Stochastic Consentability of Linear Systems With Time Delays and Multiplicative Noises

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***Abstract*—This paper develops stochastic consentability of linear multiagent systems with time delays and multiplicative noises. First, the stochastic stability for stochastic differential delay equations driven by multiplicative noises is examined, and the existence of the positive definite solution for a class of generalized algebraic Riccati equations (GAREs) is established. Then, sufficient conditions are deduced for the mean square and almost sure consentability and stabilization based on the developed stochastic stability and GAREs. Consensus protocols are designed for linear multiagent systems with undirected and leader-following topologies. It is revealed that multiagent consentability depends on certain characterizing system parameters, including linear system dynamics, communication graph, channel uncertainties, and time delay of the deterministic term. It is shown that a second-order integrator multiagent system is unconditionally mean square and almost surely consentable for any given noise intensities and time delay, and that the mean square and almost sure consensus can be achieved by carefully choosing the control gain according to certain explicit conditions.**

***Index Terms*—Consentability, consensus, multiagent system, multiplicative noise, time delay.**

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I. Introduction

Multiagent systems have attracted much attention in recent years. The research progress in such systems has provided valuable insights and benefits in understanding, designing, and implementing distributed controllers for such systems; see [1]–[3]. To date, the consensus problem of perfect models, which assume that each agent can obtain its neighbor information timely and precisely, has reached a reasonable degree of maturity; see [4] for example. However, networked systems in practical applications often operate in uncertain communication environments and are inevitably subjected to communication latency and measurement noises [5]–[7]. Hence, time delays and measurement noises should be taken into consideration for examining multiagent consensus problems.

In the literature, Olfati-Saber and Murray [7] gave optimal delay bounds for consensus under undirected graphs. Bliman and Ferrari-Trecate [8] studied average consensus problems for undirected networks with constant, time varying and nonuniform time delays, and presented some sufficient conditions for average consensus. Different types of time delays (communication delay, identical self-delay and different self-delay) were investigated in [9] for output consensus. These works have focused on first-order multiagent systems. For second-order multiagent systems, Yu *et al.* [10] gave necessary and sufficient consensus conditions related to time delays. Zhou and Lin [11] showed that consensus problems of linear multiagent systems with input and communication delays could also be solved by truncated predictor feedback protocols.

For multiplicative noises (or state-dependent noise), Ni and Li [12] studied consensus problems of continuous-time systems in which the noise intensity functions are the absolute values of relative states. Li *et al.* [13] revealed that multiplicative noises may enhance the almost sure consensus, but may have damaging effect on the mean square consensus, indicating a distinct feature from additive noises which are always destabilizing factors. These results have been established for first-order multiagent systems. For linear discrete-time systems, Li and Chen [14] gave the mean square consensus analysis by solving a modified algebraic Riccati equation. When the time delay and multiplicative noise coexist, stochastic consensus conditions were examined in [15] for second-order discrete-time models. However, little is

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known about the consensus for general continuous-time linear multiagent systems with time delays and multiplicative noises.

All the aforementioned works have focused on the issue of convergence to consensus, which is concerned with conditions on the given control protocols under which the agents can achieve certain agreements asymptotically. However, they do not address the issue of consentability. Consentability is concerned with conditions on system parameters under which a consensus protocol exists, and is of great importance, both theoretically and practically, for cooperative control protocol development such as flocking behavior, agent rendezvous, and robot coordination. Note that Ma and Zhang [16] considered general linear continuous-time multiagent systems and obtained necessary and sufficient conditions on system parameters for consentability. This problem was then generalized in [17] to the case with input constraints and uncertain initial conditions in terms of linear matrix inequalities (LMIs). For discrete-time models, Zhang and Tian [18] investigated second-order multiagent systems with Markov-switching topologies, and proved that such networked systems are mean square consentable under linear consensus protocols if and only if the union of the graphs in the switching topology has globally reachable nodes. For linear multiagent systems, You and Xie [19] obtained necessary and sufficient conditions for single-input multiagent consentability and revealed how the agent dynamic, network topology, and communication data rate affect consentability. Gu *et al.* [20] employed a properly designed dynamic filter into local control protocols to relax consensusability conditions. For the issue of structural controllability of multiagent systems, we refer the readers to [21] and references therein. At present, consentability for multiagent systems with multiplicative noises, even for the delay-free case, remains an open problem, due to the lack of suitable techniques to treat intrinsic complications from general linear systems.

This paper aims to establish the mean square and almost sure consentability and develop the consensus protocol design of linear multiagent systems with time delays and multiplicative noises. Our results accommodate different time delays in deterministic (drift) and stochastic (diffusion) terms. Departing from the feedback structure of [22], [23], where each agent's state feedback was used, our consensus protocol requires only local relative-state measurements. In this case, each agent's state is not locally stabilized, leading to a more difficult consensus analysis problem. It can be verified that a consensus problem is equivalent to the stability problem of the corresponding closed-loop system. Hence, for consensus analysis under measurement noises, the key is to establish stochastic stability of the corresponding stochastic equations driven by additive or multiplicative noises, see [12]–[15], [24]. However, for linear multiagent systems of orders higher than one, multiplicative noises are often degenerate and the existing stochastic stability theory does not involve the issues for such stochastic differential equations. Time delays add further difficulty in deriving consentability conditions and consensus protocols of linear multiagent systems with multiplicative noises. New ideas and techniques are needed to resolve these difficulties. In this paper, stochastic stability and a generalized algebraic Riccati equation (GARE) are established for obtaining multiagent consentability and designing control protocols under time delays and measurement noises. The contribution of this paper is detailed as follows.

1) Two new techniques (Theorems II.1 and II.2) are developed to study the multiagent consentability and consensus under measurement noises and time delays. We first exploit the degenerate Lyapunov functional [25] to establish the mean square and almost sure exponential stability criteria of stochastic differential delay equations (SDDEs). We show that the stability is independent of the time delay with stochastic influence. Then, the existence of positive definite solutions to GARE is obtained. The GARE plays an important role in stochastic linear-quadratic control problems [26]–[28]. Here, both sufficient and necessary conditions are presented for guaranteeing the existence of a positive definite solution to the GARE. As a byproduct, necessary and sufficient conditions are given for the existence of stochastic feedback stabilization control laws. That is, it is sufficient that the open-loop dynamics (A, B) is controllable and the product of the sum of real parts of unstable open-loop poles and the square of noise intensity is less than $1/2$, while it is necessary that (A, B) is stabilizable and the product of the maximal real part of the open-loop poles and the square of noise intensity is less than $1/2$.
2) The necessary and sufficient conditions for multiagent consentability and consensus are revealed. We first derive the mean square and almost sure consentability conditions for leader-free linear multiagent systems. a) It is proved that if the agent dynamics (A, B) is controllable, $4\lambda_0^u \frac{N-1}{N}\bar{\sigma}^2 < \lambda_2$ and the time delay $\tau_1 < \tau_1^*$, then the linear multiagent system is mean square and almost surely consentable, where λ_0^u denotes the sum of the real parts of the unstable eigenvalues of A, N is the number of agents, λ_2 is the algebraic connectivity of the graph, $\bar{\sigma}$ is the bound of the noise intensities, and τ_1^* is a bound of the time delay in the deterministic term. Especially, some special cases are given to show that the stabilizability of (A, B) and the restriction on noise intensity are necessary for the mean square consentability. b) For second-order integrator multiagent systems under undirected graphs, we show that it is mean square and almost surely consentable regardless of noises and time delays. Some necessary conditions on the mean square consensus are also obtained for the delay-free case. Then, the consentability and consensus results are extended to leader-following multiagent systems which include stochastic stabilization by delayed noisy feedback as the special case.

The rest of the paper is structured in five parts. Section II develops the main tools such as stochastic stability and GARE for examining the consentability of linear multiagent systems. Section III investigates consentability conditions and pursues consensus protocols of leader-free linear multiagent systems under undirected graphs. Section IV extends our investigation to the leader-following case and stochastic stabilization with delayed noisy feedback. Section V presents simulation results to verify the theoretical analysis. Section VI concludes the paper with some remarks that outline possible directions for future research in this field. For clarity of presentation, the proofs of most technical results are relegated to the appendix.

Notation: The symbol $\mathbf{1}_N$ denotes the N-dimensional column vector with all ones; $\eta_{N,i}$ denotes the N-dimensional column vector with the ith element being 1 and others being zero; $J_N = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$; I_N denotes the N-dimensional identity matrix. For a given matrix or vector A, its transpose is denoted by A^T, its Euclidean norm is denoted by $\|A\|$, and its maximum real part of the eigenvalues is denoted by $\max(Re(\lambda(A)))$. For a real symmetric matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalues, respectively. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. For symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). For $a, b \in \mathbb{R}$, $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ denotes $\min\{a, b\}$. Let $(\mathbf{\Omega}, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. For a given random variable or vector X, the mathematical expectation of X is denoted by $\mathbb{E}X$. For continuous martingales $M(t)$ and $N(t)$, their quadratic variation is denoted by $\langle M, N\rangle(t)$ (see [29]). We also denote $\langle M\rangle(t) := \langle M, M\rangle(t)$. For the fixed $\tau > 0$, we use $C([-\tau, 0], \mathbb{R}^n)$ to denote the space of all continuous $\mathbb{R}^n$-valued functions φ defined on $[-\tau, 0]$ with the norm $\|\varphi\|_C = \sup_{t\in[-\tau,0]} \|\varphi(t)\|$.

II. Stochastic Stability and GARE

When the control of multiagent systems with the delayed and noisy measurements is studied, the consensus problem of the closed-loop systems is transformed into the stability problem of SDDEs. Hence, the stochastic stability theorem (see Theorem II.1) should be first established. After the stability theorem, we need to fix how to design the feedback control based on the delayed and noisy measurements. The solution will resort to a GARE (see Theorem II.2). This section is to establish the two tools.

A. Stochastic Stability of SDDEs

This section aims to establish the mean square and almost sure exponential stability of the following SDDE:

$$dy(t) = [A_0 y(t) + A_1 y(t - \tau_1)]dt + dM(t) \tag{1}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}$, $M(t) = \sum_{i=1}^d \int_0^t f_i(y(s - \tau_2))dw_i(s)$, $\tau_1, \tau_2 \geq 0$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $d > 0$, $\{w_i(t)\}_{i=1}^d$ are independent Brownian motions. The functions $\{f_i(x)\}_{i=1}^d$ satisfy the following assumption.

Assumption II.1: $f_i(0) = 0$, $i = 1, \ldots, d$, and there exist positive constants $\{\varrho_i\}_{i=1}^d$ such that for any $y_1, y_2 \in \mathbb{R}^n$, $\|f_i(y_1) - f_i(y_2)\| \leq \varrho_i\|y_1 - y_2\|$, $i = 1, \ldots, d$.

We also give the initial data $y(t) = \varphi(t)$ for $t \in [-\tau, 0]$, $\tau = \tau_1 \vee \tau_2$, and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$. Here, we assume that for each $P_0 > 0$, there exists a $D_{P_0} \geq 0$ such that

$$\sum_{i=1}^d f_i^T(y)P_0 f_i(y) \leq y^T D_{P_0} y. \tag{2}$$

Theorem II.1: Suppose Assumption II.1 and condition (2) hold. If there exists a matrix $P > 0$ such that

$$\bar{A}^T P + P\bar{A} + (\bar{A}^T P\bar{A} + A_1^T PA_1)\tau_1 + D_P < 0 \tag{3}$$

where $\bar{A} = A_0 + A_1$, then there exist $C_0, \gamma_0 > 0$ such that

$$\mathbb{E}\|y(t)\|^2 \leq C_0 e^{-\gamma_0 t},\ \limsup_{t\to\infty} \frac{1}{t}\log\|y(t)\| < -\frac{\gamma_0}{2},\ \text{a.s.} \tag{4}$$

That is, the trivial solution to SDDE (1) is mean square and almost surely exponentially stable.

In applications, the diffusion $f_i(y(s - \tau_2))dw_i(s)$ may have various forms (see Sections III and IV). Note that $\langle M, P_0 M\rangle(t) = \sum_{i=1}^d \int_0^t f_i^T(y(s - \tau_2))P_0 f_i(y(s - \tau_2))ds$. So we can use the following condition to replace (2):

$$d\langle M, P_0 M\rangle(t) \leq y^T(t - \tau_2)D_{P_0} y(t - \tau_2)dt \tag{5}$$

which is a direct consequence of (2) that does not involve the concrete form of diffusion. In fact, the proof of Theorem II.1 in Appendix is based on (5). Theorem II.1 also leads to the following corollaries.

Corollary II.1: Suppose Assumption II.1 and condition (2) hold. If there exists a positive definite matrix P such that $T_P := \bar{A}^T P + P\bar{A} + D_P < 0$, then for any $\tau_1 \in [0, \tau_1(P))$, the trivial solution to SDDE (1) is mean square and almost surely exponentially stable, where $\tau_1(P) := \frac{\lambda_{\min}(-T_P)}{\|\bar{A}^T P\bar{A} + A_1^T PA_1\|}$.

Corollary II.2: Suppose that $\bar{A}$ is symmetric, $\tau_1 = 0$, and $f_i(y) = \varrho_i y$, $i = 1, \ldots, d$. Then, the trivial solution to SDDE (1) is mean square exponentially stable if and only if $2\bar{A} + \sum_{i=1}^d \varrho_i^2 I_n < 0$.

Remark II.1: Concerning Theorem II.1, first it can yield an explicit delay bound (see Corollary II.1). Second, it provides a delay dominated stability criterion and improves the LMI stability theorems in [30], which takes the SDDE $dy(t) = -y(t - \tau_1)dt + y(t)dw(t)$ for example, and gave delay bound $\tau_1^* < 0.1339$. Our Theorem II.1 yields a better delay bound $\tau_1^* < 0.5$. Third, it shows that the stability of stochastic delay systems does not necessarily depend on the time-delay in diffusion. In fact, the necessary condition of the mean square stability is independent of such delay in Corollary II.2.

Remark II.2: Corollary II.1 shows that the mean square stable linear time-invariant SDEs can tolerate a certain time delay, which depends on the original system parameters. Especially, for the scalar SDDE: $dy(t) = (a_0 y(t) + a_1 y(t - \tau_1))dt + \sigma y(t - \tau_2)dw(t)$, we have a time delay bound $\tau_1^* = -\frac{2(a_0+a_1)+\sigma^2}{(a_0+a_1)^2+a_1^2}$ for mean square stability if the delay-free SDE $dy(t) = (a_0 + a_1)y(t)dt + \sigma y(t)dw(t)$ is mean square stable, which is equivalent to $2(a_0 + a_1) + \sigma^2 < 0$.

Remark II.3: Note that the sufficient conditions for the mean square and almost sure exponential stability in Theorem II.1 do not involve time delay τ_2 in the diffusion term. Corollary II.2 also shows that the mean square stability is independent of time delay τ_2 when $\tau_1 = 0$, which is consistent with the scalar case in [31]. Theorem II.1 and Corollary II.2 also give that if $\bar{A}$ is symmetric and $f_i(y) = \varrho_i y$, $i = 1, \ldots, d$, then the trivial solution to SDDE (1) is mean square and almost surely exponentially stable if $\tau_1 \leq \frac{\lambda_{\min}(-(2\bar{A}+\sum_{i=1}^d \varrho_i^2 I_n))}{\|\bar{A}^T\bar{A}+A_1^T A_1\|}$. Compared with [32, Th. 2.1], not only does it improve the stability criterion, but also avoids solving a complex quadratic equation.

B. GARE

In this section, we develop the existence and uniqueness of the solution $P > 0$ to the following GARE:

$$A^T P + PA - 2\alpha PB(R + B^T PB)^{-1} B^T P + Q = 0 \quad (6)$$

where $\alpha > 0$, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $R > 0$, and $Q > 0$. We first derive a necessary condition for the existence and uniqueness of the solution $P > 0$.

Lemma II.1: Assume that $\max(\mathrm{Re}(\lambda(A))) \geq 0$ and (A, B) is controllable. Then, the GARE (6) has a unique solution $P = P(\alpha) > 0$ only if $\alpha > \max(\mathrm{Re}(\lambda(A)))$.

To find the sufficient conditions, we first note that the problem above, related to (6), can be considered as the following stochastic linear-quadratic control problem in [26] with $D = \sigma B$, that is, $\mathrm{Minimize}_u \mathbb{E} \int_0^\infty [x^T(t)Qx(t) + \sigma^2 u^T(t)Ru(t)]dt$, subject to

$$dx(t) = (Ax(t) + Bu(t))dt + \sigma Bu(t)dw(t) \quad (7)$$

$x(0) = x_0 \in \mathbb{R}^n$, where $\sigma = (2\alpha)^{-1/2}$ and $u(t) = Kx(t)$ with K to be designed. It is proved in [26, Corollary 4] that if (7) can be mean square stabilized by $u(t)$ for certain K, then the GARE (6) has a solution $P > 0$ and the linear-quadratic control problem is solved by the optimal control gain matrix $K = 2\alpha(R + B^T PB)^{-1} B^T P$. Therefore, we need to establish the mean square stabilizability of (7) before examining the positive definite solution of (6).

Let $\{\lambda_i^u(A)\}_i$ denote the unstable eigenvalues of A, that is, $\mathrm{Re}(\lambda_i^u(A)) \geq 0$. Define $\lambda_0^u = \sum_i \mathrm{Re}(\lambda_i^u(A))$.

Lemma II.2: System (7) can be mean square stabilized by $u(t) = Kx(t)$ for certain K if (A, B) is controllable and $\lambda_0^u \sigma^2 < 1/2$, and only if (A, B) is stabilizable and $\max(\mathrm{Re}(\lambda(A)))\sigma^2 < 1/2$.

Remark II.4: The "if" part of Lemma II.2 is based on the relationship between the approximate solution and the exact solution of SDEs in [43], and on the existence of the positive definite solution to the discrete-time ARE in [33]–[36], where a similar choice of the parameter α has been examined for stabilization.

Remark II.5: To see the stabilizability clearly, we here consider the linear scalar system $dx(t) = (ax(t) + bu(t))dt + \sigma bu(t)dw(t)$, $b > 0$, with the stabilization control law $u(t) = kx(t)$. It can be proved that the closed-loop system

$$dx(t) = (a + bk)x(t)dt + \sigma bkx(t)dw(t) \quad (8)$$

has the property that $\mathbb{E}\|x(t)\|^2 = \mathbb{E}\|x(0)\|^2 \exp\{(2(a + bk) + (\sigma^2 b^2 k^2)t\}$. Then, the mean square stabilizability requires that there exists a constant $k \in \mathbb{R}$ such that $2(a + bk) + \sigma^2 b^2 k^2 < 0$, which is equivalent to $a\sigma^2 < 1/2$. Hence, it is easy to see that 1) if $a \leq 0$, there must exists a $k \in \mathbb{R}$ such that the closed-loop system (8) is mean square stable; 2) If $a > 0$, $a\sigma^2 < 1/2$ is necessary for the mean square stabilizability.

Remark II.6: The works [37], [38] considered the control problem of the following stochastic system with control dependent noise:

$$dx(t) = [Ax(t) + Bu(t)]dt + \sigma B_0 u(t)dw(t) \quad (9)$$

where $u(t) = Kx(t)$ is the input control. The authors proved that the stochastic system (9) can be stabilized for any given noise intensity $\sigma > 0$ if and only if (A, B) is stabilizable and the columns of input coefficient matrix B_0 ($\mathrm{Rank}(B_0) < n$) in the noise term belong to the subspace of A spanned by its eigenvectors corresponding the eigenvalues with nonpositive real parts. Here, we need to remark that this robust stabilizability for any given $\sigma > 0$ above do not fall in the case of unstable A and $B = B_0$ (Hence, Lemma II.2 is consistent with the results in [37] and [38]). It suffices to show that the stabilizability of (A, B) implies that the columns of B must not belong to the subspace of A spanned by its eigenvectors. To see it, we consider the two cases: a) (A, B) is controllable; b) (A, B) is not controllable, but is stabilizable. For the case (a), $\mathrm{Rank}(B, AB, ..., A^{n-1}B) = n$, which is impossible if the columns of B belong to the subspace of A spanned by its eigenvectors. For the case (b), there is an invertible matrix S such that

$$SAS^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \ SB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where (A_{11}, B_1) is controllable and A_{22} has eigenvalues with strictly negative real parts; see [39, Propositions 2.1.6 and 2.2.3]. It is easily seen that if the columns of B belong to the subspace of A spanned by its eigenvectors, then the columns of B_1 belong to the subspace of A_{11} spanned by its eigenvectors, which is in conflict with the controllability of (A_{11}, B_1).

By using Lemmas II.1 and II.2, we can have the following the existence and uniqueness.

Theorem II.2: Assume that $\max(\mathrm{Re}(\lambda(A))) \geq 0$ and (A, B) is controllable. Then, the GARE (6) has a unique solution $P = P(\alpha) > 0$ if $\alpha > \lambda_0^u$, and only if $\alpha > \max(\mathrm{Re}(\lambda(A)))$. In particular, if B is invertible, then GARE (6) has a unique solution $P = P(\alpha) > 0$ if and only if $\alpha > \max(\mathrm{Re}(\lambda(A)))$.

Remark II.7: If B is full column rank, then $B^T PB > 0$ for any $P > 0$. In this case, one can obtain the following GARE:

$$A^T P + PA - 2\alpha PB(B^T PB)^{-1} B^T P + Q = 0 \quad (10)$$

with $\alpha > \lambda_0^u$. Especially, if the matrix B is invertible, then (10) has the form of $A(\alpha)^T P + PA(\alpha) + Q = 0$, which admits a unique positive definite solution P given by $P = \int_0^\infty e^{A(\alpha)^T t} Q e^{A(\alpha)t} dt$, where $A(\alpha) = A - \alpha I_n$ is Hurwitz.

For simplicity, we consider $R = I_m$ and $Q = I_n$. That is, the GARE (6) has the form of

$$A^T P + PA - 2\alpha PB(I_m + B^T PB)^{-1} B^T P + I_n = 0. \quad (11)$$

In what follows, we assume that B is not invertible ($\mathrm{Rank}(B) < n$), unless otherwise specified. For the case with $\mathrm{Rank}(B) = n$, λ_0^u in Theorems III.1, IV.1–IV.4, and Corollaries III.1, IV.1, IV.2 will be replaced by $\max(\mathrm{Re}(\lambda(A)))$.

In the following sections, we apply Theorems II.1 and II.2 to establish multiagent consentability, consensus, and stochastic stabilization.

III. Leader-Free Linear Multiagent Systems

A. Linear Multiagent Systems

We consider a system consisting of N ($N \geq 2$) agents where the agents are indexed by $1, 2, \ldots, N$, respectively. The dynamics of the N agents are described by the continuous-time systems:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t),\ t \in \mathbb{R}_+ \tag{12}$$

where $i = 1, 2, ..., N$, $x_i(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $u_i(t) \in \mathbb{R}^m$ is the input control of the ith agent, respectively. Denote $x(t) = [x_1^T(t), ..., x_N^T(t)]^T$ and $u(t) = [u_1^T(t), ..., u_N^T(t)]^T$. Here, the input control $u(t)$ is to be designed. The information flow structures among different agents are modeled as a connected undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, ..., N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ denotes the set of directed edges and $\mathcal{A} = [a_{ij}]\in\mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly. Also, N_i denotes the set of the node i's neighbors, that is, for $j \in N_i, a_{ij} = 1$, and $\deg_i = \sum_{j=1}^N a_{ij}$ is called the degree of i. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\deg_1, ..., \deg_N)$. It is obvious that $\mathcal{L}$ is a symmetric matrix and admits a zero eigenvalue, denoted by λ_1; other eigenvalues $0 < \lambda_2 \leq \ldots \leq \lambda_N$ are positive due to the connectivity of $\mathcal{G}$.

We consider the distributed protocol in the following form:

$$u_i(t) = K \sum_{j\in N_i} z_{ji}(t) \tag{13}$$

where $K \in \mathbb{R}^{m\times n}$ is the feedback gain matrix to be designed

$$z_{ji}(t) = x_j(t-\tau_1) - x_i(t-\tau_1) + \sum_{l=1}^{d} f_{lji}(x_j(t-\tau_2) - x_i(t-\tau_2))\xi_{lji}(t),\ j \in N_i \tag{14}$$

is the state measurement of the agent i from its neighbor agent j, τ_1, τ_2 are the time delays, $\xi_{ji}(t) = (\xi_{1ji}(t), \ldots, \xi_{dji}(t))^T \in \mathbb{R}^d$ is the measurement noise, $f_{lji}(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the noise intensity function. In this paper, $x_j(t-\tau_1) - x_i(t-\tau_1)$ is called the deterministic term and $\sum_{l=1}^d f_{lji}(x_j(t-\tau_2) - x_i(t-\tau_2))\xi_{lji}(t)$ is called the stochastic term. We assume that the noises are independent Gaussian white noises. To be exact, they satisfy the following assumption.

Assumption III.1: The noise process $\xi_{ji}(t) = (\xi_{1ji}(t), \ldots, \xi_{dji}(t))^T \in \mathbb{R}^d$ satisfies $\int_0^t \xi_{ji}(s)ds = w_{ji}(t)$, $t \geq 0, i = 1, 2, \ldots, N, j \in N_i$, where $\{w_{ji}(t), i = 1, 2, \ldots, N, j \in N_i\}$ are independent d-dimensional Brownian motions defined on the complete probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbb{P})$.

Note that the update of state $x(t)$ depends on the past states $x(s)$, $s \in [t-\tau, t)$ with $\tau = \tau_1 \vee \tau_2$. We need to define the initial function on $[-\tau, 0]$. We assume that $x(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and $\varphi \in C([-\tau, 0]; \mathbb{R}^{Nn})$ is deterministic. Let $f.(\cdot)$ denotes the family of noise intensity functions $\{f_{lji}(\cdot)$, $i = 1, 2, ..., N,\ j \in N_i,\ l = 1, 2, ..., d\}$. The collection of all admissible distributed protocols with different control gain matrices is denoted by

$$\mathcal{U}(\tau_1, \tau_2, f.(\cdot)) = \left\{ u(t) | u_i(t) = K \sum_{j=N_i} z_{ji}(t), t \geq 0, K \in \mathbb{R}^{m\times n}, i = 1, \ldots, N. \right\}. \tag{15}$$

Under the measurement noise, the consensus definitions are diversified due to the different asymptotic behaviors in the probability sense, where the mean square and the almost sure consensus are two important topics. Here, we give the definitions on the mean square and the almost sure consentability and consensus.

Definition III.1: We say that the linear systems (12) are mean square (or almost surely) consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f.(\cdot))$, if there exists a protocol $u \in \mathcal{U}(\tau_1, \tau_2, f.(\cdot))$ solving the mean square (or almost sure) consensus, that is, it makes the agents have the property that for any initial data $\varphi \in C([-\tau, 0]; \mathbb{R}^{Nn})$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x_j(t)\|^2 = 0$ (or $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, a.s.).

Remark III.1: The delayed measurement ($f_{lji}(\cdot) \equiv 0$) or the noisy measurement without delay ($\tau_1 = \tau_2 = 0$) was investigated in [7] and [13], respectively. In fact, in the complicated environment, the information communication is often subject to time-delays and measurement noises simultaneously. The general measurement (14) is to describe this phenomenon and has been examined in [15] for the first-order multiagent consensus. It is worth noting that the current results are still new even for the delay-free case ($\tau_1 = \tau_2 = 0$).

Remark III.2: Multiagent consensus and consentability based on precise (noise and delay free) relative state measurements were well studied in [7] and [18], [40]. Consensus based on relative state measurements require less information than that based on both absolute and relative state measurements, and for many realistic cases, each agent may not have the ability to get the absolute state information due to the limited perception capacity and the lack of global coordinates. Multiagent consentability has not been well studied in the delayed and/or noisy environment. Here, the admissible protocol (15) can be viewed as a natural extension from the precise relative state measurements to the delayed noisy version. The skills developed in this paper can also be used for the case with absolute state feedback.

Applying Theorems II.1 and II.2 produces the following theorem, which gives the consentability conditions and the consensus protocol design for linear systems (12) under $\max(\mathrm{Re}(\lambda(A))) \geq 0$.

Theorem III.1: Suppose that Assumption III.1 holds, $f_{lji}(x) = \sigma_{lji}x$, $\sigma_{lji} \geq 0$, and $\max(\mathrm{Re}(\lambda(A))) \geq 0$. Then, linear systems (12) are mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f.(\cdot))$ if

1) (A, B) is controllable;
2) $4\lambda_0^u \frac{N-1}{N}\bar{\sigma}^2 < \lambda_2$;

3) $\tau_1 \in [0, \tau_1^*)$,

where $\tau_1^* = \frac{1}{2\|A\|^2\|P\|} \wedge \frac{\lambda_2 - 4\frac{N-1}{N}\bar{\sigma}^2(\lambda_0^u+\epsilon)}{6(\lambda_0^u+\epsilon)\lambda_N}$, $P > 0$ is the solution to the GARE (11) with $\alpha \in (\lambda_0^u, \lambda_0^u + \epsilon)$, $\epsilon \in (0, \frac{\lambda_2 - 4\frac{N-1}{N}\bar{\sigma}^2\lambda_0^u)}{4\frac{N-1}{N}\bar{\sigma}^2})$, $\bar{\sigma}^2 = \sum_{l=1}^d \max_{i,j=1}^N \sigma_{lji}^2$. In particular, the protocol (13) with $K = k(I_m + B^T PB)^{-1}B^T P$, $k \in (\underline{k}, \bar{k})$, solves the mean square and almost sure consensus, where $\underline{k} = [\lambda_2 - \sqrt{\lambda_2^2 - 2\alpha\rho}]/\rho$, $\bar{k} = [\lambda_2 + \sqrt{\lambda_2^2 - 2\alpha\rho}]/\rho$, $\rho = (2\frac{N-1}{N}\bar{\sigma}^2 + 3\lambda_N\tau_1)\lambda_2$.

Remark III.3: If $A = 0$, then condition 1) in Theorem III.1 implies that B is invertible. In this case, for any given τ_1, τ_2, and σ_{lji}, $i, j = 1, \ldots, N, l = 1, \ldots, d$, the linear multiagent systems (12) are mean square consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$. In fact, $K = -kB^{-1}$ with $k \in (0, \frac{1}{\lambda_N\tau_1 + \frac{N-1}{N}\bar{\sigma}^2})$ can be used to achieve the stochastic consensus.

If A is stable ($\max(Re(\lambda(A))) < 0$), then deterministic consensus of $\dot{x}_i = Ax_i$ follows because each $x_i(t)$ tends to zero. Moreover, we can prove the following theorem, which shows that small control gain does not affect the consensus of the original system $\dot{x}_i = Ax_i$.

Theorem III.2: Suppose that Assumption III.1 holds, $f_{lji}(x) = \sigma_{lji}x$, $\sigma_{lji} \geq 0$, and $\max(\mathrm{Re}(\lambda(A))) < 0$. Then, the linear systems (12) are mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$ for any given $\tau_1, \tau_2, \sigma_{lji} > 0$. Especially, the choice $K = kB^T P$ can guarantee the mean square and almost sure consensus of the linear system (12), where $k \in \mathbb{R}$ satisfies $|k|\lambda_N\|PBB^T P\|(2 + \bar{\sigma}^2\frac{N-1}{N}\|BB^T P\||k|) < 1$, $P = \int_0^\infty e^{A^T t}e^{At}dt$.

When τ_1 vanishes, we can obtain the following corollary, which is a direct consequence of Theorem III.1.

Corollary III.1: Suppose that Assumption III.1 holds, $f_{lji}(x) = \sigma_{lji}x$, $\sigma_{lji} \geq 0$, $\tau_1 = 0$ and $\max(Re(\lambda(A))) \geq 0$. Then, linear systems (12) are mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$ if conditions 1) and 2) in Theorem III.1 hold. Moreover, the protocol (13) with $K = k(I_m + B^T PB)^{-1}B^T P$, $k \in (\underline{k}, \bar{k})$, solves the mean square and almost sure consensus, where $P > 0$ is the solution to GARE (11) with $\alpha \in (\lambda_0^u, \frac{\lambda_2^2}{2\rho})$, $\underline{k} = [\lambda_2 - \sqrt{\lambda_2^2 - 2\alpha\rho}]/\rho$, $\bar{k} = [\lambda_2 + \sqrt{\lambda_2^2 - 2\alpha\rho}]/\rho$, $\rho = 2\frac{N-1}{N}\bar{\sigma}^2\lambda_2$.

Theorem III.1 and Corollary III.1 focus on the sufficient conditions on stochastic consentability. We now examine the necessary conditions for the mean square consentability.

Theorem III.3: Suppose that Assumption III.1 holds and $\tau_1 = 0$. Then, linear systems (12) are mean square consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$ only if (A, B) is stabilizable.

The following example gives the necessity of certain restriction on the noise intensities for the mean square consentability of the two-dimensional dynamics.

Theorem III.4: Consider the multiagent systems (12) with

$$n = 2, \tau_1 = \tau_2 = 0, A = \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix}$$

and $B = (0, 1)^T$. Suppose that Assumption III.1 holds with $d = 2$, $\beta > 0$ and $f_{1ji}(x) = (\sigma_{ji}x_1, 0)^T$, $f_{2ji}(x) = (0, \sigma_{ji}x_2)^T$, $\sigma_{ji} > 0$. Then, linear systems (12) are mean square consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$ only if $\lambda_N > 4\beta\frac{N-1}{N}\underline{\sigma}^2$, where $\underline{\sigma} = \min_{i,j=1}^N \sigma_{ji}$.

B. Second-Order Integrator Multiagent Systems

In view of Theorems III.1 and III.4, we can see that if $\max(\mathrm{Re}(\lambda(A))) > 0$, the mean square consentability of linear systems (12) might be destroyed by the large noise intensities. If $\max(\mathrm{Re}(\lambda(A))) = 0$, then the consentability of linear systems (12) holds for any given noise intensities if $\tau_1 = 0$. However, we need to solve the GARE (11) in order for finding the consensus protocol. Moreover, we do not know whether the restriction on the time delay can be removed if $\max(\mathrm{Re}(\lambda(A))) = 0$. These motivate us to give further investigation.

In this section, we consider the second-order integrator systems, that is, $x_i(t) = [y_i(t), v_i(t)]^T \in \mathbb{R}^2$ with the dynamics

$$\dot{y}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t),\ i = 1, 2, ..., N \tag{16}$$

which can be considered as the special case of the linear system (12) with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $B = (0, 1)^T$. Here, $y_i(t)$ and $v_i(t)$ can be considered as the position and the velocity of the ith agent, respectively. It is easy to see that (A, B) is controllable and $\lambda(A) = 0$. We will show that second-order integrator systems (16) must be mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f(\cdot))$ for any given system parameters. Here, we focus on the choice of the protocol for solving stochastic consensus.

For this case, we consider the different uncertainties in each position and velocity communication. Then, the protocol (13) can be rewritten as follows:

$$u_i(t) = k_1 \sum_{j \in N_i} z_{1ji}(t) + k_2 \sum_{j \in N_i} z_{2ji}(t) \tag{17}$$

where $K = [k_1, k_2] \in \mathbb{R}^{1\times 2}$, $z_{1ji}(t) = y_j(t - \tau_1) - y_i(t - \tau_1) + f_{1ji}(y_j(t - \tau_2) - y_i(t - \tau_2))\xi_{1ji}(t)$, $z_{2ji}(t) = v_j(t - \tau_1) - v_i(t - \tau_1) + f_{2ji}(v_j(t - \tau_2) - v_i(t - \tau_2))\xi_{2ji}(t)$ are the position and the velocity measurements of the agent i from its neighbor $j \in N_i$, $\xi_{ji}(t) = [\xi_{1ji}, \xi_{2ji}]^T$ is the 2-D Gaussian white noise, that is, ξ_{1ji} and ξ_{2ji} are the scalar independent Gaussian white noise, and $f_{1ji}(\cdot)$ and $f_{2ji}(\cdot)$ satisfy the following assumption.

Assumption III.2: For each (j, i), $f_{1ji}(0) = f_{2ji}(0) = 0$ and there exist nonnegative constants $\bar{\sigma}_{1ji}$ and $\bar{\sigma}_{2ji}$ such that $\|f_{1ji}(y_1) - f_{1ji}(y_2)\| \leq \bar{\sigma}_{1ji}\|y_1 - y_2\|$, $\|f_{2ji}(y_1) - f_{2ji}(y_2)\| \leq \bar{\sigma}_{2ji}\|y_1 - y_2\|$ for all $y_1, y_2 \in \mathbb{R}$.

By using Theorem II.1 and choosing appropriate P in (3), we can obtain the following consensus conditions.

Theorem III.5: Suppose that Assumptions III.1 and III.2 hold, and $d = 2$. Then, the protocol (17) solves the mean square and almost sure consensus of the multiagent system (16) if $k_1 > 0, k_2 > 0$, and

$$\begin{aligned} &k_1\left(2\lambda_N\tau_1 + \bar{\sigma}_1^2\frac{N-1}{N}\right)(2 + k_1\lambda_N\tau_1) + k_1\lambda_2\tau_1 \\ &< 2\lambda_2 k_2\left(1 - k_2\bar{\sigma}_2^2\frac{N-1}{N}\right) - 4k_2^2\lambda_N^2\tau_1 \end{aligned} \tag{18}$$

where $\bar{\sigma}_1 = \max_{i,j=1}^N \bar{\sigma}_{1ji}$ and $\bar{\sigma}_2 = \max_{i,j=1}^N \bar{\sigma}_{2ji}$

Remark III.4: Theorem III.5 shows that for any time delays τ_1, τ_2 and $\{\sigma_{1ji}, \sigma_{2ji}\}_{i,j=1}^N$, we can find the appropriate control gains k_1, k_2 to guarantee the mean square and almost sure consensus. In fact, for any time delays τ_1, τ_2 and $\{\sigma_{1ji}, \sigma_{2ji}\}_{i,j=1}^N$, we first choose k_2 such that $k_2 < \frac{\lambda_2}{2\lambda_N^2\tau_1 + \lambda_2\bar{\sigma}_2^2\frac{N-1}{N}}$. Let k_2 be fixed. Then, we have $p_1 := 2\lambda_2 k_2(1 - k_2\bar{\sigma}_2^2\frac{N-1}{N}) - 4k_2^2\lambda_N^2\tau_1 > 0$. Based on the given k_2, we choose k_1 such that $k_1 p_2(k_1) < p_1$, where $p_2(k_1) = (2\lambda_N\tau_1 + \bar{\sigma}_1^2\frac{N-1}{N})(2 + k_1\lambda_N\tau_1) + \lambda_2\tau_1$. Hence, condition (18) holds and consensus is achieved.

We next aim to obtain necessary conditions for mean square consensus of the second-order multiagent systems (16) without time delay. We consider the following assumption.

Assumption III.3: For each (j, i), $f_{1ji}(0) = f_{2ji}(0) = 0$ and there exist some constants $\bar{\sigma}_{1ji}, \bar{\sigma}_{2ji}, \underline{\sigma}_{1ji}, \underline{\sigma}_{2ji} > 0$ such that $\bar{\sigma}_{1ji}\|y_1 - y_2\| \geq \|f_{1ji}(y_1) - f_{1ji}(y_2)\| \geq \underline{\sigma}_{1ji}\|y_1 - y_2\|, \bar{\sigma}_{2ji}\|y_1 - y_2\| \geq \|f_{2ji}(y_1) - f_{2ji}(y_2)\| \geq \underline{\sigma}_{2ji}\|y_1 - y_2\|$ for all $y_1, y_2 \in \mathbb{R}$,

By Theorem III.5 and stochastic stability theorem of stochastic ordinary differential equations, we can prove the following necessary conditions and sufficient conditions.

Theorem III.6: Suppose that Assumptions III.1 and III.3 hold, $d = 2$, and $\tau_1 = \tau_2 = 0$. Then, the protocol (17) solves the mean square consensus if

$$k_1 > 0,\ k_1\bar{\sigma}_1^2\frac{N-1}{N} < k_2\lambda_2 - k_2^2\bar{\sigma}_2^2\frac{N-1}{N}\lambda_2 \tag{19}$$

and only if

$$0 < k_1\underline{\sigma}_1^2\frac{N-1}{N} < k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\lambda_N \tag{20}$$

where $\underline{\sigma}_1 = \min_{i,j=1}^N \underline{\sigma}_{1ji}$ and $\underline{\sigma}_2 = \min_{i,j=1}^N \underline{\sigma}_{2ji}$.

Remark III.5: The approach used in obtaining Theorems III.5 and III.6 can be applied to examine the stability of the second-order stochastic differential equation

$$\begin{cases} dy(t) = v(t)dt \\ dv(t) = -\mu_1 y(t)dt - \mu_2 v(t)dt \\ \qquad\qquad + \sigma_1 y(t)dw_1(t) + \sigma_2 v(t)dw_2(t) \end{cases} \tag{21}$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$ are fixed constants, $w_1(t)$ and $w_2(t)$ are two scalar Brownian motions. We also obtain that the second-order SDE (21) is mean square exponentially stable if and only if $\mu_1, \mu_2 > 0$ and $\sigma_1^2 < (2\mu_2 - \sigma_2^2)\mu_1$, and the second-order SDE (21) is almost surely exponentially stable if $\mu_1, \mu_2 > 0$ and $\sigma_1^2 < (2\mu_2 - \sigma_2^2)\mu_1$. This is a new finding for the scalar second-order SDEs.

The stochastic stability and the GARE developed in Section II help us to find the mean square and almost sure consentability conditions and consensus protocols for the linear multiagent systems under the undirected graph. These sufficient conditions are explicit and easy to be verified. Note that for the case of general digraph, even for the balanced graph, it is a difficult task to obtain the explicit consentability conditions for the linear systems with multiplicative noises. However, for the linear leader-following multiagent systems, our skills can be used to solve the stochastic leader-following consentability and consensus problems, where we only need the subgraphs formed by the followers to be undirected. This is the following section's intention.

IV. Leader-Following Multiagent Systems

A. Leader-Following Linear Multiagent Systems

In this section, we consider a leader-following multiagent system consisting of $N + 1$ agents where an agent indexed by 0 acts as the leader and the other agents indexed by $1, 2, \ldots, N$, respectively, act as the followers. Generally, the behavior of the leader is independent of the followers. Here, x_0 denotes the state of the leader and is assumed to have linear dynamic as

$$\dot{x}_0 = Ax_0. \tag{22}$$

For the ith follower, the dynamics is described as (12) with $u_i(t)$ defined by (13). Note that this is different from Section III-A since for each agent i, its neighbors N_i may contain the leader 0. Considering the information flow from the leader to the followers, we denote the topology by $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{V}}, \widetilde{\mathcal{A}}\}$ with $\widetilde{\mathcal{V}} = \{0, 1, 2, \ldots, N\}$ and

$$\widetilde{\mathcal{A}} = \begin{pmatrix} 0 & 0_{N\times N} \\ a_0 & \mathcal{A} \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

where $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, $a_0 = [a_{10}, \ldots, a_{N0}]^T$, $a_{i0} = 1$ if $0 \in N_i$, otherwise $a_{i0} = 0$. Let $D_0 = \mathrm{diag}(a_{10}, \ldots, a_{N0})$. We use $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ to represent the subgraph formed by the N followers, where $\mathcal{V} = \widetilde{\mathcal{V}} \setminus \{0\}$.

Assumption IV.1: Assume that the graph $\widetilde{\mathcal{G}}$ contains a spanning tree and $\mathcal{G}$ is an undirected graph.

Under Assumption IV.1, $\mathcal{L}_0 = \mathcal{L} + D_0$ is symmetric, and all the eigenvalues of $\mathcal{L}_0$ are positive ([40]), denoted by $\{\lambda_{0i}\}_{i=1}^N$. Hence, there exists a unitary matrix Φ such that $\Phi^T\mathcal{L}_0\Phi = \mathrm{diag}(\lambda_{01}, \ldots, \lambda_{0N}) =: \Lambda_0$. Without loss of generality, we assume $0 < \lambda_{01} \leq \ldots \leq \lambda_{0N}$. In what follows, the main results are stated as theorems. Because the proofs are similar to that of Section III, the verbatim proofs of the theorems are omitted due to page limitation.

Theorem IV.1: Suppose that Assumptions III.1 and IV.1 hold, $f_{lji}(x) = \sigma_{lji}x$, $\sigma_{lji} \geq 0$, and $\max(\mathrm{Re}(\lambda(A))) \geq 0$. Then, linear systems (12) and (22) are mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f.(\cdot))$ if condition 1) and

2′) $4\lambda_0^u\bar{\sigma}^2 < \lambda_{01}$,

3′) $\tau_1 \in [0, \tau_1^*)$ with $\tau_1^* = \frac{1}{2\|A\|^2\|P\|} \wedge \frac{\lambda_{01} - 4\bar{\sigma}^2(\lambda_0^u + \epsilon)}{6(\lambda_0^u + \epsilon)\lambda_{0N}}$,

where $P > 0$ is the solution to the GARE (11) with $\alpha \in (\lambda_0^u, \lambda_0^u + \epsilon)$, $\epsilon \in (0, \frac{\lambda_{01} - 4\bar{\sigma}^2\lambda_0^u}{4\bar{\sigma}^2})$, $\bar{\sigma}^2 = \sum_{l=1}^d \max_{i=1,j=0}^N \sigma_{lji}^2$. Especially, the protocol (13) with $K = k(I_m + B^TPB)^{-1}B^TP$, $k \in (\underline{k}, \bar{k})$, solves the mean square and almost sure consensus, where $\underline{k} = [\lambda_{01} - \sqrt{\lambda_{01}^2 - 2\alpha\rho}]/\rho$, $\bar{k} = [\lambda_{01} + \sqrt{\lambda_{01}^2 - 2\alpha\rho}]/\rho$, $\rho = (2\bar{\sigma}^2 + 3\lambda_{0N}\tau_1)\lambda_{01}$.

Especially, if the overall network $\widetilde{\mathcal{G}}$ forms a star topology, then we have the following result.

Theorem IV.2: Suppose that Assumption III.1 holds, $\widetilde{\mathcal{G}}$ forms a star topology, $f_{lji}(x) = \sigma_{lji}x$, $\sigma_{lji} \geq 0$, and $\max(Re(\lambda(A))) \geq 0$. Then, linear systems (12) and (22) are mean square and almost surely consentable w.r.t. $\mathcal{U}(\tau_1, \tau_2, f_\cdot(\cdot))$ if condition 1) holds and

2″) $\lambda_0^u \bar{\sigma}^2 < 1/2$,

3″) $\tau_1 \in [0, \tau_1^*)$ with $\tau_1^* = \frac{1}{2\|A\|^2\|P\|} \wedge \frac{1-2\bar{\sigma}^2(\lambda_0^u+\epsilon)}{6(\lambda_0^u+\epsilon)}$,

where $P > 0$ is the solution to the GARE (11) with $\alpha \in (\lambda_0^u, \lambda_0^u + \epsilon)$, $\epsilon \in (0, \frac{1-2\bar{\sigma}^2\lambda_0^u}{2\bar{\sigma}^2})$, $\bar{\sigma}^2 = \sum_{l=1}^d \max_{i=1}^N \sigma_{l0i}^2$. Especially, the protocol (13) with $K = k(I_m + B^TPB)^{-1}B^TP$, $k \in (\underline{k}, \bar{k})$, solves the mean square and almost sure consensus, where $\underline{k} = [1 - \sqrt{1-2\alpha\rho}]/\rho$, $\bar{k} = [1 + \sqrt{1-2\alpha\rho}]/\rho$, $\rho = (\bar{\sigma}^2 + 3\tau_1)$.

If we consider the special case: $x_0(t) \equiv 0$ and $N = 1$ (one follower). Then, the stochastic consentability problem in Theorem IV.2 is actually the stochastic stabilization problem of (12), which is concluded as the following corollary and improves the case with single and linear diffusion in [46].

Corollary IV.1: Suppose that Assumption III.1 holds, $x_0(t) \equiv 0$, $N = 1$, $f_{l01}(x) = \sigma_{l01}x$, $\sigma_{l01} \geq 0$, and $\max(Re(\lambda(A))) \geq 0$. Then, linear systems (12) is mean square and almost surely stabilizable if conditions 1), 2”), 3”) hold with $P > 0$ being the solution to the GARE (11) with $\alpha \in (\lambda_0^u, \lambda_0^u + \epsilon)$, $\epsilon \in (0, \frac{1-2\bar{\sigma}^2\lambda_0^u}{2\bar{\sigma}^2})$, $\bar{\sigma}^2 = \sum_{l=1}^d \sigma_{l01}^2$. Especially, the choice K defined in Theorem IV.2 stabilizes the linear system (12).

B. Leader-Following Second-Order Integrator Multiagent Systems

Consider the ith follower's state $x_i(t) = [y_i(t), v_i(t)]^T \in \mathbb{R}^2$ with the dynamics (16), and the leader's state $x_0(t) = [y_0(t), v_0(t)]^T \in \mathbb{R}^2$ with the dynamic

$$\dot{y}_0(t) = v_0(t) = v_0 \tag{23}$$

where v_0 is a constant velocity known to all followers. The protocol is designed as (17), which includes the leader's information. Similarly to the linear case above, we have the following theorems and corollary.

Theorem IV.3: Suppose that Assumptions III.1, III.2, and IV.1 hold, and $d = 2$. Then, the protocol (17) solves the mean square and almost sure consensus of (16) and (23) if $k_1 > 0, k_2 > 0$, and $k_1(2\lambda_{0N}\tau_1 + \bar{\sigma}_1^2)(2 + k_1\lambda_{0N}\tau_1) + k_1\lambda_{01}\tau_1 < 2\lambda_{01}k_2(1 - k_2\bar{\sigma}_2^2) + 4k_2^2\lambda_{0N}^2\tau_1$, where $\bar{\sigma}_1 = \max\{\bar{\sigma}_{1ji}, i = 1, \ldots, N, j \in N_i\}$ and $\max\{\bar{\sigma}_{2ji}, i = 1, \ldots, N, j \in N_i\}$.

Theorem IV.4: Suppose that Assumptions III.1 and III.2 hold, $\widetilde{\mathcal{G}}$ forms a star topology, and $d = 2$. Then, the protocol (17) solves the mean square and almost sure consensus of (16) and (23) if $k_1 > 0, k_2 > 0$, and $k_1(2\tau_1 + 0.5\bar{\sigma}_1^2)(2 + k_1\tau_1) + k_1\tau_1 < 2k_2(1 - 0.5k_2\bar{\sigma}_2^2) + 4k_2^2\tau_1$, where $\bar{\sigma}_1 = \max_{i=1}^N \bar{\sigma}_{10i}$ and $\bar{\sigma}_2 = \max_{i=1}^N \bar{\sigma}_{20i}$.

Corollary IV.2: Suppose that Assumptions III.1 and III.2 hold, $x_0(t) \equiv 0$, $N = 1$, and $d = 2$. Then, the second-order system (16) can be mean square and almost surely stabilized by (17) with the same choice of k_1 and k_2 as that in Theorem IV.4.

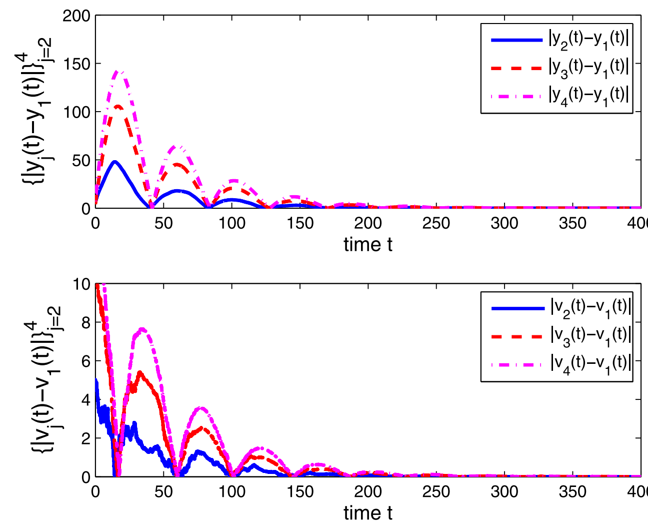

Fig. 1. Relative state errors of the four agents: $k_1 = 0.01$, $k_2 = 0.065$.

V. Simulations

We consider the almost sure and mean square consensus for a second-order integrator multiagent system (16) composed of four scalar agents. The control input $u_i(t)$ has the form of (17). We will choose the appropriate control gains k_1 and k_2 to guarantee the mean square and almost sure consensus.

Consider $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E} = \{(1,2), (2,3), (3,4), (4,3), (3,2), (2,1)\}$ and $\mathcal{A} = [a_{ij}]_{4\times 4}$ with $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = 1$ and other being zero. It can be obtained the eigenvalues of the Laplacian matrix $\mathcal{L}$: $\lambda_1 = 0, \lambda_2 = 0.5858, \lambda_3 = 2, \lambda_4 = 3.4142$. The initial states are given by $y(s) = [2, -4, -2, -5]^T$ and $v(s) = [7, 2, -4, -8]^T$, $s \in [-(\tau_1 \vee \tau_2), 0]$, where $\tau_1 = 0.2$ and $\tau_2 = 0.5$. Assume that the noise intensity functions $f_{1ji}(x) = \bar{\sigma}_{1ji}x = 0.5x$ and $f_{2ji}(x) = \bar{\sigma}_{2ji}x = 0.3x$, $i, j = 1, 2, 3, 4$.

We now use Remark III.4 to choose the control gains k_1, k_2 such that the four agents achieve the mean square and almost sure consensus. We compute that $k_2^* = \frac{\lambda_2}{2\lambda_N^2\tau_1 + \lambda_2\bar{\sigma}_2^2\frac{N-1}{N}} = 0.1246$. Therefore, we choose $k_2 = 0.065 < k_2^*$. Then, we have $p_1 := 2\lambda_2k_2(1 - k_2\bar{\sigma}_2^2\frac{N-1}{N}) - 4k_2^2\lambda_N^2\tau_1 = 0.0365$. Based on the given k_2, we choose $k_1 = 0.01$, and then $0.0323 = k_1p_2(k_1) < p_1 = 0.0365$, where $p_2(k_1) = (2\lambda_N\tau_1 + \bar{\sigma}_1^2\frac{N-1}{N})(2 + k_1\lambda_N\tau_1) + \lambda_2\tau_1$. Hence, the four agents achieve the mean square and almost sure consensus (see Theorem III.5).

In order to simulate the mean square and almost sure consensus, we consider the behaviors of the relative states $\{|y_i(t) - y_1(t)|\}_{i=2,3,4}$. Generally, the almost sure asymptotic behavior is reflected by one sample path. Here, by choosing one sample path, we obtain that the relative states $\{|y_i(t) - y_1(t)|\}_{i=2,3,4}$ tend to zero, which is revealed in Fig. 1. That is, four agents achieve the almost sure consensus. By considering each agent's behavior and the sample path, we have Fig. 2, which shows that the agents' velocities achieve the almost sure strong consensus, that is, the agents' velocities tend to a common value. For the mean square consensus, we generate 10^4 sample paths. Then,

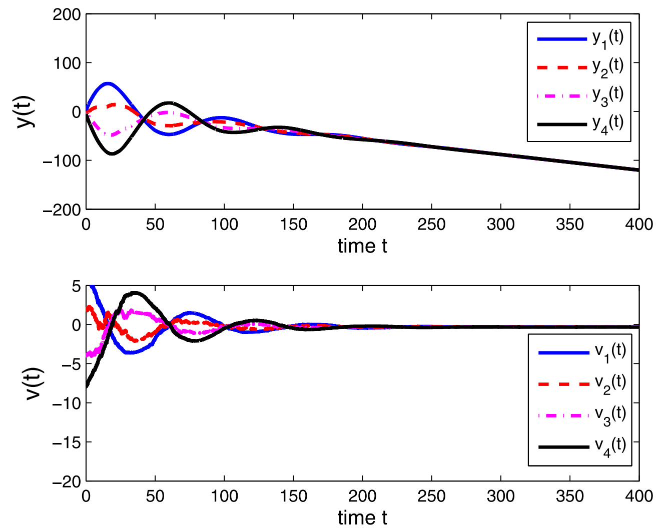

Fig. 2. States of the four agents: $k_1 = 0.01$, $k_2 = 0.065$.

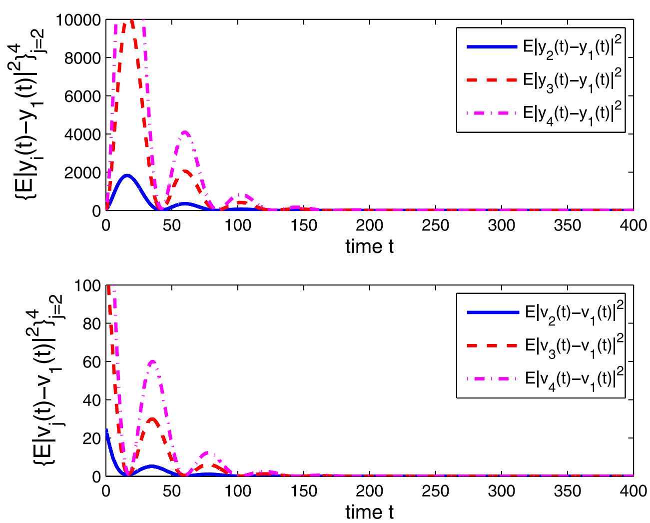

Fig. 3. Mean square errors of the relative states: $k_1 = 0.01$, $k_2 = 0.065$.

taking the mean square average, we obtain Fig. 3, which shows that the four agents achieve the mean square consensus.

VI. Concluding Remarks

In this paper, we developed consentability for linear multiagent systems with multiplicative noises and time delays. By establishing stochastic stability theorem of SDDEs and the positive definite solution of GARE, we obtained the stochastic consentability conditions and consensus protocols for the linear multiagent systems. Our results reveal that the stochastic consentability depends on the essential properties of the original systems, including (A, B), the graph, the noise intensities, and the time delay in the deterministic term. In particular, the second-order integrator system must be mean square and almost surely consentable regardless of the values of time delays and noise intensities.

There are still many interesting topics deserving further investigation. We have not obtained the optimal time delay bound since it is difficult work to find the necessary and sufficient condition for the stability of SDDEs. It can be observed that the consensus analysis with measurement noises and time delays under the general directed graphs is more difficult than the case under undirected graphs, especially for the linear (including second-order) multiagent systems. Moreover, the consensus analysis of multiagent systems with nonlinear dynamics, switching topology, time-varying delays, and measurement noises has not be given. Effort can also be directed to extend our analysis to the heterogeneous multiagent systems with time delays and measurement noises.

Appendix A
Proof of Theorems in Section II-A

Lemma A.1: ([41]) For any positive definite matrix P and positive constant τ, the following inequality holds:

$$\left(\int_{t-\tau}^{t} x(s)ds\right)^T P \left(\int_{t-\tau}^{t} x(s)ds\right) \le \tau \int_{t-\tau}^{t} x^T(s)Px(s)ds$$

provided that the integrals above are well defined.

Proof of Theorem II.1: Let $z(t) = y(t) + A_1 \int_{t-\tau_1}^{t} y(s)ds$. We choose the degenerate Lyapunov functional (see [25]) $V(y_t) = V_1(t) + V_2(t)$, where $y_t = \{y(t+\theta) : \theta \in [-\tau_1, 0]\}$, $V_1(t) = z^T(t)Pz(t)$, $V_2(t) = \int_{-\tau_1}^{0}\int_{t+s}^{t} y^T(\theta) A_1^T PA_1 y(\theta)d\theta ds$. Note that the elementary inequality: $2x^T Oy \le x^T Ox + y^T Oy$, for any positive definite matrix $O \in \mathbb{R}^{n\times n}$ and $x, y \in \mathbb{R}^n$. Using Itô's formula and Lemma A.1, we have

$$\begin{aligned} dV_1(t) &= y^T(t)[\bar{A}^T P + P\bar{A}]y(t)dt + d\langle M, PM\rangle(t) \\ &\quad + 2y^T(t)\bar{A}^T PA_1 \int_{t-\tau_1}^{t} y(s)dsdt + dM_1(t) \\ &\le y^T(t)S_1 y(t)dt + \int_{t-\tau_1}^{t} y^T(s)A_1^T PA_1 y(s)dsdt \\ &\quad + d\langle M, PM\rangle(t) + dM_1(t) \end{aligned} \tag{24}$$

where $S_1 = \bar{A}^T P + P\bar{A} + \tau_1 \bar{A}^T P\bar{A}$, $\langle M, PM\rangle(t) = \sum_{i=1}^{d} \int_0^t f_i^T(y(s-\tau_2))Pf_i(y(s-\tau_2))ds$, $M_1(t) = 2\int_0^t z^T(s)P\, dM(s)$. Then, we get from (24) and condition (2) that $dV(y_t) \le y^T(t)S_2 y(t)dt + F(t-\tau_2)dt + dM_1(t)$, where $S_2 = \bar{A}^T P + P\bar{A} + (\bar{A}^T P\bar{A} + A_1^T PA_1)\tau_1$, $F(t) = y^T(t)D_P y(t)$. Applying the Itô formula to $e^{\gamma t}V(y_t)$, for any $\gamma > 0$, we have

$$\begin{aligned} d[e^{\gamma t}V(y_t)] &= \gamma e^{\gamma t}V(y_t)dt + e^{\gamma t}dV(y_t) \\ &\le \gamma e^{\gamma t}V(y_t)dt + e^{\gamma t}y^T(t)S_2 y(t)dt \\ &\quad + e^{\gamma t}F(t-\tau_2)dt + e^{\gamma t}dM_1(t). \end{aligned}$$

Integrating both parts of the above inequality and taking the expectations yield

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(y_t) &\le \mathbb{E}V(y_0) + \mathbb{E}\int_0^t e^{\gamma s}y^T(s)S_2 y(s)ds \\ &\quad + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(y_s)ds + \mathbb{E}\int_0^t e^{\gamma s}F(s-\tau_2)ds. \end{aligned} \tag{25}$$

Note that $\int_0^t e^{\gamma s}F(s-\tau_2)ds \le e^{\gamma\tau_2}\int_{-\tau_2}^0 F(s)ds + e^{\gamma\tau_2}\int_0^t e^{\gamma s}F(s)ds$. Then, we get

$$e^{\gamma t}\mathbb{E}V(y_t) \le C_1(\gamma) + \int_0^t e^{\gamma s}\mathbb{E}y^T(s)S_3 y(s)ds + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(y_s)ds \tag{26}$$

where $C_1(\gamma) = V(y_0) + e^{\gamma\tau_2}\int_{-\tau_2}^0 F(s)ds$, $S_3(\gamma) = S_2 + e^{\gamma\tau_2}D_P$. By the definition of the functional $V(y_t)$ and the elementary inequality $(x+y)^TQ(x+y) \le 2x^TQx + 2y^TQy$, $x,y\in\mathbb{R}^n$, $Q>0$, we have $V(y_s) \le C_2\int_{s-\tau_1}^s \|y(u)\|^2du + 2\|P\|\|y(s)\|^2$, where $C_2 = 3\tau_1\|A_1\|^2\|P\|$. Substituting this into (26) yields

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(y_t) \le{}& C_1(\gamma) + \int_0^t e^{\gamma s}\mathbb{E}y^T(s)S_3(\gamma)y(s)ds \\ &+ C_2\int_0^t \gamma e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|y(u)\|^2duds \\ &+ 2\|P\|\int_0^t \gamma e^{\gamma s}\mathbb{E}\|y(s)\|^2ds. \end{aligned} \tag{27}$$

Note that

$$\begin{aligned} &\int_0^t e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|y(u)\|^2duds \\ &\le \int_{-\tau_1}^0 \mathbb{E}\|y(u)\|^2\int_u^{u+\tau_1} e^{\gamma s}dsdu \\ &\quad + \int_0^t \mathbb{E}\|y(u)\|^2\int_u^{u+\tau_1} e^{\gamma s}dsdu \\ &\le \tau_1^2 e^{\gamma\tau_1}\|\varphi\|_C^2 + \tau_1 e^{\gamma\tau_1}\int_0^t e^{\gamma u}\mathbb{E}\|y(u)\|^2du. \end{aligned}$$

Hence, we get $e^{\gamma t}\mathbb{E}V(y_t) \le C_3(\gamma) + \int_0^t e^{\gamma s}\mathbb{E}y^T(s)S_3(\gamma) y(s)ds + C_4(\gamma)\gamma\int_0^t e^{\gamma s}\mathbb{E}\|y(s)\|^2ds$, where $C_3(\gamma) = C_1(\gamma) + C_4(\gamma)\gamma\tau_1^2e^{\gamma\tau_1}\|\varphi\|_C^2$ and $C_4(\gamma) = C_2\tau_1e^{\gamma\tau_1} + 2\|P\|$. Let $S_4(\gamma) = S_3(\gamma) + \gamma C_4(\gamma)I_n$. Then, it is obtained that

$$e^{\gamma t}\mathbb{E}V(y_t) \le C_3(\gamma) + \int_0^t e^{\gamma s}\mathbb{E}y^T(s)S_4(\gamma)y(s)ds. \tag{28}$$

Note that $S_4(0) < 0$ under (3). Therefore, if (3) holds, then there must exists a $\gamma^* > 0$ such that for any $\gamma < \gamma^*$, $S_4(\gamma) = \bar{A}^TP + P\bar{A} + (\bar{A}^TP\bar{A} + A_1^TPA_1)\tau_1 + e^{\gamma\tau_2}D_P + \gamma C_4(\gamma)I_n < 0$, which together with (28) implies $\int_0^\infty e^{\gamma s}\mathbb{E}y^T(s)(-S_4(\gamma))y(s)ds < C_3(\gamma)$. This also produces $\mathbb{E}\|y(t)\|^2 \le C_0e^{-\gamma_0 t}$. Under Assumption II.1, the mean square exponential stability implies the almost sure exponential stability (see [42]). That is, (4) holds. ∎

Proof of Corollary II.2: Let $\varrho = \sqrt{\sum_{j=1}^d \varrho_j^2}$. Note that $\bar{A}$ is symmetric, then there exists a unitary matrix Φ such that $\Phi^T = \Phi^{-1}$ and $\Phi^T\bar{A}\Phi = \mathrm{diag}(\lambda_1,\lambda_3,\cdots,\lambda_n) =: \Lambda$ with $\lambda_{\min}(\bar{A}) = \lambda_1 \le \cdots \le \lambda_n = \lambda_{\max}(\bar{A})$. In view of the transformation, $T_A := 2\bar{A} + \varrho^2 I_n < 0$ if and only if $2\lambda_i + \varrho^2 < 0$, $i = 1,\ldots,d$. Letting $x(t) = \Phi^{-1}y(t)$, then we get $dx_i(t) = \lambda_i x_i(t)dt + \sum_{j=1}^d \varrho_j x_i(t-\tau_2)dw_j(t)$, $i = 1,\ldots,n$. Note that $w(t) = \sum_{j=1}^d \varrho_j dw_j(t)/\varrho$ is still a scalar Brownian motion. Then, we can rewrite the equation above as

$$dx_i(t) = \lambda_i x_i(t)dt + \varrho x_i(t-\tau_2)dw(t),\ i = 1,\ldots,n. \tag{29}$$

Hence, the trivial solution to SDDE (1) is mean square stable if and only if (29) is mean square stable, which is equivalent to that $2\lambda_i + \varrho^2 < 0$ (see [31]). ∎

Appendix B
Proof of Theorems in Section II-B

Proof of Lemma II.1: Let $A(\alpha) := A - \alpha I_n$. If $A^TP + PA - 2\alpha PB(R + B^TPB)^{-1}B^TP + Q = 0$ has positive definite solution $P > 0$, then $A(\alpha)^TP + PA(\alpha) = -2\alpha(P^{-1} + BR^{-1}B^T)^{-1} - Q < 0$, where $2\alpha[P - PB(R + B^TPB)^{-1}B^TP] = 2\alpha(P^{-1} + BR^{-1}B^T)^{-1}$ is used. Hence, $A(\alpha)$ is Hurwitz, which implies $\alpha > \max(\mathrm{Re}(\lambda(A)))$. ∎

Proof of Lemma II.2: We first prove that the corresponding discrete-time system is mean square stabilizable with a exponential convergence rate under the given conditions, then we get the sufficiency by applying the fact [43] that the mean square exponential stability of the discrete-time models yields the mean square exponential stability of the continuous-time models. Consider the following discrete-time linear system:

$$x_{i+1} = x_i + Ax_i\Delta + BKx_i\Delta + \sigma BKx_i\Delta w_i, \tag{30}$$

which can be considered as the Euler–Maruyama approximation to (7), where $\Delta > 0$ is time-stepsize and Δw_k is the Brownian increment. Let $A_\Delta = I + A\Delta$, $B_\Delta = B\Delta$. Note that (A, B) is controllable, then there is a $\Delta_1^* > 0$ such that for any $\Delta < \Delta_1^*$, (A_Δ, B) is controllable. It can be proved that

$$\lim_{\Delta\to 0}\left[1 - \frac{1}{\Pi_i|\bar{\lambda}_i^u(A_\Delta)|^2}\right]/(2\Delta) = \sum_i Re(\lambda_i^u(A)) = \lambda_0^u$$

where $\{\bar{\lambda}_i^u(A)\}_i$ denotes A's eigenvalue(s) larger than one in absolute value. Therefore, if $2\sigma^2\lambda_0^u < 1$, then there exist a $\Delta^* < \Delta_1^*$ such that for any $\Delta \in (0,\Delta^*)$, $\frac{1}{\sigma^2}\Delta > [1 - \frac{1}{\Pi_i|\bar{\lambda}_i^u(A_\Delta)|^2}]$. Hence, for any $\Delta \in (0,\Delta^*)$, the following ARE has a unique positive definite solution P (see [36]):

$$A_\Delta^TPA_\Delta - \frac{1}{\sigma^2}\Delta A_\Delta^TPB(R + B^TPB)^{-1}B^TPA_\Delta + Q\Delta = P \tag{31}$$

for $R > 0$ and $Q > 0$. That is, the choice $K = \sigma^{-2}(R + B^TPB)^{-1}B^TPA_\Delta$ guarantees the mean square stability of (30). Note the mean square asymptotical stability and the mean square exponential stability are equivalent for the linear time-invariant SDE. By the mean square exponential stability of the numerical and exact solutions (see [43]), we can see that the linear system (7) is mean square stabilizable.

Now we show that necessity of conditions that (A, B) is stabilizable and $2\max(\mathrm{Re}(\lambda(A)))\sigma^2 < 1$ for the mean square stabilization. If (A, B) is unstabilizable, then for any K, $A - BK$ is not Hurwitz. Let $A_K := A - BK$. By the matrix theorem, there exists a complex invertible matrix Q such that $QA_KQ^{-1} = J$, Here, J is the Jordan normal form of A_K, i.e., $J = \mathrm{diag}(J_{\mu_1,n_1}, J_{\mu_2,n_2},\ldots,J_{\mu_l,n_l})$, $\sum_{k=1}^l n_k = n$, where

$\mu_1, \mu_2, \ldots, \mu_l$ are all the eigenvalues of A_K and J_{μ_k, n_k} is the corresponding Jordan block of size n_k with eigenvalue μ_k. Let $Y(t) = Qx(t)$ and $M(t) = \sigma Q \int_0^t BKx(s)dw(s)$. Then, we have from (7) that $dY(t) = JY(t)dt + dM(t)$. Note that A_K is not Hurwitz, then there must exist an eigenvalue, denoted by μ_k, with the nonnegative real part ($\text{Re}(\mu_k) \geq 0$). Considering the kth Jordan block and its corresponding component $\zeta_k(t) = [\zeta_{k,1}(t), \ldots, \zeta_{k,n_k}(t)]^T$ and $M(k,t) = [M_{k,1}(t), \ldots, M_{k,n_k}(t)]^T$, where $\zeta_{k,j}(t) = Y_{k_j}(t)$ and $M_{k,j}(t) = M_{k_j}(t)$ with $k_j = \sum_{i=1}^{k-1} n_i + j$, we have $d\zeta_{k,n_k}(t) = \mu_k \zeta_{k,n_k}(t)dt + dM_{k,n_k}(t)$, which together with the variation of constants formula implies $\zeta_{k,n_k}(t) = e^{\mu_k t}\zeta_{k,n_k}(0) + \int_0^t e^{\mu_k(t-s)} dM_{k,n_k}(s)$. Hence, we get $\mathbb{E}\|\zeta_{k,n_k}(t)\|^2 = e^{\text{Re}(\mu_k)t}\|\zeta_{k,n_k}(0)\|^2 + \mathbb{E}\|\int_0^t e^{\mu_k(t-s)}dM_{k,n_k}(s)\|^2 \geq e^{\text{Re}(\mu_k)t}\|\zeta_{k,n_k}(0)\|^2$. This is in contradiction with the mean square stability. Hence, (A, B) must be stabilizable.

By [38, Th. 1], if the linear system (7) is mean square stabilizable, then there exists a positive definite solution to (6) with $2\alpha = \sigma^{-2}$. This together with Lemma II.1 implies $2\max(\text{Re}(\lambda(A)))\sigma^2 < 1$. ■

Proof of Theorem II.2: The "only if" part has been proved in Lemma II.1. Hence, we need only prove the "if" part in the two assertions, respectively.

By Lemma II.2, we know that the system (7) is mean square stabilizable. This together with Corollary 4 in [26] can yield the first "if" part by letting $\sigma = (2\alpha)^{-1/2}$.

If B is invertible and $\alpha > \max(Re(\lambda(A)))$, we do not know whether the linear system is mean square stabilizable. Therefore, [26, Corollary 4] cannot be used to prove the second "if" part. Here, we will develop a matrix approximate sequence to get the desired assertion. Let $\Gamma_P := A^TP + PA - 2\alpha PB(R + B^TPB)^{-1}B^TP + Q$. It is easy to see that $R + B^TPB > 0$ for $P \geq 0$ since $R > 0$. Note that $A(\alpha) = A - \alpha I_n$ is Hurwitz. Then, there exists a $P_0 > 0$ such that $A(\alpha)^TP_0 + P_0A(\alpha) + Q = 0$. We have

$$\Gamma_{P_0} = A(\alpha)^TP_0 + P_0A(\alpha) + M(P_0) + Q > 0 \quad (32)$$

where $M(P) = 2\alpha[P - PB(R + B^TPB)^{-1}B^TP] = 2\alpha(P^{-1} + BR^{-1}B^T)^{-1}$. Let P_0 be fixed, then there exists a $P_1 > 0$ satisfying $A(\alpha)^TP_1 + P_1A(\alpha) + Q + M(P_0) = 0$. We can see that $A(\alpha)^T(P_1 - P_0) + (P_1 - P_0)A(\alpha) = -M(P_0) < 0$, which implies $P_1 > P_0$. Then, $M(P_1) > M(P_0)$ and $\Gamma_{P_1} > 0$. Assume that we have obtained $P_k > P_{k-1} > \cdots > P_0$, $i = 0, 1, \ldots, k$, we now define P_{k+1} as follows:

$$A(\alpha)^TP_{k+1} + P_{k+1}A(\alpha) + Q + M(P_k) = 0. \quad (33)$$

Note that $M(P_k) > M(P_{k-1})$, then we have $A(\alpha)^T(P_{k+1} - P_k) + (P_{k+1} - P_k)A(\alpha) = -[M(P_k) - M(P_{k-1})] < 0$, which implies $P_{k+1} > P_k$. Repeating the same procedure iteratively, we can find an increasing sequence $\{P_i\}_{i=0}^{\infty}$. It can be seen that $M(P_k) \leq 2\alpha(BR^{-1}B^T)^{-1}$ and then

$$P_{k+1} = \int_0^\infty e^{A(\alpha)^Tt}[Q + M(P_k)]e^{A(\alpha)t}dt$$
$$\leq \int_0^\infty e^{A(\alpha)^Tt}[Q + 2\alpha(BR^{-1}B^T)^{-1}]e^{A(\alpha)t}dt < \infty.$$

Therefore, the exists a $P^* > P_0 > 0$ such that $P^* = \lim_{i\to\infty} P_i$. Taking the limit in (33) yields the existence of the positive definite solution. This together with [44, Th. 4.2] gives that the positive definite solution to GARE (6) is unique. ■

APPENDIX C
PROOF OF THEOREMS IN SECTION III

For the linear multiagent systems, we denote $\delta(t) = [(I_N - J_N) \otimes I_n]x(t)$. Let $\delta(t) = [\delta_1^T(t), ..., \delta_N^T(t)]^T$, where $\delta_i(t) \in \mathbb{R}^n$, $i = 1, ..., N$. Define the unitary matrix $T_{\mathcal{L}} = [\frac{\mathbf{1}_N}{\sqrt{N}}, \phi_2, ..., \phi_N]$, where ϕ_i is the unit eigenvector of $\mathcal{L}$ associated with the eigenvalue λ_i, that is, $\phi_i^T\mathcal{L} = \lambda_i\phi_i^T$, $\|\phi_i\| = 1$, $i = 2, ..., N$. Denote $\phi = [\phi_2, ..., \phi_N]$. Let $\delta(t) = (T_{\mathcal{L}} \otimes I_n)\widetilde{\delta}(t)$ and $\widetilde{\delta}(t) = [\widetilde{\delta}_1^T(t), ..., \widetilde{\delta}_N^T(t)]^T$, then it can be verified that $\widetilde{\delta}_1(t) \equiv 0$. Denote $\overline{\delta}(t) = [\widetilde{\delta}_2^T(t), ..., \widetilde{\delta}_N^T(t)]^T$ and $\Lambda = \text{diag}(\lambda_2, \lambda_3, \cdots, \lambda_N)$.

Proof of Theorem III.1: It suffices to show that the protocol (13) with $K = k(I_m + B^TPB)^{-1}B^TP$ solves the mean square and almost sure consensus. The consensus problems will be first transformed into the stability problems of a SDDE, then Theorems II.1 and II.2 will be applied to solve the stability problems, which also solve the consensus problems.

Note that conditions 2) and 3) and the definition of ϵ imply that $\lambda_2^2 > 2(\lambda_0^u + \epsilon)\rho$, which guarantees that $\underline{k}$ and $\bar{k}$ are well defined for $\alpha \in (\lambda_0^u, \lambda_0^u + \epsilon)$. With the protocol (13), the closed-loop system takes the form $dx(t) = (I_N \otimes A)x(t)dt - (\mathcal{L} \otimes BK)x(t - \tau_1)dt + d\widetilde{M}_1(t)$, where $\widetilde{M}_1(t) = \sum_{l=1}^d \sum_{i,j=1}^N a_{ij}\sigma_{lji} \int_0^t [S_{i,j} \otimes BK]\delta(s - \tau_2) dw_{lji}(s)$, $S_{i,j} = [s_{kl}]_{N\times N}$ is an $N \times N$ matrix with $s_{ii} = -a_{ij}, s_{ij} = a_{ij}$ and all other elements being zero, $i, j = 1, 2, \ldots, N$. By the definition of $\delta(t)$, we have $d\delta(t) = (I_N \otimes A)\delta(t)dt - (\mathcal{L} \otimes BK)\delta(t - \tau_1)dt + d\widetilde{M}_2(t)$, where $\widetilde{M}_2(t) = \sum_{l=1}^d \sum_{i,j=1}^N a_{ij}\sigma_{lji} \int_0^t [(I_N - J_N)S_{i,j} \otimes BK] \quad \delta(s - \tau_2)dw_{lji}(s)$. This together with the definition of $\overline{\delta}(t)$ implies

$$d\overline{\delta}(t) = A_0\overline{\delta}(t)dt + A_1\overline{\delta}(t - \tau_1)dt + d\widetilde{M}_3(t) \quad (34)$$

where $A_0 = I_{N-1} \otimes A$, $A_1 = -\Lambda \otimes BK$, $\widetilde{M}_3(t) = \sum_{l=1}^d \sum_{i,j=1}^N a_{ij}\sigma_{lji} \int_0^t [(\phi^T(I_N - J_N)S_{i,j}\phi) \otimes BK] \quad \overline{\delta}(s - \tau_2) dw_{1ji}(s)$. This together with the definition of $\overline{\delta}(t)$ yields that the consensus problems equal to the stability of (34).

Hence, we need to prove the stability of (34) under conditions 1)-3). Let $\bar{P} = I_{N-1} \otimes P$. Note that $\langle \widetilde{M}_3, \bar{P}\widetilde{M}_3\rangle(t) = \sum_{l=1}^d \sum_{i,j=1}^N a_{ij}\sigma_{lji}^2 \int_0^t \overline{\delta}^T(s - \tau_2)[((\phi^T(I_N - J_N)S_{i,j}\phi)^T(\phi^T(I_N - J_N)S_{i,j}\phi)) \otimes K^TB^TPBK] \overline{\delta}(s - \tau_2)ds$, $\phi^T(I_N - J_N)S_{i,j}\phi = \phi^TS_{i,j}\phi$ and that $\sum_{i,j=1}^N a_{ij}(\phi^TS_{i,j}\phi)^T(\phi S_{i,j}\phi) = \frac{2(N-1)}{N}\Lambda$. Then, we have

$$d\langle \widetilde{M}_3, \bar{P}\widetilde{M}_3\rangle(t) \leq \bar{\sigma}^2\overline{\delta}^T(t - \tau_2)D\overline{\delta}(t - \tau_2)dt \quad (35)$$

where $D = 2\frac{N-1}{N}\Lambda \otimes (K^TB^TPBK)$. We now show that conditions 2) and 3) under $K = k(I_m + B^TPB)^{-1}B^TP$ imply $\bar{A}^T\bar{P} + \bar{P}\bar{A} + (\bar{A}^T\bar{P}\bar{A} + A_1^T\bar{P}A_1)\tau_1 + \bar{\sigma}^2D < 0$, where

$\bar{A} = A_0 + A_1$. Note that this can be guaranteed by

$$W_I^T P + PW_i + W_i^T PW_i\tau_1 + \left(\lambda_i^2\tau_1 + 2\lambda_i\frac{N-1}{N}\bar{\sigma}^2\right)K^T(B^TPB)K < 0 \quad (36)$$

$W_i = A - \lambda_i BK,\ i = 2,\dots,N$. By the elementary inequality $(x+y)^TQ(x+y) \le 2x^TQx + 2y^TQy,\ x,y\in\mathbb{R}^n,\ Q > 0$, then $W_i^TPW_i \le 2A^TPA + 2\lambda_i^2K^TB^TPBK$. Substituting this above into (36) and letting $K = k(I_m + B^TPB)^{-1}B^TP$, we can observe that (36) is assured by

$$\Gamma_i := A^TP + PA + 2\tau_1A^TPA - \zeta_iPB(I_m + B^TPB)^{-1}B^TP < 0, i = 2,\dots,N \quad (37)$$

where $\zeta_i = 2k\lambda_i - (2\frac{N-1}{N}\bar{\sigma}^2 + 3\lambda_N\tau_1)\lambda_ik^2$. Note that $P > 0$ is the solution to GARE (11). By condition 3), we can easily get from (37) that

$$\Gamma_i < (2\alpha - \zeta_i)PB(I_m + B^TPB)^{-1}B^TP, \quad (38)$$

Note that for $k\in(\underline{k},\bar{k})$, $2\alpha - \zeta_i < 0$. Then, we get from (38) that $\Gamma_i < 0$. Hence, by Theorem II.1, there exist $C_0 > 0$ and $\gamma > 0$, $\mathbb{E}\|\overline{\delta}(t)\|^2 \le C_0e^{-\gamma t}, \limsup_{t\to\infty}\frac{1}{t}\log\|\overline{\delta}(t)\| \le -\frac{\gamma}{2}$, a.s., which together with the definition of $\overline{\delta}(t)$ implies the mean square and almost sure consensus. ■

Proof of Theorem III.2: Note that $\max(Re(\lambda(A))) < 0$. Then, there exists $P > 0$ such that $A^TP + PA = -I_n$. In fact, $P = \int_0^\infty e^{A^Tt}e^{At}dt$. Letting $K = kB^TP$ in (34) and applying Itô's formula to $V(t) = \overline{\delta}^T(t)(I_{N-1}\otimes P)\overline{\delta}(t)$ yield

$$\begin{aligned} dV(t) &= (2k\overline{\delta}^T(t)(\Lambda\otimes PBB^TP)\overline{\delta}(t-\tau_1) - \|\overline{\delta}(t)\|^2)dt \\ &\quad + d\langle\widetilde{M}_3, (I_{N-1}\otimes P)\widetilde{M}_3\rangle(t) + dm(t) \\ &\le (|k|b_p - 1)\|\overline{\delta}(t)\|^2dt + |k|b_p\|\overline{\delta}(t-\tau_1)\|^2dt \\ &\quad + \sigma(b,p)k^2\|\overline{\delta}(t-\tau_2)\|^2dt + dm(t) \end{aligned}$$

where $m(t) = 2\int_0^t\overline{\delta}^T(s)(I_{N-1}\otimes P)d\widetilde{M}_3(s)$, $b_p = \lambda_N\|PBB^TP\|$, $\sigma(b,p) = \bar{\sigma}^2\frac{N-1}{N}b_p\|BB^TP\|$. Hence, for any $\gamma > 0$, we have $d(e^{\gamma t}V(t)) \le (|k|b_p - 1)e^{\gamma t}\ \|\overline{\delta}(t)\|^2dt + e^{\gamma t}dm(t) + |k|b_pe^{\gamma t}\|\overline{\delta}(t-\tau_1)\|^2dt + \gamma e^{\gamma t}V(t)dt + \sigma(b,p)k^2e^{\gamma t}\|\overline{\delta}(t-\tau_2)\|^2dt$. Taking expectations, we obtain

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(t) &\le V(0) + (|k|b_p - 1)\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s)\|^2ds \\ &\quad + |k|b_p\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s-\tau_1)\|^2ds \\ &\quad + \sigma(b,p)k^2\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s-\tau_2)\|^2ds \\ &\quad + \gamma\int_0^t e^{\gamma s}\mathbb{E}V(s)ds. \end{aligned} \quad (39)$$

Note that $V(s) \le \lambda_{\max}(P)\|\overline{\delta}(s)\|^2$ and for $i = 1,2$, $\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s-\tau_i)\|^2ds \le \tau_ie^{\gamma\tau_i}\sup_{s\in[-\tau_i,0]}\mathbb{E}\|\overline{\delta}(s)\|^2 + e^{\gamma\tau_i}\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s)\|^2ds$. Then from (39), we obtain

$$e^{\gamma t}\mathbb{E}V(t) \le C_0(\gamma) + h(\gamma)\int_0^t e^{\gamma s}\mathbb{E}\|\overline{\delta}(s)\|^2ds \quad (40)$$

where $C_0(\gamma) = V(0) + (|k|b_p\tau_1e^{\gamma\tau_1} + \sigma(b,p)k^2\tau_2e^{\gamma\tau_2})\sup_{s\in[-\tau,0]}\mathbb{E}\|\overline{\delta}(s)\|^2$, $h(\gamma) = |k|b_p - 1 + |k|b_pe^{\gamma\tau_1} + \sigma(b,p)k^2e^{\gamma\tau_2} + \lambda_{\max}(P)\gamma$. Note that $h(0) = 2|k|b_p + \sigma(b,p)k^2 - 1 < 0$, and $h(\gamma_1) \ge |k|b_p > 0$, where γ_1 satisfies $kb_pe^{\gamma_1\tau_1} = 1$. These produce that there exists $\gamma^* > 0$ such that $h(\gamma^*) = 0$. Therefore, we obtain from (40) that $e^{\gamma^*t}\mathbb{E}\|\overline{\delta}(t)\|^2 \le C_0(\gamma^*)$. Note that mean square exponential stability implies almost sure exponential stability under a linear growth condition (see [42]). Then, the desired assertions follow. ■

Proof of Theorem III.3: For the general f_{lji}, we can obtain form the definition of $\overline{\delta}(t)$ that $d\overline{\delta}(t) = [(I_{N-1}\otimes A) - (\Lambda\otimes BK)]\overline{\delta}(t)dt + d\widehat{M}_3(t)$, where $\widehat{M}_3(t) = \sum_{l=1}^d\sum_{i,j=1}^N a_{ij}\int_0^t[Q(i)\otimes BKf_{lji}(x_j(s-\tau_2) - x_i(s-\tau_2))dw_{lji}(s)]$. Note that $\overline{\delta}(t) = [\widetilde{\delta}_2^T(t),\dots,\widetilde{\delta}_N^T(t)]^T$, $d\widetilde{\delta}_i(t) = (A - \lambda_iBK)\widetilde{\delta}_i(t)dt + d\widetilde{M}_{4,i}(t)$, $i = 2,\dots,N$, where $\widetilde{M}_{4,i}(t) = (\eta_{N,i}\otimes I_n)\widehat{M}_3(t)$. In the following, we fix certain i, and show that $\lim_{t\to\infty}\mathbb{E}\|\widetilde{\delta}_i(t)\|^2 > 0$ for $\widetilde{\delta}_i(0)\ne 0$ without the stabilizability condition of (A,B). If (A,B) is not stabilizable, then for any $\kappa\ne 0$ and K, $A - \kappa BK$ is not Hurwitz. Let $\kappa = \lambda_i$ and $A_K := A - \lambda_iBK$. Then, the desired assertion can be proved by the similar skills used in the proof of Lemma II.2. ■

For the 2-D dynamics (including the second-order integrator systems in Section III-B), we denote $x_i(t) = [y_i(t), v_i(t)]^T\in\mathbb{R}^2$. Then, we use another variable transformation. Let $y(t) = [y_1(t),\dots,y_N(t)]^T$ and $v(t) = [v_1(t),\dots,v_N(t)]^T$. Denote $\delta_g(t) = (I_N - J_N)g(t)$, $g = y, v$, and $\delta(t) = [\delta_y^T(t),\delta_v^T(t)]^T$. Let $\delta_g(t) = [\delta_{g1}(t),\dots,\delta_{gN}(t)]^T$, where $\delta_{gi}(t)\in\mathbb{R}$, $i = 1,2,\dots,N$. Let $\delta_q(t) = T_{\mathcal{L}}\widetilde{\delta}_q(t)$ and $\widetilde{\delta}_q(t) = [\widetilde{\delta}_{q1}(t),\dots,\widetilde{\delta}_{qN}(t)]^T$, $T_{\mathcal{L}}$ is defined before Proof of Theorem III.1, then it can be verified that $\widetilde{\delta}_{q1}(t)\equiv 0$, $q = y, v$. Denote $\overline{\delta}_q(t) = [\widetilde{\delta}_{q2}(t),\dots,\widetilde{\delta}_{qN}(t)]^T$ and $\overline{\delta}(t) = [\overline{\delta}_y^T(t),\overline{\delta}_v^T(t)]^T$.

Proof of Theorem III.4: It is sufficient to show that without the condition $\lambda_N > 4\beta\frac{N-1}{N}\underline{\sigma}^2$, the mean square consensus cannot be achieved for any $K = [k_1,k_2]\in\mathbb{R}^2$. We first show that for any given $K = [k_1,k_2]\in\mathbb{R}^2$, the mean square consensus implies $k_1 > 0, k_2\Lambda - \beta I_{N-1} > 0$. Note that $dv(t) = -k_1\mathcal{L}y(t)dt - (k_2\mathcal{L} - \beta I_N)v(t)dt + d\bar{M}(t)$, where $\bar{M}(t) = k_1\sum_{i,j=1}^N a_{ij}\eta_{N,i}\int_0^t\sigma_{ji}(y_j(s) - y_i(s))\ dw_{1ji}(s) + k_2\sum_{i,j=1}^N a_{ij}\eta_{N,i}\int_0^t\sigma_{ji}(v_j(s) - v_i(s))dw_{2ji}(s)$. By the definitions of $\overline{\delta}_y$ and $\overline{\delta}_v$ and $\tau_1 = \tau_2 = 0$, it suffices to examine the mean square behavior of the following second-order SDE:

$$\begin{cases} d\overline{\delta}_y(t) = \overline{\delta}_v(t)dt \\ d\overline{\delta}_v(t) = -k_1\Lambda\overline{\delta}_y(t)dt + d\bar{M}_1(t) \\ \qquad\qquad - (k_2\Lambda - \beta I_{N-1})\overline{\delta}_v(t)dt + d\bar{M}_2(t) \end{cases} \quad (41)$$

where $\bar{M}_1(t) = k_1\sum_{i,j=1}^N a_{ij}\int_0^t\overline{Q}(i)\sigma_{ji}(\delta_{yj}(s) - \delta_{yi}(s))dw_{1ji}(s)$, $\bar{M}_2(t) = k_2\sum_{i,j=1}^N a_{ij}\int_0^t\overline{Q}(i)\,\sigma_{ji}(\delta_{vj}(s) - \delta_{vi}(s))$

$dw_{2ji}(s)$ and $\overline{Q}(i) = \phi^T(I_N - J_N)\eta_{N,i}$. Note that the mean square consensus is equivalent to the mean square stability of (41). However, the fact is that Brownian motions can not contribute positively to the mean square stability of the closed-loop system, that is, the unstable system can not be mean square stabilized by Brownian motions (see [45]). Hence, in order for the mean square stability of (41), the deterministic part must be stable. Hence, in order for the mean square stability of the SDE (41), the matrix $\bar{L} = \begin{pmatrix} 0 & I_{N-1} \\ -k_1\Lambda & \beta I_{N-1} - k_2\Lambda \end{pmatrix}$ must be Hurwitz, which implies $k_1 > 0$ and $L_{22} := k_2\Lambda - \beta I_{N-1} > 0$.

Then, based on $k_1 > 0$ and $L_{22} := k_2\Lambda - \beta I_{N-1} > 0$, we will show that the mean square consensus implies there is a i such that $h_i(k_2) := \beta - k_2\lambda_i + k_2^2\underline{\sigma}^2\frac{N-1}{N}\lambda_i < 0$. In fact, we can choose the energy function as follows:

$$V_1(t) = k_1\overline{\delta}_y^T(t)\Lambda\overline{\delta}_y(t) + \|\overline{\delta}_v(t)\|^2. \tag{42}$$

Applying the Itô formula to $V_1(t)$, we have

$$\begin{aligned} dV_1(t) = & -2\overline{\delta}_v^T(t)L_{22}\overline{\delta}_v(t)dt + d\langle\bar{M}_1\rangle(t) + d\langle\bar{M}_2\rangle(t) \\ & + 2\overline{\delta}_v^T(t)d[\bar{M}_1(t) + \bar{M}_2(t)] \end{aligned} \tag{43}$$

where $\langle\bar{M}_1\rangle(t) = k_1^2\sum_{i,j=1}^N a_{ij}\int_0^t \|b_{1ij}(s)\|^2 ds$ and $\langle\bar{M}_2\rangle(t) = k_2^2\sum_{i,j=1}^N a_{ij}\int_0^t \|b_{2ij}(s)\|^2 ds$, $b_{1ij}(t) = \overline{Q}(i)\sigma_{ji}(\delta_{yj}(t) - \delta_{yi}(t)))$, $b_{2ij}(t) = \overline{Q}(i)\sigma_{ji}(\delta_{vj}(t) - \delta_{vi}(t))$. Noting that $(I_N - J_N)(I_N - J_N) = I_N - J_N$, then we have $\overline{Q}^T(i)\overline{Q}(i) = \eta_{N,i}^T(I_N - J_N)\eta_{N,i} = \frac{N-1}{N}$. By the properties of undirected graphs, we have the sum-of-squares (SOS) property ([7]): $\delta_q^T(t)\mathcal{L}\delta_q(t) = \frac{1}{2}\sum_{i,j=1}^N a_{ij}\|\delta_{qj}(t) - \delta_{qi}(t)\|^2$. Then, we get

$$\begin{aligned} d\langle\bar{M}_1\rangle(t) &\geq k_1^2\underline{\sigma}^2\frac{N-1}{N}\sum_{i,j=1}^N a_{ij}\|\delta_{yj}(t) - \delta_{yi}(t)\|^2 dt \\ &= 2k_1^2\underline{\sigma}^2\frac{N-1}{N}\overline{\delta}_y^T(t)\Lambda\overline{\delta}_y(t)dt \end{aligned} \tag{44}$$

and similarly,

$$d\langle\bar{M}_2\rangle(t) \geq 2k_2^2\underline{\sigma}^2\frac{N-1}{N}\overline{\delta}_v^T(t)\Lambda\overline{\delta}_v(t)dt. \tag{45}$$

Hence, taking expectations on the both sides of (43) and noting that $\langle\bar{M}_1\rangle(t) \geq 0$, we obtain

$$\mathbb{E}V_1(t) \geq V_1(0) + 2\mathbb{E}\int_0^t \overline{\delta}_v^T(s)h(k_2,\Lambda)\overline{\delta}_v(s)ds$$

where $h(k_2,\Lambda) := \beta I_{N-1} - k_2\Lambda + k_2^2\underline{\sigma}^2\frac{N-1}{N}\Lambda$. Hence, if $h_i(k_2) \geq 0$ for all $i = 1,\ldots,N$, then $h(k_2,\Lambda) \geq 0$, and we must have $\liminf_{t\to 0}\mathbb{E}V_1(t) \geq V_1(0) > 0$ for $\overline{\delta}(0) \neq 0$. This is in conflict with the definition of the mean square consensus.

Finally, we show that $h_i(k_2) < 0$ for certain i and k_2 implies $\lambda_N > 4\beta\frac{N-1}{N}\underline{\sigma}^2$. Otherwise, $\lambda_N \leq 4\beta\frac{N-1}{N}\underline{\sigma}^2$, then we must have $h_i(k_2) := \beta - k_2\lambda_i + k_2^2\underline{\sigma}^2\frac{N-1}{N}\lambda_i \geq 0$ for all $i = 2,\ldots,N$ and $k_2 \in \mathbb{R}$, which is a contradiction. ■

Proof of Theorem III.5: We first transform the consensus problem into the stability problem of a SDDE having the form of (1). Then, we use Theorem II.1 to get the stability under the given conditions by choosing a appropriate matrix $P > 0$.

Similar to (41), we have

$$\begin{cases} d\overline{\delta}_y(t) = \overline{\delta}_v(t)dt \\ d\overline{\delta}_v(t) = -k_1\Lambda\overline{\delta}_y(t-\tau_1)dt + d\bar{M}_1(t) \\ \qquad\qquad -k_2\Lambda\overline{\delta}_v(t-\tau_1)dt + d\bar{M}_2(t) \end{cases} \tag{46}$$

where $\bar{M}_1(t) = k_1\sum_{i,j=1}^N a_{ij}\int_0^t \overline{Q}(i)f_{1ji}(\delta_{yj}(s-\tau_2) - \delta_{yi}(s-\tau_2))dw_{1ji}(s)$ and $\bar{M}_2(t) = k_2\sum_{i,j=1}^N a_{ij}\int_0^t \overline{Q}(i) f_{2ji}(\delta_{vj}(s-\tau_2) - \delta_{vi}(s-\tau_2))dw_{2ji}(s)$. Let $L = L_0 + L_1$ with

$$L_0 = \begin{bmatrix} 0 & I_{N-1} \\ 0 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 \\ -k_1\Lambda & -k_2\Lambda \end{bmatrix}.$$

Then, we have the following transformed SDDE:

$$d\overline{\delta}(t) = L_0\overline{\delta}(t)dt + L_1\overline{\delta}(t-\tau_1)dt + d\bar{M}_3(t) + d\bar{M}_4(t) \tag{47}$$

where $\bar{M}_3(t) = k_1\sum_{i,j=1}^N a_{ij}\int_0^t B_{1ij}(s-\tau_2)dw_{1ji}(s)$, $\bar{M}_4(t) = k_2\sum_{i,j=1}^N a_{ij}\int_0^t B_{2ij}(s-\tau_2)dw_{2ji}(s)$, $B_{1ij}(t) = [0, b_{1ij}(t)]^T$, $b_{1ij}(t) = \overline{Q}(i)f_{1ji}(\delta_{yj}(t) - \delta_{yi}(t)))$, $B_{2ij}(t) = [0, b_{2ij}(t)]^T$, $b_{2ij}(t) = \overline{Q}(i)f_{2ji}(\delta_{vj}(t) - \delta_{vi}(t))$. Hence, in the following, we need to prove the mean square and almost sure stability of SDDE (47).

To apply Theorem II.1, we choose

$$P = \begin{bmatrix} \mu\Lambda & \theta I_{N-1} \\ \theta I_{N-1} & I_{N-1} \end{bmatrix} \tag{48}$$

with $\mu,\theta > 0$ to be designed. In fact, we need $\theta^2 < \mu\lambda_2$ to guarantee the positive definiteness of P. Let $\Theta_1(t) = \langle\bar{M}_3, P\bar{M}_3\rangle(t)$, $\Theta_2(t) = \langle\bar{M}_4, P\bar{M}_4\rangle(t)$, and $\bar{M}_{34}(t) = \bar{M}_3(t) + \bar{M}_4(t)$. By the properties of the sum-of-squares (SOS) property ([7]), the similar estimations as (44) and (45), and Assumption III.2, we get

$$\begin{aligned} d\Theta_1(t) &= k_1^2\sum_{i,j=1}^N a_{ij}\|b_{1ij}(t-\tau_2)\|^2 dt \\ &\leq 2k_1^2\bar{\sigma}_1^2\frac{N-1}{N}\overline{\delta}_y^T(t-\tau_2)\Lambda\overline{\delta}_y(t-\tau_2)dt \end{aligned}$$

and similarly, $d\Theta_2(t) \leq 2k_2^2\bar{\sigma}_2^2\frac{N-1}{N}\overline{\delta}_v^T(t-\tau_2)\Lambda\overline{\delta}_v(t-\tau_2)dt$. Hence, $d\langle\bar{M}_{34}, P\bar{M}_{34}\rangle(t) \leq \overline{\delta}^T(t-\tau_2)U\overline{\delta}(t-\tau_2)dt$, where

$$U = 2\begin{bmatrix} k_1^2\bar{\sigma}_1^2\frac{N-1}{N}\Lambda & 0 \\ 0 & k_2^2\bar{\sigma}_2^2\frac{N-1}{N}\Lambda \end{bmatrix}.$$

By Theorem II.1, the mean square and almost sure stability of (47) can be assured by $\bar{S} := L^TP + PL + (L^TPL + L_1^TPL_1)\tau_1 + U < 0$. It is easy to see

$$L^TP + PL = \begin{bmatrix} -2k_1\theta\Lambda & (\mu - k_1 - \theta k_2)\Lambda \\ (\mu - k_1 - \theta k_2)\Lambda & 2(\theta I_{N-1} - k_2\Lambda) \end{bmatrix}$$

and $L^TPL + L_1^TPL_1 =$

$$\begin{bmatrix} 2k_1^2\Lambda^2 & 2k_1k_2\Lambda^2 - k_1\theta\Lambda \\ 2k_1k_2\Lambda^2 - k_1\theta\Lambda & (\mu - k_2\theta)\Lambda - k_2\theta\Lambda + 2k_2^2\Lambda^2 \end{bmatrix}.$$

Let $\mu = k_1 + k_2\theta + k_1\theta\tau_1$. Note that

$$\begin{bmatrix} 0 & 2k_1k_2\Lambda^2 \\ 2k_1k_2\Lambda^2 & 0 \end{bmatrix} \le \begin{bmatrix} 2k_1^2\Lambda^2 & 0 \\ 0 & 2k_2^2\Lambda^2 \end{bmatrix}.$$

Therefore, we have

$$\bar{S} \le \begin{bmatrix} s_{11}(\theta) & 0 \\ 0 & s_{22}(\theta) \end{bmatrix},$$

where $s_{11}(\theta) = -2k_1\theta\Lambda + 4k_1^2\Lambda^2\tau_1 + 2k_1^2\frac{N-1}{N}\bar{\sigma}_1^2\Lambda$, $s_{22}(\theta) = 2(\theta I_{N-1} - k_2\Lambda) + (\mu - k_2\theta)\Lambda\tau_1 - k_2\theta\Lambda\tau_1 + 4k_2^2\Lambda^2\tau_1 + 2k_2^2\bar{\sigma}_2^2\frac{N-1}{N}\Lambda$. Then, we need $s_{11}(\theta) < 0$ and $s_{22}(\theta) < 0$. It is easy to verify that $\theta > \theta_1 := 2\lambda_N k_1\tau_1 + k_1\frac{N-1}{N}\bar{\sigma}_1^2$ implies $s_{11} < 0$, and $\theta < \theta_2 := \frac{2k_2\lambda_2(1-k_2\bar{\sigma}_2^2\frac{N-1}{N})-(k_1\lambda_2+4k_2^2\lambda_N^2)\tau_1}{2+k_1\lambda_N\tau_1^2}$ implies $s_{22}(\theta) < 0$. Note that condition (18) guarantees $\theta_1 < \theta_2$, and then $s_{11}(\theta) < 0$ and $s_{22}(\theta) < 0$ for $\theta \in (\theta_1, \theta_2)$.

We now show that the choice $\mu = k_1 + k_2\theta + k_1\theta\tau_1$ with $\theta \in (\theta_1, \theta_2)$ can still guarantee the matrix P to be positive definite. From $\theta^2 < \mu\lambda_2$ and $\mu = k_1 + \theta k_2 + k_1\theta\tau_1$, it is enough to show that for $\theta \in (\theta_1, \theta_2)$, $\theta^2 - \theta\lambda_2(k_2 + k_1\tau_1) - k_1\lambda_2 < 0$. This can be guaranteed under the condition $0 < \theta < \theta^*$, where

$$\theta^* = \left[\lambda_2(k_2 + k_1\tau_1) + \sqrt{\lambda_2^2(k_2 + k_1\tau_1)^2 + 4k_1\lambda_2}\right]/2. \quad (49)$$

It is easy to see that $\theta_2 < \theta^*$. That is, for any $\theta \in (\theta_1, \theta_2)$, we have $P > 0$ and $\bar{S} < 0$. Hence, from Theorem II.1 and the definition of $\overline{\delta}(t)$, the mean square and almost sure consensus follow. ■

Proof of Theorem III.6: The sufficiency follows directly from Theorem III.5. Note that condition (20) contains three parts: 1) $k_1 > 0, k_2 > 0$; 2) The choice of k_2 should satisfy $1 - k_2\underline{\sigma}_2^2\frac{N-1}{N} > 0$; 3) Based on the choice of k_2, k_1 must obey $k_1\underline{\sigma}_1^2\frac{N-1}{N} < k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}$. The proof will be given according to the three cases.

We first show that the necessity of $k_1 > 0, k_2 > 0$ for the mean square consensus. Similarly to (41), in order for the mean square stability of (47), the matrix L must be Hurwitz, which implies $k_1 > 0$ and $k_2 > 0$. Then, we examine the necessity of $1 - k_2\underline{\sigma}_2^2\frac{N-1}{N} > 0$. By the definitions of $\overline{\delta}_y(t)$ and $\overline{\delta}_v(t)$ given by (46) and the Itô formula to $V_1(t)$ given by (42), we have

$$\begin{aligned} dV_1(t) = & -2k_2\overline{\delta}_v^T(t)\Lambda\overline{\delta}_v(t)dt + d\langle\bar{M}_1\rangle(t) + d\langle\bar{M}_2\rangle(t) \\ & + 2\overline{\delta}_v^T(t)d[\bar{M}_1(t) + \bar{M}_2(t)] \end{aligned} \quad (50)$$

where $\langle\bar{M}_1\rangle(t) = k_1^2\sum_{i,j=1}^N a_{ij}\int_0^t \|b_{1ij}(s)\|^2 ds$ and $\langle\bar{M}_2\rangle(t) = k_2^2\sum_{i,j=1}^N a_{ij}\int_0^t \|b_{2ij}(s)\|^2 ds$. Using the similar estimations of (44) and (45) and Assumption III.3, we can obtain

$$d\langle\bar{M}_1\rangle(t) \ge 2k_1^2\underline{\sigma}_1^2\frac{N-1}{N}\overline{\delta}_y^T(t)\Lambda\overline{\delta}_y(t)dt \quad (51)$$

and

$$d\langle\bar{M}_2\rangle(t) \ge 2k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\overline{\delta}_v^T(t)\Lambda\overline{\delta}_v(t)dt. \quad (52)$$

Hence, integrating and taking expectations on the both sides of (50) yield

$$\mathbb{E}V_1(t) \ge V_1(0) - 2k_2\left(1 - k_2\underline{\sigma}_2^2\frac{N-1}{N}\right)\mathbb{E}\int_0^t \overline{\delta}_v^T(s)\Lambda\overline{\delta}_v(s)ds$$

since $\langle\bar{M}_1\rangle(t) \ge 0$ and $\langle\bar{M}_2\rangle(t) \ge 0$. Therefore, if $1 - k_2\underline{\sigma}_2^2\frac{N-1}{N} \le 0$, we must have $\liminf_{t\to\infty}\mathbb{E}V_1(t) \ge V_1(0) > 0$ for $\overline{\delta}(0) \ne 0$. This together with the definitions of $\overline{\delta}(t)$ and the mean square consensus gives the necessity of the condition $1 - k_2\underline{\sigma}_2^2\frac{N-1}{N} > 0$.

Finally, we prove the necessity of $k_1\underline{\sigma}_1^2\frac{N-1}{N} < k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}$. We choose the Lyapunov function

$$V_2(t) = \overline{\delta}^T(t)P\overline{\delta}(t) \quad (53)$$

where P is defined by (48). Applying the Itô formula to $V_2(t)$ with $\overline{\delta}(t)$ being defined by (47), and combining (51) and (52), we can get

$$\begin{aligned} \mathbb{E}V_2(t) \ge & V_2(0) + 2h_1(\mu,\theta)\mathbb{E}\int_0^t \overline{\delta}_y^T(s)\Lambda\overline{\delta}_v(s)ds \\ & + 2\mathbb{E}[\int_0^t \overline{\delta}_y^T(s)\bar{H}_2(\theta)\overline{\delta}_y(s)ds + \int_0^t \overline{\delta}_v^T(s)\bar{H}_3(\theta)\overline{\delta}_v(s)ds] \end{aligned} \quad (54)$$

where $h_1(\mu,\theta) = \mu - k_1 - \theta k_2$, $\bar{H}_2(\theta) = k_1^2\underline{\sigma}_1^2\frac{N-1}{N}\Lambda - k_1\theta\Lambda$ and $\bar{H}_3(\theta) = \theta I_{N-1} - k_2\Lambda + k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\Lambda$. Note that $k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\lambda_N > 0$, which is proved above. Let $\bar{\theta}_1 := k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\lambda_N (> 0)$ and $\bar{\theta}_2 := k_1\underline{\sigma}_1^2\frac{N-1}{N} \wedge \theta^*$, where θ^* is defined in (49) with $\tau_1 = 0$. It is easy to see that $\theta^* > \bar{\theta}_1$. If $k_1\underline{\sigma}_1^2\frac{N-1}{N} < k_2\lambda_N - k_2^2\underline{\sigma}_2^2\frac{N-1}{N}$ fails, we would have that for any $\theta_0 \in [\bar{\theta}_1, \bar{\theta}_2]$, $\bar{H}_2(\theta_0) \ge 0$ and $\bar{H}_3(\theta_0) \ge 0$. Let $\mu = k_1 + k_2\theta_0$, which together with $\theta_0 < \theta^*$ gives $P > 0$. Then, it is easy to deduce from (54) that $\liminf_{t\to\infty}\mathbb{E}V_2(t) \ge V_2(0) > 0$ for $\delta(0) \ne 0$, which is in conflict with the definition of the mean square consensus. That is, condition (20) is necessary and the proof is completed. ■

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