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SOCIAL OPTIMA IN MEAN FIELD LINEAR-QUADRATIC-GAUSSIAN MODELS WITH MARKOV JUMP PARAMETERS*

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Abstract. This paper investigates social optima of mean field linear-quadratic-Gaussian (LQG) control models with Markov jump parameters. The common objective of the agents is to minimize a social cost—the cost average of the whole society. In the cost functions there are coupled mean field terms. First, we consider the centralized case and get a parameterized equation of mean field effect. Then, we design a set of distributed strategies by solving a limiting optimal control problem in an augmented state space subject to the consistency requirement for mean field approximation. It is shown that the set of distributed strategies is asymptotically team-optimal, and the asymptotically optimal social cost value can be obtained explicitly. The optimal social average cost is compared with the optimal individual cost in mean field games by virtue of the explicit expressions, and the difference is further illustrated by a numerical example.

Key words. mean field model, team decision problem, social optimum, LQG control, Markov jump parameter

AMS subject classifications. 91B06, 93E20, 68M14, 91B70

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1. Introduction.

In recent years, mean field games and control have drawn a lot of attention from the control community [11]. Mean field models have broad backgrounds in many areas including economics, finance, communication engineering, biology, and medicine [12, 9, 7, 3]. Such models have been investigated by researchers from a variety of perspectives [16, 21, 42, 44, 37]. In mean field models, each agent is affected by the average interaction of all the other agents, while the individual influence of each agent is negligible. From the relationship between population macroscopic behavior and individual behavior, one can get that the population aggregate (mean field) effect satisfies a fixed-point equation. Then by tackling the fixed-point equation and the single-agent optimal control problem, decentralized asymptotic Nash equilibria are obtained [17, 22, 37]. For more literature, readers are referred to [22] for mean field games with stochastic time-averaged costs, [15, 28, 38] for mean field models with a major player, [40] for mean field Stackelberg games, [14] for mean field games with model uncertainty, [36] for application to dynamic production output adjustment with sticky prices, and [43, 41, 4, 6] for various control examples of multiagent systems.

In practical financial markets, ecological systems, and social systems, the ambient environment is constantly changing. For instance, the change rates of prices in a financial market may be very different for different time slots. A powerful tool

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depicting abrupt environmental changes is the Markov jump model [25, 8]. Wang and Zhang [37, 38, 39] investigated mean field games for multiagent systems with Markov jump parameters and gave distributed asymptotical Nash equilibrium strategies.

This paper investigates social optima in mean field linear-quadratic-Gaussian (LQG) control with Markov jump parameters, where the common objective is to minimize the cost average of all the agents. If we take all the agents as a society, then the cost average is a social cost. This problem is usually regarded as a type of team decision problem which has a long research history [26, 30, 13, 33, 34] but most of the works focus on static decisions. In team decision problems, all the agents cooperate to achieve a common objective, while different agents have different measurements or information structures [13]. A team-optimal strategy is necessarily person-by-person optimal; under some convexity conditions, the person-by-person optimal strategy is also team-optimal [1]. Tembine et al. [32] considered the asymptotic behavior for mean field Markov team decision problems and focused on seeking the stationary policy which belongs to a static optimization problem. Huang, Caines, and Malhame [18] investigated social optima in mean field LQG control problems and gave centralized and decentralized team-optimal solutions. Compared with [32, 18], we will study a kind of dynamic team decision problems with Markov jump parameters, which are closely related to practical backgrounds [25, 8].

Different from previous works [16, 17, 18, 37, 38, 39], dynamics and costs of all the agents in this paper are driven by a common continuous-time Markov chain, which is a form of common noise [5]. In the previous works, the parameters of agents are constants (random variables) or a sequence of independent Markov chains and the mean field effect is a deterministic function which can be calculated by tackling a fixed-point equation. In contrast, the mean field effect in our model is a stochastic process depending on the Markov jump parameter due to the impact of the common random parameter. Thus, the above approach cannot be entirely applied to handle the problem in this paper. Also, the socially optimal problem involving optimization of the average of individual costs has very high computational complexity. We achieve the control synthesis by the parametric approach and the state space augmentation. By analyzing the centralized strategy, we get a parameterized equation of mean field effect and a transformation of the original social optimum problem. From these we construct an optimal control problem with the indefinite state weight in the augmented state space. By solving this problem with the consistency requirement for mean field approximation, we obtain a set of consistency equations, from which a set of distributed strategies is designed. By constructing stochastic Lyapunov functions, we show that the closed-loop system is uniformly stable, and the set of distributed strategies is asymptotically team-optimal. Furthermore, we provide an explicit expression of the asymptotically optimal social cost by ergodic limiting theory of Markov chains. Meanwhile, we get an explicit expression of the optimal individual cost in mean field games with the help of the results in our previous works [35, 37]. By comparing these two expressions, one can obtain the difference of the optimal average costs within team and game formulations, which shows the efficiency of cooperation. This is further illustrated by a numerical example.

The organization of the paper is as follows. Section 2 formulates the socially optimal team problem of mean field LQG models with Markov jump parameters. Section 3 provides the form of mean field effect and a transformation of the social optimum problem. Section 4 presents a set of distributed strategies by solving a limiting optimal control problem subject to the consistency requirement for mean field approximation. Section 5 shows the asymptotic optimality of distributed strategies.

Section 6 gives the asymptotically optimal social cost and the optimal individual cost in mean field games. Section 7 provides a numerical example to verify the result. Section 8 concludes the paper.

The following notation will be used in the paper. $\|\cdot\|$ denotes the Euclidean vector norm or matrix norm induced by the Euclidean vector norm; I_n denotes an n-dimensional identity matrix. For a given matrix A, A^T denotes its transpose, and $A>0$ means that A is positive definite. For any vector x with proper dimensions and symmetric matrix $Q\geq 0$, $\|x\|_Q=(x^TQx)^{1/2}$. For an n-dimensional matrix P, $\lambda_{min}(P)$ denotes the minimum eigenvalue of P, and $\lambda_{max}(P)$ denotes maximum eigenvalue of P. I_B denotes the indicator function of set B. $C_b([0,\infty),\mathbb{R}^n)$ denotes the class of n-dimensional bounded continuous functions in $[0,\infty)$.

2. Problem formulation. Consider a multiagent system of the form

$$\begin{aligned} dx_i(t) &= A_{\theta(t)}x_i(t)dt + B_{\theta(t)}u_i(t)dt + h(t)dt \\ &\quad + D_{\theta(t)}dW_i(t),\ 1\leq i\leq N, \end{aligned} \tag{2.1}$$

where $x_i\in\mathbb{R}^n$ and $u_i\in\mathbb{R}^r$ are the state and input of the ith agent, and $\{W_i(t),1\leq i\leq N\}$ is a family of independent d-dimensional Brownian motions. The underlying filtered probability space is $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$, where $(\mathcal{F}_t)_{t\geq 0}$ is a collection of nondecreasing σ-algebras. $h\in C_b([0,\infty),\mathbb{R}^n)$ is a deterministic external disturbance, reflecting the impact of the environment. $\{\theta(t)\}$ is a continuous-time Markov chain taking value in $S=\{1,2,\dots,m\}$ with transition rate matrix (infinitesimal generator) $\Lambda=\{\lambda_{ij},i,j=1,\dots,m\}$. The cost of the ith agent is

$$J_i(u)=\limsup_{T\to\infty}\frac{1}{T}E\int_0^T\Big\{\big\|x_i(t)-\Phi[x^{(N)}(t)]\big\|^2_{Q_{\theta(t)}}+\|u_i(t)\|^2_{R_{\theta(t)}}\Big\}dt, \tag{2.2}$$

where $Q_j\geq 0$ and $R_j>0$, $j=1,\dots,m$. $\Phi: x\mapsto Hx+\alpha$, $H\in\mathbb{R}^{n\times n},\alpha\in\mathbb{R}^n$. $x^{(N)}(t)=\frac{1}{N}\sum_{j=1}^N x_j(t)$, and $u=\{u_1,\dots,u_i,\dots,u_N\}$.

The main goal is to seek a set of distributed strategies that optimizes the social (average) cost $J^{(N)}(u)$ for the system (2.1)–(2.2), where

$$J^{(N)}(u)=\frac{1}{N}\sum_{i=1}^N J_i(u).$$

Specifically, each agent makes decisions to minimize the social cost $J^{(N)}$ over the distributed strategy set

$$\begin{aligned}\mathcal{U}_{d,i}=\Big\{&u_i\mid u_i(t)\text{ is adapted to }\sigma(x_i(0),W_i(s),\theta(s),s\leq t)\\ &\text{and }\limsup_{T\to\infty}\frac{1}{T}E\int_0^T\|u_i(t)\|^2dt<\infty\Big\},\ i=1,\dots,N.\end{aligned}$$

This is a type of team decision problem with the common objective $J^{(N)}$. As a contrast, we introduce the game problem: agent i makes decisions to minimize the individual cost J_i over the distributed strategy set $\mathcal{U}_{d,i}, i=1,\dots,N$.

Remark 2.1. The social optimum is a kind of team problem where all the agents cooperate to minimize the cost average of the society. On the contrary, the (noncooperative) game involves competitive agents; each agent is only concerned with its own cost.

For the convenience of reference, we list some assumptions as follows:

(A1) $x_i(0), 1 \leq i \leq N$, are independent random variables; $\{x_i(0)\}$, $\{W_i(t), 1 \leq i \leq N\}$ and $\{\theta(t)\}$ are independent of each other. $Ex_i(0) = x_0, 1 \leq i \leq N$; $\max_{1\leq i\leq N} E\|x_i(0)\|^2 < \infty$.

(A2) The Markov jump system

$$dx(t) = A_{\theta(t)}x(t)dt + B_{\theta(t)}u(t)dt \tag{2.3}$$

is mean square stabilizable.[1]

3. Mean field approximation and transformation of socially optimal problems. In this section, we first get asymptotic behavior of population state average $x^{(N)}$ from heuristic derivation. Next, we provide a transformation of the original social optimum problem by analyzing an open-loop centralized strategy.

3.1. Heuristic derivation of mean field effect. To design the distributed strategy, the key step is to analyze the asymptotic behavior of population state average $x^{(N)}$. In the previous works on mean field models with time-invariant parameters, $x^{(N)}$ in the closed-loop system converges to a deterministic function, which is called the mean field effect. In contrast, for the model (2.1)–(2.2), the mean field effect (population aggregate behavior) is stochastic due to the impact of the random parameter. Specifically, $x^{(N)}(t)$ collapses into a stochastic process depending on the jump parameter $\theta(s), s \leq t$, as the number of agents grows to infinity.

We now get the form of the mean field effect by heuristic examination of $x^{(N)}$ under centralized strategies. If each agent can get the state information of all agents, then we can take the problem (2.1)–(2.2) as the standard Markov jump LQG optimal control problem. By solving a set of coupled dynamic programming equations, one can obtain a centralized feedback control for the each agent [25, 2]:

$$u_i(t) = \tilde{P}_{\theta(t)}x_i(t) + \sum_{j\neq i} \bar{P}_{\theta(t)}x_j(t) + \bar{s}_{\theta(t)}(t),$$

where $1 \leq i \leq N$, $\tilde{P}(\cdot), \bar{P}(\cdot) \in \mathbb{R}^{n\times n}$, and $\bar{s}(\cdot) \in \mathbb{R}^n$. In the control strategy above, we suppose that the coefficient matrix of each state only depends on $\theta(t)$. The main reason is as follows: (i) for the single-agent Markov jump optimal control problem, the control gain is only dependent on $\theta(t)$ [25]; (ii) as $\theta(t) \equiv j_0$, the original problem is reduced into the standard LQG optimal control problem, and in this case the control gain is a constant matrix [2]. Since there is a time-varying term $h(t)$ in the state equation (2.1), $\bar{s}$ is dependent not only on $\theta(t)$ but also on t explicitly. We can get $\bar{s}$ by solving a set of coupled differential equations. Substituting the above strategy into the state equation (2.1) leads to the closed-loop equation:

$$\begin{aligned} dx_i(t) = (A_{\theta(t)} + B_{\theta(t)}\tilde{P}_{\theta(t)})x_i(t)dt + B_{\theta(t)} \sum_{j\neq i} \bar{P}_{\theta(t)}x_j(t)dt + B_{\theta(t)}\bar{s}_{\theta(t)}(t) \\ +h(t)dt + D_{\theta(t)}dW_i(t), \quad 1 \leq i \leq N. \end{aligned}$$

[1] Assumption (A2) holds if and only if for any given positive definite matrices $N_1, \ldots, N_m$, there exists a unique set of positive definite solutions $\{P_j, j = 1, \ldots, m\}$ to the following coupled Riccati equations: $A_j^T P_j + P_j A_j - P_j B_j B_j^T P_j + \sum_{k=1}^m \lambda_{jk} P_k = -N_j, \quad j = 1, \ldots, m$. See [25, 19] for more conditions that ensure mean square stabilizability of the system (2.3).

Summing up the above equation from $i = 1$ to N, and dividing by N, we get

$$dx^{(N)}(t) = [A_{\theta(t)} + B_{\theta(t)}(\tilde{P}_{\theta(t)} + (N-1)\bar{P}_{\theta(t)})]x^{(N)}(t)dt + B_{\theta(t)}\bar{s}_{\theta(t)}(t) + h(t)dt + d\left[\frac{1}{N}\sum_{j=1}^{N} D_{\theta(t)}W_j(t)\right]. \tag{3.1}$$

Owing to the coupling coefficient between two agents being $1/N$, we can roughly assume that the magnitude of $\bar{P}_{\theta(t)}$ is $1/N$, and $(N-1)\bar{P}_{\theta(t)}$ converges to a matrix-valued function as $N \to \infty$. Notice that

$$\frac{1}{N}\sum_{j=1}^{N} D_{\theta(t)}W_j(t) \to 0 \text{ a.s.}$$

Thus, letting $N \to \infty$, from (3.1) the mean field effect satisfies

$$dz(t) = M_{\theta(t)}z(t)dt + s_{\theta(t)}(t)dt,\ z(0) = x_0, \tag{3.2}$$

where $M(\cdot) \in \mathbb{R}^{n\times n}$ and $s(\cdot) \in \mathbb{R}^n$ are parameters to be determined. It is worth pointing out that the above form of the mean field effect is obtained only by heuristic arguments and the relevant hypotheses need to be verified further.

3.2. Transformation of socially optimal problems. Let

$$\mathcal{U}_c = \big\{u_i | u_i(t) \text{ is adapted to } \sigma\{x_i(0), W_i(s), \theta(s), s \le t, i = 1,\dots,N\}\big\}$$

be the centralized strategy set. Then, a so-called team-optimal solution is referred to a set of strategies u^* satisfying $J^{(N)}(u^*) \le J^{(N)}(u)$ for any u. Notice that all the agents share a common social cost. Seeking the team-optimal centralized strategy is equivalent to solving a multivariate optimal control problem.

As we know, the team-optimal strategy is necessarily person-by-person optimal [1]. We provide a transformation of problem (2.1)–(2.2) by using an argument similar to Lemma 3.5 in [18].

LEMMA 3.1. *If* $\hat{u} = \{\hat{u}_i, i = 1,\dots,N | \hat{u}_i \in \mathcal{U}_c\}$ *is the team-optimal strategy of problem* (2.1)–(2.2)*, then* $\hat{u}_i$ *is necessarily an optimal strategy of the following control problem:*

$$\text{(P0)}\quad dx_i(t) = A_{\theta(t)}x_i(t)dt + B_{\theta(t)}u_i(t)dt + h(t)dt + D_{\theta(t)}dW_i(t),\ 1 \le i \le N, J_i^0(u_i) = \limsup_{T\to\infty} \frac{1}{T}E\int_0^T L(x_i(t), \hat{x}_{-i}^{(N)}(t), u_i(t))dt,$$

where $J_i^0(u_i)$ *is the cost to be minimized over* $u_i \in \mathcal{U}_c$, $\hat{x}_{-i}^{(N)} \triangleq \frac{1}{N}\sum_{j\neq i}\hat{x}_j$, *and*

$$\begin{aligned} L(x_i, \hat{x}_{-i}^{(N)}, u_i) = {}& x_i^T\left[\Big(I - \frac{H}{N}\Big)^T Q_\theta \Big(I - \frac{H}{N}\Big) + \frac{(N-1)H^TQ_\theta H}{N^2}\right]x_i \\ & - 2(H\hat{x}_{-i}^{(N)} + \alpha)^T Q_\theta \left(I - \frac{H}{N}\right)x_i \\ & - 2\left\{\left[I - \left(1 - \frac{1}{N}\right)H\right]\hat{x}_{-i}^{(N)} - \left(1 - \frac{1}{N}\right)\alpha\right\}^T Q_\theta H x_i + \|u_i\|_{R_\theta}^2. \end{aligned} \tag{3.3}$$

The above lemma shows that the terms in $J^{(N)}$ affected by u_i appear only in (3.3) when $\hat{u}_{-i} = (\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots, \hat{u}_N)$ is given. By analyzing (3.3) we obtain the cost for agent i only depends on x_i, u_i and $\hat{x}_{-i}^{(N)}$, where $\hat{x}_{-i}^{(N)}$ does not change with individual strategy u_i. Thus, to get the solution of the social optimum problem within population limit by mean field approximation, we can first solve an optimal tracking problem with an unknown exogenous signal. Lemma 3.1 acts on guidance of constructing distributed control strategies.

4. Distributed strategy design. From the previous heuristic derivation, as $N \to \infty$, $x^{(N)}(t)$ may be approximated by a stochastic process $z(t)$, which depends on $\theta(t)$ and satisfies (3.2). By Lemma 3.1, we construct the following auxiliary optimal control problem:

$$dx_i(t) = A_{\theta(t)} x_i(t)dt + B_{\theta(t)} u_i(t)dt + h(t)dt + D_{\theta(t)} dW_i(t),\ 1 \le i \le N,$$

$$J_i^1(u_i) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T L^1(x_i(t), z(t), u_i(t))dt,$$

where

$$L^1(x_i, z, u_i) = x_i^T Q_\theta x_i - 2(Hz+\alpha)^T Q_\theta x_i - 2\left[(I_n - H)z - \alpha\right]^T Q_\theta H x_i + \|u_i\|^2_{R_\theta},$$

$$dz(t) = M_{\theta(t)} z(t)dt + s_{\theta(t)}(t)dt. \tag{4.1}$$

Taking (x_i, z) as a $2n$-dimensional state, the above problem can be rewritten as follows:

$$\text{(P1)}\quad d\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A_{\theta(t)} & 0 \\ 0 & M_{\theta(t)} \end{pmatrix}\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} dt + \begin{pmatrix} B_{\theta(t)} \\ 0 \end{pmatrix} u_i(t)dt$$

$$+ \begin{pmatrix} h(t) \\ s_{\theta(t)}(t) \end{pmatrix} dt + \begin{pmatrix} D_{\theta(t)} \\ 0 \end{pmatrix} dW_i(t),\ 1 \le i \le N, \tag{4.2}$$

$$J_i^1(u_i) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \Bigg\{ (x_i(t)\ z(t)) \begin{pmatrix} Q_{\theta(t)} & H^Q_{\theta(t)} \\ H^Q_{\theta(t)} & 0 \end{pmatrix}\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix}$$

$$- \left[2\alpha^T Q_{\theta(t)}(I_n - H), 0\right]\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} + \|u_i(t)\|^2_{R_{\theta(t)}} \Bigg\} dt, \tag{4.3}$$

where

$$H_j^Q \triangleq H^T Q_j H - H^T Q_j - Q_j H,\ j = 1, \ldots, m.$$

Let

$$\bar{A}_j = \begin{pmatrix} A_j & 0 \\ 0 & M_j \end{pmatrix}, \qquad \bar{B}_j = \begin{pmatrix} B_j \\ 0 \end{pmatrix}, \qquad \bar{D}_j = \begin{pmatrix} D_j \\ 0 \end{pmatrix}.$$

We introduce the coupled algebra Riccati equations

$$\bar{A}_j^T K_j + K_j \bar{A}_j + \sum_{l=1}^m \lambda_{jl} K_l - K_j \bar{B}_j R_j^{-1} \bar{B}_j^T K_j + \begin{pmatrix} Q_j & H_j^Q \\ H_j^Q & 0 \end{pmatrix} = 0 \tag{4.4}$$

and the coupled ordinary differential equations

$$\frac{dr_j(t)}{dt} + G_j^T r_j(t) + \sum_{l=1}^m \lambda_{jl} r_l(t) + K_j \begin{pmatrix} h(t) \\ s_j(t) \end{pmatrix} - \left[\alpha^T Q_j (I_n - H), 0\right]^T = 0, \tag{4.5}$$

where

$$G_j \triangleq \bar{A}_j - \bar{B}_j R_j^{-1} \bar{B}_j^T K_j, \ j = 1, \ldots, m.$$

If the corresponding conditions in Lemma A.1 are satisfied, then the optimal control law of problem (P1) is given by

$$u_i(t) = -R_{\theta(t)}^{-1} \bar{B}_{\theta(t)}^T [K_{\theta(t)} (x_i^T(t), z^T(t))^T + r_{\theta(t)}(t)]. \tag{4.6}$$

Substituting the control law (4.6) into the dynamic equation (4.2), we get

$$\begin{aligned} d \begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} &= G_{\theta(t)} \begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} dt + \bar{B}_{\theta(t)} R_{\theta(t)}^{-1} \bar{B}_{\theta(t)}^T r_{\theta(t)}(t) dt \\ &+ \begin{pmatrix} h(t) \\ s_{\theta(t)}(t) \end{pmatrix} dt + \begin{pmatrix} D_{\theta(t)} \\ 0 \end{pmatrix} dW_i(t). \end{aligned} \tag{4.7}$$

Let

$$G_j = \begin{pmatrix} G_{j,1} & G_{j,2} \\ G_{j,3} & G_{j,4} \end{pmatrix}, \qquad j = 1, \ldots, m.$$

Then, from (4.7) it follows that

$$\begin{aligned} dx_i(t) &= G_{\theta(t),1} x_i(t) dt + G_{\theta(t),2} z(t) dt \\ &\quad - B_{\theta(t)} R_{\theta(t)}^{-1} \bar{B}_{\theta(t)}^T r_{\theta(t)}(t) dt + h(t) dt + D_{\theta(t)} dW_i(t). \end{aligned}$$

Summing up the above equation from $i = 1$ to N, and then, dividing by N, we have

$$\begin{aligned} dx^{(N)}(t) &= G_{\theta(t),1} x^{(N)}(t) dt + G_{\theta(t),2} z(t) dt - B_{\theta(t)} R_{\theta(t)}^{-1} \bar{B}_{\theta(t)}^T r_{\theta(t)}(t) dt \\ &\quad + h(t) dt + \frac{1}{N} \sum_{i=1}^N D_{\theta(t)} dW_i(t). \end{aligned} \tag{4.8}$$

From the law of large numbers, it follows that

$$E \Big\| \frac{1}{N} \sum_{j=1}^N D_{\theta(t)} W_j(t) \Big\|^2 \to 0.$$

By the mean field methodology [17, 37], (4.8) in the infinite population limit should be consistent with (3.2) in the sense that as $N \to \infty$, $x^{(N)}$ in (4.8) can be approximated by z in (3.2). Thus, to proceed with the design of distributed strategies, we impose the following consistency condition:

$$G_{\theta(t),1} + G_{\theta(t),2} = M_{\theta(t)}, \tag{4.9}$$

$$h(t) - B_{\theta(t)} R_{\theta(t)}^{-1} \bar{B}_{\theta(t)}^T r_{\theta(t)}(t) = s_{\theta(t)}(t). \tag{4.10}$$

Combining (4.4), (4.5), (4.9), and (4.10) leads to the system of consistency equations:

$$\begin{cases} \bar{A}_j^T K_j + K_j \bar{A}_j + \sum_{l=1}^m \lambda_{jl} K_l - K_j \bar{B}_j R_j^{-1} \bar{B}_j^T K_j + \begin{pmatrix} Q_j & H_j^Q \\ H_j^Q & 0 \end{pmatrix} = 0, \\ G_{j,1} + G_{j,2} = M_j, \\ \frac{dr_j(t)}{dt} + G_j^T r_j(t) + \sum_{l=1}^m \lambda_{jl} r_l(t) + K_j \begin{pmatrix} h(t) \\ s_j(t) \end{pmatrix} - \begin{pmatrix} (I_n - H^T) Q_j \alpha \\ 0 \end{pmatrix} = 0, \\ h(t) - B_j R_j^{-1} \bar{B}_j^T r_j(t) = s_j(t), \quad j = 1, \ldots, m. \end{cases} \tag{4.11}$$

DEFINITION 4.1. $(K_j, M_j, r_j(t), s_j(t), j = 1, \ldots, m)$ *is called a stabilizing solution to* (4.11) *if* $(K_j, M_j, r_j(t), s_j(t), j = 1, \ldots, m)$ *satisfies* (4.11)*, and for any* $1 \leq j \leq m, r_j(t) \in C_b([0, \infty), \mathbb{R}^n)$*, the system*

$$dy(t) = G_{\theta(t)} y(t) dt \tag{4.12}$$

is mean-square stable.

Remark 4.1. If for any initial value y_0 we have $E\|y(t)\|^2 \to 0 (t \to \infty)$, then the system (4.12) is mean-square stable. By [19, 10], the system (4.12) is mean-square stable if and only if for any positive definite matrix $N_j, j = 1, \ldots, m$, the equation

$$P_j G_j + G_j^T P_j + \sum_{k=1}^{m} \lambda_{jk} P_k = -N_j$$

admits a unique set of positive definite solutions $P_j, j = 1, \ldots, m$.

We now examine under what conditions there exists a stabilizing solution to (4.11). Note

$$\begin{pmatrix} I_n & 0 \end{pmatrix} G_j \begin{pmatrix} I_n \\ I_n \end{pmatrix} = G_{j,1} + G_{j,2}$$

and

$$\begin{pmatrix} I_n & 0 \end{pmatrix} \bar{A}_j \begin{pmatrix} I_n \\ I_n \end{pmatrix} = A_j.$$

Equation (4.9) can be equivalently transformed into

$$A_j - \begin{pmatrix} I_n & 0 \end{pmatrix} \bar{B}_j R^{-1} \bar{B}_j^T K_j \begin{pmatrix} I_n \\ I_n \end{pmatrix} = M_j, \ j = 1, \ldots, m. \tag{4.13}$$

Let

$$K_j = \begin{pmatrix} K_{j,1} & K_{j,2} \\ K_{j,3} & K_{j,4} \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} I_n & 0 \end{pmatrix} \bar{B}_j R_j^{-1} \bar{B}_j^T K_j \begin{pmatrix} I_n \\ I_n \end{pmatrix} = B_j R_j^{-1} B_j^T (K_{j,1} + K_{j,2}).$$

This together with (4.13) implies that (4.9) is equivalent to

$$M_j = A_j - B_j R_j^{-1} B_j^T (K_{j,1} + K_{j,2}). \tag{4.14}$$

By partitioning the matrices $\bar{A}_j$ and K_j, (4.4) can be transformed into

$$\begin{aligned} &\begin{pmatrix} A_j^T K_{j,1} & A_j^T K_{j,2} \\ M_j^T K_{j,3} & M_j^T K_{j,4} \end{pmatrix} + \begin{pmatrix} K_{j,1} A_j & K_{j,2} M_j \\ K_{j,3} A_j & K_{j,4} M_j \end{pmatrix} \\ &+ \begin{pmatrix} \sum_{l=1}^m \lambda_{jl} K_{l,1} & \sum_{l=1}^m \lambda_{jl} K_{l,2} \\ \sum_{l=1}^m \lambda_{jl} K_{l,3} & \sum_{l=1}^m \lambda_{jl} K_{l,4} \end{pmatrix} + \begin{pmatrix} Q_j & H_j^Q \\ H_j^Q & 0 \end{pmatrix} \\ &- \begin{pmatrix} K_{j,1} B_j R_j^{-1} B_j^T K_{j,1} & K_{j,1} B_j R_j^{-1} B_j^T K_{j,2} \\ K_{j,3} B_j R_j^{-1} B_j^T K_{j,1} & K_{j,3} B_j R_j^{-1} B_j^T K_{j,2} \end{pmatrix} = 0. \end{aligned}$$

Substituting (4.14) into the above equation yields the following four matrix equations:

$$A_j^T K_{j,1} + K_{j,1} A_j + \sum_{l=1}^{m} \lambda_{jl} K_{l,1} - K_{j,1} B_j R_j^{-1} B_j^T K_{j,1} + Q_j = 0; \tag{4.15}$$

$$\begin{aligned} &A_j^T K_{j,2} + K_{j,2}\left[A_j - B_j R_j^{-1} B_j^T \left(K_{j,1} + K_{j,2}\right)\right] + \sum_{l=1}^{m} \lambda_{jl} K_{l,2} \\ &-K_{j,1} B_j R_j^{-1} B_j^T K_{j,2} + H_j^Q = 0; \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\left[A_j - B_j R_j^{-1} B_j^T \left(K_{j,1} + K_{j,2}\right)\right]^T K_{j,3} + K_{j,3} A_j + \sum_{l=1}^{m} \lambda_{jl} K_{l,3} \\ &-K_{j,3} B_j R_j^{-1} B_j^T K_{j,1} + H_j^Q = 0; \end{aligned} \tag{4.17}$$

$$M_j^T K_{j,4} + K_{j,4} M_j + \sum_{l=1}^{m} \lambda_{jl} K_{l,4} - K_{j,3} B_j R_j^{-1} B_j^T K_{j,2} = 0. \tag{4.18}$$

In what follows, we will analyze the existence of solutions to (4.15)–(4.18). From assumption (A2) and [19, Theorem 5], (4.15) has a unique positive definite solution and the system

$$dy(t) = (A_{\theta(t)} - B_{\theta(t)} R_{\theta(t)}^{-1} B_{\theta(t)}^T K_{\theta(t),1}) y(t) dt \tag{4.19}$$

is mean-square stable. By direct computations, (4.16) can be equivalently transformed to

$$\begin{aligned} &(A_j - B_j R_j^{-1} B_j^T K_{j,1})^T K_{j,2} + K_{j,2}(A_j - B_j R_j^{-1} B_j^T K_{j,1}) + \sum_{l=1}^{m} \lambda_{jl} K_{l,2} \\ &- K_{j,2} B_j R_j^{-1} B_j^T K_{j,2} + H_j^Q = 0, \quad j = 1, \ldots, m. \end{aligned} \tag{4.20}$$

Since the system (4.19) is mean-square stable, then it is also mean-square stabilizable. If $H_j^Q, j = 1, \ldots, m$ are positive definite, then by Remark 4.1, (4.20) has a unique set of positive definite solutions $K_{j,2}, j = 1, \ldots, m$, and the system

$$dy(t) = \left[A_{\theta(t)} - B_{\theta(t)} R_{\theta(t)}^{-1} B_{\theta(t)}^T \left(K_{\theta(t),1} + K_{\theta(t),2}\right)\right] y(t) dt \tag{4.21}$$

is mean-square stable. However, $K_{j,2}$ is not necessarily positive definite or even symmetric since it is not a diagonal block. Hence, we do not have to require that $H_j^Q, j = 1, \ldots, m$, are positive definite.

Let's look at the following example.

Example 4.1. Take the parameters in (2.1)–(2.2) as follows. $A_1 = 0.2, A_2 = 0.8$, $B_j = Q_j = R_j = 1, j = 1, 2, H = 0.25$,

$$\Lambda = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In this case, $H_j^Q = -0.4375 < 0$. From the computation with MATLAB, the (maximal) solution to (4.15) is $K_{1,1} = 1.4251, K_{2,1} = 1.8859$, and the (maximal) solution to (4.16) is $K_{1,2} = -0.2012, K_{2,2} = -0.2161$. It can be verified that the system (4.19) is mean-square stable. Meanwhile, by solving (4.18), we have $K_{1,4} = 0.07595, K_{2,4} = 0.08054$. It can be verified that both K_1 and K_2 are positive definite.

For further analysis, we introduce the following assumption:

(A3) There exists a set $(K_{1,2},\dots,K_{m,2})$ satisfying (4.20) such that the system (4.21) is mean-square stable.

Remark 4.2. From [23], assumption (A3) holds if there exists a $\bar{K}_2 = (K_{1,2},\dots,K_{m,2})$ such that $\mathcal{R}_j(\bar{K}_2) > 0$, where

$$\mathcal{R}_j(\bar{K}_2) = (A_j - B_jR_j^{-1}B_j^TK_{j,1})^TK_{j,2} + K_{j,2}(A_j - B_jR_j^{-1}B_j^TK_{j,1}) + \sum_{l=1}^m \lambda_{jl}K_{l,2}$$
$$-K_{j,2}B_jR_j^{-1}B_j^TK_{j,2} + H_j^Q, \quad j = 1,\dots,m.$$

In this case, $K_{j,2}$ is taken as the maximal solution such that the system (4.21) is mean-square stable. The so-called maximal solution is referred to as a solution $(P_1,\dots,P_m)$ to (4.21) if for any $(\bar{P}_1,\dots,\bar{P}_m)$ with $\mathcal{R}_j(\bar{P}_1,\dots,\bar{P}_m) \geq 0$, it holds that $P_j - \bar{P}_j \geq 0$. On the other hand, if $H_j^Q \geq 0$, then (A3) holds necessarily.

Note that (4.17) can be deformed equivalently to

$$\begin{aligned}&(A_j - B_jR_j^{-1}B_j^TK_{j,1})^TK_{j,3}\\&+K_{j,3}(A_j - B_jR_j^{-1}B_j^TK_{j,1}) + \sum_{l=1}^m \lambda_{jl}K_{l,3}\\&-K_{j,2}^TB_jR_j^{-1}B_j^TK_{j,3} + H_j^Q = 0.\end{aligned} \tag{4.22}$$

Compared with (4.20), (4.22) holds when $K_{j,3}$ is taken identically as $K_{j,2}$. Hence, the solution $K_{j,2}$ to (4.16) is also a solution to (4.17). Last, noticing that the system (4.21) is mean-square stable, the coupled Lyapunov equation (4.18) has a unique symmetric solution $K_{j,4}$.

So far, we have shown that under assumptions (A2)–(A3) there exists a set of solutions to (4.15)–(4.18).

THEOREM 4.2. *Under assumptions* (A2)–(A3), (4.4) *and* (4.9) *admit a set of solutions* $\{K_j,\ M_j, j = 1,\dots,m\}$, *and the system* (4.12) *is mean-square stable.*

Proof. We only need to prove that the system (4.12) is stable in the mean-square sense. See Appendix B for the proof. ∎

We now provide that there exists a unique set of $r_j(t), s_j(t), j = 1,\dots,m$, satisfying (4.5) and (4.10). Substituting (4.10) into (4.5) gives

$$\frac{dr_j(t)}{dt} + (G_j^T - S_j)r_j(t) + \sum_{l=1}^m \lambda_{jl}r_l(t) + K_j\begin{pmatrix} h(t) \\ h(t)\end{pmatrix} - \begin{pmatrix}(I_n - H^T)Q_j\alpha \\ 0\end{pmatrix} = 0, \tag{4.23}$$

where

$$S_j = \begin{pmatrix} K_{j,2}B_jR_j^{-1}B_j^T & 0 \\ K_{j,4}B_jR_j^{-1}B_j^T & 0\end{pmatrix}, \quad j = 1,\dots,m.$$

THEOREM 4.3. *Under assumptions* (A2)–(A3), (4.23) *has a unique set of solutions* $r_j(t), j = 1,\dots,m,$ *in* $C_b([0,\infty),\mathbb{R}^{2n})$.

Proof. See Appendix B for the proof. ∎

By Theorems 4.2 and 4.3, there exists a stabilizing solution to the system of consistency equations (4.11). This implies that we can get a set of distributed strategies:

$$\hat{u}_i(t) = -R_{\theta(t)}^{-1}B_{\theta(t)}^T[K_{\theta(t),1}x_i(t) + K_{\theta(t),2}z(t) + r_{\theta(t),1}(t)],\ 1 \leq i \leq N, \tag{4.24}$$

where $(K_j, r_j, j = 1,\dots,m)$ is a stabilizing solution to (4.11), and $r_{j,1} \triangleq (I_n, 0)r_j$.

5. Analysis of asymptotic optimality. Applying the control strategy (4.24) into the system (4.2) gives the closed-loop system equation

$$d\begin{pmatrix} \hat{x}_i(t) \\ \hat{z}(t) \end{pmatrix} = G_{\theta(t)} \begin{pmatrix} \hat{x}_i(t) \\ \hat{z}(t) \end{pmatrix} dt + \begin{pmatrix} s_{\theta(t)}(t) \\ s_{\theta(t)}(t) \end{pmatrix} dt + \begin{pmatrix} D_{\theta(t)} \\ 0 \end{pmatrix} dW_i(t). \tag{5.1}$$

We first provide the uniform stability of the closed-loop system.

THEOREM 5.1. *Under assumptions* (A1)–(A3), *there exists a constant* C_0 *independent of* N *such that the strategy* (4.24) *and the closed-loop system* (5.1) *satisfy*

$$\max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T} E\int_0^T (\|\hat{x}_i(t)\|^2 + \|\hat{z}(t)\|^2 + \|\hat{u}_i(t)\|^2) dt \le C_0. \tag{5.2}$$

Proof. See Appendix C for the proof. ∎

Below we provide a result of the approximation error, which is useful to analyze asymptotic optimality of the set of strategies (4.24).

LEMMA 5.2. *If assumptions* (A1)–(A3) *hold, then under* (4.24), *the closed-loop system satisfies*

$$\limsup_{T\to\infty} \frac{1}{T} E\int_0^T \left\|\hat{x}^{(N)}(t) - \hat{z}(t)\right\|^2 dt = O\left(\frac{1}{N}\right). \tag{5.3}$$

Proof. It follows from the closed-loop dynamics (5.1) that

$$\begin{aligned} d\hat{x}^{(N)}(t) &= G_{\theta(t),1}\hat{x}^{(N)}(t)dt + G_{\theta(t),2}\hat{z}(t)dt \\ &\quad + s(\theta(t),t)dt + \frac{1}{N}\sum_{i=1}^N D_{\theta(t)} dW_i(t), \end{aligned} \tag{5.4}$$

$$d\hat{z}(t) = M_{\theta(t)}\hat{z}(t)dt + s(\theta(t),t)dt. \tag{5.5}$$

Notice that $M_{\theta(t)} = G_{\theta(t),1} + G_{\theta(t),2}$. Then, by (5.4) and (5.5) we have

$$d\xi^{(N)}(t) = G_{\theta(t),1}\xi^{(N)}(t)dt + D_{\theta(t)} d\left[\frac{1}{N}\sum_{i=1}^N W_i(t)\right], \tag{5.6}$$

where $\xi^{(N)}(t) = \hat{x}^{(N)}(t) - \hat{z}(t)$. Since $\{W_i(t), i = 1, \dots, N\}$ is a family of independent standard Brownian motions, then we have that for all $t > s$,

$$E\Big\{\frac{1}{N}\sum_{i=1}^N [W_i(t) - W_i(s)]\big|\mathcal{F}_s\Big\} = 0,$$

and

$$\begin{aligned} &E\Big\{\Big[\frac{1}{N}\sum_{i=1}^N (W_i(t) - W_i(s))\Big]^T \frac{1}{N}\sum_{i=1}^N (W_i(t) - W_i(s))\big|\mathcal{F}_s\Big\} \\ &= \frac{1}{N^2} E\Big\{\sum_{i=1}^N\sum_{j=1}^N (W_i(t) - W_i(s))^T (W_j(t) - W_j(s))\big|\mathcal{F}_s\Big\} \\ &= \frac{1}{N^2} E\Big\{\sum_{i=1}^N (W_i(t) - W_i(s))^T (W_i(t) - W_i(s))\big|\mathcal{F}_s\Big\} = \frac{1}{N}(t-s). \end{aligned}$$

Hence, the process $\frac{1}{N}\sum_{i=1}^{N} W_i(t)$ is also a Brownian motion [20]. Let $V_1(j,x) = x^T P_{j,1} x, j = 1,\dots,m, x \in \mathbb{R}^n$, and $\mathcal{A}^{(N)}$ denote the infinitesimal generator of $\{\theta(t), \xi^{(N)}(t)\}$ in (5.6). Then, from (B.2) it follows that

$$\begin{aligned}
&\mathcal{A}^{(N)} V_1(j,x)\\
&\quad= \lim_{s\to 0}\frac{1}{s}\big\{E[V_1(\theta(t+s),\xi^{(N)}(t+s))|\theta(t)=j,\xi^{(N)}(t)=x] - V_1(j,x)\big\}\\
&\quad= \frac{\partial V}{\partial x}(j,x)G_{j,1}x + \sum_{k=1}^{m}\lambda_{jk}V(k,x) + \frac{1}{N}\mathrm{tr}(D_j^T P_{j,1} D_j)\\
&\quad= x^T(G_{j,1}^T P_{j,1} + P_{j,1}G_{j,1} + \lambda_{jk}P_{k,1})x + \frac{1}{N}\mathrm{tr}(D_j^T P_{j,1} D_j)\\
&\quad= -x^T x + \frac{1}{N}\mathrm{tr}(D_j^T P_{j,1} D_j).
\end{aligned}$$

By Dynkin's formula [29], we have

$$\begin{aligned}
E\mathcal{A}^{(N)}V_1(\theta(t),\xi^{(N)}(t)) &= -E\|\xi^{(N)}(t)\|^2 + \frac{1}{N}\mathrm{tr}(D_j^T P_{j,1} D_j)\\
&\le -\frac{1}{\bar\lambda}EV_1(\theta(t),\xi^{(N)}(t)) + \frac{1}{N}\max_{1\le j\le m}\mathrm{tr}(D_{\theta(t)}^T P_{\theta(t),1} D_{\theta(t)}),
\end{aligned}$$

where $\bar\lambda = \max_{1\le j\le m}(\lambda_{max}(P_{j,1}))$. This together with the comparison theorem implies

$$EV_1(\theta(t),\xi^{(N)}(t)) \le EV_1(\theta(0),\xi^{(N)}(0))e^{-\frac{t}{\bar\lambda}} + \frac{\bar\lambda}{N}\max_{1\le j\le m}\mathrm{tr}(D_j^T P_{j,1} D_j)(1-e^{-\frac{t}{\bar\lambda}}).$$

Thus, we have

$$\begin{aligned}
E[\|\xi^{(N)}(t)\|^2] \le \frac{1}{\min_j(\lambda_{\min}(P_j))}\Big[&\frac{1}{N}\max_{1\le j\le m}\|P_{j,1}\|\max_{1\le i\le N}E\|\hat x_i(0)\|^2\exp(-t/\bar\lambda)\\
&+\frac{\bar\lambda}{N}\max_{1\le j\le m}\mathrm{tr}(D_j^T P_{j,1} D_j)(1-\exp(-t/\bar\lambda))\Big],
\end{aligned} \tag{5.7}$$

which implies

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}E\int_0^T \left\|\hat x^{(N)}(t) - \hat z(t)\right\|^2 dt\\
&\le \limsup_{T\to\infty}\frac{1}{T\min_{1\le j\le m}(\lambda_{min}(P_{j,1}))}\\
&\quad\times\Big[\frac{1}{N}\max_{1\le j\le m}\|P_{j,1}\|\max_{1\le i\le N}E\|\hat x_i(0)\|^2\int_0^T e^{-\frac{t}{\bar\lambda}}dt + \frac{T\bar\lambda}{N}\max_{1\le j\le m}\mathrm{tr}(D_j^T P_{j,1} D_j)\Big]\\
&= \frac{\bar\lambda}{N\min_j(\lambda_{min}(P_{j,1}))}\max_{1\le j\le m}\mathrm{tr}(D_j^T P_{j,1} D_j) = O\left(\frac{1}{N}\right).
\end{aligned}$$

∎

Now we are in a position to state the main result on asymptotic optimality of distributed strategies.

THEOREM 5.3. *If* (A1)–(A3) *hold, and* $I-H$ *is nonsingular, then the set of strategies* (4.24) *is an asymptotically team-optimal solution to the problem* (2.1)–(2.2), *i.e., for any* $u = \{u_i, i = 1, \dots, N | u_i \in \mathcal{U}_c\}$,

$$J^{(N)}(u) \geq J^{(N)}(\hat{u}) - \varepsilon, \tag{5.8}$$

where $\varepsilon = O(1/\sqrt{N})$.

Proof. See Appendix C for the proof. ∎

6. Asymptotically optimal costs. In this section, we give closed forms of the asymptotic optimal social cost and the optimal individual cost in mean field games. The difference of the optimal costs within the team and game formulations can be obtained from the two closed-form expressions.

We now give a closed-form expression of the optimal social average cost within the population limit case.

THEOREM 6.1. *If* (A1)–(A3) *hold and* $I - H$ *is nonsingular, then the asymptotic optimal social cost is given by*

$$\lim_{N\to\infty} \inf_{u\in\mathcal{U}_c} J^{(N)}(u) = \lim_{N\to\infty} J^{(N)}(\hat{u}) \tag{6.1}$$

$$= \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \Big[\mathrm{tr}(K_{\theta,1} D_\theta D_\theta^T) + 2r_\theta^T(t)[h^T(t), s_\theta^T(t)]^T \tag{6.2}$$

$$-\|\bar{B}_\theta^T r_\theta(t)\|^2_{R_\theta^{-1}} - \|\hat{z}(t)\|^2_{H_\theta^Q} + \|\alpha\|^2_{Q_\theta}\Big] dt, \tag{6.3}$$

where $\hat{z}$ *is determined by* (5.1). *Furthermore, if* $\theta(t)$ *is an ergodic Markov chain with stationary distribution* $\{\pi_j, j = 1, \dots, m\}$, *then the optimal cost is given by*

$$\lim_{N\to\infty} J^{(N)}(\hat{u}) = \sum_{j=1}^m \pi_j \limsup_{T\to\infty} \frac{1}{T} \int_0^T \Big[\mathrm{tr}(K_{j,1} D_j D_j^T)$$

$$+2r_j^T(t)[h^T(t), s_j^T(t)]^T - r_{j,1}^T(t) B_j R_j^{-1} B_j^T r_{j,1}(t) + \|\alpha\|^2_{Q_j}\Big] dt$$

$$-\sum_{j=1}^m \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{tr}(H_j^Q \bar{Z}_j(t)) dt, \tag{6.4}$$

where $\bar{Z}_j(0) = p_j(0) x_0 x_0^T$ *and*

$$\frac{d\bar{Z}_j(t)}{dt} = M_j \bar{Z}_j(t) + \bar{Z}_j(t) M_j^T + \sum_{k=1}^m \lambda_{kj} \bar{Z}_k(t) + s_j(t) \bar{z}_j^T(t) + \bar{z}_j(t) s_j^T(t).$$

Here $p_j(t) \triangleq P(\theta(t) = j)$ *and*

$$\frac{d\bar{z}_j(t)}{dt} = M_j \bar{z}_j(t) + \sum_{k=1}^m \lambda_{kj} \bar{z}_k(t) + p_j(t) s_j(t), \;\; \bar{z}_j(0) = p_j(0) x_0.$$

Proof. From Lemma 5.2, Theorem 5.1, and Schwarz's inequality, it follows that

$$\max_{1\leq i\leq N} \left| J_i(\hat{u}) - \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \hat{L}(\hat{x}_i, \hat{z}, \hat{u}_i) dt \right| = O\left(\frac{1}{\sqrt{N}}\right), \tag{6.5}$$

where

$$\hat{L}(\hat{x}_i, \hat{z}, \hat{u}_i) = \hat{x}_i^T Q_\theta \hat{x}_i - 2(H\hat{z}+\alpha)^T Q_\theta \hat{x}_i + \|H\hat{z}+\alpha\|^2_{Q_\theta} + \|\hat{u}_i\|^2_{R_\theta}.$$

Let $J^1(\hat{u}_i)$ be the optimal cost of problem (P1). By (4.1), (6.5), and direct computations, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{NT}\sum_{i=1}^N E\int_0^T \hat{L}(\hat{x}_i, \hat{z}, \hat{u}_i)dt \\
&= \frac{1}{N}\sum_{i=1}^N J^1(\hat{u}_i) + \limsup_{T\to\infty} \frac{1}{NT}\sum_{i=1}^N E\int_0^T 2\left[(I_n - H)\hat{z} - \alpha\right]^T Q_\theta H\hat{x}_i dt \\
&\quad + \limsup_{T\to\infty} \frac{1}{NT}\sum_{i=1}^N E\int_0^T \|H\hat{z}+\alpha\|^2_{Q_\theta} dt \\
&= \frac{1}{N}\sum_{i=1}^N J^1(\hat{u}_i) + \limsup_{T\to\infty} \frac{1}{T}E\int_0^T \|H\hat{z}+\alpha\|^2_{Q_\theta} dt \\
&\quad + \limsup_{T\to\infty} \frac{1}{T}E\int_0^T 2[(H^TQ_\theta - H^TQ_\theta H)\hat{z} - H^TQ_\theta\alpha]^T \hat{x}^{(N)} dt,
\end{aligned} \tag{6.6}$$

where $\hat{z}$ is determined by (5.1). From Schwarz's inequality, Lemma 5.2, and Theorem 5.1, it follows that

$$\begin{aligned}
&\left|\limsup_{T\to\infty} \frac{1}{T}E\int_0^T [(H^TQ_\theta - H^TQ_\theta H)\hat{z} - H^TQ_\theta\alpha]^T(\hat{x}^{(N)} - \hat{z})dt\right|^2 \\
&\le \limsup_{T\to\infty} \frac{1}{T}E\int_0^T \|C_1\hat{z}+C_2\|^2 dt \cdot \limsup_{T\to\infty} \frac{1}{T}E\int_0^T \|\hat{x}^{(N)} - \hat{z}\|^2 dt \le O(1/N).
\end{aligned}$$

This together with (6.6) leads to

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{NT}\sum_{i=1}^N E\int_0^T \hat{L}(\hat{x}_i, \hat{z}, \hat{u}_i)dt = \frac{1}{N}\sum_{i=1}^N J^1(\hat{u}_i) \\
&+ \limsup_{T\to\infty} \frac{1}{T}E\int_0^T \Big[- \hat{z}^T H_\theta^Q \hat{z} + \|\alpha\|^2_{Q_\theta}\Big] dt,
\end{aligned}$$

which together with Lemma A.1 and Theorem 5.3 gives (6.1).

Denote $\bar{z}_j(t) = E[\hat{z}(t)I_{[\theta(t)=j]}]$ and $\bar{Z}_j(t) = E[\hat{z}(t)\hat{z}^T(t)I_{[\theta(t)=j]}]$. Then, by the generalized Itô's formula [31] we have

$$\begin{aligned}
\frac{d\bar{z}_j(t)}{dt} &= M_j\bar{z}_j(t) + \sum_{k=1}^m \lambda_{kj}\bar{z}_k(t) + p_j(t)s_j(t), \\
\frac{d\bar{Z}_j(t)}{dt} &= M_j\bar{Z}_j(t) + \bar{Z}_j(t)M_j^T + \sum_{k=1}^m \lambda_{kj}\bar{Z}_k(t) + s_j(t)\bar{z}_j^T(t) + \bar{z}_j(t)s_j^T(t),
\end{aligned}$$

where $p_j(t) = P(\theta(t) = j)$. Noticing that $\hat{z}(0) = Ex_i(0) = x_0$, one can get $\bar{z}_j(0) = p_j(0)x_0$ and $\bar{Z}_j(0) = p_j(0)x_0x_0^T$. If $\theta(t)$ is ergodic and the stationary distribution is

π, then it follows that $\lim_{t\to\infty} p_j(t) = \pi_j, j = 1,\dots,m$, i.e., for any $\epsilon > 0$, there exists T_0 such that $|p_j(t) - \pi_j| < \epsilon, t \geq T_0$. Noticing that

$$Er_\theta(t) = \sum_{j=1}^{m} r_j(t)p_j(t),$$

by (6.1) we have (6.4). ◻

Remark 6.1. If for all $t \geq 0$, $\theta(t) \equiv \theta$, then system (2.1)–(2.2) degenerates to the (uniform) model in [18]. In this case, the asymptotically optimal cost in (6.1) coincides with the one given by Theorem 6.2 in [18]. Actually, $\hat{z}$ in (5.1) plays the same role as $\bar{x}$ in [18], since both of them are approximations of $\hat{x}^{(N)}$ in the large population case.

The above theorem provides an explicit expression of the asymptotically optimal social cost. In previous works [35, 37], we investigated mean field games with Markov jump parameters. To make a comparison of the optimal average costs within team and game problems, we review and unite the previous results on mean field games (see [35, 37] for details). The dynamics and the cost of each agent are also given by (2.1) and (2.2). The design of decentralized strategies is implemented as follows. First, the equation of mean field effect is presumed as

$$\dot{z}(t) = M_{\theta(t)} + s_{\theta(t)}(t),\ z(0) = x_0,$$

where M and s are to be determined. Then, we construct an auxiliary optimal control problem:

$$d\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A_{\theta(t)} & 0 \\ 0 & M_{\theta(t)} \end{pmatrix}\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} dt + \begin{pmatrix} B_{\theta(t)} \\ 0 \end{pmatrix} u_i(t)dt + \begin{pmatrix} h(t) \\ s_{\theta(t)}(t) \end{pmatrix} dt + \begin{pmatrix} D_{\theta(t)} \\ 0 \end{pmatrix} dW_i(t),\ 1 \leq i \leq N,$$

$$J_i(u_i) = \limsup_{T\to\infty} \frac{1}{T} E\int_0^T \left\{ \left\| (I\ \ -H)\begin{pmatrix} x_i(t) \\ z(t) \end{pmatrix} - \alpha \right\|^2_{Q_{\theta(t)}} + \|u_i(t)\|^2_{R_{\theta(t)}} \right\} dt.$$

By solving the above optimal control problem and using a similar construction as that in section 4, we get a set of consistency equations:

$$(6.7)\quad \begin{cases} \bar{A}_j^T K_j + K_j \bar{A}_j + \sum_{l=1}^m \lambda_{jl} K_l - K_j \bar{B}_j R_j^{-1} \bar{B}_j^T K_j \\ +(I_n\ \ -H)^T Q_j (I_n\ \ -H) = 0, \\ G_{j,1} + G_{j,2} = M_j, \\ \frac{dr_j(t)}{dt} + G_j^T r_j(t) + \sum_{l=1}^m \lambda_{jl} r_l(t) + K_j \begin{pmatrix} h(t) \\ s_j(t) \end{pmatrix} - (I_n\ \ -H)^T Q_j \alpha = 0, \\ h(t) - B_j R_j^{-1} \bar{B}_j^T r_j(t) = s_j(t), \quad j = 1,\dots,m. \end{cases}$$

Combining [35, Theorem 5] and [37, Theorem 3.1], we have the following result.

THEOREM 6.2. *For the game problem* (2.1)–(2.2)*, if* (A1)–(A2) *hold and* (6.7) *admits a stabilizing solution, then we have* (i) *the set of strategies* $(\hat{u}_i, 1 \leq i \leq N)$ *given by*

$$\hat{u}_i(t) = -R_{\theta(t)}^{-1} B_{\theta(t)}^T [K_{\theta(t),1} x_i(t) + K_{\theta(t),2} z(t) + r_{\theta(t),1}(t)]$$

is an ε-Nash equilibrium, where $\varepsilon = 1/\sqrt{N}$; (ii) *the optimal cost is given by*

$$\lim_{N\to\infty} J_i(\hat{u}) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \Big[\mathrm{tr}(K_{\theta,1} D_\theta D_\theta^T) + 2r_\theta^T(t)[h^T(t), s_\theta^T(t)]^T - \|\bar{B}_\theta^T r_\theta(t)\|^2_{R_\theta^{-1}} + \|\alpha\|^2_{Q_\theta}\Big] dt.$$

Furthermore, if $\theta(t)$ is an ergodic Markov chain with stationary distribution $\{\pi_j, j = 1, \dots, m\}$, then the optimal cost is given by

$$\lim_{N\to\infty} J_i(\hat{u}) = \sum_{j=1}^m \pi_j \limsup_{T\to\infty} \frac{1}{T} \int_0^T \Big[\mathrm{tr}(K_{j,1} D_j D_j^T) + 2r_j^T(t)[h^T(t), s_j^T(t)]^T - r_{j,1}^T(t) B_j R_j^{-1} B_j^T r_{j,1}(t) + \|\alpha\|^2_{Q_j}\Big] dt.$$

Remark 6.2. The closed-form expressions of optimal average costs in team and game problems are provided in Theorems 6.1 and 6.2, respectively. The reduction of the optimal average cost incurred from the game formulation to the team formulation can be obtained and analyzed by virtue of the closed-form expressions above. From a formal perspective, the reduction term is $\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|\hat{z}(t)\|^2_{H_\theta^Q} dt$, although the sets of consistency equations are different (but similar) within different formulations. The difference of optimal average costs in two formulations shows the efficiency of cooperation in mean field control.

7. Numerical example. In this section, we illustrate the consistency of mean field approximation and the asymptotical optimality of distributed strategies via a numerical example. Furthermore, we provide a comparison of the optimal average costs between team and game problems.

Without loss of generality, we consider the following simple case. Agent i evolves by

$$dx_i(t) = a(\theta(t))x_i(t)dt + (u_i(t) + 1)dt + 0.5dw_i(t),$$

where $x_i, u_i \in \mathbb{R}, 1 \le i \le N$, and $\{\theta(t)\}$ is a Markov chain taking value in $\{1, 2\}$ with the transition rate matrix

$$\Lambda = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$\{w_i(t)\}$ is a sequence of independent Brownian motions. The cost of agent i is described by

$$J_i(u) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \Big\{[x_i - (hx^{(N)} + \alpha)]^2 + u_i^2\Big\} dt.$$

Take $a_1 = 0.2, a_2 = 0.8, \alpha = 5, h = 0.25$. Then the Riccati equation system (4.15) has the maximal solution $k_{1,1} = 1.4251, k_{2,1} = 1.8859$ and (4.16) has the maximal solution $k_{1,2} = -0.2012, k_{2,2} = -0.2161$. Take $\{x_i(0)\}$ as a sequence of independent random variables with the normal distribution $N(0, 3)$. $\{\theta(0)\}$ is a random variable with $P(\theta(0) = 1) = 0.5$ and $P(\theta(0) = 2) = 0.5$. It can be verified that assumptions (A1)–(A3) hold.

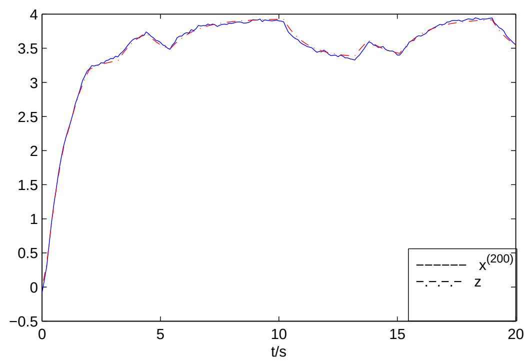

FIG. 1. *Curves of $x^{(200)}$ and z.*

Let $\bar{r}_1(t) = [r_{1,1}(t), r_{2,1}(t)]^T$. Then, from (4.23) we have

$$\frac{d\bar{r}_1(t)}{dt} + \left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \Lambda \right] \bar{r}_1(t) - \bar{\eta}_1(t) = 0,$$

where $m_1 = a_1 - k_{1,1} - k_{1,2} = -1.0239$, $m_2 = a_2 - k_{2,1} - k_{2,2} = -0.8698$, and

$$\bar{\eta}_1(t) = [(1-h)\alpha - k_{1,1} - k_{1,2}, (1-h)\alpha - k_{2,1} - k_{2,2}]^T = [2.5261, 2.0802]^T.$$

Noticing that $\left(\begin{smallmatrix} ccm_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right) + \Lambda$ is stable, and $\bar{r}_1(t) \in C_b([0,\infty), \mathbb{R}^n)$, we can get

$$\bar{r}_1(t) = -\int_0^\infty \exp\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} t + \Lambda t \right] \bar{\eta}_1(t) dt = [-2.4435, -2.4194]^T.$$

Thus, from (4.24) we obtain the following distributed strategies:

$$\hat{u}_i(t) = -\sum_{j=1}^{2} I_{[\theta(t)=j]}(k_{j,1}x_i(t) + k_{j,2}z(t) + r_{j,1}(t)), \; 1 \le i \le N, \tag{7.1}$$

where z satisfies

$$\frac{dz(t)}{dt} = m_{\theta(t)}z(t) + 1 - r_{\theta,1}(t), \; z(0) = x_0.$$

Figure 1 depicts the curves of z^* and $x^{(N)}$ when the number of agents is 200. It can be seen from Figure 1 that the curves of z and $x^{(N)}$ coincide well as $N = 200$, which illustrates the consistency of mean field approximation. Figure 2 shows the curve of ε when the number of agents grows from 1 to 200, where

$$\varepsilon(N) = \left(\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \left\| x^{(N)} - z \right\|^2 dt \right)^{\frac{1}{2}}.$$

It can be seen that ε is very small when N is 200, which implies that the set of distributed strategies (7.1) is asymptotically optimal.

By Theorems 6.1 and 6.2, we compute the optimal average costs of team and game problems within population limit for $h = -1.2, -0.8, -0.4, -0.25, 0.25, 0.4, 0.8, 1.2$, respectively. Comparison of both optimal costs is shown in Table 1 and Figure 3 (J_1 denotes the optimal cost of the game problem, and $J^{(\infty)}$ denotes the optimal cost of the team problem). It can be seen that the optimal cost of the game problem J_1 is greater than that of the team problem $J^{(\infty)}$, especially when the interaction intensity h is large.

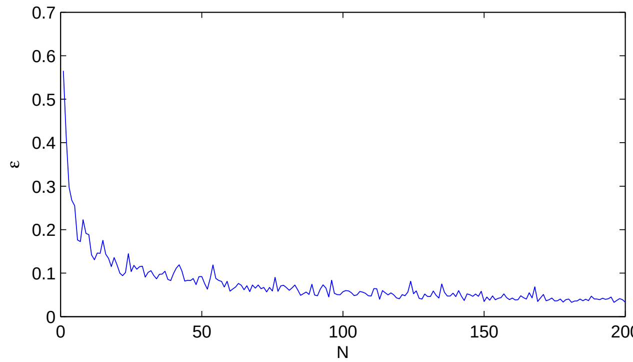

FIG. 2. *Curve of* ε *with respect to* N.

TABLE 1
Optimal average costs of game and team problems.

h	–1.2	–0.8	–0.4	–0.25	0.25	0.4	0.8	1.2
J_1	5.8741	6.8968	8.5641	8.9450	15.3032	18.8444	35.4553	619.2769
$J^{(\infty)}$	5.6397	6.7377	8.5171	8.7822	14.9478	17.6496	26.1194	26.1194

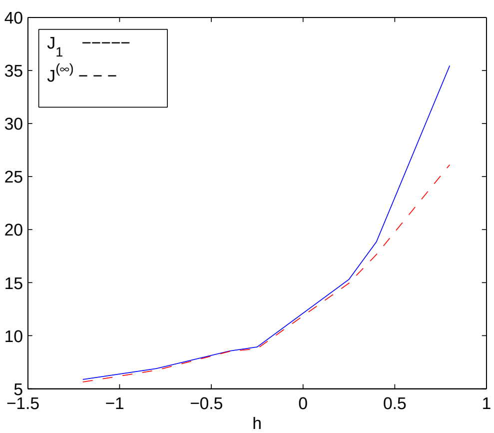

FIG. 3. *Curves of* J_1 *and* $J^{(\infty)}$ *with respect to* h.

8. Concluding remarks. In this paper, social optima have been investigated in mean field LQG control problems with Markov jump parameters. By the parametric approach and state space augmentation, a set of distributed strategies is designed and further shown to be asymptotically team-optimal. An explicit expression of the asymptotically optimal social cost is given. The difference of the optimal costs within team and game formulations can be obtained from the explicit expressions and further illustrated by a numerical example. The result of this paper can be generalized to the case where Markov jump parameters are unknown or the impact of private and public information is included [27]. Also, the model in this paper can be applied to economic growth or pollution backgrounds.

Appendix A. Markov jump optimal control with indefinite weight. Consider

$$dx(t) = A_{\theta(t)}x(t)dt + B_{\theta(t)}u(t)dt + h(t)dt + D_{\theta(t)}dW(t), \quad t \geq 0, \tag{A.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $f \in \mathcal{C}_b([0,\infty), \mathbb{R}^n)$, and $W(t)$ is a d-dimensional standard Brownian motion. The initial value $E\|x(0)\|^2 < \infty$. $\theta(t)$ is a continuous-time Markov chain taking value in $\{1, 2, \ldots, m\}$ with transition rate matrix $\Lambda = (\lambda_{ij})$. The admissible control set is

$$\mathcal{U} = \Big\{ u \mid u(t) \text{ is adapted to } \sigma(x(s), \theta(s), s \leq t), E\|x(T)\| = o(\sqrt{T}) \Big\}.$$

Let the cost function be given by

$$J(u) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \left[x^T(t)Q_{\theta(t)}x(t) - 2g^T_{\theta(t)}(t)x(t) + u^T(t)R_{\theta(t)}u(t) \right] dt, \tag{A.2}$$

where for any $i = 1, \ldots, m$, Q_i is symmetric, $R_i > 0$, and $g_i \in C_b([0,\infty), \mathbb{R}^n)$. Define a set of coupled AREs,

$$K_iA_i + A_i^TK_i + \sum_{j=1}^m \lambda_{ij}K_j - K_iB_iR_i^{-1}B_i^TK_i + Q_i = 0, \quad i = 1, \ldots, m, \tag{A.3}$$

and a set of coupled ODEs,

$$\frac{dr_i(t)}{dt} + G_i^Tr_i(t) + \sum_{j=1}^m \lambda_{ij}r_j(t) + K_ih(t) - g_i(t) = 0, \quad i = 1, \ldots, m, \tag{A.4}$$

where $G_i = A_i - B_iR_i^{-1}B_i^TK_i$ and the initial condition $r_i(0)$ is arbitrarily given.

Lemma A.1. *For the optimal control problem* (A.1)–(A.2), *assume* (i) (A.3) *has a symmetric solution such that the system*

$$dy(t) = G_{\theta(t)}y(t)dt$$

is mean-square stable; (ii) $h \in C_b([0,\infty), \mathbb{R}^n)$ *and* $g_i \in C_b([0,\infty), \mathbb{R}^n), i = 1, \ldots, m$. *Then we have*

(1) *there exists a unique set of solutions* $r_i \in C_b([0,\infty), \mathbb{R}^n), i = 1, \ldots, m$ *to* (A.4)*;*

(2) *the optimal control is* $\hat{u}(t) = -R^{-1}_{\theta(t)}B^T_{\theta(t)}[K_{\theta(t)}x(t) + r_{\theta(t)}(t)]$*;*

(3) *the optimal cost is given by*

$$\begin{aligned} J(\hat{u}) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T &\Big\{ \mathrm{tr}(K_{\theta(t)}D_{\theta(t)}D^T_{\theta(t)}) \\ &- \|B^T_{\theta(t)}r_{\theta(t)}(t)\|^2_{R^{-1}_{\theta(t)}} + 2r_{\theta(t)}(t)h(t) \Big\} dt. \end{aligned} \tag{A.5}$$

Particularly, if $\theta(t)$ *is an ergodic Markov chain with stationary distribution* $\{\pi_i, i = 1, \ldots, m\}$*, then the optimal cost* $J(\hat{u})$ *is given by*

$$\sum_{i=1}^m \pi_i \limsup_{T\to\infty} \frac{1}{T} \int_0^T \Big\{ \mathrm{tr}(K_iD_iD_i^T) - \|B_i^Tr_i(t)\|^2_{R_i^{-1}} + 2r_i(t)h(t) \Big\} dt. \tag{A.6}$$

Proof. We can show part (1) by using the similar argument in [37, Theorem 3.1(iii)]. For part (2) we prove it by using a completion of squares technique. Noticing that Q_i is merely symmetric, the integrand in (A.2) cannot be written in the square form $\|x-g_1\|^2_{Q_\theta}+h_1$ as in [37]. However, by a similar derivation as in [37, Theorem 3.1] we can get that for $u\in\mathcal{U}$,

$$\begin{aligned}
&J(u)\\
&=\limsup_{T\to\infty}\frac{1}{T}E\{x^T(0)K_{\theta(0)}x(0)-x^T(T)K_{\theta(T)}x(T)+2r^T_{\theta(0)}(0)x(0)-2r^T_{\theta(T)}(T)x(T)\}\\
&\quad+\limsup_{T\to\infty}\frac{1}{T}E\int_0^T\Big\{\|u(t)+R^{-1}_{\theta(t)}B^T_{\theta(t)}[K_{\theta(t)}x(t)+r_{\theta(t)}(t)]\|^2_{R_{\theta(t)}}\\
&\quad-\|B^T_{\theta(t)}r_{\theta(t)}(t)\|^2_{R^{-1}_{\theta(t)}}+2r_{\theta(t)}(t)h(t)+\mathrm{tr}(K_{\theta(t)}D_{\theta(t)}D^T_{\theta(t)})\Big\}dt\\
&=\limsup_{T\to\infty}\frac{1}{T}E\int_0^T\Big\{\|u(t)+R^{-1}_{\theta(t)}B^T_{\theta(t)}[K_{\theta(t)}x(t)+r_{\theta(t)}(t)]\|^2_{R_{\theta(t)}}\\
&\quad-\|B^T_{\theta(t)}r_{\theta(t)}(t)\|^2_{R^{-1}_{\theta(t)}}+2r_{\theta(t)}(t)h(t)+\mathrm{tr}(K_{\theta(t)}D_{\theta(t)}D^T_{\theta(t)})\Big\}dt\\
&\geq\limsup_{T\to\infty}\frac{1}{T}E\int_0^T\Big\{\mathrm{tr}(K_{\theta(t)}D_{\theta(t)}D^T_{\theta(t)})-\|B^T_{\theta(t)}r_{\theta(t)}(t)\|^2_{R^{-1}_{\theta(t)}}+2r_{\theta(t)}(t)h(t)\Big\}dt.
\end{aligned}$$

The last equality holds if and only if $u(t)=\hat u(t)\triangleq -R^{-1}_{\theta(t)}B^T_{\theta(t)}[K_{\theta(t)}x(t)+r_{\theta(t)}(t)]$. Noticing that under the control $\hat u$, the system

$$dy(t)=G_{\theta(t)}y(t)dt$$

is mean-square stable, we have $E\|x(T)\|=o(\sqrt{T})$. Thus, we get $\hat u\in\mathcal{U}$, and the optimal cost is given by (A.5). If $\theta(t)$ is ergodic and the stationary distribution is $\{\pi_i, i=1,\dots,m\}$, then it follows that $\lim_{t\to\infty}P(\theta(t)=i)=\pi_i, i=1,\dots,m$. Since the cost is the infinite-time average, the optimal cost is given by (A.6). ∎

Appendix B. Proofs of Theorems 4.2 and 4.3.

Proof of Theorem 4.2. Noticing that $G_j=\bar A_j-\bar B_jR_j^{-1}\bar B_j^TK_j$, it follows by the matrix block multiplication that

$$G_j=\begin{pmatrix} A_j-B_jR_j^{-1}B_j^TK_{j,1} & -B_jR_j^{-1}B_j^TK_{j,2}\\ 0 & A_j-B_jR_j^{-1}B_j^T(K_{j,1}+K_{j,2})\end{pmatrix}. \tag{B.1}$$

Then $G_{j,1}=A_j-B_jR_j^{-1}B_j^TK_{j,1}$, $G_{j,2}=-B_jR_j^{-1}B_j^TK_{j,2}$, $G_{j,4}=M_j=A_j-B_jR_j^{-1}B_j^T(K_{j,1}+K_{j,2})$. Since the system (4.19) is mean-square stable, there exists a unique set of positive definite matrices $P_{j,1}, j=1,\dots,m$, such that

$$G^T_{j,1}P_{j,1}+P_{j,1}G_{j,1}+\sum_{k=1}^m\lambda_{jk}P_{k,1}=-I_n. \tag{B.2}$$

Since the system (4.21) is mean-square stable, there also exists a unique set of positive definite matrices $P_{j,2}, j=1,\dots,m$, such that

$$G^T_{j,4}P_{j,2}+P_{j,2}G_{j,4}+\sum_{k=1}^m\lambda_{jk}P_{k,2}=-I_n-G^T_{j,2}P^2_{j,1}G_{j,2}. \tag{B.3}$$

Let

$$P_j = \begin{pmatrix} P_{j,1} & 0 \\ 0 & P_{j,2} \end{pmatrix}, \ j = 1, \dots, m.$$

Then, P_j is positive-definite. From (B.2) and (B.3) it follows that

$$G_j^T P_j + P_j G_j + \sum_{k=1}^{m} \lambda_{jk} P_k = \begin{pmatrix} -I_n & P_{j,1} G_{j,2} \\ G_{j,2}^T P_{j,1} & -I_n - G_{j,2}^T P_{j,1}^2 G_{j,2} \end{pmatrix}. \tag{B.4}$$

Noticing

$$\begin{pmatrix} I_n & 0 \\ G_{j,2}^T P_{j,1} & I_n \end{pmatrix} \begin{pmatrix} -I_n & P_{j,1} G_{j,2} \\ G_{j,2}^T P_{j,1} & -I_n - G_{j,2}^T P_{j,1}^2 G_{j,2} \end{pmatrix} \begin{pmatrix} I_n & P_{j,1} G_{j,2} \\ 0 & I_n \end{pmatrix}$$
$$= \begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

we get that

$$\begin{pmatrix} -I_n & P_{j,1} G_{j,2} \\ G_{j,2}^T P_{j,1} & -I_n - G_{j,2}^T P_{j,1}^2 G_{j,2} \end{pmatrix}$$

is negative definite. By (B.4) and Remark 4.1, the system (4.12) is stable in the mean-square sense. ∎

Proof of Theorem 4.3. From (4.14) and (B.1), it follows that

$$G_j^T - S_j = \begin{pmatrix} M_j & G_{j,5} \\ 0 & M_j \end{pmatrix}^T, \tag{B.5}$$

where $G_{j,5} = -B_j R_j^{-1} B_j^T (K_{j,2} + K_{j,4})$. Note that $M_j = A_j - B_j R_j^{-1} B_j^T (K_{j,1} + K_{j,2})$. Since the system (4.21) is mean-square stable, there exist positive matrices $P_{j,3}$ and $P_{j,4}$ such that the following equations hold:

$$M_j^T P_{j,3} + P_{j,3} M_j + \sum_{k=1}^{m} \lambda_{jk} P_{k,3} = -I_n, \tag{B.6}$$

$$M_j^T P_{j,4} + P_{j,4} M_j + \sum_{k=1}^{m} \lambda_{jk} P_{k,4} = -I_n - G_{j,5}^T P_{j,3}^2 G_{j,5}. \tag{B.7}$$

Let

$$\tilde{P}_j = \begin{pmatrix} P_{j,3} & 0 \\ 0 & P_{j,4} \end{pmatrix}, \quad j = 1, \dots, m.$$

From (B.5)–(B.7) it follows that

$$(G_j - S_j^T)^T \tilde{P}_j + \tilde{P}_j (G_j - S_j^T) + \sum_{k=1}^{m} \lambda_{jk} \tilde{P}_k = \begin{pmatrix} -I_n & P_{j,3} G_{j,5} \\ G_{j,5}^T P_{j,3} & -I_n - G_{j,5}^T P_{j,3}^2 G_{j,5} \end{pmatrix}.$$

By the matrix's elementary transformation, we know

$$\begin{pmatrix} -I_n & P_{j,3} G_{j,5} \\ G_{j,5}^T P_{j,3} & -I_n - G_{j,5}^T P_{j,3}^2 G_{j,5} \end{pmatrix}$$

is negative definite. Thus, it follows from Remark 4.1 that

$$dy(t) = (G_{\theta(t)} - S^T_{\theta(t)})y(t)dt \tag{B.8}$$

is mean-square stable.

Let $\varphi_j(t) = I_{[\theta(t)=j]}$. Then, from [24] we have

$$\varphi_j(t) = \varphi_j(0) + \sum_{k=1}^{m} \int_0^t \lambda_{kj}\varphi_k(s)ds + m_j(t), \quad j = 1, \ldots, m,$$

where $m_j(t)$ is a mean-square integrable martingale. Using Itô's formula, we can obtain

$$\begin{aligned} y(t)\varphi_j(t) = &\int_0^t [G_{\theta(s)} - S^T_{\theta(s)}]\varphi_j(s)y(s)ds \\ &+ \int_0^t y(s)\sum_{k=1}^{m} \lambda_{kj}\varphi_k(s)ds + \int_0^t y(s)dm_j(s). \end{aligned} \tag{B.9}$$

Let $Y_i(t) = E(y(t)\varphi_i(t))$ and $Y(t) = (Y_1^T(t),\ldots, Y_m^T(t))^T$. Then, by taking expectations on both sides of (B.9), we have

$$Y_i(t) = \int_0^t (G_i - S_i^T)Y_i(s)ds + \int_0^t \sum_{j=1}^{m} \lambda_{ji}Y_j(s)ds,$$

i.e.,

$$\frac{dY(t)}{dt} = (G - S^T + \Lambda^T \otimes I_{2n})Y(t), \tag{B.10}$$

where $G = \mathrm{diag}\{G_1, \ldots, G_m\}$ and $S = \mathrm{diag}\{S_1, \ldots, S_m\}$. Since the system (B.8) is mean-square stable, the system (B.10) is asymptotically stable, which implies that $G - S^T + \Lambda^T \otimes I_{2n}$ is stable.

Let

$$\begin{aligned} r(t) &= [r_1^T(t), \ldots, r_m^T(t)]^T, \\ V &= G - S^T + \Lambda^T \otimes I_{2n}, \\ \eta(t) &= \Big[(\alpha^T Q_1(I_n - H), 0) - (h^T(t), h^T(t))K_1^T, \ldots, \\ &\qquad (\alpha^T Q_m(I_n - H), 0) - (h^T(t), h^T(t))K_m^T\Big]^T. \end{aligned}$$

Then, (4.23) can be written as

$$\frac{dr(t)}{dt} = -V^T r(t) + \eta(t).$$

The general solution of this equation can be expressed as

$$r(t) = e^{-V^T t}\left[r(0) + \int_0^t e^{V^T s}\eta(s)ds\right].$$

Since V^T is stable and $\eta(t) \in C_b([0,\infty), \mathbb{R}^{2mn})$, when $r(0) = -\int_0^\infty e^{V^T s}\eta(s)ds \triangleq r^*(0)$, we have

$$r^*(t) = -\int_t^\infty e^{V^T(s-t)}\eta(s)ds \in C_b([0,\infty), \mathbb{R}^{2mn}).$$

When the perturbation of the initial value $\Delta r(0) \neq 0$, the corresponding solution is given by $r(t) = e^{-V^T t}\Delta r(0) + r^*(t)$. Since all the eigenvalues of $-V^T$ have positive real parts, and $r^*(t) \in C_b([0,\infty), \mathbb{R}^{2mn})$, then $r(t)$ is unbounded. Hence, (4.23) admits a unique solution in $C_b([0,\infty), \mathbb{R}^{2mn})$. ◻

Appendix C. Proof of Theorems 5.1 and 5.3.

Proof of Theorem 5.1. Let $\bar{x}_i(t) = \begin{pmatrix} \hat{x}_i(t) \\ \hat{z}(t) \end{pmatrix}$, $i = 1, \dots, N$. Then, (5.1) can be written as

$$d\bar{x}_i(t) = G_{\theta(t)}\bar{x}_i(t)dt + \begin{pmatrix} s_{\theta(t)}(t) \\ s_{\theta(t)}(t) \end{pmatrix} dt + \begin{pmatrix} D_{\theta(t)} \\ 0 \end{pmatrix} dW_i(t). \tag{C.1}$$

From (4.10) and Theorem 4.3, it follows that

$$\max_{1\le j\le m} \|s_j(t)\|_\infty \le \|h\|_\infty + \max_{1\le j\le m} \|B_j R_j^{-1} B_j^T\| \|r_{j,1}(t)\|_\infty \triangleq C_s.$$

Let $V(j,\bar{x}) = \bar{x}^T P_j \bar{x}$, $j = 1, \dots, m$, $\bar{x} \in \mathbb{R}^{2n}$, and $\mathcal{A}_i$ denote the infinitesimal generator of $\{\theta(t), \bar{x}_i(t)\}$ in (C.1). Then, it follows from (B.4) that

$$\begin{aligned}
&\mathcal{A}_i V(j,\bar{x}) \\
&= \lim_{s\to 0} \frac{1}{s}\big\{E[V(\theta(t+s), \bar{x}_i(t+s))|\theta(t) = j, \bar{x}_i(t) = \bar{x}] - V(j,\bar{x})\big\} \\
&= \frac{\partial V}{\partial \bar{x}}(j,\bar{x})[G_j\bar{x} + (s_j^T(t)\ s_j^T(t))^T] + \sum_{k=1}^m \lambda_{jk} V(k,\bar{x}) + \mathrm{tr}[(D_j^T\ 0)P_j(D_j^T\ 0)^T] \\
&= \bar{x}^T(G_j^T P_j + P_j G_j + \sum_{k=1}^m \lambda_{jk} P_k)\bar{x} - 2\bar{x}^T P_j (s_j^T(t)\ s_j^T(t))^T + \mathrm{tr}(D_j^T P_{j,1} D_j) \\
&= -\bar{x}^T \bar{N}_j \bar{x} - 2\bar{x}^T P_j (s_j^T(t)\ s_j^T(t))^T + \mathrm{tr}(D_j^T P_{j,1} D_j),
\end{aligned}$$

where $\bar{N}_j = \begin{pmatrix} I_n & -P_{j,1}G_{j,2} \\ -G_{j,2}^T P_{j,1} & I_n + G_{j,2}^T P_{j,1}^2 G_{j,2} \end{pmatrix} > 0$. By Dynkin's formula [29], we have

$$\begin{aligned}
\frac{dV(\theta(t), \bar{x}_i(t))}{dt} &= E\mathcal{A}_i V(\theta(t), \bar{x}_i(t)) \\
&= -E[\bar{x}^T(t) N_{\theta(t)} \bar{x}_i(t)] + \mathrm{tr}(D_{\theta(t)}^T P_{\theta(t),1} D_{\theta(t)}) \\
&\quad - 2\bar{x}^T(t) P_{\theta(t)} \big(s_{\theta(t)}^T(t)\ s_{\theta(t)}^T(t)\big)^T \\
&\le -a_1 EV(\theta(t), \bar{x}_i(t)) + \mathrm{tr}(D_{\theta(t)}^T P_{\theta(t),1} D_{\theta(t)}) \\
&\quad + 2 \max_{1\le j\le m} \sqrt{2EV(\theta(t), \bar{x}_i(t)) \|P_{\theta(t)}\|} \|s(\theta(t), t)\| \\
&\le -\frac{1}{2} a_1 EV(\theta(t), \bar{x}_i(t)) + b_1,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \min_{1\le j\le m} \frac{\lambda_{min}(\bar{N}_j)}{\lambda_{max}(P_j)}, \\
b_1 &= 4 \max_{1\le j\le m} \lambda_{max}(P_j) \big(\|B_j R_j^{-1} B_j^T\| C_s + \|h\|_\infty\big)^2 / a_1 + \max_{1\le j\le m} \mathrm{tr}(D_j^T P_{j,1} D_j).
\end{aligned}$$

This together with the comparison theorem implies that

$$EV(\theta(t), \bar{x}_i(t)) \leq EV(\theta(0), \bar{x}_i(0))e^{-\frac{1}{2}a_1 t} + \frac{2b_1}{a_1}(1 - e^{-\frac{1}{2}a_1 t}).$$

Thus, we have

$$E[\|\hat{x}_i(t)\|^2 + \|\hat{z}(t)\|^2] \leq \frac{EV(\theta(t), \bar{x}_i(t))}{\min_{1\leq j\leq m}(\lambda_{min}(P_j))}$$

$$\text{(C.2)} \qquad \leq \frac{\max_j \lambda_{max}(P_j)\big(\max_{1\leq i\leq N} E\|\bar{x}_i(0)\|^2 + \|x_0\|^2\big) + 2b_1/a_1}{\min_{1\leq j\leq m}(\lambda_{min}(P_j))} \triangleq C_1,$$

which together with (4.24) implies

$$\text{(C.3)} \qquad E\|\hat{u}_i(t)\|^2 \leq \max_{1\leq j\leq m} \|R_j^{-1}B_j\|^2\big[4 \max_{1\leq j\leq m}(\|K_{j,1}\|^2 + \|K_{j,2}\|^2)C_1 + 2\|r\|_\infty^2\big].$$

Note that C_s and C_1 are independent of i, N, and take

$$C_0 = C_1 + \max_{1\leq j\leq m} \|R_j^{-1}B_j\|^2\big[4 \max_{1\leq j\leq m}(\|K_{j,1}\|^2 + \|K_{j,2}\|^2)C_1 + 2\|r\|_\infty^2\big].$$

Then, by (C.2) and (C.3) we have (5.2). ∎

To prove Theorem 5.3, we present an auxiliary lemma, which can be shown by a similar argument for proving Lemma C.1 in [18], and hence the proof is omitted here.

LEMMA C.1. *If* (A1)–(A3) *hold, and* $I - H$ *is nonsingular, then* $J^{(N)}(u) < \infty$ *implies*

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_i(t)\|^2 + \|x^{(N)}(t)\|^2 + \|u_i(t)\|^2 dt < \infty, \qquad 1 \leq i \leq N.$$

Proof of Theorem 5.3. Notice $\inf_{u_j\in\mathcal{U}_c, j=1,\ldots,N} J_i(u) \leq J_i(\hat{u})$. Then, to obtain (5.8), we need only to prove that for any

$$u \in \mathcal{U}_c' \triangleq \{u_i \in \mathcal{U}_c, i = 1, \ldots, N | J^{(N)}(u) \leq J^{(N)}(\hat{u})\},$$

the following inequality holds:

$$\text{(C.4)} \qquad J^{(N)}(u) \geq J^{(N)}(\hat{u}) - \varepsilon.$$

By Lemma C.1, we have for $u \in \mathcal{U}_c'$,

$$\text{(C.5)} \qquad \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_i\|^2 + \|x^{(N)}\|^2 + \|u_i\|^2 dt < \infty.$$

Let $\tilde{x}_i = x_i - \hat{x}_i$ and $\tilde{u}_i = u_i + R_\theta^{-1}B_\theta^T(K_{\theta,1}x_i + K_{\theta,2}z + r_{\theta,1})$. Then, it follows from (5.1) and (5.2) that

$$d\tilde{x}_i = G_{\theta,1}\tilde{x}_i + B_\theta\tilde{u}_i,$$

$$\text{(C.6)} \qquad \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|\tilde{x}_i\|^2 + \|\tilde{u}_i\|^2 dt < \infty.$$

Let $\hat{x}^{(N)} = 1/N \sum_{j=1}^{N} \hat{x}_j$ and $\tilde{x}^{(N)} = 1/N \sum_{j=1}^{N} \tilde{x}_j$. Then, we have

$$\begin{aligned}
&NJ^{(N)}(u)\\
&= \sum_{i=1}^{N} \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|\hat{x}_i - H\hat{x}^{(N)} - \alpha + \tilde{x}_i - H\tilde{x}^{(N)}\|^2_{Q_\theta}\\
&+\|\hat{u}_i + \tilde{u}_i - R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i\|^2_{R_\theta} dt\\
&= \sum_{i=1}^{N} \big(J_i(\hat{u}) + \tilde{J}_i(\tilde{u}) + 2I_i\big), \qquad \text{(C.7)}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_i(\tilde{u}) &\triangleq \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|\tilde{x}_i - H\tilde{x}^{(N)}\|^2_{Q_\theta} + \|\tilde{u}_i - R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i\|^2_{R_\theta} dt,\\
I_i &\triangleq \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \big(\hat{x}_i - H\hat{x}^{(N)} - \alpha\big)^T Q_\theta \big(\tilde{x}_i - H\tilde{x}^{(N)}\big)\\
&\quad +\hat{u}_i^T R_\theta (\tilde{u}_i - R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i) dt.
\end{aligned}$$

Notice $\tilde{J}_i(\tilde{u}) \ge 0$. To complete the proof, we need only to show $\big|\frac{1}{N}\sum_{i=1}^{N} I_i\big| \le \frac{\varepsilon}{2}$.

To obtain $\big|\frac{1}{N}\sum_{i=1}^{N} I_i\big| \le \frac{\varepsilon}{2}$, we first make the following deformation:

$$\begin{aligned}
&\sum_{i=1}^{N} \big(\hat{x}_i - H\hat{x}^{(N)} - \alpha\big)^T Q_\theta \big(\tilde{x}_i - H\tilde{x}^{(N)}\big)\\
&= \sum_{i=1}^{N} \big(\hat{x}_i - Hz - \alpha\big)^T Q_\theta \big(\tilde{x}_i - H\tilde{x}^{(N)}\big) - \sum_{i=1}^{N} (\hat{x}^{(N)} - z) H^T Q_\theta (\tilde{x}_i - H\tilde{x}^{(N)})\\
&= \sum_{i=1}^{N} \big(\hat{x}_i - Hz - \alpha\big)^T Q_\theta \tilde{x}_i - \big(\hat{x}^{(N)} - Hz - \alpha\big)^T Q_\theta H \sum_{j=1}^{N} \tilde{x}_j\\
&\quad - N\,(\hat{x}^{(N)} - z)^T (H^T Q_\theta - H^T Q_\theta H)\tilde{x}^{(N)}\\
&= \sum_{i=1}^{N} \tilde{x}_i^T \big[Q_\theta \hat{x}_i - (Q_\theta H + H^T Q_\theta - H^T Q_\theta H) z - Q_\theta \alpha + H^T Q_\theta \alpha\big]\\
&\quad - N\,(\hat{x}^{(N)} - z)^T (Q_\theta H + H^T Q_\theta - H^T Q_\theta H)\tilde{x}^{(N)}. \qquad \text{(C.8)}
\end{aligned}$$

Notice

$$\hat{u}_i^T R_\theta (\tilde{u}_i - R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i) = (K_{\theta,1}\hat{x}_i + K_{\theta,2} z + r_{\theta,1})^T B_\theta (R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i - \tilde{u}_i).$$

Then, by (C.8) we get

$$\sum_{i=1}^{N} I_i = \sum_{i=1}^{N} \beta_i + \zeta,$$

where

$$\begin{aligned}
\beta_i &= \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \tilde{x}_i^T \big[Q_\theta \hat{x}_i - (Q_\theta H + H^T Q_\theta - H^T Q_\theta H) z - Q_\theta \alpha + H^T Q_\theta \alpha\big]\\
&\quad +\ (K_{\theta,1}\hat{x}_i + K_{\theta,2} z + r_{\theta,1})^T B_\theta (R_\theta^{-1} B_\theta^T K_{\theta,1}\tilde{x}_i - \tilde{u}_i) dt,\\
\zeta &= \limsup_{T\to\infty} \frac{1}{T} E \int_0^T N(z - \hat{x}^{(N)})^T (Q_\theta H + H^T Q_\theta - H^T Q_\theta H)\tilde{x}^{(N)} dt.
\end{aligned}$$

Applying the generalized Itô's formula [31] to $\tilde{x}_i^T(K_{\theta,1}\hat{x}_i + K_{\theta,2}z + r_{\theta,1})$, from (4.5), (4.15), and (4.16) we obtain

$$
\begin{aligned}
&E\left[\tilde{x}_i^T(T)(K_{\theta,1}\hat{x}_i(T) + K_{\theta,2}z(T) + r_{\theta,1}(T))\right] \\
&= E\int_0^T \tilde{x}_i^T\big[Q_\theta\hat{x}_i - (Q_\theta H + H^TQ_\theta - H^TQ_\theta H)z \\
&\quad -Q_\theta\alpha + H^TQ_\theta\alpha\big] + (K_{\theta,1}\hat{x}_i + K_{\theta,2}z + r_{\theta,1})^TB_\theta(R_\theta^{-1}B_\theta^TK_{\theta,1}\tilde{x}_i - \tilde{u}_i)dt. \qquad \text{(C.9)}
\end{aligned}
$$

Let $\tilde{V}_1(j,\tilde{x}) = \tilde{x}^TP_{j,1}\tilde{x}$, $j = 1,\ldots,m$, $\tilde{x}\in\mathbb{R}^n$. Then, by (B.2) and (C.5) there exists $a_2 > 0$ and $b_2 > 0$ such that

$$
\frac{dE\tilde{V}_1(\theta(t),\tilde{x}_i(t))}{dt} \le -a_2E\tilde{V}_1(\theta(t),\tilde{x}_i(t)) + 2b_2\tilde{u}_i(t)\sqrt{\tilde{V}_1(\theta(t),\tilde{x}_i(t))},
$$

and hence,

$$
\begin{aligned}
E\tilde{V}_1(\theta(t),\tilde{x}_i(t)) &\le \left[\int_0^t e^{-\frac{a_2}{2}(t-s)}b_2u_i(s)ds\right]^2 \\
&\le \int_0^t e^{-a_2(t-s)}b_2^2ds\cdot\int_0^t \|u_i(s)\|^2ds \le C\int_0^t \|u_i(s)\|^2ds,
\end{aligned}
$$

which together with (C.6) implies

$$
E\|\tilde{x}_i(T)\|^2 \le C\int_0^T \|u_i(s)\|^2 \le O(T). \qquad \text{(C.10)}
$$

Moreover, it follows from (C.2) that

$$
E\|K_{\theta,1}\hat{x}_i(T) + K_{\theta,2}z(T)\|^2 \le C.
$$

This together with (C.9), (C.10), and $r_{\theta,1}\in\mathcal{C}_b([0,\infty),\mathbb{R}^n)$ gives

$$
\begin{aligned}
\beta_i &\le \limsup_{T\to\infty}\frac{1}{T}\left\{E\|\tilde{x}_i(T)\|^2E\|K_{\theta,1}\hat{x}_i(T) + K_{\theta,2}z(T) + r_{\theta,1}(T)\|^2\right\}^{1/2} \\
&\le \limsup_{T\to\infty} O\Big(\frac{1}{\sqrt{T}}\Big) = 0, \qquad i = 1,2,\ldots,N.
\end{aligned}
$$

On the other hand, by Theorem 5.1 we can get directly

$$
\limsup_{T\to\infty}\frac{1}{T}E\int_0^T \|\hat{x}^{(N)}\|^2dt \le C_0.
$$

Moreover, it follows from Lemma C.1 that $\limsup_{T\to\infty}\frac{1}{T}E\int_0^T \|\tilde{x}^{(N)}\|^2dt \le C$. This together with the Schwarz's inequality and Lemma 5.2 renders

$$
|\zeta| \le NC\Big(\frac{1}{T}E\int_0^T \|z - \hat{x}^N\|^2dt\Big)^{1/2}\Big(\frac{1}{T}E\int_0^T \|\tilde{x}^N\|^2dt\Big)^{1/2} \le C\sqrt{N}.
$$

Hence,

$$
\left|\frac{1}{N}\sum_{i=1}^N I_i\right| = \frac{1}{N}\left(\sum_{i=1}^N \beta_i + \zeta\right) \le \varepsilon = O\left(\frac{1}{\sqrt{N}}\right).
$$

□

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