N-2,x;(0,u_{N-1}^{N-2,x,*})). \end{aligned} \tag{45}$$

In the above, $Z_{N-1}^{N-2,0}$ is computed via

$$\begin{cases} Z_\ell^{N-2,0} = A_{N-2,\ell}^T \mathbb{E}_\ell Z_{\ell+1}^{N-2,0} + \bar{A}_{N-2,\ell}^T \mathbb{E}_{N-2} Z_{\ell+1}^{N-2,0} \\ \qquad + \sum_{i=1}^p \big[(C_{N-2,\ell}^i)^T \mathbb{E}_\ell (Z_{\ell+1}^{N-2,0} w_\ell^i) \\ \qquad + (\bar{C}_{N-2,\ell}^i)^T \mathbb{E}_{N-2}(Z_{\ell+1}^{N-2,0} w_\ell^i)\big] \\ \qquad + Q_{N-2,\ell} X_\ell^{N-2,0} + \bar{Q}_{N-2,\ell} \mathbb{E}_{N-2} X_\ell^{N-2,0} \\ \qquad + q_{N-2,\ell} \\ Z_N^{N-2,0} = G_{N-2} X_N^{N-2,0} + \bar{G}_{N-2} \mathbb{E}_{N-2} X_N^{N-2,0} \\ \qquad + g_{N-2} \\ \ell \in \{N-2,N-1\} \end{cases}$$

where $X^{N-2,0}$ is given by

$$\begin{cases} X_N^{N-2,0} = \big(A_{N-2,N-1}X_{N-1}^{N-2,0} \\ \qquad + \bar{A}_{N-2,N-1}\mathbb{E}_{N-2}X_{N-2}^{N-1,0} \\ \qquad + B_{N-2,N-1}u_{N-1}^{N-2,x,*} \\ \qquad + \bar{B}_{N-2,N-1}\mathbb{E}_{N-2}u_{N-1}^{N-2,x,*} \\ \qquad + f_{N-2,N-1}\big) \\ \qquad + \sum_{i=1}^p \big[C_{N-2,N-1}^i X_{N-1}^{N-2,0} \\ \qquad + \bar{C}_{N-2,N-1}^i \mathbb{E}_{N-2}X_{N-2}^{N-1,0} \\ \qquad + D_{N-2,N-1}^i u_{N-1}^{N-2,x,*} \\ \qquad + \bar{D}_{N-2,N-1}^i \mathbb{E}_{N-2}u_{N-1}^{N-2,x,*} \\ \qquad + d_{N-2,N-1}^i\big] w_{N-1}^i \\ X_{N-1}^{N-2,0} = \big(\mathcal{A}_{N-2,N-2}X_{N-2}^{N-2,0} + f_{N-2,N-2}\big) \\ \qquad + \sum_{i=1}^p \big[\mathcal{C}_{N-2,N-2}^i X_{N-2}^{N-2,0} \\ \qquad + d_{N-2,N-2}^i\big] w_{N-2}^i \\ X_{N-2}^{N-2,0} = x \\ \ell \in \{N-2, N-1\} \end{cases} \tag{46}$$

and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^m$, and

$$M_{N-2}(Z_{N-1}^{N-2,0}) = \rho_{N-2,N-2} + \mathcal{B}_{N-2,N-2}^T \mathbb{E}_{N-2} Z_{N-1}^{N-2,0} + \sum_{i=1}^p (\mathcal{D}_{N-2,N-2}^i)^T \mathbb{E}_{N-1}(Z_{N-1}^{N-2,0} w_{N-2}^i).$$

If we "select"

$$u_{N-2}^{N-2,x,*} = -\mathbb{W}_{N-2}^{-1} M_{N-2}(Z_{N-1}^{N-2,0}) \tag{47}$$

then the following inequality

$$J(N-2, x; (u_{N-2}^{N-2,x,*}, u_{N-1}^{N-2,x,*})) \le J(N-2, x; (u_{N-2}, u_{N-1}^{N-2,x,*}))$$

seems to hold.

However, it should be mentioned that it is questionable about (47). If $u^{N-2,x,*}$ exists, we should have

$$u_{N-1}^{N-2,x,*} = -\mathcal{W}_{N-1}^{-1}\mathcal{H}_{N-1}X_{N-1}^{N-2,x,*} - \mathcal{W}_{N-1}^{-1}\beta_{N-1}$$

and

$$X_{N-1}^{N-2,x,*} = \mathcal{A}_{N-2,N-2}x + \mathcal{B}_{N-2,N-2}u_{N-2}^{N-2,x,*} + \sum_{i=1}^p \big(\mathcal{C}_{N-2,N-2}^i x + \mathcal{D}_{N-2,N-2}^i u_{N-2}^{N-2,x,*}\big) w_{N-2}^i.$$

Hence, $u_{N-1}^{N-2,x,*}$ depends on $u_{N-2}^{N-2,x,*}$, and it is so for $Z_{N-1}^{N-2,0}$. Therefore, the right-hand side of (47) is a functional of $u_{N-2}^{N-2,x,*}$, and it cannot be concluded that (47) makes sense under the assumption $\mathbb{W}_{N-2} \succ 0$. Recall that $\{\{T_{k,\ell}, \ell \in \mathcal{T}_k\}, k \in \mathbb{T}\}$ is also needed to characterize the open-loop equilibrium control, and that for $k \in \mathbb{T}$

$$\mathcal{W}_k = \mathbb{W}_k + \mathcal{B}_{k,k}^T \mathcal{T}_{k,k+1}\mathcal{B}_{k,k} + \sum_{i,j=1}^p (\mathcal{D}_{k,k}^i)^T \mathcal{T}_{k,k+1}\mathcal{D}_{k,k}^j.$$

Note that elements in $\{\{\mathcal{T}_{k,\ell}, \ell \in \mathbb{T}_k\}, k \in \mathbb{T}\}$ are generally nonsymmetric. So far, it is not known now whether or not $\mathbb{W}_k \succ 0$ could ensure the nonsingularity of $\mathcal{W}_k$. Therefore, we have to check case by case the solvability of (39), (40) [by validating (41)].

V. Multiperiod Mean–Variance Portfolio Selection

Consider a capital market consisting of one riskless asset and n risky assets within a time horizon N. Let $s_k (> 1)$ be a given deterministic return of the riskless asset at time period k and $e_k = (e_k^1, \ldots, e_k^n)^T$ the vector of random returns of the n risky assets at period k. We assume that vectors $e_k, k = 0, 1, \ldots, N-1$, are statistically independent and the only information known about the random return vector e_k is its first two moments: its mean $\mathbb{E}(e_k) = (\mathbb{E}e_k^1, \mathbb{E}e_k^2, \ldots, \mathbb{E}e_k^n)^T$ and its covariance $\mathrm{Cov}(e_k) = \mathbb{E}[(e_k - \mathbb{E}e_k)(e_k - \mathbb{E}e_k)^T]$. Clearly, $\mathrm{Cov}(e_k)$ is nonnegative definite, i.e., $\mathrm{Cov}(e_k) \succeq 0$.

Let X_k be the wealth of the investor at the beginning of the kth period, and let u_k^i, $i = 1, 2, \ldots, n$, be the amount invested in the ith risky asset at period k. Then, $X_k - \sum_{i=1}^n u_k^i$ is the amount invested in the riskless asset at period k, and the wealth at the beginning of the $(k+1)$th period [22] is given by

$$X_{k+1} = \sum_{i=1}^n e_k^i u_k^i + \left(X_k - \sum_{i=1}^n u_k^i\right) s_k = s_k X_k + O_k^T u_k \tag{48}$$

where O_k is the excess return vector of risky assets [22] defined as

$$\begin{aligned} O_k &= (O_k^1, O_k^2, \ldots, O_k^n)^T \\ &= (e_k^1 - s_k, e_k^2 - s_k, \ldots, e_k^n - s_k)^T. \end{aligned} \tag{49}$$

Clearly, $X_k \in \mathbb{R}$, $k \in \mathbb{T}$. In this section, we consider the case where short selling of stocks is allowed, i.e., $u_k^i, i = 1, ..., k$, could take values in $\mathbb{R}$, which leads to an unconstrained mean–variance portfolio selection formulation.

Let

$$\mathcal{F}_k = \sigma(e_\ell, \ell = 0, 1, \ldots, k-1)$$

which contains $\mathcal{F}'_k = \sigma(X_\ell, \ell = 0, 1, \ldots, k)$. Then, the time-inconsistent version of multiperiod mean–variance problem [22] can be formulated as follows.

Problem (MV): Letting $t \in \mathbb{T}$ and $x \in l_{\mathcal{F}}^2(t; \mathbb{R}^n)$, find $u^* \in l_{\mathcal{F}}^2(\mathbb{T}_t; \mathbb{R}^n)$ such that

$$J_m(t, x; u^*) = \inf_{u \in l_{\mathcal{F}}^2(\mathbb{T}_t; \mathbb{R}^n)} J_m(t, x; u).$$

Here

$$J_m(t, x; u) = \lambda \mathbb{E}_t(X_N - \mathbb{E}_t X_N)^2 - \mathbb{E}_t X_N$$

which is subject to

$$\begin{cases} X_{k+1} = s_k X_k + O_k^T u_k \\ X_k = x \end{cases}$$

with $\lambda > 0$ the tradeoff parameter between the mean and the variance of the terminal wealth.

It is noted that some nondegenerate assumptions are posed in [2], [6], [9], [10], [17], and [18]. Specifically, the volatilities of the stocks in [2], [6], [17], and [18] and the return rates of the risky securities in [9], [10], and [22] are assumed to be nondegenerate. In this section, we do not pose the nondegenerate constraint on $\mathrm{Cov}(e_k), \mathrm{Cov}(O_k), k \in \mathbb{T}$, and want to see what is the weakest condition on the existence of open-loop equilibrium portfolio control of Problem (MV)

To solve Problem (MV), we shall transform (48) into a linear controlled system of form (4), by which the general theory in above sections will work. Precisely, define

$$\begin{cases} w_k^i = e_k^i - s_k - \mathbb{E}(e_k^i - s_k) \\ D_k^i = (0, \dots, 0, 1, 0, \dots, 0) \\ i = 1, \cdots, n, \ k = 0, 1, \cdots, N-1 \end{cases}$$

where the ith entry of D_k^i is 1. Then, $\{w_k = (w_k^1, ..., w_k^n)^T, k \in \mathbb{T}\}$ is a martingale difference sequence as $e_k, k = 1, .., N-1$, are statistically independent. Furthermore

$$\mathbb{E}_k[w_k w_k^T] = \mathbb{E}[w_k w_k^T] = \mathrm{Cov}(e_k) = (\gamma_k^{ij})_{n\times n}.$$

This leads to

$$\begin{cases} X_{k+1} = (s_k X_k + (\mathbb{E}O_k)^T u_k) + \sum_{i=1}^n D_k^i u_k w_k^i \\ X_k = x. \end{cases} \quad (50)$$

Due to Theorem IV.1, we have the following result.

Theorem V.1: The following statements are equivalent.

1) For any $t \in \mathbb{T}$ and any $x \in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem (MV) admits an open-loop equilibrium portfolio control.
2) $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{T}$.

Under any of the above conditions

$$u_k^{t,x,*} = -\mathcal{W}_k^\dagger \beta_k, \ \ k \in \mathbb{T}_t \quad (51)$$

is an open-loop equilibrium portfolio control for the initial pair (t, x), where

$$\begin{cases} \mathcal{W}_k = P_{k+1}\mathrm{Cov}(O_k) \\ \beta_k = \pi_{k+1}\mathbb{E}O_k \\ k \in \mathbb{T} \end{cases} \quad (52)$$

with

$$\begin{cases} P_k = s_k^2 P_{k+1} \\ \pi_k = s_k \pi_{k+1} \\ P_N = \lambda, \ \ \pi_N = -\frac{1}{2} \\ k \in \mathbb{T}. \end{cases}$$

Proof: In this case, (38)–(40) become to

$$\begin{cases} P_k = s_k^2 P_{k+1} \\ \mathcal{P}_k = s_k^2 \mathcal{P}_{k+1} \equiv 0 \\ P_N = \lambda, \ \ \mathcal{P}_N = 0 \\ \mathbb{W}_k = \sum_{i,j=1}^n \gamma_k^{ij} (D_k^i)^T P_{k+1} D_k^j \\ \quad\ = P_{k+1}\mathrm{Cov}(O_k) \succeq 0 \\ k \in \mathbb{T} \end{cases} \quad (53)$$

$$\begin{cases} T_k = s_k^2 T_{k+1} - s_k (P_{k+1} + T_{k+1})(\mathbb{E}O_k)^T \mathcal{W}_k^\dagger \mathcal{H}_k \\ \mathcal{T}_k = \mathcal{T}_{k+1}\left[s_k^2 - s_k(\mathbb{E}O_k)^T \mathcal{W}_k^\dagger \mathcal{H}_k\right] \equiv 0 \\ T_N = 0, \ \ \mathcal{T}_N = 0 \\ \mathcal{W}_k \mathcal{W}_k^\dagger \mathcal{H}_k - \mathcal{H}_k = 0 \\ k \in \mathbb{T} \end{cases} \quad (54)$$

and

$$\begin{cases} \pi_k = s_k \pi_{k+1} \\ \pi_N = -\frac{1}{2} \\ \mathcal{W}_k \mathcal{W}_k^\dagger \beta_k - \beta_k = 0 \\ k \in \mathbb{T} \end{cases} \quad (55)$$

where

$$\begin{cases} \mathcal{W}_k = \sum_{i,j=1}^n \gamma_k^{ij} (D_k^i)^T \left(P_{k+1} + T_{k+1}\right) D_k^j \\ \quad\ = \left(P_{k+1} + T_{k+1}\right)\mathrm{Cov}(O_k) \\ \mathcal{H}_k = 0 \\ \beta_k = \pi_{k+1}\mathbb{E}O_k \\ k \in \mathbb{T}. \end{cases} \quad (56)$$

Noting that $P_{k+1} > 0, \mathcal{H}_k = 0, k \in \mathbb{T}$, (53) and (54) are solvable, and $T_k = 0, k \in \mathbb{T}$. As $\pi_{k+1} \neq 0, k \in \mathbb{T}$, the solvability of (55) is then equivalent to the fact $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{T}$. By Theorem IV.1, we achieve the conclusion. ■

Concerned with the uniqueness of open-loop equilibrium control, we have the following result.

Theorem V.2: The following statements are equivalent.

1) For any $t \in \mathbb{T}$ and any $x \in l^2_{\mathcal{F}}(t;\mathbb{R})$, Problem (MV) admits a unique open-loop equilibrium portfolio control.
2) $\mathrm{Cov}(O_k) \succ 0, k \in \mathbb{T}$.

Under any of the above conditions, the unique open-loop equilibrium portfolio control for the initial pair (t, x) is given by

$$u_k^{t,x,*} = -\mathcal{W}_k^{-1}\beta_k, \ \ k \in \mathbb{T}_t \quad (57)$$

with $\mathcal{W}_k, \beta_k, k \in \mathbb{T}$ given in (52).

Proof: The proof follows from Theorem III.5, Proposition IV.1 and Theorem V.1. ■

Note that $\mathrm{Cov}(O_k) \succ 0, k \in \mathbb{T}$ is a common assumption in multiperiod mean–variance portfolio selection [9], [11], [22]. In this situation, the open-loop equilibrium portfolio control for the initial pair (t, x) is

$$u_k^{t,x,*} = -\frac{1}{2\lambda s_{k+1}\cdots s_{N-1}}(\mathrm{Cov}(O_k))^{-1}\mathbb{E}O_k, \ \ k \in \mathbb{T}_t.$$

This section just studied the simplest dynamic mean–variance model [22]. In the future, dynamic mean–variance portfolio optimizations are much desirable for the more general models.

VI. Conclusion

In this paper, the open-loop time-consistent equilibrium control is investigated for a kind of mean-field stochastic LQ problem, where both the system matrices and the weighting matrices are depending on the initial time, and the conditional expectations of the control and state enter quadratically into the cost functional. Necessary and sufficient conditions are presented for both the case with a fixed initial pair and the case with all the initial pairs. Furthermore, a set of constrained GDREs and two sets of constrained LDEs are introduced to characterize the open-loop equilibrium control. Note that this paper is concerned with the time consistency of open-loop control. For future research, the time consistency of the strategy should be studied.

APPENDIX

A. Proof of Lemma III.1

Let us replace u_k with $u_k + \lambda\bar{u}_k$ in the forward SΔE of (15), and denote its solution by $X^{k,\lambda}$. Then, we have

$$\begin{cases} \frac{X^{k,\lambda}_{\ell+1}-X^k_{\ell+1}}{\lambda} = \Big(A_{k,\ell}\frac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda} + \bar{A}_{k,\ell}\frac{\mathbb{E}_k X^{k,\lambda}_\ell - \mathbb{E}_k X^k_\ell}{\lambda}\Big) \\ \quad + \sum_{i=1}^p \Big(C^i_{k,\ell}\frac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda} + \bar{C}^i_{k,\ell}\frac{\mathbb{E}_k X^{k,\lambda}_\ell - \mathbb{E}_k X^k_\ell}{\lambda}\Big)w^i_\ell \\ \frac{X^{k,\lambda}_{k+1}-X^k_{k+1}}{\lambda} = \Big(A_{k,k}\frac{X^{k,\lambda}_k - X^k_k}{\lambda} + \bar{A}_{k,k}\frac{\mathbb{E}_k X^{k,\lambda}_k - \mathbb{E}_k X^k_k}{\lambda} \\ \quad + B_{k,k}\bar{u}_k + \bar{B}_{k,k}\bar{u}_k\Big) + \sum_{i=1}^p \Big(C^i_{k,k}\frac{X^{k,\lambda}_k - X^k_k}{\lambda} \\ \quad + \bar{C}^i_{k,k}\frac{\mathbb{E}_k X^{k,\lambda}_k - \mathbb{E}_k X^k_k}{\lambda} + D^i_{k,k}\bar{u}_k + \bar{D}^i_{k,k}\mathbb{E}_k\bar{u}_k\Big)w^i_k \\ \frac{X^{k,\lambda}_k - X^k_k}{\lambda} = 0, \quad \ell \in \mathbb{T}_{k+1}. \end{cases}$$

Denoting $\frac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda}$ by $Y^{k,\bar{u}_k}_\ell$, we get (14). Note that $X^{k,\lambda}_\ell = X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell, \forall \ell \in \mathbb{T}_k$. Then, we have

$$\begin{aligned} & J(k,\zeta;(u_k+\lambda\bar{u}_k, u|_{\mathbb{T}_{k+1}})) - J(k,\zeta;u) \\ &= \sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell)^T Q_{k,\ell}(X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell) \\ &\quad + [\mathbb{E}_k(X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell)]^T\bar{Q}_{k,\ell}\mathbb{E}_k(X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell) \\ &\quad + 2q^T_{k,\ell}(X^k_\ell + \lambda Y^{k,\bar{u}_k}_\ell) - (X^k_\ell)^T Q_{k,\ell}X^k_\ell \\ &\quad - [\mathbb{E}_k X^k_\ell]^T\bar{Q}_{k,\ell}\mathbb{E}_\ell X^k_\ell - 2q^T_{k,\ell}X^k_\ell\Big] \\ &\quad + (u_k+\lambda\bar{u}_k)^T(R_{k,k}+\bar{R}_{k,k})(u_k+\lambda\bar{u}_k) \\ &\quad + 2\rho^T_{k,\ell}(u_k+\lambda\bar{u}_k) - u^T_k(R_{k,k}+\bar{R}_{k,k})u_k - 2\rho^T_{k,\ell}u_k \\ &\quad + [\mathbb{E}_k(X^k_N + \lambda Y^{k,\bar{u}_k}_N)]^T\bar{G}_k\mathbb{E}_k(X^k_N + \lambda Y^{k,\bar{u}_k}_N) \\ &\quad + \mathbb{E}_k\big[(X^k_N + \lambda Y^{k,\bar{u}_k}_N)^T G_k(X^k_N + \lambda Y^{k,\bar{u}_k}_N)\big] \\ &\quad + 2\mathbb{E}_k[g^T_k(X^k_N + \lambda Y^{k,\bar{u}_k}_N)] \\ &\quad - \mathbb{E}_k\big[(X^k_N)^T G_k X^k_N\big] - (\mathbb{E}_k X^k_N)^T\bar{G}_k\mathbb{E}_k X^k_N \\ &\quad - 2\mathbb{E}_k g^T_k X^k_N \\ &= 2\lambda\Bigg\{\sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(X^k_\ell)^T Q_{k,\ell}Y^{k,\bar{u}_k}_\ell + q^T_{k,\ell}Y^k_\ell \\ &\quad + [\mathbb{E}_k X^k_\ell]^T\bar{Q}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big] + u^T_k(R_{k,k}+\bar{R}_{k,k})\bar{u}_k \\ &\quad + \rho^T_{k,\ell}\bar{u}_k + \mathbb{E}_k\big[(X^{k,u_k}_N)^T G_k Y^{k,\bar{u}_k}_N\big] + \mathbb{E}_k[g^T_k Y^{k,\bar{u}_k}_N] \\ &\quad + [\mathbb{E}_k X^k_N]^T\bar{G}_k\mathbb{E}_k Y^{k,\bar{u}_k}_N\Bigg\} \\ &\quad + \lambda^2\Bigg\{\sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(Y^{k,\bar{u}_k}_\ell)^T Q_{k,\ell}Y^{k,\bar{u}_k}_\ell \\ &\quad + (\mathbb{E}_k Y^{k,\bar{u}_k}_\ell)^T\bar{Q}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big] \\ &\quad + \mathbb{E}_k\big[\bar{u}^T_k(R_{k,k}+\bar{R}_{k,k})\bar{u}_k\big] \\ &\quad + \mathbb{E}_k\big[(Y^{k,\bar{u}_k}_N)^T G_k Y^{k,\bar{u}_k}_N\big] \\ &\quad + (\mathbb{E}_k Y^{k,\bar{u}_k}_N)^T\bar{G}_k\mathbb{E}_k Y^{k,\bar{u}_k}_N\Bigg\}. \end{aligned} \tag{58}$$

From (15), it holds that $\mathbb{E}_k Z^k_N = \mathcal{G}_k\mathbb{E}_k X^k_N + g_k$ and $Z^k_N - \mathbb{E}_k Z^k_N = G_k(X^k_N - \mathbb{E}_k X^k_N)$. Noting $Y^{k,\bar{u}_k}_k = 0$, then, we have

$$\begin{aligned} & \sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(X^k_\ell)^T Q_{k,\ell}Y^{k,\bar{u}_k}_\ell + q^T_{k,\ell}Y^{k,\bar{u}_k}_\ell \\ & + [\mathbb{E}_k X^k_\ell]^T\bar{Q}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big] + u^T_k(R_{k,k}+\bar{R}_{k,k})\bar{u}_k \\ & + \rho^T_{k,\ell}\bar{u}_k + \mathbb{E}_k\big[(X^{k,u_k}_N)^T G_k Y^{k,\bar{u}_k}_N\big] \\ & + [\mathbb{E}_k X^k_N]^T\bar{G}_k\mathbb{E}_k Y^{k,\bar{u}_k}_N + \mathbb{E}_k[g^T_k Y^{k,\bar{u}_k}_N] \\ &= \sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[\Big(Q_{k,\ell}(X^{k,u_k}_\ell - \mathbb{E}_k X^{k,u_k}_\ell) \\ &\quad + A^T_{k,\ell}(\mathbb{E}_\ell Z^k_{\ell+1} - \mathbb{E}_k Z^k_{\ell+1}) \\ &\quad + \sum_{i=1}^p (C^i_{k,\ell})^T\big(\mathbb{E}_\ell(Z^k_{\ell+1}w^i_\ell) - \mathbb{E}_k(Z^k_{\ell+1}w^i_\ell)\big) \\ &\quad - (Z^k_\ell - \mathbb{E}_k Z^k_\ell)\Big)^T(Y^{k,\bar{u}_k}_\ell - \mathbb{E}_k Y^{k,\bar{u}_k}_\ell) \\ &\quad + \Big(\mathcal{Q}_{k,\ell}\mathbb{E}_k X^{k,u_k}_\ell + q_{k,\ell} + \mathcal{A}^T_{k,\ell}\mathbb{E}_k Z^k_{\ell+1} \\ &\quad + \sum_{i=1}^p (\mathcal{C}^i_{k,\ell})^T\mathbb{E}_k(Z^k_{\ell+1}w^i_\ell) - \mathbb{E}_k Z^k_\ell\Big)^T\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big] \\ &\quad + \Big[\mathcal{R}_{k,k}u_k + \mathcal{B}^T_{k,k}\mathbb{E}_k Z^k_{k+1} \\ &\quad + \sum_{i=1}^p (\mathcal{D}^i_{k,k})^T\mathbb{E}_k(Z^k_{k+1}w^i_k) + \rho_{k,k}\Big]^T\bar{u}_k \\ &= \Big[\mathcal{R}_{k,k}u_k + \mathcal{B}^T_{k,k}\mathbb{E}_k Z^k_{k+1} \\ &\quad + \sum_{i=1}^p (\mathcal{D}^i_{k,k})^T\mathbb{E}_k(Z^k_{k+1}w^i_k) + \rho_{k,k}\Big]^T\bar{u}_k. \end{aligned}$$

This together with (58) implies the conclusion. ■

B. Proof of Theorem III.1

i)⇒ii). Let $u^{t,x,*}$ be an open-loop equilibrium control. As (18) is a decoupled FBSΔE, (18) is solvable. From (12), we have

$$\begin{aligned}&J(k, X_k^{t,x,*}; (u_k^{t,x,*} + \lambda \bar{u}_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})) - J(k, X_k^{t,x,*}; u^{t,x,*})\\&= 2\lambda\Big[\mathcal{R}_{k,k} u_k^{t,x,*} + \mathcal{B}_{k,k}^T \mathbb{E}_k Z_{k+1}^{k,t,x} + \sum_{i=1}^p (\mathcal{D}_{k,k}^i)^T \mathbb{E}_k (Z_{k+1}^{k,t,x} w_k^i)\\&\quad + \rho_{k,k}\Big]^T \bar{u}_k + \lambda^2 \widehat{J}(k, 0; \bar{u}_k)\\&\geq 0. \qquad (59)\end{aligned}$$

Noting that (59) holds for any $\lambda \in \mathbb{R}$ and $\bar{u}_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$, we have (16) and (17). In fact, if (16) was not satisfied, then there would be a $\bar{u}_k$ such that $\lim_{\lambda\mapsto\infty} \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*} + \lambda\bar{u}_k) - \bar{J}(k, X_k^{t,x,*}; u_k^{t,x,*}) = -\infty$. This is impossible. Furthermore, if for some $k_0 \in \mathbb{T}_t$

$$\begin{aligned}\gamma_{k_0} &= \mathcal{R}_{k_0,k_0} u_{k_0}^{t,x,*} + \mathcal{B}_{k_0,k_0}^T \mathbb{E}_{k_0} Z_{k_0+1}^{k_0,t,x}\\&\quad + \sum_{i=1}^p (\mathcal{D}_{k_0,k_0}^i)^T \mathbb{E}_{k_0} (Z_{k_0+1}^{k_0,t,x} w_{k_0}^i) + \rho_{k_0,k_0}\\&\neq 0\end{aligned}$$

we let $\bar{u}_{k_0} = \gamma_{k_0}$. Then, (59) implies that

$$2\lambda|\gamma_{k_0}|^2 + \lambda^2 \widehat{J}(k_0, 0; \gamma_{k_0}) \geq 0$$

holds for any $\lambda \in \mathbb{R}$. However, for negative number λ with sufficient small magnitude, it holds that

$$2\lambda|\gamma_{k_0}|^2 + \lambda^2 \widehat{J}(k_0, 0; \gamma_{k_0}) < 0$$

and contradiction arises. Therefore, γ_{k_0} must be 0, and (17) holds.

ii)⇒i). In this case, for any $\lambda \in \mathbb{R}$ and $\bar{u}_k \in L^2_{\mathcal{F}}(k; \mathbb{R}^m)$ we have

$$\begin{aligned}&J(k, X_k^{t,x,*}; (u_k^{t,x,*} + \lambda \bar{u}_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})) - J(k, X_k^{t,x,*}; u^{t,x,*})\\&= \lambda^2 \widehat{J}(k, 0; \bar{u}_k)\\&\geq 0.\end{aligned}$$

Hence, $u^{t,x,*}$ is an open-loop equilibrium control. ■

C. Proof of Lemma III.3

It is assumed that $u_\ell^{t,x,*} = \Psi_\ell X_\ell^{t,x,*} + \alpha_\ell, \ell \in \mathbb{T}_\tau$. Then, we have

$$\begin{aligned}X_N^{k,t,x} &= A_{k,N-1} X_{N-1}^{k,t,x} + \bar{A}_{k,N-1} \mathbb{E}_k X_{N-1}^{k,t,x}\\&\quad + B_{k,N-1} \Psi_{N-1} X_{N-1}^{t,x,*} + \bar{B}_{k,N-1} \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*}\\&\quad + \mathcal{B}_{k,N-1} \alpha_{N-1} + f_{k,N-1}\\&\quad + \sum_{i=1}^p \Big[C_{k,N-1}^i X_{N-1}^{k,t,x} + \bar{C}_{k,N-1}^i \mathbb{E}_k X_{N-1}^{k,t,x}\\&\quad + D_{k,N-1}^i \Psi_{N-1} X_{N-1}^{t,x,*} + \bar{D}_{k,N-1}^i \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*}\\&\quad + \mathcal{D}_{k,N-1}^i \alpha_{N-1} + d_{k,N-1}^i\Big] w_{N-1}^i.\end{aligned}$$

To calculate $Z_{N-1}^{k,t,x}$, we need some preparations. Noting that

$$Z_N^{k,t,x} = G_k X_N^{k,t,x} + \bar{G}_k \mathbb{E}_k X_N^{k,t,x} + g_k$$

we get

$$\begin{aligned}&A_{k,N-1}^T \mathbb{E}_{N-1} Z_N^{k,t,x}\\&= A_{k,N-1}^T \mathbb{E}_{N-1} \big[G_k X_N^{k,t,x} + \bar{G}_k \mathbb{E}_k X_N^{k,t,x} + g_k\big]\\&= A_{k,N-1}^T G_k A_{k,N-1} X_{N-1}^{k,t,x} + \big[A_{k,N-1}^T G_k \bar{A}_{k,N-1}\\&\quad + A_{k,N-1}^T \bar{G}_k \mathcal{A}_{k,N-1}\big] \mathbb{E}_k X_{N-1}^{k,t,x}\\&\quad + A_{k,N-1}^T G_k B_{k,N-1} \Psi_{N-1} X_{N-1}^{t,x,*}\\&\quad + A_{k,N-1}^T \big[G_k \bar{B}_{k,N-1} + \bar{G}_k \mathcal{B}_{k,N-1}\big] \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*}\\&\quad + A_{k,N-1}^T \mathcal{G}_k \big[\mathcal{B}_{k,N-1} \alpha_{N-1} + f_{k,N-1}\big] + A_{k,N-1}^T g_k\end{aligned}$$

and

$$\begin{aligned}&\bar{A}_{k,N-1}^T \mathbb{E}_k Z_N^{k,t,x}\\&= \bar{A}_{k,N-1}^T g_k + \bar{A}_{k,N-1}^T \mathcal{G}_k \mathcal{A}_{k,N-1} \mathbb{E}_k X_{N-1}^{k,t,x}\\&\quad + \bar{A}_{k,N-1}^T \mathcal{G}_k \mathcal{B}_{k,N-1} \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*}\\&\quad + \bar{A}_{k,N-1}^T \mathcal{G}_k (\mathcal{B}_{k,N-1} \alpha_{N-1} + f_{k,N-1}).\end{aligned}$$

Furthermore, it holds that

$$\begin{aligned}&(C_{k,N-1}^i)^T \mathbb{E}_{N-1} (Z_N^{k,t,x} w_{N-1}^i)\\&= (C_{k,N-1}^i)^T G_k \sum_{j=1}^p \gamma_{N-1}^{ij} \Big[C_{k,N-1}^j X_{N-1}^{k,t,x}\\&\quad + \bar{C}_{k,N-1}^j \mathbb{E}_k X_{N-1}^{k,t,x} + D_{k,N-1}^j \Psi_{N-1} X_{N-1}^{t,x,*}\\&\quad + \bar{D}_{k,N-1}^j \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*} + \mathcal{D}_{k,N-1}^j \alpha_{N-1} + d_{k,N-1}^j\Big]\end{aligned}$$

and

$$\begin{aligned}&(C_{k,N-1}^i)^T \mathbb{E}_k (Z_N^{k,t,x} w_{N-1}^i)\\&= (\bar{C}_{k,N-1}^i)^T G_k \sum_{j=1}^p \gamma_{N-1}^{ij} \Big[\mathcal{C}_{k,N-1}^j \mathbb{E}_k X_{N-1}^{k,t,x}\\&\quad + \mathcal{D}_{k,N-1}^j \Psi_{N-1} \mathbb{E}_k X_{N-1}^{t,x,*} + \mathcal{D}_{k,N-1}^j \alpha_{N-1} + d_{k,N-1}^j\Big].\end{aligned}$$

Therefore

$$\begin{aligned}&Z_{N-1}^{k,t,x}\\&= \Big\{Q_{k,N-1} + A_{k,N-1}^T G_k A_{k,N-1}\\&\quad + \sum_{i,j=1}^p \gamma_{N-1}^{ij} (C_{k,N-1}^i)^T G_k C_{k,N-1}^j\Big\} X_{N-1}^{k,t,x}\\&\quad + \Big\{\bar{Q}_{k,N-1} + A_{k,N-1}^T G_k \bar{A}_{k,N-1}\\&\quad + A_{k,N-1}^T \bar{G}_k \mathcal{A}_{k,N-1} + \bar{A}_{k,N-1}^T \mathcal{G}_k \mathcal{A}_{k,N-1}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{p}\gamma_{N-1}^{ij}\big[(C_{k,N-1}^{i})^{T}G_k\bar{C}_{k,N-1}^{j}\\
&+(\bar{C}_{k,N-1}^{i})^{T}G_k\mathcal{C}_{k,N-1}^{j}\big]\Big\}\mathbb{E}_kX_{N-1}^{k,t,x}\\
&+\Big\{A_{k,N-1}^{T}G_kB_{k,N-1}\\
&+C_{k,N-1}^{T}G_kD_{k,N-1}\Big\}\Psi_{N-1}X_{N-1}^{t,x,*}\\
&+\Big\{A_{k,N-1}^{T}\big[G_k\bar{B}_{k,N-1}+\bar{G}_k\mathcal{B}_{k,N-1}\big]\\
&+\bar{A}_{k,N-1}^{T}\mathcal{G}_k\mathcal{B}_{k,N-1}+\sum_{i,j=1}^{p}\gamma_{N-1}^{ij}\big[(C_{k,N-1}^{i})^{T}G_k\bar{D}_{k,N-1}^{j}\\
&+(\bar{C}_{k,N-1}^{i})^{T}G_k\mathcal{D}_{k,N-1}^{j}\big]\Big\}\Psi_{N-1}\mathbb{E}_kX_{N-1}^{t,x,*}\\
&+\mathcal{A}_{k,N-1}^{T}\mathcal{G}_k\big[\mathcal{B}_{k,N-1}\alpha_{N-1}+f_{k,N-1}\big]\\
&+\sum_{i,j=1}^{p}\gamma_{N-1}^{ij}\mathcal{C}_{k,N-1}^{T}G_k\big[\mathcal{D}_{k,N-1}\alpha_{N-1}+d_{k,N-1}\big]\\
&+\mathcal{A}_{k,N-1}^{T}g_k+q_{k,N-1}\\
=\;&P_{k,N-1}X_{N-1}^{k,t,x}+\bar{P}_{k,N-1}\mathbb{E}_kX_{N-1}^{k,t,x}+T_{k,N-1}X_{N-1}^{t,x,*}\\
&+\bar{T}_{k,N-1}\mathbb{E}_kX_{N-1}^{t,x,*}+\pi_{k,N-1}.
\end{aligned}
$$

We now calculate $Z_{N-2}^{k,t,x}$. Note that

$$
\begin{aligned}
&A_{k,N-2}^{T}\mathbb{E}_{N-2}Z_{N-1}^{k,t,x}\\
=\;&A_{k,N-2}^{T}P_{k,N-1}A_{k,N-2}X_{N-2}^{k,t,x}\\
&+\big(A_{k,N-2}^{T}P_{k,N-1}\bar{A}_{k,N-2}+A_{k,N-2}^{T}\bar{P}_{k,N-1}\mathcal{A}_{k,N-2}\big)\\
&\times\mathbb{E}_kX_{N-2}^{k,t,x}+\big[A_{k,N-2}^{T}P_{k,N-1}B_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}T_{k,N-1}\big(\mathcal{A}_{N-2,N-2}+\mathcal{B}_{N-2,N-2}\Psi_{N-2}\big)\big]X_{N-2}^{t,x,*}\\
&+\big[A_{k,N-2}^{T}P_{k,N-1}\bar{B}_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}\bar{P}_{k,N-1}\mathcal{B}_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}\bar{T}_{k,N-1}\big(\mathcal{A}_{N-2,N-2}+\mathcal{B}_{N-2,N-2}\Psi_{N-2}\big)\big]\\
&\times\mathbb{E}_kX_{N-2}^{t,x,*}+A_{k,N-2}^{T}\mathcal{P}_{k,N-1}\big(\mathcal{B}_{k,N-2}\alpha_{N-2}+f_{k,N-2}\big)\\
&+A_{k,N-2}^{T}\mathcal{T}_{k,N-1}\big(\mathcal{B}_{N-2,N-2}\alpha_{N-2}+f_{N-2,N-2}\big)\\
&+A_{k,N-2}^{T}\pi_{k,N-1}
\end{aligned}
$$

and similar expressions for $C_{k,N-2}^{T}\mathbb{E}_{N-2}(Z_{N-1}^{k,t,x}w_{N-2})$, $\bar{A}_{k,N-2}^{T}\mathbb{E}_kZ_{N-1}^{k,t,x}$, and $\bar{C}_{k,N-2}^{T}\mathbb{E}_k\big(Z_{N-1}^{k,t,x}w_{N-2}\big)$. Then, from (18) we have

$$
\begin{aligned}
&Z_{N-2}^{k,t,x}\\
=\;&\Big\{Q_{k,N-2}+A_{k,N-2}^{T}P_{k,N-1}A_{k,N-2}\\
&+\sum_{i,j=1}^{p}\gamma_{N-1}^{ij}(C_{k,N-2}^{i})^{T}P_{k,N-1}C_{k,N-2}^{j}\Big\}X_{N-2}^{k,t,x}\\
&+\Big\{\bar{Q}_{k,N-2}+A_{k,N-2}^{T}P_{k,N-1}\bar{A}_{k,N-2}\\
&+A_{k,N-2}^{T}\bar{P}_{k,N-1}\mathcal{A}_{k,N-2}+\bar{A}_{k,N-2}^{T}\mathcal{P}_{k,N-1}\mathcal{A}_{k,N-2}\\
&+\sum_{i,j}^{p}\gamma_{N-1}^{ij}\big[(C_{k,N-2}^{i})^{T}P_{k,N-1}\bar{C}_{k,N-2}^{j}\\
&+(\bar{C}_{k,N-2}^{i})^{T}P_{k,N-1}\mathcal{C}_{k,N-2}^{j}\big]\Big\}\mathbb{E}_kX_{N-2}^{k,t,x}\\
&+\Big\{A_{k,N-2}^{T}P_{k,N-1}B_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}T_{k,N-1}\big(\mathcal{A}_{N-2,N-2}+\mathcal{B}_{N-2,N-2}\Psi_{N-2}\Big\}\\
&+\sum_{i,j=1}^{p}\gamma_{N-2}^{ij}\big[(C_{k,N-2}^{i})^{T}P_{k,N-1}D_{k,N-2}^{j}\Psi_{N-1}\\
&+(C_{k,N-2}^{i})^{T}T_{k,N-1}\big(\mathcal{C}_{N-2,N-2}^{j}+\mathcal{D}_{N-2,N-2}^{j}\Psi_{N-2})\big]\Big\}\\
&\times X_{N-2}^{t,x,*}+\Big\{A_{k,N-2}^{T}P_{k,N-1}\bar{B}_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}\bar{P}_{k,N-1}\mathcal{B}_{k,N-2}\Psi_{N-2}\\
&+A_{k,N-2}^{T}\bar{T}_{k,N-1}\big(\mathcal{A}_{N-2,N-2}+\mathcal{B}_{N-2,N-2}\Psi_{N-2}\big)\\
&+\sum_{i,j=1}^{p}\gamma_{N-2}^{ij}(C_{k,N-2}^{i})^{T}P_{k,N-1}\bar{D}_{k,N-2}^{j}\Psi_{N-2}\\
&+\bar{A}_{k,N-2}^{T}\mathcal{P}_{k,N-1}\mathcal{B}_{k,N-2}\Psi_{N-2}+\bar{A}_{k,N-2}^{T}\mathcal{T}_{k,N-1}\\
&\times\big(\mathcal{A}_{N-2,N-2}+\mathcal{B}_{N-2,N-2}\Psi_{N-2}\big)\\
&+\sum_{i,j=1}^{p}\gamma_{N-2}^{ij}\big[(\bar{C}_{k,N-2}^{i})^{T}P_{k,N-1}\mathcal{D}_{k,N-2}^{j}\Psi_{N-1}\\
&+(\bar{C}_{k,N-2}^{i})^{T}T_{k,N-1}\big(\mathcal{C}_{N-2,N-2}^{j}+\mathcal{D}_{N-2,N-2}^{j}\Psi_{N-2})\big]\Big\}\\
&\times\mathbb{E}_kX_{N-2}^{t,x,*}+\mathcal{A}_{k,N-2}^{T}\mathcal{P}_{k,N-1}\big(\mathcal{B}_{k,N-2}\alpha_{N-2}+f_{k,N-2}\big)\\
&+\mathcal{A}_{k,N-2}^{T}\mathcal{T}_{k,N-1}\big(\mathcal{B}_{N-2,N-2}\alpha_{N-2}+f_{N-2,N-2}\big)\\
&+\sum_{i,j=1}^{p}\gamma_{N-2}^{ij}\big[(\mathcal{C}_{k,N-2}^{i})^{T}P_{k,N-1}\big(\mathcal{D}_{k,N-2}^{j}\alpha_{N-2}+d_{k,N-2}^{j}\big)\\
&+(\mathcal{C}_{k,N-2}^{i})^{T}T_{k,N-1}\big(\mathcal{D}_{N-2,N-2}^{j}\alpha_{N-2}+d_{N-2,N-2}^{j})\big]\\
&+\mathcal{A}_{k,N-2}^{T}\pi_{k,N-2}+q_{k,N-2}\\
=\;&P_{k,N-2}X_{N-2}^{k,t,x}+\bar{P}_{k,N-2}\mathbb{E}_kX_{N-2}^{k,t,x}+T_{k,N-2}X_{N-2}^{t,x,*}\\
&+\bar{T}_{k,N-2}\mathbb{E}_kX_{N-2}^{t,x,*}+\pi_{k,N-2}.
\end{aligned}
$$

By deduction, we achieve the conclusion. ■

D. Proof of Theorem III.3

i)⇒ii). Let $u^{t,x,*}$ be the one that satisfies the condition (17). Noting that $X_k^{k,t,x} = X_k^{t,x,*}$ of (19), we have

$$X_{k+1}^{k,t,x} = X_{k+1}^{t,x,*}, \quad k \in \mathbb{T}_t \tag{60}$$

as

$$Z_N^{N-1,t,x} = G_{N-1}X_N^{N-1,t,x} + \bar{G}_{N-1}\mathbb{E}_{N-1}X_N^{N-1,t,x} + g_{N-1}$$

we have

$$\begin{aligned}\mathbb{E}_{N-1}Z_N^{N-1,t,x} &\\ &= \mathcal{G}_{N-1}\mathcal{A}_{N-1,N-1}X_{N-1}^{t,x,*}\\ &\quad + \mathcal{G}_{N-1}\mathcal{B}_{N-1,N-1}u_{N-1}^{t,x,*}\\ &\quad + \mathcal{G}_{N-1}f_{N-1,N-1} + g_{N-1}.\end{aligned} \tag{61}$$

Similarly, it holds that

$$\begin{aligned}&\mathbb{E}_{N-1}(Z_N^{N-1,t,x}w_{N-1}^i)\\ &= \sum_{j=1}^p \gamma_{N-1}^{ij}G_{N-1}\mathcal{C}_{N-1,N-1}^jX_{N-1}^{t,x,*}\\ &\quad + \sum_{j=1}^p \gamma_{N-1}^{ij}G_{N-1}\mathcal{D}_{N-1,N-1}^ju_{N-1}^{t,x,*}\\ &\quad + \sum_{j=1}^p \gamma_{N-1}^{ij}G_{N-1}d_{N-1,N-1}^j.\end{aligned} \tag{62}$$

From (17), (61), and (62), we have

$$0 = \mathcal{W}_{N-1}u_{N-1}^{t,x,*} + \mathcal{H}_{N-1}X_{N-1}^{t,x,*} + \beta_{N-1} \tag{63}$$

where

$$\begin{cases}\mathcal{W}_{N-1} = \mathcal{R}_{N-1,N-1} + \mathcal{B}_{N-1,N-1}^T\mathcal{G}_{N-1}\mathcal{B}_{N-1,N-1}\\ \qquad + \sum_{i,j=1}^p \gamma_{N-1}^{ij}(\mathcal{D}_{N-1,N-1}^i)^TG_{N-1}\mathcal{D}_{N-1,N-1}^j\\ \mathcal{H}_{N-1} = \mathcal{B}_{N-1,N-1}^T\mathcal{G}_{N-1}\mathcal{A}_{N-1,N-1}\\ \qquad + \sum_{i,j=1}^p \gamma_{N-1}^{ij}(\mathcal{D}_{N-1,N-1}^i)^TG_{N-1}\mathcal{C}_{N-1,N-1}^j\\ \beta_{N-1} = \mathcal{B}_{N-1,N-1}^T\left[\mathcal{G}_{N-1}f_{N-1,N-1} + g_{N-1}\right]\\ \qquad + \sum_{i,j=1}^p \gamma_{N-1}^{ij}(\mathcal{D}_{N-1,N-1}^i)^TG_{N-1}d_{N-1,N-1}^j\\ \qquad + \rho_{N-1,N-1}.\end{cases}$$

Note that $X_{N-1}^{t,x,*}$ is not influenced by $u_{N-1}^{t,x,*}$. From Lemma III.4, $u_{N-1}^{t,x,*}$ can be selected as

$$\begin{aligned}u_{N-1}^{t,x,*} &= -\mathcal{W}_{N-1}^\dagger\mathcal{H}_{N-1}X_{N-1}^{t,x,*} - \mathcal{W}_{N-1}^\dagger\beta_{N-1}\\ &\triangleq \Psi_{N-1}X_{N-1}^{t,x,*} + \alpha_{N-1}\end{aligned} \tag{64}$$

and

$$\left(I - \mathcal{W}_{N-1}\mathcal{W}_{N-1}^\dagger\right)\left(\mathcal{H}_{N-1}X_{N-1}^{t,x,*} + \beta_{N-1}\right) = 0$$

holds, which is equivalent to

$$\mathcal{H}_{N-1}X_{N-1}^{t,x,*} + \beta_{N-1} \in \mathrm{Ran}(\mathcal{W}_{N-1}).$$

Moving to the case $k = N-2$ and by deduction, we then have (28) and (33).

ii)⇒i). Let $X^{t,x,*}$ and $u^{t,x,*}$ be given in (32) and (33). From (28), we have

$$0 = \mathcal{W}_ku^{t,x,*} + \mathcal{H}_kX_k^{t,x,*} + \beta_k, \quad k \in \mathbb{T}_t. \tag{65}$$

Furthermore, from (33) and Lemma III.3, then, (25), equivalently (34), holds. Similarly to (61) and (62), we have

$$\begin{aligned}\mathbb{E}_kZ_{k+1}^{k,t,x} &= (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})\mathcal{A}_{k,k}X_k^{t,x,*}\\ &\quad + (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})\mathcal{B}_{k,k}u_k^{t,x,*}\\ &\quad + (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})f_{k,k} + \pi_{k,k+1}\end{aligned} \tag{66}$$

and

$$\begin{aligned}\mathbb{E}_k(Z_{k+1}^{k,t,x}w_k^i) &= \sum_{j=1}^p \gamma_k^{ij}\left(P_{k,k+1} + T_{k,k+1}\right)\mathcal{C}_{k,k}^jX_k^{t,x,*}\\ &\quad + \sum_{j=1}^p \gamma_k^{ij}\left(P_{k,k+1} + T_{k,k+1}\right)\mathcal{D}_{k,k}^ju_k^{t,x,*}\\ &\quad + \sum_{j=1}^p \gamma_k^{ij}\left(P_{k,k+1} + T_{k,k+1}\right)d_{k,k}^j.\end{aligned} \tag{67}$$

Combining (65)–(67), we have the stationary condition (17). ∎

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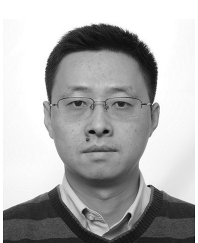

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