IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 61, NO. 11, NOVEMBER 2016

Indefinite Mean-Field Stochastic Linear-Quadratic Optimal Control: From Finite Horizon to Infinite Horizon

Yuan-Hua Ni, Xun Li, and Ji-Feng Zhang, *Fellow, IEEE*

***Abstract*—In this paper, the finite-horizon and the infinite-horizon indefinite mean-field stochastic linear-quadratic optimal control problems are studied. Firstly, the open-loop optimal control and the closed-loop optimal strategy for the finite-horizon problem are introduced, and their characterizations, difference and relationship are thoroughly investigated. The open-loop optimal control can be defined for a fixed initial state, whose existence is characterized via the solvability of a linear mean-field forward-backward stochastic difference equation with stationary conditions and a convexity condition. On the other hand, the existence of a closed-loop optimal strategy is shown to be equivalent to any one of the following conditions: the solvability of a couple of generalized difference Riccati equations, the finiteness of the value function for all the initial pairs, and the existence of the open-loop optimal control for all the initial pairs. It is then proved that the solution of the generalized difference Riccati equations converges to a solution of a couple of generalized algebraic Riccati equations. By studying another generalized algebraic Riccati equation, the existence of the maximal solution of the original ones is obtained together with the fact that the stabilizing solution is the maximal solution. Finally, we show that the maximal solution is employed to express the optimal value of the infinite-horizon indefinite mean-field linear-quadratic optimal control. Furthermore, for the question whether the maximal solution is the stabilizing solution, the necessary and the sufficient conditions are presented for several cases.**

***Index Terms*—Indefinite linear-quadratic optimal control, mean-field theory, stochastic system.**

I. Introduction

In THIS paper, a kind of discrete-time stochastic linear-quadratic (LQ) optimal control of mean-field type is investigated. Compared with the classical stochastic LQ optimal control, an important feature of the problem is that both the objective functional and the dynamics involve the states and the controls as well as their expected values. In this case, the system equation is a discrete-time stochastic difference equation (SDE) of McKean-Vlasov type, which is also referred as the mean-field SDE (MF-SDE). As a feature of such a class of SDEs, the dynamics depend on the statistical distribution of the solution, which provides simple but effective techniques for studying large systems by reducing their dimension and complexity. This new feature roots itself in the category of the mean-field theory, which is developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamical systems. According to the mean-field theory, the interactions among agents are modeled by a mean-field term. When the number of individuals goes to infinity, the mean-field term approaches the expected value.

The past few years have witnessed many successful applications of the mean-field formulation in various fields of engineering, games, finance and economics; and the mean-field control theory has attracted much attention from the mathematics and control communities. The investigation of continuous-time mean-field stochastic differential equations can be traced back to the 1960s [32]. In [3], to cope with the possible time-inconsistency of the optimal control, an extended version of dynamic programming principle is derived by using the Nisio nonlinear operator semigroup. Recently, stochastic maximum principles of mean-field type are extensively studied in several works [2], [22], [30], and [41], which specify the necessary conditions for the optimality. The results range from the case of a convex action space to the case of a general action space. As applications, the Markowitz mean-variance portfolio selection and a class of mean-field LQ problems are studied in [2] and [30] by using the stochastic maximum principle. In [41], the stochastic maximum principle under partial information is investigated, while [22] presents the maximum principle for the controlled mean-field forward-backward stochastic differential equations with Poisson jumps. In [43], the definite mean-field LQ control with a finite time horizon is systemically studied by using a variational method and a decoupling technique. It is shown that the optimal control is of linear feedback form and that the gains are represented by the solutions of two coupled differential Riccati equations. In [20], the discrete-time definite mean-field LQ problem is formulated as an operator stochastic LQ optimal control problem. By the kernel-range decomposition representation of the expectation operator and its pseudo-inverse, an optimal control is obtained based on the

Manuscript received May 28, 2015; revised October 6, 2015; accepted December 8, 2015. Date of publication December 17, 2015; date of current version October 25, 2016. This work was supported by the National Natural Science Foundation of China under Grants 11471242 and 61227902, by the China Postdoctoral Science Foundation, by the Hong Kong RGC Grants 520412 and 15209614, and by the National Key Basic Research Program of China (973 Program) under Grant 2014CB845301. Recommended by Associate Editor H. L. Trentelman.

Y-H. Ni is with the Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin 300387, China, and is also with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (e-mail: yhni@amss.ac.cn).

X. Li is with the Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong (e-mail: malixun@polyu.edu.hk).

J-F. Zhang is with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (e-mail: jif@iss.ac.cn).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2015.2509958

0018-9286 © 2015 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.

solutions of two Riccati difference equations. Furthermore, the closed-loop formulation is also investigated. Later, [23] and [33] generalize results in [20] and [43] to the case with an infinite time horizon.

As pointed out above, the mean-field stochastic differential equations were first studied in 1960s. A recent study on controlled mean-field stochastic differential and difference equations is partially relighted by a surge of interest in mean-field games (see [10]–[15], [24]–[26], [28], [31], [38], and [40]). Compared with the topic of this paper, mean-field games use decentralized controls, that is, the controls are selected to achieve each individual's own goal by using local information. In [24]–[26], Huang, Caines and Malhame investigate large population stochastic dynamical games with mean-field terms. Independently, Lasry and Lions [28] introduce similar problems from the viewpoint of the mean-field theory. Now both the Huang-Caines-Malhame and the Lasry-Lions formulations are termed as mean-field games. Subsequently, for the large population multi-agent systems, [31] and [40] deal with the asymptotically optimal decentralized control problem and the problem with Markov jump parameters, respectively. In [10] and [11], LQ N-person games and mean-field games with ergodic costs are intensively explored via studying the Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Planck (HJB-KFP) equations; the contents range from the existence of affine Nash equilibria of the LQ N-person game, the asymptotical property of the HJB-KFP equations, and examples with explicit solutions. Risk-sensitive mean-field games are firstly formulated in [38] together with several interesting aspects: the equation that the mean-field value satisfies, an explicit solution of the mean-field best response and an equivalent mean-field risk-neutral formulation. In [13], LQ mean-field games are investigated via the adjoint equation approach; several sufficient conditions are presented to ensure the existence and uniqueness of equilibrium strategy. Interestingly, a mean-field LQ optimal control with finite horizon is also studied using the same approach. Concerned with the difference between mean-field games and mean-field optimal control, "mean field games can be reduced to a standard control problem and an equilibrium, and mean field type control is a nonstandard control problem" [12]. We can refer to [12], [14], [15] and other related works for more extensive contents of the backgrounds, results, methodologies and more advanced questions of these two classes of mean-field type problems.

Mean-field stochastic LQ optimal control with indefinite cost weighting matrices is studied in this paper, which is referred as the indefinite mean-field stochastic LQ optimal control. Indefinite stochastic LQ optimal control without mean-field terms was first studied at the end of last century. It is found that an indefinite stochastic LQ problem may still be well-posed, which challenges the standard belief about LQ problems [4]–[7]. It is further shown that the indefinite stochastic LQ problems are closely related to Markowitz's mean-variance portfolio selection problems in financial investment [29], [45]. As pointed out in [29] and [45], when the expectations of state and control appear nonlinearly in the cost functional, the corresponding problems are nonseparable in the sense that the standard dynamic-programming-based methodology fails to work. In [34], for an inhomogeneous version of Problem (MF-LQ) with $L_k, \bar{L}_k = 0, k \in \mathbb{T}$ (see in Section II), the authors propose a modified backward recursive technique, and get around the nonseparability using the method of completing the square. Moreover, it is shown that the well-posedness and the solvability of the mean-field LQ problem are both equivalent to the solvability of a couple of generalized difference Riccati equations (GDREs) and a constrained linear recursive equation. As an application, the multi-period mean-variance portfolio selection is well studied, and the obtained results extend those in [29] to the case that the return rates of the risky securities are possibly degenerate.

In this paper, we shall investigate both the finite-horizon and the infinite-horizon indefinite mean-field stochastic LQ optimal control problems. For the finite-horizon poblem, we give a more detailed and deeper investigation than that in [34]. Specifically, we introduce the open-loop optimal control and the closed-loop optimal strategy, and investigate their characterizations, difference and relationship. The open-loop optimal control can be defined for a fixed initial state, whose existence is characterized via the solvability of a linear mean-field forward-backward stochastic difference equation (MF-FBSDE) with stationary conditions and a convexity condition. Note that the open-loop optimal control may depend on the initial state. In contrast to this, the closed-loop optimal strategy is required to be independent of all the initial pairs. It is then shown that the existence of the closed-loop optimal strategy is equivalent to the solvability of a couple of GDREs, the finiteness of the value function for all the initial pairs, and the existence of the open-loop optimal control for all the initial pairs.

To study the infinite-horizon mean-field LQ optimal control, we first review the properties of the stability of the MF-SDEs and introduce a couple of generalized algebraic Riccati equations (GAREs). To study the GAREs, we construct another GARE, by which we easily obtain the existence and the properties of the maximal solution of the original GAREs. It is then shown that the maximal solution to the GAREs is employed to express the optimal value of the infinite-horizon mean-field LQ optimal control. Finally, the problem "when is the maximal solution of the GAREs a stabilizing solution?" is partially addressed. This relates to an open problem raised by the seminal work [42] of finding an optimal control by the primal-dual semidefinite programming (SDP) technique.

As mentioned above, the discrete-time linear MF-FBSDEs are introduced in this paper. Compared with the continuous-time backward stochastic differential equations and forward-backward stochastic differential equations, the discrete-time case has more compact forms and are easier to validate the well-posedness of the existence of the solutions. Concerned with the differences, we consider the infinite-horizon backward stochastic difference and differential equations. The infinite-horizon backward differential equations (without mean-field terms) are intensively studied in [35] and [37]; it is shown that the L^2-stable solution exists under mild conditions. To the best of our knowledge, there are some difficulties to establish similar results for the infinite-horizon backward stochastic difference equations. Unlike the Brownian motion that drives the continuous-time backward differential equations in [35] and [37], the noise $\{w_k\}$ in this paper is assumed to be any martingale difference with properties (2) given below. Generally speaking, the process $\{v_k = \sum_{l=0}^{k} w_l\}$ does not have the so-called martingale characterization, as the one for the Brownian motion. Due to this fact, in the study of the infinite-horizon mean-field LQ optimal control we do not touch the infinite-horizon discrete-time MF-FBSDEs, which will be left for future research.

The remainder of this paper is organized as follows. Sections II and III deal with the finite-horizon and the infinite-horizon mean-field LQ optimal control, respectively. In Section IV, several examples are presented. Section V gives some concluding remarks.

II. Indefinite Mean-Field Stochastic LQ Optimal Control Over a Finite Horizon

A. Open-Loop Optimal Control

Consider the following dynamic system:

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) \\ \qquad\quad + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k) w_k \\ x_0 = \zeta, k \in \mathbb{T} \end{cases} \tag{1}$$

where $A_k, \bar{A}_k, C_k, \bar{C}_k \in \mathbb{R}^{n\times n}$, and $B_k, \bar{B}_k, D_k, \bar{D}_k \in \mathbb{R}^{n\times m}$ are given deterministic matrices; $\mathbb{T}$ denotes the set $\{0, 1, \ldots, N-1\}$. In (1), $\{x_k, k \in \bar{\mathbb{T}}\}$, $\{u_k, k \in \mathbb{T}\}$ and $\{w_k, k \in \mathbb{T}\}$ are the state, control and disturbance process, respectively, with $\bar{\mathbb{T}} = \{0, 1, \ldots, N\}$; $\{w_k\}$ is assumed to be a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and

$$\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0, \quad \mathbb{E}\left[(w_{k+1})^2|\mathcal{F}_k\right] = 1 \tag{2}$$

with $\mathcal{F}_k$ being the σ-algebra generated by $\{x_0, w_l, l = 0, 1, \ldots, k\}$. For convenience, $\mathcal{F}_{-1}$ denotes $\sigma(x_0)$. The initial value ζ is assumed to be square integrable. The cost functional associated with (1) is

$$\begin{aligned} J(\zeta; u) = \sum_{k=0}^{N-1} \mathbb{E}\big[&x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + 2x_k^T L_k u_k \\ &+ 2(\mathbb{E}x_k)^T \bar{L}_k \mathbb{E}u_k + u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k\big] \\ &+ \mathbb{E}\left(x_N^T G_N x_N\right) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N \end{aligned} \tag{3}$$

where $Q_k, \bar{Q}_k \in \mathbb{R}^{n\times n}, R_k, \bar{R}_k \in \mathbb{R}^{m\times m}, L_k, \bar{L}_k \in \mathbb{R}^{n\times m}$, $k \in \mathbb{T}$, $G_N, \bar{G}_N \in \mathbb{R}^{n\times n}$ are deterministic symmetric matrices of appropriate dimensions.

For any $t \in \mathbb{T}$, denote $\{t, \ldots, N-1\}$ and $\{t, \ldots, N\}$ by $\mathbb{T}_t$ and $\bar{\mathbb{T}}_t$, respectively. Clearly, $\mathbb{T}_0 = \mathbb{T}$, $\bar{\mathbb{T}}_0 = \bar{\mathbb{T}}$. Let $L^2_{\mathcal{F}}(\mathbb{T}_t; \mathcal{H})$ be the set of $\mathcal{H}$-valued processes $\nu = \{\nu_k, k \in \mathbb{T}_t\}$ such that ν_k is $\mathcal{F}_{k-1}$-measurable and $\sum_{k=t}^{N-1} \mathbb{E}|\nu_k|^2 < \infty$. In addition, $L^2_{\mathcal{F}}(t; \mathcal{H})$ is the set of random variables ξ such that $\xi \in \mathcal{H}$ is $\mathcal{F}_{t-1}$-measurable and $\mathbb{E}|\xi|^2 < \infty$. Let $\mathcal{X}_0 = \{\zeta | \zeta$ is $\mathcal{F}_{-1}-$ measurable and square integrable$\}$, which denotes the set of all the initial states. Then the optimal control over $\mathbb{T}$ is stated as follows.

Problem (MF-LQ). Given $\zeta \in \mathcal{X}_0$, find a $u^* \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$ such that

$$J(\zeta; u^*) = \inf_{u \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)} J(\zeta; u). \tag{4}$$

For any admissible control $u \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$, the requirement that u_k is $\mathcal{F}_{k-1}$-measurable is parallel to the standard statement on the admissible controls of continuous-time stochastic optimal control; see [21] and [44] for details. Furthermore, by known result of probability theory, there exists a measurable function f_k such that $u_k = f_k(x_0, w_0, \ldots, w_{k-1})$. In other words, u_k is determined from $\{k, x_0, w_0, \ldots, w_{k-1}\}$ only, irrespective of how the state process x of (1) develops. From this and the standard arguments about the open-loop control [9], any u in $L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$ can be viewed as an open-loop control. We then call u^* satisfying (4) an open-loop optimal control for the initial state ζ. Here, the "open-loop optimal" is due to the fact that we minimize $J(\zeta; u)$ over $L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$.

Introduce an inner product in $L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^p)$ $(p = n, m)$

$$\left\langle y^{(1)}, y^{(2)} \right\rangle = \sum_{k=0}^{N-1} \mathbb{E}\left(\left(y_k^{(1)}\right)^T y_k^{(2)}\right)$$

for $y^{(1)}, y^{(2)} \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^p)$, and use the convention

$$\begin{cases} (Qx)(\cdot) = Q.x., \quad \forall\, x \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^n) \\ (\bar{Q}\varphi)(\cdot) = \bar{Q}.\varphi., \quad \forall\, \varphi = (\varphi_0, \ldots, \varphi_{N-1}) \\ \qquad \text{with } \varphi_k \in \mathbb{R}^n \text{ such that } \sum_{k=0}^{N-1} |\varphi_k|^2 < \infty \\ (L^T x)(\cdot) = L^T_. x., \quad \forall\, x \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^n) \\ (\bar{L}^T \varphi)(\cdot) = \bar{L}^T_. \varphi., \quad \forall\, \varphi = (\varphi_0, \ldots, \varphi_{N-1}) \\ \qquad \text{with } \varphi_k \in \mathbb{R}^n \text{ such that } \sum_{k=0}^{N-1} |\varphi_k|^2 < \infty \\ (Ru)(\cdot) = R.u., \quad \forall\, u \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m) \\ (\bar{R}\psi)(\cdot) = \bar{R}.\psi., \quad \forall\, \psi = (\psi_0, \ldots, \psi_{N-1}) \text{ with } \psi_k \in \mathbb{R}^m \\ \qquad \text{such that } \sum_{k=0}^{N-1} |\psi_k|^2 < \infty \\ (\mathbb{E}\nu)(\cdot) = \mathbb{E}\nu., \nu \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^p). \end{cases}$$

For $L^2_{\mathcal{F}}(N-1; \mathbb{R}^p)$ $(p = n, m)$, an inner product is defined as $\left\langle y^{(1)}, y^{(2)} \right\rangle = \mathbb{E}\left(y^{(1)T} y^{(2)}\right)$, $\quad y^{(1)}, y^{(2)} \in L^2_{\mathcal{F}}(N-1; \mathbb{R}^p)$.

The cost functional $J(\zeta; u)$ is then represented as

$$\begin{aligned} J(\zeta; u) = &\langle Qx, x\rangle + \langle \bar{Q}\mathbb{E}x, \mathbb{E}x\rangle + 2\langle L^T x, u\rangle + 2\langle \bar{L}^T \mathbb{E}x, \mathbb{E}u\rangle \\ &+ \langle Ru, u\rangle + \langle \bar{R}\mathbb{E}u, \mathbb{E}u\rangle + \langle Gx_N, x_N\rangle \\ &+ \langle \bar{G}\mathbb{E}x_N, \mathbb{E}x_N\rangle. \end{aligned} \tag{5}$$

Fixing ζ, $J(\zeta; u)$ is a quadratic functional of u, and x is the internal state, which is induced by ζ and u.

In what follows, we shall calculate the first order and second order directional derivatives (when they exist), and intend to characterize the existence of the optimal control via the two derivatives. Denote by x^λ the solution to (1) with control $u + \lambda\bar{u}$. Hence

$$\begin{cases} \frac{x^\lambda_{k+1} - x_{k+1}}{\lambda} = \left[A_k \frac{x^\lambda_k - x_k}{\lambda} + \bar{A}_k \frac{\mathbb{E}x^\lambda_k - \mathbb{E}x_k}{\lambda} + B_k \bar{u}_k + \bar{B}_k \mathbb{E}\bar{u}_k\right] \\ \quad + \left[C_k \frac{x^\lambda_k - x_k}{\lambda} + \bar{C}_k \frac{\mathbb{E}x^\lambda_k - \mathbb{E}x_k}{\lambda} + D_k \bar{u}_k + \bar{D}_k \mathbb{E}\bar{u}_k\right] w_k \\ \frac{x^\lambda_0 - x_0}{\lambda} = 0. \end{cases}$$

As $(x_0^\lambda - x_0)/\lambda = 0$, $((x_{k+1}^\lambda - x_{k+1})/\lambda)$ is then independent of u and λ. For $k \in \mathbb{T}$, denote $(x_{k+1}^\lambda - x_{k+1})/\lambda$ by y_{k+1}, which satisfies

$$\begin{cases} y_{k+1} = [A_k y_k + \bar{A}_k \mathbb{E} y_k + B_k \bar{u}_k + \bar{B}_k \mathbb{E} \bar{u}_k] \\ \qquad\quad + [C_k y_k + \bar{C}_k \mathbb{E} y_k + D_k \bar{u}_k + \bar{D}_k \mathbb{E} \bar{u}_k] w_k \\ y_0 = 0, \quad k \in \mathbb{T}. \end{cases} \tag{6}$$

Clearly, for any $k \in \bar{\mathbb{T}}$, it holds that $x_k^\lambda = x_k + \lambda y_k$. To obtain the first order directional derivative, we need some preparations. Simple calculations show that

$$\lim_{\lambda \downarrow 0} \frac{\langle R(u+\lambda \bar{u}), u + \lambda \bar{u}\rangle - \langle Ru, u\rangle}{\lambda} = 2\langle Ru, \bar{u}\rangle + \lim_{\lambda \downarrow 0} \lambda \langle R\bar{u}, \bar{u}\rangle = 2\langle Ru, \bar{u}\rangle.$$

Similarly, we have

$$\lim_{\lambda \downarrow 0} \frac{\langle \bar{R}\mathbb{E}(u+\lambda\bar{u}), \mathbb{E}(u+\lambda\bar{u})\rangle - \langle \bar{R}\mathbb{E}u, \mathbb{E}u\rangle}{\lambda} = 2\langle \bar{R}\mathbb{E}u, \mathbb{E}\bar{u}\rangle$$
$$\lim_{\lambda \downarrow 0} \frac{\langle Qx^\lambda, x^\lambda\rangle - \langle Qx, x\rangle}{\lambda} = 2\langle Qx, y\rangle$$
$$\lim_{\lambda \downarrow 0} \frac{\langle \bar{Q}\mathbb{E}x^\lambda, \mathbb{E}x^\lambda\rangle - \langle \bar{Q}\mathbb{E}x, \mathbb{E}x\rangle}{\lambda} = 2\langle \bar{Q}\mathbb{E}x, \mathbb{E}y\rangle$$
$$\lim_{\lambda \downarrow 0} \frac{\langle L^T x^\lambda, u+\lambda\bar{u}\rangle - \langle L^T x, u\rangle}{\lambda} = \langle L^T x, \bar{u}\rangle + \langle L^T y, u\rangle$$
$$\lim_{\lambda \downarrow 0} \frac{\langle \bar{L}^T \mathbb{E}x^\lambda, \mathbb{E}(u+\lambda\bar{u})\rangle - \langle \bar{L}^T\mathbb{E}x, \mathbb{E}u\rangle}{\lambda} = \langle \bar{L}^T\mathbb{E}x, \mathbb{E}\bar{u}\rangle + \langle \bar{L}^T\mathbb{E}y, \mathbb{E}u\rangle.$$

Therefore, the first order directional derivative with the direction $\bar{u}$ is given by

$$\begin{aligned} dJ(\zeta;u;\bar{u}) &= \lim_{\lambda \downarrow 0} \frac{J(\zeta; u+\lambda\bar{u}) - J(\zeta;u)}{\lambda} \\ &= 2\langle Qx, y\rangle + 2\langle \bar{Q}\mathbb{E}x, \mathbb{E}y\rangle + 2\langle L^T x, \bar{u}\rangle + 2\langle L^T y, u\rangle \\ &\quad + 2\langle \bar{L}^T\mathbb{E}x, \mathbb{E}\bar{u}\rangle + 2\langle \bar{L}^T\mathbb{E}y, \mathbb{E}u\rangle + 2\langle Ru, \bar{u}\rangle \\ &\quad + 2\langle \bar{R}\mathbb{E}u, \mathbb{E}\bar{u}\rangle + 2\langle Gx_N, y_N\rangle + 2\langle \bar{G}\mathbb{E}x_N, \mathbb{E}y_N\rangle. \end{aligned}$$

Similarly, we can derive the second order directional derivative with the directions $\bar{u}$ and $\hat{u}$

$$\begin{aligned} & d^2 J(\zeta;u;\bar{u};\hat{u}) \\ &= \lim_{\lambda \downarrow 0} \frac{dJ(\zeta; u+\lambda\hat{u};\bar{u}) - dJ(\zeta;u;\bar{u})}{\lambda} \\ &= 2\langle Q\hat{y}, y\rangle + 2\langle \bar{Q}\mathbb{E}\hat{y}, \mathbb{E}y\rangle + 2\langle L^T \hat{y}, \bar{u}\rangle + 2\langle L^T y, \hat{u}\rangle \\ &\quad + 2\langle L^T\mathbb{E}\hat{y}, \mathbb{E}\bar{u}\rangle + 2\langle L^T\mathbb{E}y, \mathbb{E}\hat{u}\rangle + 2\langle Ru, \bar{u}\rangle \\ &\quad + 2\langle \bar{R}\mathbb{E}u, \mathbb{E}\bar{u}\rangle + 2\langle G\hat{y}_N, y_N\rangle + 2\langle \bar{G}\mathbb{E}\hat{y}_N, \mathbb{E}y_N\rangle \end{aligned}$$

where

$$\begin{cases} \hat{y}_{k+1} = [A_k \hat{y}_k + \bar{A}_k \mathbb{E}\hat{y}_k + B_k \hat{u}_k + \bar{B}_k \mathbb{E}\hat{u}_k] \\ \qquad\quad + [C_k \hat{y}_k + \bar{C}_k \mathbb{E}\hat{y}_k + D_k \hat{u}_k + \bar{D}_k \mathbb{E}\hat{u}_k] w_k \\ \hat{y}_0 = 0. \end{cases}$$

If $\hat{u} = \bar{u}$, then it holds that

$$d^2 J(\zeta;u;\bar{u};\bar{u}) = 2J(0;\bar{u}) \tag{7}$$

which is independent of u and ζ. Furthermore, we can show that $J(\zeta;u)$ is infinitely differentiable in the sense that the directional derivatives of all orders exist.

By classical results on convex analysis [19], the convexity of the map $u \mapsto J(\zeta;u)$ can be fully characterized via $d^2 J(\zeta;u;\bar{u};\bar{u})$, and we have the following result by combining the property (7).

Lemma 2.1: The following statements are equivalent.

(i) The map $u \mapsto J(0;u)$ is convex.
(ii) $\inf_{u \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)} J(0;u) \geq 0$.
(iii) The map $u \mapsto J(\zeta;u)$ is convex.

The following theorem gives several equivalent characterizations on the existence of the open-loop optimal control of Problem (MF-LQ).

Theorem 2.1: Given $\zeta \in \mathcal{X}_0$, the following statements are equivalent.

(i) There exists an open-loop optimal control of Problem (MF-LQ).
(ii) There exists a u^* in $L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$ such that
$$\forall\, \bar{u} \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m), dJ(\zeta;u^*;\bar{u}) = 0 \tag{8}$$
and $\inf_{u \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)} J(0;u) \geq 0$.
(iii) There exists a u^* in $L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$ such that the following MF-FBSDE admits a solution (x, z):
$$\begin{cases} x_{k+1} = \left(A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k^* + \bar{B}_k \mathbb{E}u_k^*\right) \\ \qquad\quad + \left(C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k^* + \bar{D}_k \mathbb{E}u_k^*\right) w_k \\ z_k = A_k^T \mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) + \bar{A}_k^T \mathbb{E}z_{k+1} \\ \qquad + C_k^T \mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) + \bar{C}_k^T \mathbb{E}(z_{k+1}w_k) \\ \qquad + Q_k x_k + \bar{Q}_k \mathbb{E}x_k + L_k u_k^* + \bar{L}_k \mathbb{E}u_k^* \\ x_0 = \zeta, z_N = Gx_N + \bar{G}\mathbb{E}x_N, \quad k \in \mathbb{T} \end{cases} \tag{9}$$
with the following stationary conditions
$$\begin{cases} 0 = B_k^T \mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) + \bar{B}_k^T \mathbb{E}z_{k+1} \\ \qquad + D_k^T \mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) + \bar{D}_k^T \mathbb{E}(z_{k+1}w_k) \\ \qquad + L_k^T x_k + \bar{L}_k^T \mathbb{E}x_k + R_k u_k^* + \bar{R}_k \mathbb{E}u_k^* \\ k \in \mathbb{T}. \end{cases} \tag{10}$$
Moreover, the following holds:
$$\inf_{u \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)} J(0;u) \geq 0. \tag{11}$$

Under any of above conditions, u^* is an optimal control and the optimal value of Problem (MF-LQ) at ζ is given by

$$V(\zeta) = \inf_{u \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)} J(\zeta;u) = \mathbb{E}\left[z_0^T \zeta\right]. \tag{12}$$

Proof: Since $J(\zeta;u)$ is infinitely differentiable with respect to u and $d^2 J(\zeta;u;\bar{u};\bar{u})$ is independent of u, the minimizing point u^* of $J(\zeta;u)$ is characterized by the first order and second order derivatives: $dJ(\zeta;u^*;\bar{u}) = 0$ and $d^2 J(\zeta;u^*;\bar{u};\bar{u}) \geq 0$ for any $\bar{u}$ in $L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$. Due to this and Lemma 2.1, the equivalence between (i) and (ii) follows. We now prove the equivalence between (ii) and (iii).

(iii)$\Rightarrow$ (ii). From (6), we have

$$\begin{cases} \mathbb{E}y_{k+1} = (A_k + \bar{A}_k)\mathbb{E}y_k + (B_k + \bar{B}_k)\mathbb{E}\bar{u}_k \\ \mathbb{E}y_0 = 0, k \in \mathbb{T} \end{cases}$$

$$\begin{cases} y_{k+1} - \mathbb{E}y_{k+1} = [A_k(y_k - \mathbb{E}y_k) + B_k(\bar{u}_k - \mathbb{E}\bar{u}_k)] \\ \qquad\qquad + \left[C_k(y_k - \mathbb{E}y_k) + (C_k + \bar{C}_k)\mathbb{E}y_k \right. \\ \qquad\qquad\quad \left. + D_k(\bar{u}_k - \mathbb{E}\bar{u}_k) + (D_k + \bar{D}_k)\mathbb{E}\bar{u}_k\right] w_k \\ y_0 - \mathbb{E}y_0 = 0, k \in \mathbb{T}. \end{cases}$$

Furthermore, from (9), it holds that

$$\begin{cases} z_k - \mathbb{E}z_k = A_k^T \left(\mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) - \mathbb{E}z_{k+1}\right) \\ \qquad + C_k^T \left(\mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) - \mathbb{E}(z_{k+1}w_k)\right) \\ \qquad + Q_k(x_k - \mathbb{E}x_k) + L_k\left(u_k^* - \mathbb{E}u_k^*\right) \\ z_N - \mathbb{E}z_N = G(x_N - \mathbb{E}x_N), \quad k \in \mathbb{T} \end{cases} \tag{13}$$

$$\begin{cases} \mathbb{E}z_k = (A_k + \bar{A}_k)^T \mathbb{E}z_{k+1} \\ \qquad +(C_k + \bar{C}_k)^T \mathbb{E}(z_{k+1}w_k) \\ \qquad +(Q_k + \bar{Q}_k)\mathbb{E}x_k + (L_k + \bar{L})\mathbb{E}u_k^* \\ \mathbb{E}z_N = (G + \bar{G})\mathbb{E}x_N, \quad k \in \mathbb{T}. \end{cases} \tag{14}$$

Therefore, we have for any $\bar{u} \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$

$$\begin{aligned} &\frac{1}{2} dJ(\zeta; u^*; \bar{u}) \\ &= \sum_{k=0}^{N-1} \mathbb{E}\Big\{\big[A_k^T \left(\mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) - \mathbb{E}z_{k+1}\right) \\ &\quad + C_k^T \left(\mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) - \mathbb{E}(z_{k+1}w_k)\right) \\ &\quad + Q_k(x_k - \mathbb{E}x_k) + L_k\left(u_k^* - \mathbb{E}u_k^*\right) - (z_k - \mathbb{E}z_k)\big]^T \\ &\quad \cdot (y_k - \mathbb{E}y_k) + \big[B_k^T \left(\mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) - \mathbb{E}z_{k+1}\right) \\ &\quad + D_k^T \left(\mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) - \mathbb{E}(z_{k+1}w_k)\right) \\ &\quad + L_k^T(x_k - \mathbb{E}x_k) + R_k\left(u_k^* - \mathbb{E}u_k^*\right)\big]^T \left(\bar{u}_k - \mathbb{E}\bar{u}_k\right)\Big\} \\ &\quad + \sum_{k=0}^{N-1} \Big\{\big[(A_k + \bar{A}_k)^T \mathbb{E}z_{k+1} - \mathbb{E}z_k \\ &\quad + (C_k + \bar{C}_k)^T \mathbb{E}(z_{k+1}w_k) \\ &\quad + (Q_k + \bar{Q}_k)\mathbb{E}x_k + (L_k + \bar{L}_k)\mathbb{E}u_k^*\big]^T \mathbb{E}y_k \\ &\quad + \big[(B_k + \bar{B}_k)^T \mathbb{E}z_{k+1} + (L_k + \bar{L}_k)^T \mathbb{E}x \\ &\quad + (D_k + \bar{D}_k)^T \mathbb{E}(z_{k+1}w_k) \\ &\quad + (R_k + \bar{R}_k)\mathbb{E}u_k^*\big]^T \mathbb{E}\bar{u}_k\Big\}. \end{aligned} \tag{15}$$

Due to (10) and (13)–(15), we have

$$dJ(\zeta; u^*; \bar{u}) = 0, \quad \forall\, \bar{u} \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m).$$

Therefore, (ii) holds.

(ii)⇒ (iii). Let u^* be the optimal control of Problem (MF-LQ). Substituting this u^* into (9), the MF-FBSDE (9) is then a partially decoupled one, i.e., the backward state z does not appear in the forward MF-SDE (equation for x). Therefore, given (x, u^*), the mean-field backward stochastic difference equation (MF-BSDE)

$$\begin{cases} z_k = A_k^T \mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) + \bar{A}_k^T \mathbb{E}z_{k+1} \\ \qquad + C_k^T \mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) + \bar{C}_k^T \mathbb{E}(z_{k+1}w_k) \\ \qquad + Q_k x_k + \bar{Q}_k \mathbb{E}x_k + L_k u_k^* + \bar{L}_k \mathbb{E}u_k^* \\ z_N = G x_N + \bar{G}\mathbb{E}x_N, \quad k \in \mathbb{T} \end{cases}$$

is well-defined. This means that the MF-FBSDE (9) admits a solution (x, z). We then have the expressions (13) and (14), which imply the desired result (15). As $\bar{u}$ can be arbitrarily selected in $L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$, (10) holds from (8) and (15).

Finally, similar to (15), we can get the result (12). This completes the proof. □

Let

$$\begin{aligned} &J(t, \hbar; u) \\ &= \sum_{k=t}^{N-1} \mathbb{E}\big[x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + 2x_k^T L_k u_k \\ &\qquad + 2(\mathbb{E}x_k)^T \bar{L}_k \mathbb{E}u_k + u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k\big] \\ &\quad + \mathbb{E}\left(x_N^T G_N x_N\right) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N \end{aligned}$$

where $t \in \mathbb{T}$ and $\hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^m)$. We then state the following problem.

Problem $(\textbf{MF-LQ})_t$. Given $\hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^m)$, find a $u^* \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$ such that

$$J(t, \hbar; u^*) = \inf_{u \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)} J(t, \hbar; u).$$

When $t = 0$, Problem $(\text{MF-LQ})_0$ is a version of Problem (MF-LQ). We then have the following corollary.

Corollary 2.1: Let Problem (MF-LQ) admit an open-loop optimal control u^* and $(x(\cdot; \zeta, u^*), z(\cdot; \zeta, u^*))$ be the solution of the MF-FBSDE (9). Then for any $t \in \mathbb{T}$, the optimal value of Problem $(\text{MF-LQ})_t$ with the initial state $x(t; \zeta, u^*)$ is finite, and $u^*|_{\mathbb{T}_t}$, the restriction of u^* on $\mathbb{T}_t$, is the optimal control of Problem $(\text{MF-LQ})_t$ for the initial state $x(t; \zeta, u^*)$.

Proof: It holds that the restriction of (9) on $\bar{\mathbb{T}}_t$ admits a solution with initial forward state $x(t; \zeta, u^*)$. Furthermore, versions of (10) and (11) are satisfied. We thus achieve the conclusions. □

B. Closed-Loop Optimal Strategy

We now introduce a type of closed-loop optimal strategy for Problem (MF-LQ). This notion is motivated by those in [20] and [36].

Definition 2.1: (i) Let $u = \{\mathbb{K}_k x_k + \bar{\mathbb{K}}_k \mathbb{E}x_k, k \in \mathbb{T}\}$ be a control of (1) with $\mathbb{K}_k, \bar{\mathbb{K}}_k \in \mathbb{R}^{m\times n}, k \in \mathbb{T}$, being deterministic matrices. If $\mathbb{K}_k, \bar{\mathbb{K}}_k, k \in \mathbb{T}$, are independent of all the initial pairs $(t, \hbar)$, $t \in \mathbb{T}, \hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^m)$, we then call $(\mathbb{K}, \bar{\mathbb{K}}) = \{(\mathbb{K}_k, \bar{\mathbb{K}}_k), k \in \mathbb{T}\}$ a closed-loop strategy of (1).

(ii) A closed-loop strategy $(\mathbb{K}^*, \bar{\mathbb{K}}^*) = \{(\mathbb{K}_k^*, \bar{\mathbb{K}}_k^*), k \in \mathbb{T}\}$ of (1) with $\mathbb{K}_k^*, \bar{\mathbb{K}}_k^* \in \mathbb{R}^{m\times n}, k \in \mathbb{T}$, is called a closed-loop optimal strategy of Problem (MF-LQ) if

$$J\left(t, \hbar; (\mathbb{K}^* x^* + \bar{\mathbb{K}}^* \mathbb{E}x^*)|_{\mathbb{T}_t}\right) \le J(t, \hbar; u) \tag{16}$$

holds for all $t \in \mathbb{T}$, $\hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^n)$ and $u \in L^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$. Here, $\mathbb{K}^* x^* + \bar{\mathbb{K}}^* \mathbb{E}x^*$ is understood as the control $\{\mathbb{K}_k^* x_k^* + \bar{\mathbb{K}}_k^* \mathbb{E}x_k^*, k \in \mathbb{T}\}$ with x^* being the corresponding state of (1), and $(\mathbb{K}^* x^* + \bar{\mathbb{K}}^* \mathbb{E}x^*)|_{\mathbb{T}_t}$ is the restriction of $\mathbb{K}^* x^* + \bar{\mathbb{K}}^* \mathbb{E}x^*$ on $\mathbb{T}_t$.

Remark 2.1: Note that the open-loop optimal control can be defined for a given initial pair, which may be viewed as a local property. In contrast to this, we shall refer to the property that closed-loop strategies are independent of all the initial pairs as being a global property. Furthermore, in Definition 2.1, (16) holds for all the $t \in \mathbb{T}$ and $\hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^n)$, which is different from that in [36] and [37]. This is because of the following fact. If (16) holds at $t = 0$ for any $\hbar \in L^2_{\mathcal{F}}(0; \mathbb{R}^n)$, we cannot assure that (16) holds at any $t \in \mathbb{T}_1$ and any $\hbar \in L^2_{\mathcal{F}}(t; \mathbb{R}^n)$.

Introduce a control $u^v = \{\mathbb{K}^* x_k + \bar{\mathbb{K}}^* \mathbb{E}x_k + v_k, k \in \mathbb{T}\}$ with $v \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$, under which (1) becomes

$$\begin{cases} x_{k+1} = \big[(A_k + B_k\mathbb{K}^*_k)x_k + B_k v_k + \bar{B}_k \mathbb{E}v_k \\ \qquad + (\bar{A}_k + B_k\bar{\mathbb{K}}^*_k + \bar{B}_k(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k))\mathbb{E}x_k\big] \\ \qquad + \big[(C_k + D_k\mathbb{K}^*_k)x_k + D_k v_k + \bar{D}_k \mathbb{E}v_k \\ \qquad + \big(\bar{C}_k + D_k\bar{\mathbb{K}}^*_k + \bar{D}_k(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k)\big)\mathbb{E}x_k\big] w_k \\ x_0 = \zeta, k \in \mathbb{T}. \end{cases} \tag{17}$$

Furthermore, let

$$\widehat{J}(t,\hbar; v|_{\mathbb{T}_t}) = J\left(t,\hbar; (\mathbb{K}^* x + \bar{\mathbb{K}}^* \mathbb{E}x + v)|_{\mathbb{T}_t}\right)$$

where x is the state of (17). Clearly, under the control u^v, the state x of (17) is in $L^2_{\mathcal{F}}(\bar{\mathbb{T}};\mathbb{R}^n)$. Therefore, (16) is equivalent to that

$$\widehat{J}(t,\hbar;0|_{\mathbb{T}_t}) \le \widehat{J}(t,\hbar;v|_{\mathbb{T}_t})$$

holds for any $t \in \mathbb{T}$, $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$ and $v \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$. Therefore, $0|_{\mathbb{T}_t}$ is the optimal control of Problem $(\text{MF-LQ})_t$. Similar to the proof of Theorem 2.1, we can obtain the following result.

Proposition 2.1: $(\mathbb{K}^*, \bar{\mathbb{K}}^*)$ is a closed-loop optimal strategy of Problem (MF-LQ) if and only if for any $t \in \mathbb{T}$ and $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$ the following MF-FBSDE admits a solution (x, z):

$$\begin{cases} x_{k+1} = \left[(A_k + B_k\mathbb{K}^*_k)\, x_k \right. \\ \qquad + \left(\bar{A}_k + B_k\bar{\mathbb{K}}^*_k + \bar{B}_k\left(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k\right)\right)\mathbb{E}x_k\big] \\ \qquad + \big[(C_k + D_k\mathbb{K}^*_k)\, x_k \\ \qquad + \big(\bar{C}_k + D_k\bar{\mathbb{K}}^*_k + \bar{D}_k\left(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k\right)\big)\mathbb{E}x_k\big] w_k \\ z_k = A_k^T \mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) + \bar{A}_k^T \mathbb{E}z_{k+1} \\ \qquad + C_k^T \mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) + \bar{C}_k^T \mathbb{E}(z_{k+1}w_k) \\ \qquad + (Q_k + L_k\mathbb{K}^*_k)\, x_k \\ \qquad + \left(\bar{Q}_k + L_k\bar{\mathbb{K}}^*_k + \bar{L}_k\left(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k\right)\right)\mathbb{E}x_k \\ x_t = \hbar, z_N = Gx_N + \bar{G}\mathbb{E}x_N, \quad k \in \mathbb{T}_t \end{cases} \tag{18}$$

such that the following stationary conditions hold, a.s.,

$$\begin{cases} 0 = (B_k + \bar{B}_k)^T \mathbb{E}z_{k+1} + (D_k + \bar{D}_k)^T \mathbb{E}(z_{k+1}w_k) \\ \qquad + \left[(L_k + \bar{L}_k)^T + (R_k + \bar{R}_k)\left(\mathbb{K}^*_k + \bar{\mathbb{K}}^*_k\right)\right]\mathbb{E}x_k \\ 0 = B_k^T\left(\mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) - \mathbb{E}z_{k+1}\right) \\ \qquad + D_k^T\left(\mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) - \mathbb{E}(z_{k+1}w_k)\right) \\ \qquad + \left(L_k^T + R_k\mathbb{K}^*_k\right)(x_k - \mathbb{E}x_k) \\ k \in \mathbb{T}_t \end{cases} \tag{19}$$

and for any $t \in \mathbb{T}$, $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, $(\mathbb{K}^*, \bar{\mathbb{K}}^*)|_{\mathbb{T}_t}$ is required to be independent of the initial pair $(t, \hbar)$, and the following is satisfied:

$$\inf_{v \in L^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)} J(0; \mathbb{K}^* x + \bar{\mathbb{K}}^* \mathbb{E}x + v) \ge 0.$$

Introduce a couple of GDREs

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - H_k^T W_k^\dagger H_k \\ T_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T P_{k+1}(C_k + \bar{C}_k) \\ \qquad + (A_k + \bar{A}_k)^T T_{k+1}(A_k + \bar{A}_k) - \bar{H}_k^T \bar{W}_k^\dagger \bar{H}_k \\ P_N = G_N, T_N = G_N + \bar{G}_N \\ W_k, \bar{W}_k \ge 0, W_k W_k^\dagger H_k - H_k = 0 \\ \bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0, \quad k \in \mathbb{T} \end{cases} \tag{20}$$

where

$$\begin{cases} W_k = R_k + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k \\ H_k = B_k^T P_{k+1} A_k + D_k^T P_{k+1} C_k + L_k^T \\ \bar{W}_k = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1}(B_k + \bar{B}_k) \\ \qquad + (D_k + \bar{D}_k)^T P_{k+1}(D_k + \bar{D}_k) \\ \bar{H}_k = (B_k + \bar{B}_k)^T T_{k+1}(A_k + \bar{A}_k) \\ \qquad + (D_k + \bar{D}_k)^T P_{k+1}(C_k + \bar{C}_k) + L_k^T + \bar{L}_k^T \\ k \in \mathbb{T} \end{cases} \tag{21}$$

and $W_k^\dagger$, $\bar{W}_k^\dagger$ denote the pseudo-inverses of W_k and $\bar{W}_k$, respectively. Note that $W_k W_k^\dagger H_k - H_k = 0$ and $\bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0$ are required for any $k \in \mathbb{T}$. Then (20) is a set of constrained equations. The GEREs (20) are called solvable if all the constrained equations $W_k W_k^\dagger H_k - H_k = 0$, $\bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0$, $k \in \mathbb{T}$, are satisfied.

Similar to [34, Corollary 4.1], we have the following result.

Lemma 2.2: Under the conditions that

$$\begin{bmatrix} Q_k & L_k \\ L_k^T & R_k \end{bmatrix} \ge 0, \begin{bmatrix} Q_k + \bar{Q}_k & L_k + \bar{L}_k \\ L_k^T + \bar{L}_k^T & R_k + \bar{R}_k \end{bmatrix} \ge 0, k \in \mathbb{T} \tag{22}$$

and $G_N \ge 0, G_N + \bar{G}_N \ge 0$, the GDREs (20) are solvable.

Denote $\bar{H}_k, \bar{W}_k, H_k$ and W_k in (21) by, respectively, $\bar{H}_k(P,T), \bar{W}_k(P,T), H_k(P)$ and $W_k(P)$ to emphasize the dependence on (P_{k+1}, T_{k+1}). Furthermore, let

$$\begin{cases} \bar{J}_k(P,T) = Q_k + \bar{Q}_k \\ \qquad + (C_k + \bar{C}_k)^T P_{k+1}(C_k + \bar{C}_k) \\ \qquad + (A_k + \bar{A}_k)^T T_{k+1}(A_k + \bar{A}_k) - T_k \\ J_k(P) = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - P_k \end{cases} \tag{23}$$

$$\begin{cases} \overline{\mathcal{H}}_k(P,T) = \begin{bmatrix} \bar{J}_k(P,T) & \bar{H}_k^T(P,T) \\ \bar{H}_k(P,T) & \bar{W}_k(P,T) \end{bmatrix} \\ \mathcal{H}_k(P) = \begin{bmatrix} J_k(P) & H_k^T(P) \\ H_k(P) & W_k(P) \end{bmatrix} \end{cases} \tag{24}$$

$$\mathcal{M}_{G_N,\bar{G}_N} = \left\{ (P_k^T, T_k^T) = (P_k, T_k) \,\middle|\, \begin{array}{l} \overline{\mathcal{H}}k(P,T) \ge 0, \mathcal{H}_k(P) \ge 0, \\ k \in \mathbb{T}, P_N \le G_N, \\ T_N \le G_N + \bar{G}_N. \end{array} \right\}. \tag{25}$$

We then have the following result, which gives several equivalent characterizations on the existence of the closed-loop optimal strategy.

Theorem 2.2: The following statements are equivalent.

(i) Problem (MF-LQ) admits a closed-loop optimal strategy.
(ii) The GDREs (20) are solvable.
(iii) $\mathcal{M}_{G_N,\bar{G}_N} \ne \varnothing$.
(iv) For any $t \in \mathbb{T}$ and $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$

$$\inf_{u \in L^2(\mathbb{T}_t;\mathbb{R}^m)} J(t,\hbar;u) > -\infty. \tag{26}$$

(v) For any $t \in \mathbb{T}$ and $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, Problem $(\text{MF-LQ})_t$ with the initial state $\hbar$ admits an open-loop optimal control.

When any of the above statements is true, the LQ problem is attained by

$$u_k = -W_k^\dagger H_k(x_k - \mathbb{E}x_k) - \bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k, \quad k \in \mathbb{T} \tag{27}$$

i.e., $\{(-W_k^\dagger H_k, W_k^\dagger H_k - \bar{W}_k^\dagger \bar{H}_k), k\in\mathbb{T}\}$ is a closed-loop optimal strategy. Moreover, the optimal value is

$$V(\zeta) = \mathbb{E}\left[(\zeta-\mathbb{E}\zeta)^T P_0(\zeta-\mathbb{E}\zeta)\right] + (\mathbb{E}\zeta)^T T_0 \mathbb{E}\zeta. \quad (28)$$

Proof: (i)⇒(ii). Let $(\mathbb{K}^*, \bar{\mathbb{K}}^*)$ be a closed-loop optimal strategy of Problem (MF-LQ). From Proposition 2.1, for any initial pair $(t,\hbar)$, (18) admits a solution (x,z) and (19) holds. Since the backward state z does not appear in the forward equation, the solution of (18) is unique. Define a map Γ

$$x_t \mapsto \Gamma_t x_t = z_t.$$

We know that Γ is linear and continuous. Noting that $z_N = G_N(x_N-\mathbb{E}x_N) + (G_N+\bar{G}_N)\mathbb{E}x_N$, we let

$$z_k = P_k(x_k-\mathbb{E}x_k) + T_k\mathbb{E}x_k, \quad k\in\bar{\mathbb{T}} \quad (29)$$

with $P_N = G_N, T_N = G_N+\bar{G}_N$ and $P_k, T_k, k\in\mathbb{T}$, being deterministic and determined below. From (18), it holds that

$$\begin{cases} \mathbb{E}x_{k+1} = \left[(A_k+\bar{A}_k)\right.\\ \qquad\qquad \left.+ (B_k+\bar{B}_k)\left(\mathbb{K}_k^*+\bar{\mathbb{K}}_k^*\right)\right]\mathbb{E}x_k \\ \mathbb{E}z_k = \left(A_k^T+\bar{A}_k^T\right)\mathbb{E}z_{k+1} + (C_k+\bar{C}_k)^T\mathbb{E}(z_{k+1}w_k) \\ \qquad + \left[(Q_k+\bar{Q}_k)+(L_k+\bar{L}_k)\left(\mathbb{K}_k^*+\bar{\mathbb{K}}_k^*\right)\right]\mathbb{E}x_k \\ \mathbb{E}x_t = \mathbb{E}\hbar,\ \mathbb{E}z_N = (G+\bar{G})\mathbb{E}x_N \end{cases} \quad (30)$$

$$\begin{cases} x_{k+1}-\mathbb{E}x_{k+1} = \left[(A_k+B_k\mathbb{K}_k^*)(x_k-\mathbb{E}x_k)\right. \\ \quad + \left\{(C_k+D_k\mathbb{K}_k^*)(x_k-\mathbb{E}x_k) + \left[(C_k+\bar{C}_k)\right.\right. \\ \quad \left.\left.+ (D_k+\bar{D}_k)\left(\mathbb{K}_k^*+\bar{\mathbb{K}}_k^*\right)\right]\mathbb{E}x_k\right\} w_k \\ z_k-\mathbb{E}z_k = A_k^T\left(\mathbb{E}(z_{k+1}|\mathcal{F}_{k-1}) - \mathbb{E}z_{k+1}\right) \\ \quad + C_k^T\left(\mathbb{E}(z_{k+1}w_k|\mathcal{F}_{k-1}) - \mathbb{E}(z_{k+1}w_k)\right) \\ \quad + (Q_k+L_k\mathbb{K}_k^*)(x_k-\mathbb{E}x_k) \\ x_t-\mathbb{E}x_t = \hbar-\mathbb{E}\hbar, \quad z_N-\mathbb{E}z_N = G(x_N-\mathbb{E}x_N). \end{cases} \quad (31)$$

We have from (19), (29), (30) and (31) that

$$\begin{cases} P_t(x_t-\mathbb{E}x_t) = z_t-\mathbb{E}z_t \\ = \left[(Q_t+L_t\mathbb{K}_t^*) + A_t^T P_{t+1}(A_t+B_t\mathbb{K}_t^*)\right. \\ \quad \left.+ C_t^T P_{t+1}(C_t+D_t\mathbb{K}_t^*)\right](x_t-\mathbb{E}x_t) \\ 0 = \left[B_t^T P_{t+1}(A_t+B_t\mathbb{K}_t^*) + D_t^T P_{t+1}\right. \\ \qquad \left.\times (C_t+D_t\mathbb{K}_t^*) + \left(L_t^T+R_t\mathbb{K}_t^*\right)\right](x_t-\mathbb{E}x_t) \end{cases} \quad (32)$$

$$\begin{cases} T_t\mathbb{E}x_t = \mathbb{E}z_t \\ = \left[(Q_t+\bar{Q}_t) + (L_t+\bar{L}_t)\left(\mathbb{K}_t^*+\bar{\mathbb{K}}_t^*\right)\right. \\ \quad + A_t^T+\bar{A}_t^T)T_{t+1}(A_t+\bar{A}_t) \\ \quad + \left(A_t^T+\bar{A}_t^T\right)T_{t+1}(B_t+\bar{B}_t)\left(\mathbb{K}_t^*+\bar{\mathbb{K}}_t^*\right) \\ \quad + (C_t+\bar{C}_t)^T P_{t+1}(C_t+\bar{C}_t) \\ \quad + (C_t+\bar{C}_t)^T P_{t+1}(D_t+\bar{D}_t) \\ \quad \left.\times \left(\mathbb{K}_t^*+\bar{\mathbb{K}}_t^*\right)\right]\mathbb{E}x_t \\ 0 = \left\{(B_t+\bar{B}_t)^T T_{t+1}(A_t+\bar{A}_t) + (L_t+\bar{L}_t)^T\right. \\ \qquad + (D_t+\bar{D}_t)^T P_{t+1}(C_t+\bar{C}_t) \\ \qquad + \left[(B_t+\bar{B}_t)^T T_{t+1}(B_t+\bar{B}_t)\right. \\ \qquad\quad + (D_t+\bar{D}_t)^T P_{t+1}(D_t+\bar{D}_t) \\ \qquad\quad \left.\left.+ (R_t+\bar{R}_t)\right]\left(\mathbb{K}_t^*+\bar{\mathbb{K}}_t^*\right)\right\}\mathbb{E}x_t. \end{cases} \quad (33)$$

Due to the definition of the closed-loop optimal strategy, t and $\hbar(=x_t)$ can be arbitrarily selected. Combining this, (32) and (33), we have

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1}A_k + C_k^T P_{k+1}C_k \\ \qquad + \left(A_k^T P_{k+1}B_k + C_k^T P_{k+1}D_k + L_k\right)\mathbb{K}_k^* \\ 0 = B_k^T P_{k+1}A_k + D_k^T P_{k+1}C_k + L_k^T \\ \qquad + \left[R_k + B_k^T P_{k+1}B_k + D_k^T P_{k+1}D_k\right]\mathbb{K}_k^* \\ P_N = G, k\in\mathbb{T} \end{cases} \quad (34)$$

$$\begin{cases} T_k = Q_k+\bar{Q}_k + (A_k+\bar{A}_k)^T T_{k+1}(A_k+\bar{A}_k) \\ \qquad +(C_k+\bar{C}_k)^T P_{k+1}(C_k+\bar{C}_k) \\ \qquad + \left[(A_k+\bar{A}_k)^T T_{k+1}(B_k+\bar{B}_k)\right. \\ \qquad\quad \left.+(C_k+\bar{C}_k)^T P_{k+1}(D_k+\bar{D}_k) + L_k+\bar{L}_k\right] \\ \qquad \times\left(\mathbb{K}_k^*+\bar{\mathbb{K}}_k^*\right) \\ 0 = (B_k+\bar{B}_k)^T T_{k+1}(A_k+\bar{A}_k) + (L_k+\bar{L}_k)^T \\ \qquad +(D_k+\bar{D}_k)^T P_{k+1}(C_k+\bar{C}_k) \\ \qquad + \left[(B_k+\bar{B}_k)^T T_{k+1}(B_k+\bar{B}_k)\right. \\ \qquad\quad \left.+(D_k+\bar{D}_k)^T P_{k+1}(D_k+\bar{D}_k) + (R_k+\bar{R}_k)\right] \\ \qquad \times\left(\mathbb{K}_k^*+\bar{\mathbb{K}}_k^*\right) \\ T_N = G_N+\bar{G}_N, \quad k\in\mathbb{T}. \end{cases} \quad (35)$$

From Lemma 3.1 of [4], (20) can be constructed by (34) and (35).

(ii)⇒(iii). If the GDREs (20) have a solution $\{(P_k,T_k), k\in\bar{\mathbb{T}}\}$ with $P_N = G_N, T_N = G_N+\bar{G}_N$, then by the Extended Schur's Lemma [8] we have

$$\overline{\mathcal{H}}_k(P,T)\geq 0, \quad \mathcal{H}_k(P)\geq 0. \quad (36)$$

Hence, $\mathcal{M}_{G_N,\bar{G}_N}\neq\emptyset$.

(iii)⇒(ii). This part follows from a method of [5]. Suppose $\mathcal{M}_{G_N,\bar{G}_N}\neq\emptyset$. Let $\{(\tilde{P}_k,\tilde{T}_k), k\in\bar{\mathbb{T}}\}\in\mathcal{M}_{G_N,\bar{G}_N}$, and introduce the following GDREs:

$$\begin{cases} U_k = J_k(\tilde{P}) + A_k^T U_{k+1}A_k + C_k^T U_{k+1}C_k \\ \qquad - \tilde{H}_k^T\tilde{W}_k^\dagger\tilde{H}_k \\ V_k = \bar{J}_k(\tilde{P},\tilde{T}) + (C_k+\bar{C}_k)^T U_{k+1}(C_k+\bar{C}_k) \\ \qquad + (A_k+\bar{A}_k)^T V_{k+1}(A_k+\bar{A}_k) - \tilde{\bar{H}}_k^T\tilde{\bar{W}}_k^\dagger\tilde{\bar{H}}_k \\ U_N = G_N-\tilde{P}_N, V_N = G_N+\bar{G}_N-\tilde{T}_N \\ \tilde{W}_k, \tilde{\bar{W}}_k\geq 0, \tilde{W}_k\tilde{W}_k^\dagger\tilde{H}_k - \tilde{H}_k = 0 \\ \tilde{\bar{W}}_k\tilde{\bar{W}}_k^\dagger\bar{H}_k - \tilde{\bar{H}}_k = 0, \quad k\in\mathbb{T} \end{cases} \quad (37)$$

with

$$\begin{cases} \tilde{W}_k = W_k(\tilde{P}) + B_k^T U_{k+1}B_k + D_k^T U_{k+1}D_k \\ \tilde{H}_k = H_k(\tilde{P}) + B_k^T U_{k+1}A_k + D_k^T U_{k+1}C_k \\ \tilde{\bar{W}}_k = \bar{W}_k(\tilde{P},\tilde{T}) + (B_k+\bar{B}_k)^T V_{k+1}(B_k+\bar{B}_k) \\ \qquad + (D_k+\bar{D}_k)^T U_{k+1}(D_k+\bar{D}_k) \\ \tilde{\bar{H}}_k = \bar{H}_k(\tilde{P},\tilde{T}) + (B_k+\bar{B}_k)^T V_{k+1}(A_k+\bar{A}_k) \\ \qquad + (D_k+\bar{D}_k)^T U_{k+1}(C_k+\bar{C}_k), \quad k\in\mathbb{T}. \end{cases} \quad (38)$$

Then, the GDREs (37) are versions of (20) with Q_k, $Q_k+\bar{Q}_k$, L_k, $L_k+\bar{L}_k$, R_k, $R_k+\bar{R}_k$ replaced by $J_k(\tilde{P})$,

$\bar{J}(\tilde{P},\tilde{T}), H_k(\tilde{P}), \bar{H}_k(\tilde{P},\tilde{T}), W_k(\tilde{P}), \bar{W}_k(\tilde{P},\tilde{T})$, respectively. Since

$$\begin{cases}\overline{\mathcal{H}}_k(\tilde{P},\tilde{T}) \geq 0, \quad \mathcal{H}_k(\tilde{P}) \geq 0, \quad G_N - \tilde{P}_N \geq 0\\ G_N + \bar{G}_N - \tilde{T}_N \geq 0\end{cases}$$

the GDREs (37) are solvable by Lemma 2.2, and $U_k, V_k \geq 0, k \in \mathbb{T}$. Letting $P_k = U_k + \tilde{P}_k, T_k = V_k + \tilde{T}_k, k \in \mathbb{T}$ and by simple calculations, we can get (P, T) which solves the GDREs (20).

(iii)⇒(iv). Simple calculations show that

$$\begin{aligned}J(\zeta;u) = &\sum_{k=0}^{N-1}\left\{\begin{bmatrix}\mathbb{E}x_k\\ \mathbb{E}u_k\end{bmatrix}^T \overline{\mathcal{H}}_k(P,T)\begin{bmatrix}\mathbb{E}x_k\\ \mathbb{E}u_k\end{bmatrix}\right.\\ &\left. + \mathbb{E}\left(\begin{bmatrix}x_k - \mathbb{E}x_k\\ u_k - \mathbb{E}u_k\end{bmatrix}^T \mathcal{H}_k(P)\begin{bmatrix}x_k - \mathbb{E}x_k\\ u_k - \mathbb{E}u_k\end{bmatrix}\right)\right\}\\ &+ \mathbb{E}\Big[(x_N - \mathbb{E}x_N)^T (G_N - P_N)(x_N - \mathbb{E}x_N)\\ &\quad + (\mathbb{E}x_N)^T (G_N + \bar{G}_N - T_N)\mathbb{E}x_N\Big]\\ &+ \mathbb{E}\left[(\zeta - \mathbb{E}\zeta)^T P_0 (\zeta - \mathbb{E}\zeta)\right] + (\mathbb{E}\zeta)^T T_0 \mathbb{E}\zeta. \quad (39)\end{aligned}$$

Therefore

$$J(\zeta;u) \geq \mathbb{E}\left[(\zeta - \mathbb{E}\zeta)^T P_l(\zeta - \mathbb{E}\zeta)\right] + (\mathbb{E}\zeta)^T T_l \mathbb{E}\zeta > -\infty.$$

(ii)⇒(i). Similar to (39), we can complete the square to derive the optimal control (27). Due to space limitations, the detailed proof is omitted here and we can refer to [34, Theorem 4.3]. Clearly, $\{(-W_k^\dagger H_k, W_k^\dagger H_k - \bar{W}_k^\dagger \bar{H}_k), k \in \mathbb{T}\}$ is a closed-loop optimal strategy.

(ii)⇒(v). The proof is also by completing the square.

(v)⇒(iv). This is clear.

(iv)⇒(ii). This has been founded in the proof of [34, Theorem 4.3].

The property (28) follows naturally. This completes the proof. □

Remark 2.2: In Remark 2.1, we have compared the open-loop optimal control with the closed-loop optimal strategy. By Theorem 2.2, we know that the existence of the closed-loop optimal strategy is equivalent to that for all $t \in \mathbb{T}$ and all $\hbar \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, Problem (MF-LQ)$_t$ admits an open-loop optimal control. In [34], (ii), (iv), and (v) are shown to be equivalent by using a modified recursive method. In addition, the equivalent characterization (via MF-FBSDE) of the existence of the open-loop optimal control with a given initial state is not presented in [34]. Furthermore, the open-loop optimal control and the closed-loop optimal strategy are not clarified in [34]. Hence, Theorem 2.2 of this paper is deeper and more general than [34, Theorem 4.3].

III. Indefinite Mean-Field Stochastic LQ Optimal Control Over an Infinite Horizon

In this section, an indefinite mean-field LQ optimal control problem over an infinite time horizon is studied. Specifically, introduce

$$\begin{cases}x_{k+1} = (Ax_k + \bar{A}\mathbb{E}x_k + Bu_k + \bar{B}\mathbb{E}u_k)\\ \qquad\quad + (Cx_k + \bar{C}\mathbb{E}x_k + Du_k + \bar{D}\mathbb{E}u_k)w_k\\ x_0 = \zeta, \quad k \in \{0,1,2,\ldots\}\end{cases} \quad (40)$$

$$\begin{aligned}\widetilde{J}(\zeta;u) = \sum_{k=0}^{\infty}\mathbb{E}\big[&x_k^T Q x_k + (\mathbb{E}x_k)^T \bar{Q}\mathbb{E}x_k + 2x_k^T L u_k\\ &+ 2(\mathbb{E}x_k)^T \bar{L}\mathbb{E}u_k + u_k^T R u_k + (\mathbb{E}u_k)^T \bar{R}\mathbb{E}u_k\big]. \quad (41)\end{aligned}$$

Let $\widetilde{\mathbb{T}} = \{0,1,2,\ldots\}$, and $L^2_{\mathcal{F}}(\widetilde{\mathbb{T}};\mathcal{H})$ be the set of $\mathcal{H}$-valued processes $\nu = \{\nu_k, k \in \widetilde{\mathbb{T}}\}$ such that ν_k is $\mathcal{F}_{k-1}$-measurable and $\sum_{k=0}^{\infty}\mathbb{E}|\nu_k|^2 < \infty$. Introduce the admissible control set

$$\widetilde{\mathcal{U}}_{ad} = \left\{u \,\middle|\, u \in L^2_{\mathcal{F}}(\widetilde{\mathbb{T}};\mathbb{R}^m) \text{ and } x(\cdot;\zeta,u) \in L^2_{\mathcal{F}}(\widetilde{\mathbb{T}};\mathbb{R}^n)\right\}$$

where $x(\cdot;\zeta,u)$ denotes the state of (40) with the initial state ζ and the control u. In what follows, the system equation (40) is denoted by $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$. Furthermore, $[A,\bar{A};C,\bar{C}]$ denotes $[A,\bar{A},0,0;C,\bar{C},0,0]$, and $[A;C]$ denotes $[A,0,0,0;C,0,0,0]$. Let $\mathcal{X}$, in this section, denote the space of all $\mathbb{R}^n$-valued square-integrable random variables, endowed with a norm $\|\cdot\|$: $\|z\| = \sqrt{\mathbb{E}z^T z}$ for any $z \in \mathcal{X}$.

Problem (MF-LQI). For any $\zeta \in \mathcal{X}$, find a $u^* \in \widetilde{\mathcal{U}}_{ad}$ such that

$$\widetilde{J}(\zeta,u^*) = \inf_{u \in \widetilde{\mathcal{U}}_{ad}} \widetilde{J}(\zeta,u). \quad (42)$$

Related to the open-loop optimal control of the finite-horizon problem, in the infinite-horizon case we might investigate the conditions on the existence of the optimal control for a fixed initial state. However, until now there are two difficulties that we need to overcome. The first one is about the finiteness of $\widetilde{J}(\zeta;u)$ and the stability of $[A,\bar{A};C,\bar{C}]$ with respect to a fixed initial state ζ. To study Problem (MF-LQI), the finiteness of $\widetilde{J}(\zeta;u)$ is necessary, which corresponds to some set of admissible controls. While for a fixed initial state, it is hard to characterize the admissible control set for the infinite-horizon LQ problems. Furthermore, it is also hard to characterize the fact that $[A,\bar{A};C,\bar{C}]$ is stable from a given initial state. The second difficulty is about the stability of the solution of the (discrete-time) linear MF-BSDEs and MF-FBSDEs. In [35] and [37], the continuous-time linear BSDEs over an infinite horizon (without mean-field terms) are intensively studied. To our knowledge, it is not easy to study the stability of the infinite-horizon version of (9). Unlike the Brownian motion that drives the continuous-time FBSDEs, the noise $\{w_k\}$ in this paper is assumed to be *any* martingale difference with property (2). Generally speaking, for the process $\{v_k = \sum_{l=0}^{k} w_l\}$ we do not have the so-called martingale characterization, as that of the Brownian motion. Hence, in this paper we do not touch the infinite-horizon MF-FBSDEs. Upon the above two points, in this section we study the optimal control problem which involves all the possible initial state in $\mathcal{X}$. Clearly, u^* satisfying (42) is an open-loop optimal control, which is shown in the following to be a closed-loop one. Furthermore, it is shown in this case that the GAREs play a fundamental role in studying the optimal value and the optimal control.

A. L^2-Stability

Definition 3.1: System $[A,\bar{A};C,\bar{C}]$ is said to be L^2-exponentially stable from $\mathcal{X}$, L^2-globally summable from $\mathcal{X}$,

L^2-asymptotically stable from $\mathcal{X}$, if for any $\zeta \in \mathcal{X}$ the solution of (40) satisfies, respectively

$$\lim_{k\to\infty} e^{\lambda k}\mathbb{E}|x_k|^2 = 0, \quad \text{for some } \lambda > 0$$
$$\sum_{k=0}^{\infty}\mathbb{E}|x_k|^2 < \infty$$
$$\lim_{k\to\infty} \mathbb{E}|x_k|^2 = 0.$$

By Theorem 2.1 and Theorem 2.7 of [33], we have the following results.

Theorem 3.1: The following statements are equivalent:

(i) System $[A,\bar{A};C,\bar{C}]$ is L^2-exponentially stable from $\mathcal{X}$.
(ii) System $[A,\bar{A};C,\bar{C}]$ is L^2-globally summable from $\mathcal{X}$.
(iii) System $[A,\bar{A};C,\bar{C}]$ is L^2-asymptotically stable from $\mathcal{X}$.
(iv) For any $Y, Y+\bar{Y} > 0$, the Lyapunov equations

$$\begin{cases}\bar{X} = (A+\bar{A})^T\bar{X}(A+\bar{A}) + (C+\bar{C})^T X(C+\bar{C}) + Y + \bar{Y}\\ X = A^TXA + C^TXC + Y\end{cases}$$

admit solution $(X, X+\bar{X})$ with $X, X+\bar{X} > 0$.
(v) There exist $Y, Y+\bar{Y} > 0$, the discrete-time Lyapunov equations

$$\begin{cases}\bar{X} = (A+\bar{A})^T\bar{X}(A+\bar{A}) + (C+\bar{C})^T X(C+\bar{C})\\ \qquad + Y + \bar{Y}\\ X = A^TXA + C^TXC + Y\end{cases}$$

admit solution $(X, X+\bar{X})$ with $X, X+\bar{X} > 0$.
(vi) $[A;C]$ is L^2-stable from $\mathcal{X}$, and $A+\bar{A}$ is stable.

Remark 3.1: In what follows, the L^2-exponentially stable from $\mathcal{X}$, L^2-globally summable from $\mathcal{X}$, L^2-asymptotically stable from $\mathcal{X}$ are all called L^2-stable from $\mathcal{X}$.

Definition 3.2: System $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$ is said to be open-loop L^2-stabilizable from $\mathcal{X}$ if for any $\xi \in \mathcal{X}$, there exists $u \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{T};\mathbb{R}^m)$ such that the solution $x(\cdot;\xi,u)$ of (40) is in $L^2_{\mathcal{F}}(\widetilde{\mathbb{T}};\mathbb{R}^n)$. We then call such u an open-loop stabilizer.

Definition 3.3: System $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$ is called to be closed-loop L^2-stabilizable from $\mathcal{X}$ if there exists a pair $(K,\bar{K}) \in \mathbb{R}^{m\times n}\times\mathbb{R}^{m\times n}$ such that for any $x \in \mathcal{X}$, the closed-loop system

$$\begin{cases}x_{k+1} = \big[(A+BK)x_k + (\bar{A}+(B+\bar{B})\bar{K}-BK)\mathbb{E}x_k\big]\\ \qquad + \big[(C+DK)x_k + \big(\bar{C}+(D+\bar{D})\bar{K}\\ \qquad\qquad -DK\big)\,\mathbb{E}x_k\big]\,w_k\\ x_0 = \xi, k\in\widetilde{\mathbb{T}}\end{cases} \tag{43}$$

is L^2-stable from $\mathcal{X}$. In this case, we call that $(K,\bar{K})$ or $\{u_k = \bar{K}\mathbb{E}x_k + K(x_k\mathbb{E}x_k), k\in\widetilde{\mathbb{T}}\}$ is stabilizing or a closed-loop L^2-stabilizer from $\mathcal{X}$.

Proposition 3.1: The following statements are equivalent.

(i) $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$ is open-loop L^2-stabilizable from $\mathcal{X}$.
(ii) $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$ is closed-loop L^2-stabilizable from $\mathcal{X}$.

Remark 3.2: This result can be found in [33]. The main technique to prove above result is to construct a definite mean-field LQ problem with all the weighting matrices being positive definite. For such an LQ problem and under the open-loop L^2-stabilizability of $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$, the optimal control is closed-loop and the optimal system is L^2-stable. This means that the optimal control of this definite LQ problem is a closed-loop L^2-stabilizer from $\mathcal{X}$. Furthermore, it is worth mentioning that the open-loop stabilizability from $\mathcal{X}$ is a global property. This is because in Definition 3.2 the referred property holds for all the elements in $\mathcal{X}$.

For simplicity, the closed-loop L^2-stabilizability from $\mathcal{X}$ and the open-loop L^2-stabilizability from $\mathcal{X}$ are both called the L^2-stabilizability from $\mathcal{X}$. Throughout this section, we pose the following assumption.

(A) $[A,\bar{A},B,\bar{B};C,\bar{C},D,\bar{D}]$ is L^2-stabilizable from $\mathcal{X}$.

B. The GAREs

Let

$$\overline{\mathcal{M}} = \Big\{(P,T) = (P^T,T^T)\Big|\overline{\mathcal{H}}(P,T)\geq 0, \mathcal{H}(P)\geq 0\Big\} \tag{44}$$

where

$$\begin{cases}\overline{\mathcal{H}}(P,T) = \begin{bmatrix}\bar{J}(P,T) & \bar{H}^T(P,T)\\ \bar{H}(P,T) & \bar{W}(P,T)\end{bmatrix}\\ \mathcal{H}(P) = \begin{bmatrix}J(P) & H^T(P)\\ H(P) & W(P)\end{bmatrix}\end{cases} \tag{45}$$

with

$$\begin{cases}\bar{J}(P,T) = Q+\bar{Q}+(C+\bar{C})^TP(C+\bar{C})\\ \qquad + (A+\bar{A})^TT(A+\bar{A}) - T\\ \bar{H}(P,T) = (B+\bar{B})^TT(A+\bar{A})\\ \qquad + (D+\bar{D})^TP(C+\bar{C}) + L^T + \bar{L}^T\\ \bar{W}(P,T) = R+\bar{R}+(D+\bar{D})^TP(D+\bar{D})\\ \qquad + (B+\bar{B})^TT(B+\bar{B})\\ J(P) = Q + A^TPA + C^TPC - P\\ H(P) = B^TPA + D^TPC + L^T\\ W(P) = R + B^TPB + D^TPD.\end{cases} \tag{46}$$

The following theorem shows the convergence of the solution of the GDREs (20).

Theorem 3.2: If $\overline{\mathcal{M}} \neq \emptyset$, then for any terminal condition $P_N = \widetilde{P}, T_N = \widetilde{T}$ with $(\widetilde{P},\widetilde{T}) \in \overline{\mathcal{M}}$, the solution $\{(P_k,T_k), -\infty \leq k \leq N\}$ of the time-invariant version of GDREs (20) exists and is bounded. Moreover, P_k and T_k are monotonically nondecreasing as k decreases, and (P_k,T_k) converges to a solution (P,T) of the following GAREs:

$$\begin{cases}P = Q + A^TPA + C^TPC - H^TW^\dagger H\\ T = Q+\bar{Q}+(C+\bar{C})^TP(C+\bar{C})\\ \qquad + (A+\bar{A})^TT(A+\bar{A}) - \bar{H}^T\bar{W}^\dagger\bar{H}\\ W,\bar{W}\geq 0, WW^\dagger H - H = 0\\ \bar{W}\bar{W}^\dagger\bar{H} - \bar{H} = 0\end{cases} \tag{47}$$

with

$$\begin{cases}W = R + B^TPB + D^TPD\\ H = B^TPA + D^TPC + L^T\\ \bar{W} = R+\bar{R}+(B+\bar{B})^TT(B+\bar{B})\\ \qquad + (D+\bar{D})^TP(D+\bar{D})\\ \bar{H} = (B+\bar{B})^TT(A+\bar{A}) + (D+\bar{D})^TP(C+\bar{C})\\ \qquad + L^T + \bar{L}^T.\end{cases} \tag{48}$$

Proof: For $(\widetilde{P},\widetilde{T})\in\overline{\mathcal{M}}$, introduce the following cost functional:

$$J_{\widetilde{P},\widetilde{T}}(l,\zeta,u)=\sum_{k=l}^{N}\mathbb{E}\left\{\begin{bmatrix}\mathbb{E}x_k\\ \mathbb{E}u_k\end{bmatrix}^T\overline{\mathcal{H}}(\widetilde{P},\widetilde{T})\begin{bmatrix}\mathbb{E}x_k\\ \mathbb{E}u_k\end{bmatrix}+\begin{bmatrix}x_k-\mathbb{E}x_k\\ u_k-\mathbb{E}u_k\end{bmatrix}^T\mathcal{H}(\widetilde{P})\begin{bmatrix}x_k-\mathbb{E}x_k\\ u_k-\mathbb{E}u_k\end{bmatrix}\right\}. \quad (49)$$

By Corollary 2.2, we know that $\inf_{u\in\mathcal{U}_{ad}}J_{\widetilde{P},\widetilde{T}}(l,\zeta,u)\geq 0>-\infty$ for any (l,ζ). Furthermore, the corresponding value function is

$$V(l,\zeta)=\mathbb{E}\left[(x_l-\mathbb{E}x_l)^TU_l(x_l-\mathbb{E}x_l)\right]+(\mathbb{E}x_l)^TV_l\mathbb{E}x_l \quad (50)$$

with $x_l=\zeta$. Here, $(U_\cdot,V_\cdot)$ is the solution of the time-invariant version of (37), which is denoted as $(U_{k,N},V_{k,N})$ to emphasize the dependence on N. As $\zeta\ (=x_l)$ is arbitrarily selected, by (50) we have that for $l_1<l_2$

$$U_{l_1,N}\geq U_{l_2,N},V_{l_1,N}\geq V_{l_2,N}. \quad (51)$$

Due to the time invariance, we have $U_{l,N}=U_{0,N-l},V_{l,N}=V_{0,N-l}$. Now, for any given $x_0=\zeta\in\mathcal{X}$ and any stabilizing control $u_k=\bar{K}\mathbb{E}x_k+K(x_k-\mathbb{E}x_k)$, we have from (49) and (50)

$$\begin{aligned}&\mathbb{E}\left[(x_0-\mathbb{E}x_0)^TU_{0,N-l}(x_0-\mathbb{E}x_0)\right]+(\mathbb{E}x_0)^TV_{0,N-l}\mathbb{E}x_0\\ &\leq\sum_{k=0}^{N-l}\mathbb{E}\left\{\begin{bmatrix}\mathbb{E}x_k\\ \bar{K}\mathbb{E}x_k\end{bmatrix}^T\overline{\mathcal{H}}(\widetilde{P},\widetilde{T})\begin{bmatrix}\mathbb{E}x_k\\ \bar{K}\mathbb{E}x_k\end{bmatrix}+\begin{bmatrix}x_k-\mathbb{E}x_k\\ K(x_k-\mathbb{E}x_k)\end{bmatrix}^T\mathcal{H}(\widetilde{P})\begin{bmatrix}x_k-\mathbb{E}x_k\\ K(x_k-\mathbb{E}x_k)\end{bmatrix}\right\}\\ &\leq c\sum_{k=0}^{\infty}\mathbb{E}|x_k|^2\end{aligned} \quad (52)$$

where $c>0$ is a constant and independent of N and l. By selecting $x_0\in\mathbb{R}^n$, we have for any N,l

$$x_0^TV_{0,N-l}x_0<\infty. \quad (53)$$

On the other hand, let $x_0=\xi\varepsilon$ with $\xi\in\mathbb{R}^n$, and $P(\varepsilon=-1)=P(\varepsilon=1)=(1/2)$. For this x_0, from (52) we have

$$\mathbb{E}\left[(x_0-\mathbb{E}x_0)^TU_{l,N}(x_0-\mathbb{E}x_0)\right]=\xi^TU_{0,N-l}\xi<\infty. \quad (54)$$

This together with (51), (53), and (54), gives

$$\begin{cases}\lim\limits_{l\to-\infty}U_{l,N}=\lim\limits_{N-l\to\infty}U_{0,N-l}\equiv\bar{U}\\ \lim\limits_{l\to-\infty}V_{l,N}=\lim\limits_{N-l\to\infty}V_{0,N-l}\equiv\bar{V}\end{cases}$$

and $\bar{U},\bar{V}$ are bounded. Letting $l\to-\infty$, we have the GAREs. Let $P_k=U_k+\widetilde{P},T_k=V_k+\widetilde{T}$. Then, P_k and T_k are monotonically nondecreasing as k increases, and (P_k,T_k) converges to (P,T) with (P,T) being a solution of the GAREs (47). □

We now introduce several notions about the solution of the GAREs (47).

Definition 3.4: A solution of the GAREs (47) is called the maximal solution, denoted by (P^*,T^*), if for any solution (P,T) of the GAREs (47), $P^*\geq P,T^*\geq T$ hold.

Definition 3.5: A solution (P,T) of the GAREs (47) is called a stabilizing solution if the control

$$u_k=-\bar{W}^\dagger\bar{H}\mathbb{E}x_k-W^\dagger H(x_k-\mathbb{E}x_k),k\in\widetilde{\mathbb{T}}$$

stabilizes (40) in the L^2 sense, where, $\bar{W},\bar{H},W$, and H are given in (48).

Definition 3.6: A solution (P,T) of the GAREs (47) is called a semi-stabilizing solution if under

$$u_k=-\bar{W}^\dagger\bar{H}\mathbb{E}x_k-W^\dagger H(x_k-\mathbb{E}x_k),\quad k\in\widetilde{\mathbb{T}} \quad (55)$$

the spectral radius $\rho(\mathcal{L}_{K,\bar{K}})\leq 1$.

The operation $\mathcal{L}_{K,\bar{K}}$ in Definition 3.6 is defined as

$$\begin{aligned}\mathcal{L}_{K,\bar{K}}(M)=&(A_1+B_1K_1)^TM(A_1+B_1K_1)\\ &+(C_1+D_1K_1)^TM(C_1+D_1K_1)\\ &+(C_2+D_2K_1)^TM(C_2+D_2K_1)\end{aligned} \quad (56)$$

for $M\in\mathbb{R}^{2n\times 2n}$, where

$$A_1=\begin{pmatrix}A+\bar{A}&0\\0&A\end{pmatrix},B_1=\begin{pmatrix}B+\bar{B}&0\\0&B\end{pmatrix}$$
$$K_1=\begin{pmatrix}\bar{K}&0\\0&K\end{pmatrix},C_1=\begin{pmatrix}0&0\\0&C\end{pmatrix}$$
$$C_2=\begin{pmatrix}0&0\\C+\bar{C}&0\end{pmatrix},D_1=\begin{pmatrix}0&0\\0&D\end{pmatrix}$$
$$D_2=\begin{pmatrix}0&0\\D+\bar{D}&0\end{pmatrix}.$$

Let

$$\mathbb{P}=\begin{pmatrix}T&0\\0&P\end{pmatrix}\triangleq\mathrm{diag}\{T,P\}.$$

Then, from the GAREs (47), $\mathbb{P}$ satisfies

$$\begin{cases}\mathbb{P}=\mathbb{Q}+A_1^T\mathbb{P}A_1+C_1^T\mathbb{P}C_1+C_2^T\mathbb{P}C_2-\mathbb{H}^T\mathbb{W}^\dagger\mathbb{H}\\ \mathbb{W}\geq 0,\ \ \mathbb{W}\mathbb{W}^\dagger\mathbb{H}-\mathbb{H}=0\end{cases} \quad (57)$$

where

$$\begin{cases}\mathbb{W}=\mathbb{R}+B_1^T\mathbb{P}B_1+D_1^T\mathbb{P}D_1+D_2^T\mathbb{P}D_2\\ \mathbb{H}=B_1^T\mathbb{P}A_1+D_1^T\mathbb{P}C_1+D_2^T\mathbb{P}C_2+\mathbb{L}^T\end{cases} \quad (58)$$

with

$$\mathbb{R}=\begin{pmatrix}R+\bar{R}&0\\0&R\end{pmatrix},\mathbb{Q}=\begin{pmatrix}Q+\bar{Q}&0\\0&Q\end{pmatrix}$$
$$\mathbb{L}=\begin{pmatrix}L+\bar{L}&0\\0&L\end{pmatrix}.$$

By resorting to the GARE (57), we could easily establish the existence of the maximal solution of the GAREs (47).

Lemma 3.1: Assume that the GAREs (47) and the GARE (57) admit the maximal solutions and the stabilizing solutions. Then the following statements hold.

(i) $\mathbb{P}^*=\mathrm{diag}\{T^*,P^*\}$ is the maximal solution of the GARE (57) if and only if (P^*,T^*) is the maximal solution of the GAREs (47).

(ii) $\mathbb{P}=\mathrm{diag}\{T,P\}$ is a stabilizing solution of the GARE (57) if and only if (P,T) is a stabilizing solution of the GAREs (47).

Proof: (i) is clear. For (ii), we have from (58) that

$$\mathbb{W}^\dagger\mathbb{H} = \begin{pmatrix} \bar W^\dagger \bar H & 0 \\ 0 & W^\dagger H \end{pmatrix}.$$

Hence, from [33, Lemma 5.1], $-\mathbb{W}^\dagger\mathbb{H}$ is a stabilizing feedback gain of the following system:

$$\begin{cases} \alpha_{k+1} = (A_1\alpha_k + B_1 v_k) + (C_1\alpha_k + D_1 v_k)\mu_k \\ \qquad\quad +(C_2\alpha_k + D_2 v_k)\theta_k \\ \alpha_0 = \varsigma, \quad k \in \widetilde{\mathbb{T}} \end{cases}$$

with $\mu = \{\mu_k, k\in\widetilde{\mathbb{T}}\}$ and $\theta = \{\theta_k, k\in\widetilde{\mathbb{T}}\}$ being two mutually independently martingale difference sequences with properties similar to (2) if and only if $\{u_k = -\bar W^\dagger \bar H \mathbb{E}x_k - W^\dagger H(x_k - \mathbb{E}x_k), \quad k\in\widetilde{\mathbb{T}}\}$ is a stabilizing control of (40). Therefore, (ii) follows. □

Recalling that $\overline{\mathcal{H}}(P,T)$ and $\mathcal{H}(P)$ are given in (45), we now define

$$\mathbb{H}(\mathbb{P}) = \begin{bmatrix} \mathbb{Q} + A_1^T\mathbb{P}A_1 + C_1^T\mathbb{P}C_1 + C_2^T\mathbb{P}C_2 & \mathbb{H}^T \\ \mathbb{H} & \mathbb{W} \end{bmatrix}$$

where $\mathbb{H}$ and $\mathbb{W}$ are given in (58).

Lemma 3.2: $\mathbb{H}(\mathbb{P}) \geq 0$ if and only if $\overline{\mathcal{H}}(P,T)\geq 0$ and $\mathcal{H}(P)\geq 0$.

Proof: For any $y = [y_1^T\, y_2^T\, y_3^T\, y_4^T]^T \in \mathbb{R}^{2(n+m)}$ with $y_1, y_3 \in \mathbb{R}^n$ and $y_2, y_4 \in \mathbb{R}^m$, we have

$$\begin{aligned} y^T\mathbb{H}(\mathbb{P})y &= y_1^T\bar J(P,T)y_1 + y_1^T\bar H^T(P,T)y_3 + y_3^T\bar H(P,T)y_1 \\ &\quad + y_3^T\bar W(P,T)y_3 + y_2^T J(P)y_2 + y_2^T H^T(P)y_4 \\ &\quad + y_4^T H(P)y_2 + y_4^T W(P)y_4 \\ &= \begin{bmatrix} y_1 \\ y_3\end{bmatrix}^T \overline{\mathcal{H}}(P,T)\begin{bmatrix} y_1 \\ y_3\end{bmatrix} + \begin{bmatrix} y_2 \\ y_4\end{bmatrix}^T \mathcal{H}(P)\begin{bmatrix} y_2 \\ y_4\end{bmatrix}. \end{aligned}$$

Therefore, the conclusion follows. □

Lemma 3.3: If $\overline{\mathcal{M}} \neq \varnothing$ and (**A**) are satisfied, then the following statements hold.

(i) The GARE (57) admits the maximal solution $\mathbb{P}^*$, which is a semi-stabilizing solution.
(ii) A stabilizing solution of the GARE (57) is the maximal solution.

Proof: From Lemma 3.2, we have

$$\overline{\mathcal{M}} = \left\{(P,T) = (P^T, T^T)\middle| \mathbb{H}(\mathbb{P}) \geq 0\right\}.$$

Then we could resort to the methods in [5] and [6], of [1, Section 6.8], and [18, Theorem 5.3.1] to prove the conclusions. Due to space limitations, we omit the proof here. □

Proposition 3.2: If $\overline{\mathcal{M}} \neq \varnothing$ and (**A**) are satisfied, then the following statements hold.

(i) The GAREs (47) admit the maximal solution and at most a stabilizing solution.
(ii) If $\overline{\mathcal{M}}$ has a nonempty interior $(\widetilde P, \widetilde T)$ in the sense that $\overline{\mathcal{H}}(\widetilde P,\widetilde T) > 0, \mathcal{H}(\widetilde P) > 0$, then the GAREs (47) admit a stabilizing solution.

Proof: (i) follows from Lemma 3.1 and Lemma 3.3. (ii) follows from the method to prove [5, Theorem 4.3]. □

C. Mean-Field Stochastic LQ Optimal Control Over an Infinite Horizon

In this section, we shall show that the optimal value of Problem (MF-LQI) can be expressed via the maximal solution of the GAREs.

Theorem 3.3: If $\overline{\mathcal{M}} \neq \varnothing$ and (**A**) are satisfied, then the following statements hold.

(i) Problem (MF-LQI) is well-posed and the optimal value $\widetilde V(x_0)$ is given by $\mathbb{E}[(x_0 - \mathbb{E}x_0)^T P^*(x_0 - \mathbb{E}x_0)] + (\mathbb{E}x_0)^T T^*\mathbb{E}x_0$.
(ii) If the optimal control of Problem (MF-LQI) exists, then an optimal control is

$$u_k = -W^{*\dagger}H^*(x_k - \mathbb{E}x_k) - \bar W^{*\dagger}\bar H^*\mathbb{E}x_k, \quad k\in\widetilde{\mathbb{T}} \quad (59)$$

where $W^*, H^*, \bar W^*$ and $\bar H^*$ are those in (48) with (P,T) replaced by (P^*, T^*).

Proof: (i). Similar to the proof of (39), by completing the squares we have that for any $u\in\widetilde{\mathcal{U}}_{ad}$

$$\begin{aligned} &\widetilde J(\zeta,u) \\ &= \sum_{k=0}^\infty \mathbb{E}\Big[(x_k - \mathbb{E}x_k)^T\left(Q + A^TP^*A + C^TP^*C - P^*\right. \\ &\qquad\qquad \left. - H^{*T}W^{*\dagger}H^*\right)(x_k - \mathbb{E}x_k) \\ &\qquad\qquad + \left(u_k - \mathbb{E}u_k + W^{*\dagger}H^*(x_k - \mathbb{E}x_k)\right)^T W^* \\ &\qquad\qquad \cdot\left(u_k - \mathbb{E}u_k + W^{*\dagger}H^*(x_k - \mathbb{E}x_k)\right) \\ &\qquad\qquad + (\mathbb{E}u_k + \bar W^{*\dagger}\bar H^*\mathbb{E}x_k)^T\bar W^*(\mathbb{E}u_k + \bar W^{*\dagger}\bar H^*\mathbb{E}x_k) \\ &\qquad\qquad + (\mathbb{E}x_k)^T(Q + \bar Q + (C+\bar C)^TP^*(C+\bar C) \\ &\qquad\qquad + (A+\bar A)^TT^*(A+\bar A) - T - \bar H^{*T}\bar W^{*\dagger}\bar H^*)\mathbb{E}x_k\Big] \\ &\quad + \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^TP^*(x_0 - \mathbb{E}x_0)\right] + (\mathbb{E}x_0)^TT^*\mathbb{E}x_0 \\ &\quad - \lim_{N\to\infty}\mathbb{E}\left[(x_N - \mathbb{E}x_N)^TP^*(x_N - \mathbb{E}x_N)\right] \\ &\quad - \lim_{N\to\infty}(\mathbb{E}x_N)^TT^*\mathbb{E}x_N \\ &= \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^TP^*(x_0 - \mathbb{E}x_0)\right] + (\mathbb{E}x_0)^TT^*\mathbb{E}x_0 \\ &\quad + \sum_{k=0}^\infty \mathbb{E}\Big[\left(u_k - \mathbb{E}u_k + W^{*\dagger}H^*(x_k - \mathbb{E}x_k)\right)^T W^* \\ &\qquad\qquad \cdot\left(u_k - \mathbb{E}u_k + W^{*\dagger}H^*(x_k - \mathbb{E}x_k)\right) \\ &\qquad\qquad + (\mathbb{E}u_k + \bar W^{*\dagger}\bar H^*\mathbb{E}x_k)^T\bar W^* \\ &\qquad\qquad \times(\mathbb{E}u_k + \bar W^{*\dagger}\bar H^*\mathbb{E}x_k)\Big]. \end{aligned} \quad (60)$$

Noting that $W^*, \bar W^* \geq 0$, we have

$$\widetilde J(\zeta,u) \geq \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^TP^*(x_0 - \mathbb{E}x_0)\right] + (\mathbb{E}x_0)^TT^*\mathbb{E}x_0.$$

Hence

$$\begin{aligned} \widetilde V(x_0) &= \inf_{u\in\widetilde{\mathcal{U}}_{ad}} \widetilde J(x_0,u) \\ &\geq \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^TP^*(x_0 - \mathbb{E}x_0)\right] + (\mathbb{E}x_0)^TT^*\mathbb{E}x_0. \end{aligned} \quad (61)$$

We now prove that the converse of (61) holds. For a positive decreasing sequence $\{\varepsilon_i, i = 0,1,2,\ldots\}$, we introduce the GAREs with $Q, Q+\bar Q, R, \bar R$ replaced by $Q+\varepsilon_iI, Q+\bar Q+\varepsilon_iI, R+\varepsilon_iI, \bar R+\varepsilon_iI$, respectively. For each i, the corresponding maximal solution of the GAREs is denoted as $(P_{\varepsilon_i}, T_{\varepsilon_i})$, which is the stabilizing solution. We can see that $P_{\varepsilon_0} \geq \cdots \geq P_{\varepsilon_i} \geq P_{\varepsilon_{i+1}} \geq \cdots \geq P$, and $T_{\varepsilon_0} \geq \cdots \geq T_{\varepsilon_i} \geq T_{\varepsilon_{i+1}} \geq \cdots \geq T$. Furthermore, $\lim_{i\to\infty} P_{\varepsilon_i} = P$ and $\lim_{i\to\infty} T_{\varepsilon_i} = T$. For each i, take $u_k^i = -W_i^\dagger H_i(x_k - \mathbb{E}x_k) - \bar{W}_i^\dagger \bar{H}_i \mathbb{E}x_k$. Then, we get

$$\begin{aligned}\widetilde{V}(x_0) \leq \sum_{k=0}^{\infty} \mathbb{E}\big[&(x_k)^T(Q+\varepsilon_i I)x_k + (\mathbb{E}x_k)^T\bar{Q}\mathbb{E}x_k \\ &+ 2(x_k)^T L u_k + 2(\mathbb{E}x_k)^T \bar{L}\mathbb{E}u_k + (u_k)^T \\ &\times (R+\varepsilon_i I)u_k + (\mathbb{E}u_k)^T \bar{R}\mathbb{E}u_k\big] \\ = \mathbb{E}\big[&(x_0-\mathbb{E}x_0)^T P_{\varepsilon_i}(x_0-\mathbb{E}x_0)\big] + (\mathbb{E}x_0)^T T_{\varepsilon_i}\mathbb{E}x_0\end{aligned} \tag{62}$$

where the equality is from the analysis similar to (60). Letting $i \to \infty$ in (62) leads to $\widetilde{V}(x_0) \leq \mathbb{E}[(x_0-\mathbb{E}x_0)^T P^*(x_0-\mathbb{E}x_0)] + (\mathbb{E}x_0)^T T^*\mathbb{E}x_0$. This together with (61) leads to $\widetilde{V}(x_0) = \mathbb{E}[(x_0-\mathbb{E}x_0)^T P^*(x_0-\mathbb{E}x_0)] + (\mathbb{E}x_0)^T T^*\mathbb{E}x_0$.

The proof of (ii) follows from (i) and (60). This completes the proof. □

In Theorem 3.3, (59) is shown to be an optimal control under the condition that the optimal control of Problem (MF-LQI) exists. A question arises naturally: When does the optimal control of Problem (MF-LQI) exist? Noting (i) of Theorem 3.3, equivalently, we may ask: Is the maximal solution of the GAREs (47) a stabilizing solution? In fact, this question relates to an open problem raised by [42] of finding an LQ optimal control using the primal-dual SDP technique. To the best of our knowledge, this question has not yet been addressed. In what follows, we shall study some aspects of this open problem.

For $\mathbb{M} \in \mathbb{R}^{2m\times 2n}$, define an operator

$$\begin{aligned}\Gamma_{\mathbb{M}}(Z) &= (A_1 + B_1\mathbb{M})Z(A_1+B_1\mathbb{M})^T \\ &\quad + \sum_{i=1}^{2}\big[(C_i + D_i\mathbb{M})Z(C_i+D_i\mathbb{M})^T,\ Z \in \mathbb{R}^{2n\times 2n}.\end{aligned}$$

Clearly, $\Gamma_{\mathbb{M}}$ is a linear, compact and positive operator defined on $\mathbb{R}^{2n\times 2n}$. If $\mathbb{M}=0$, $\Gamma_{\mathbb{M}}$ will be denoted as Γ, i.e.,

$$\Gamma(Z) = A_1 Z A_1^T + \sum_{i=1}^{2} C_i Z C_i^T. \tag{63}$$

Theorem 3.4: If $\overline{\mathcal{M}} \neq \varnothing$ and (**A**) are satisfied, then the following statements hold.

(i) If 1 is an eigenvalue of Γ, then the following statements hold.

a) Let the maximal solution (P^*, T^*) of the GAREs (47) with properties $W^* > 0, \bar{W}^* > 0$ be the stabilizing solution. Then any symmetric eigenvector χ of Γ corresponding to 1 is $\mathbb{Q}$-observable in the sense that $\mathbb{Q}\chi \neq 0$.

b) If $Q, Q+\bar{Q} \geq 0, R, R+\bar{R} \geq 0$ and any symmetric eigenvector χ of Γ corresponding to 1 is $\mathbb{Q}$-observable, then the maximal solution of GAREs (47) is the stabilizing solution.

(ii) Let $Q, Q+\bar{Q} \geq 0, R, R+\bar{R} > 0$. Then the following statements hold.

a) If 1 is an eigenvalue of Γ, then the following statements are equivalent.

i′) The maximal solution of the GAREs (47) is the stabilizing solution.

ii′) Any symmetric eigenvector χ of Γ corresponding to 1 is $\mathbb{Q}$-observable.

b) If 1 is not an eigenvalue of Γ, then the maximal solution of the GAREs (47) is the stabilizing solution.

Proof: (i)-a). From Lemma 3.1, we know that the maximal solution $\mathbb{P}^*$ of the GARE (57) is the stabilizing solution with property $\mathbb{W}^* > 0$. Let $\mathbb{K}^* = -\mathbb{W}^{*\dagger}\mathbb{H}^*$. Then the GARE (57) can be rewritten as

$$\mathbb{P}^* = \mathbb{Q} + A_1^T\mathbb{P}^*A_1 + \sum_{i=1}^{2} C_i^T\mathbb{P}^*C_i - \mathbb{K}^{*T}\mathbb{W}\mathbb{K}^*. \tag{64}$$

Hence, it holds that

$$\begin{aligned}&\hat{\varphi}(\mathbb{P}^*) + (\mathbb{K}^{*T}\otimes\mathbb{K}^{*T})\hat{\varphi}(\mathbb{W}^*) \\ &\qquad = \hat{\varphi}(\mathbb{Q}) + \left[A_1^T\otimes A_1^T + \sum_{i=1}^{2} C_i^T\otimes C_i^T\right]\hat{\varphi}(\mathbb{P}^*)\end{aligned}$$

where

$$\hat{\varphi}(\mathbb{P}^*) = \left(\mathbb{P}_1^{*T}, \ldots, \mathbb{P}_n^{*T}\right)^T \in \mathbb{R}^{4n^2}$$

with $\mathbb{P}_1^*, \ldots, \mathbb{P}_n^*$ being the columns of $\mathbb{P}^*$. Assume χ_1 is a symmetric eigenvector of Γ corresponding to 1, and $\mathbb{Q}\chi_1 = 0$. Then, we have

$$\left[A_1\otimes A_1 + \sum_{i=1}^{2} C_i\otimes C_i\right]\hat{\varphi}(\chi_1) = \hat{\varphi}(\chi_1)$$

$$\begin{aligned}&\hat{\varphi}(\chi_1)^T\hat{\varphi}(\mathbb{P}^*) + \hat{\varphi}(\chi_1)^T(\mathbb{K}^{*T}\otimes\mathbb{K}^{*T})\hat{\varphi}(\mathbb{W}^*) \\ &= \hat{\varphi}(\chi_1)^T\left[A_1^T\otimes A_1^T + \sum_{i=1}^{2} C_i^T\otimes C_i^T\right]\hat{\varphi}(\mathbb{P}^*) + \hat{\varphi}(\chi_1)^T\hat{\varphi}(\mathbb{Q}).\end{aligned}$$

Hence,

$$\hat{\varphi}(\chi_1)^T(\mathbb{K}^{*T}\otimes\mathbb{K}^{*T})\hat{\varphi}(\mathbb{W}^*) = \hat{\varphi}(\chi_1)^T\hat{\varphi}(\mathbb{Q})$$

or equivalently,

$$\mathrm{Tr}[\mathbb{Q}\chi_1] = \mathrm{Tr}\left[\chi_1^{\frac{1}{2}}\mathbb{K}^{*T}\mathbb{W}^*\mathbb{K}^*\chi_1^{\frac{1}{2}}\right].$$

As $\mathbb{Q}\chi_1 = 0$ and $\mathbb{W}^* > 0$, it holds that $\mathbb{K}^*\chi_1 = 0$. From this, we then have

$$\begin{aligned}\Gamma_{\mathbb{K}^*}(\chi_1) &= (A_1 + B_1\mathbb{K}^*)\chi_1(A_1+B_1\mathbb{K}^*)^T \\ &\quad + \sum_{i=1}^{2}\big[(C_i + D_i\mathbb{K}^*)\chi_1(C_i+D_i\mathbb{K}^*)^T \\ &= \Gamma(\chi_1) = \chi_1.\end{aligned}$$

Hence, 1 is an eigenvalue of $\Gamma_{\mathbb{K}}^*$. This contradicts that $\mathbb{P}^*$ is the stabilizing solution. Therefore, any symmetric eigenvector χ of Γ corresponding to 1 is $\mathbb{Q}$-observable.

(i)-b). Let $\mathbb{P}^*$ be the maximal solution of (57) and $\mathbb{K}^* = -\mathbb{W}^{*\dagger}\mathbb{H}^*$. Then the GARE (57) can be rewritten as

$$\mathbb{P}^* = \mathbb{Q} + \mathbb{K}^{*T}\mathbb{R}\mathbb{K}^* + (A_1 + B_1\mathbb{K}^*)^T\mathbb{P}(A_1 + B_1\mathbb{K}^*) + \sum_{i=1}^{2}(C_i + D_i\mathbb{K}^*)^T\mathbb{P}^*(C_i + D_i\mathbb{K}^*).$$

Hence

$$\hat{\varphi}(\mathbb{P}^*) = \hat{\varphi}(\mathbb{Q}) + (\mathbb{K}^{*T} \otimes \mathbb{K}^{*T})\hat{\varphi}(\mathbb{R}) + \mathbb{A}_+\hat{\varphi}(\mathbb{P}^*) \tag{65}$$

where

$$\mathbb{A}_+ \triangleq (A_1 + B_1\mathbb{K}^*)^T \otimes (A_1 + B_1\mathbb{K}^*)^T + \sum_{i=1}^{2}(C_i + D_i\mathbb{K}^*)^T \otimes (C_i + D_i\mathbb{K}^*)^T.$$

Assume now that the maximal solution $\mathbb{P}^*$ of the GARE (57) is not the stabilizing solution. Then the spectral radius $\rho(\Gamma_{\mathbb{K}^*})$ equals to 1, as $\mathbb{P}^*$ is a semi-stabilizing solution. Therefore, 1 is an eigenvalue of $\Gamma_{\mathbb{K}^*}$ and there exists a matrix $\eta \geq 0$ such that

$$\mathbb{A}_+^T\hat{\varphi}(\eta) = \hat{\varphi}(\eta). \tag{66}$$

Pre-multiplying $\hat{\varphi}(\eta)^T$ in (65), we have

$$\hat{\varphi}(\eta)^T\hat{\varphi}(\mathbb{P}^*) = \hat{\varphi}(\eta)^T\hat{\varphi}(\mathbb{Q}) + \hat{\varphi}(\eta)^T(\mathbb{K}^{*T} \otimes \mathbb{K}^{*T})\hat{\varphi}(\mathbb{R}) + \hat{\varphi}(\eta)^T\mathbb{A}_+\hat{\varphi}(\mathbb{P}^*). \tag{67}$$

Combining (66) and (67) becomes

$$\begin{aligned} 0 &= \hat{\varphi}(\eta)^T\hat{\varphi}(\mathbb{Q}) + \hat{\varphi}(\eta)^T(\mathbb{K}^{*T} \otimes \mathbb{K}^{*T})\hat{\varphi}(\mathbb{R}) \\ &= \mathrm{Tr}\left[\mathbb{Q}\eta + \eta^{\frac{1}{2}}\mathbb{K}^{*T}\mathbb{R}\mathbb{K}^*\eta^{\frac{1}{2}}\right]. \end{aligned} \tag{68}$$

As $\mathbb{Q} \geq 0, \mathbb{R} \geq 0$, we then have

$$\mathbb{Q}\eta = 0. \tag{69}$$

which contradicts the condition that $\mathbb{Q}\eta \neq 0$. Therefore, the maximal solution $\mathbb{P}^*$ of the GARE (57) is a stabilizing solution. From Lemma 3.1, the maximal solution of the GAREs (47) is the stabilizing solution.

(ii). We need only to prove (ii)-b). Suppose that the maximal solution of the GARE (57) is not the stabilizing solution. By an analysis similar to above, we have from (68) and the fact $Q, Q + \bar{Q} \geq 0, R, R + \bar{R} > 0$ that $\mathbb{K}\eta = 0$. Combining the fact $\Gamma_{\mathbb{K}}^*(\eta) = \eta$, we have

$$\eta = \Gamma_{\mathbb{K}^*}(\eta) = A_1\eta A_1^T + \sum_{i=1}^{2}C_i\eta C_i^T. \tag{70}$$

Therefore, 1 is an eigenvalue of Γ, which contradicts the condition. Hence, the maximal solution of the GARE (57) is the stabilizing solution. From Lemma 3.1, the maximal solution of the GAREs (47) is the stabilizing solution. This completes the proof. □

IV. Examples

In this section, we give some numerical experiments about calculating the maximal solution of the GAREs (47).

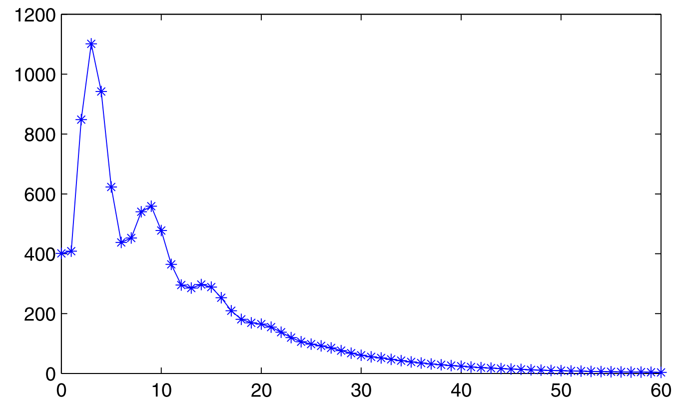


Fig. 1. Curve of $\mathbb{E}|x_k|^2$ with initial state $x_0 = [1 - 20]^T$.

Example 4.1: The system matrices and the weighting matrices are given as follows:

$$A = \begin{bmatrix} 2.0 & 1.0 \\ 0.0 & 2.0 \end{bmatrix}, \bar{A} = \begin{bmatrix} 2.0 & 1.0 \\ 0.0 & 1.0 \end{bmatrix}, B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}, C = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$D = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \bar{D} = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}, L = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$Q = \begin{bmatrix} 7.8889 & 0.8333 \\ 0.8333 & 6.2222 \end{bmatrix}, \bar{Q} = \begin{bmatrix} -3.2525 & -6.3210 \\ -6.3210 & -0.1419 \end{bmatrix}$$
$$\bar{L} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, R = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}$$
$$\bar{R} = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}.$$

Clearly, $Q > 0$, $R < 0$, $\bar{R} < 0$, and $Q + \bar{Q}$ is indefinite as the eigenvalues of $Q + \bar{Q}$ are -0.1766 and 10.8933. By standard SDP theory [42], the maximal solution (P^*, T^*) of the corresponding GAREs can be calculated by the following optimization problem:

$$\begin{aligned} &\text{maximize} \quad \mathbf{Tr}(P) + \mathbf{Tr}(T) \\ &\text{subject to} \quad (P, T) \in \overline{\mathcal{M}}. \end{aligned}$$

It is known that the SDP algorithm has a polynomial computation complexity. Solving above problem gives

$$P^* = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, T^* = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

Using this maximal solution, we construct the control

$$\begin{aligned} u_k &= -\bar{W}^{*-1}\bar{H}^*\mathbb{E}x_k - W^{*-1}H^*(x_k - \mathbb{E}x_k) \\ &\triangleq K^*\mathbb{E}x_k + \bar{K}^*(x_k - \mathbb{E}x_k), \quad k \in \widetilde{\mathbb{T}} \end{aligned} \tag{71}$$

where

$$K^* = \begin{bmatrix} -1.7778 & -0.5556 \\ -0.2778 & -1.7778 \end{bmatrix}$$
$$\bar{K}^* = \begin{bmatrix} -1.6543 & -0.3082 \\ 0.2501 & -1.3344 \end{bmatrix}.$$

The spectral radius $\rho(\mathcal{L}_{K^*, \bar{K}^*})$ is shown to be 0.9087. Hence, the maximal solution (P^*, T^*) is the stabilizing solution. A curve of $\mathbb{E}|x_k|^2$ under the control (71) is shown in Fig. 1.

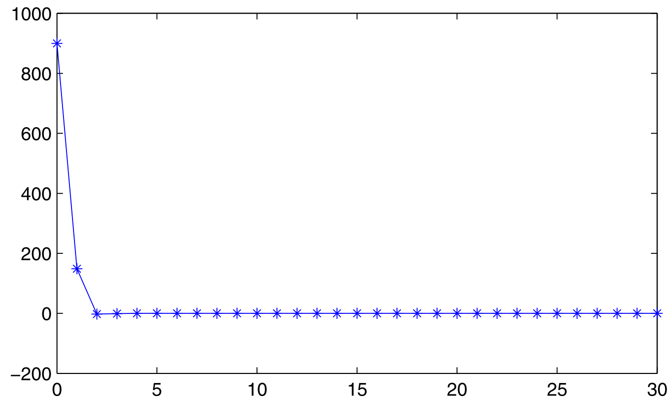


Fig. 2. Curve of $\mathbb{E}|x_k|^2$ with initial state $x_0 = [-0.1\ 30]^T$.

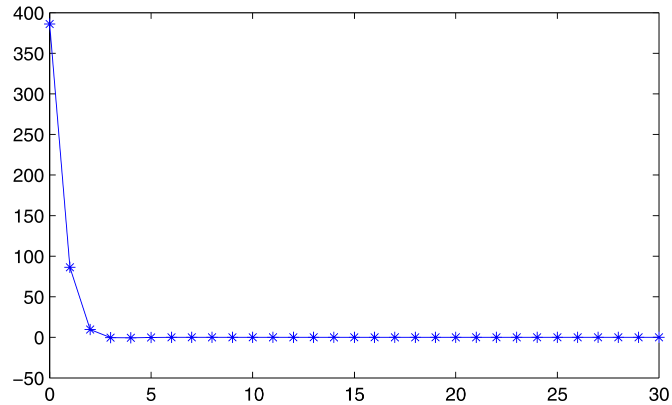


Fig. 3. Curve of $\mathbb{E}|x_k|^2$ with initial state $x_0 = [-5\ 19]^T$.

Example 4.2: The system matrices and weighting matrices are given as follows:

$$A = \begin{bmatrix} 1.0 & 0.5 \\ 0.0 & 1.0 \end{bmatrix}, \bar{A} = \begin{bmatrix} 1.0 & 0.5 \\ 0.0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}, C = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$D = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \bar{D} = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}, L = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$Q = \begin{bmatrix} 5.3182 & 0.0000 \\ 0.0000 & 5.3182 \end{bmatrix}, \bar{Q} = \begin{bmatrix} 1.0158 & -0.7543 \\ -0.7543 & 0.1561 \end{bmatrix}$$
$$\bar{L} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, R = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \bar{R} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}.$$

Here, $Q, Q + \bar{Q} > 0$ and $R, R + \bar{R} > 0$. We then have that

$$P^* = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \bar{P}^* = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$
$$K^* = \begin{bmatrix} -1.0000 & -0.2273 \\ -0.2273 & -1.0000 \end{bmatrix} \quad (72)$$
$$\bar{K}^* = \begin{bmatrix} -1.0123 & -0.1285 \\ 0.1199 & -0.8687 \end{bmatrix}. \quad (73)$$

Furthermore, the spectral radius $\rho(\mathcal{L}_{K^*,\bar{K}^*})$ is shown to be 0.1260. Hence, the maximal solution (P^*, T^*) is the stabilizing solution. A curve of $\mathbb{E}|x_k|^2$ under the control (71) [with K^* and $\bar{K}^*$ given in (72) and (73)] is shown in Fig. 2.

Example 4.3: The system matrices are same to those of Example 4.2, and the weighting matrices are given as follows:

$$Q = \begin{bmatrix} 12.6550 & 1.0819 \\ 1.0819 & 7.2602 \end{bmatrix}, \bar{Q} = \begin{bmatrix} 3.0980 & -1.9252 \\ -1.9252 & 0.9536 \end{bmatrix}$$
$$L = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \bar{L} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$
$$R = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}, \bar{R} = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}.$$

Here, $Q, Q + \bar{Q} > 0$ and $R, R + \bar{R} < 0$. We then have

$$P^* = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \bar{P}^* = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$
$$K^* = \begin{bmatrix} -1.1053 & -0.2632 \\ -0.2778 & -1.2222 \end{bmatrix} \quad (74)$$
$$\bar{K}^* = \begin{bmatrix} -1.0739 & -0.1134 \\ 0.1600 & -0.9810 \end{bmatrix}. \quad (75)$$

Furthermore, the spectral radius $\rho(\mathcal{L}_{K^*,\bar{K}^*})$ is shown to be 0.3647. Hence, the maximal solution (P^*, T^*) is the stabilizing solution. A curve of $\mathbb{E}|x_k|^2$ under the control (71) [with K^* and $\bar{K}^*$ given in (74), (75)] is shown in Fig. 3.

V. Conclusion

In this paper, the finite-horizon and the infinite-horizon mean-field LQ optimal control problems are studied. For the finite-horizon case, the open-loop optimal control and the closed-loop optimal strategy are introduced, whose relationship is carefully investigated. It is shown that the MF-FBSDEs and the GDREs play important roles in studying these two notions. In the infinite-horizon case, the maximal solution of the GAREs is employed to express the optimal value. Furthermore, the relationship between the maximal solution and the stabilizing solution is intensively investigated.

Clearly, the coefficients of the system and the cost weighting matrices are deterministic, which makes it possible for us to separately deal with the two orthogonal parts of the state and the control. This in fact plays a fundamental role to derive the main results of this paper. In the future, we should develop new methodology to study the mean-field LQ problems with random parameters. Another class of problems is about the (discrete-time) MF-BSDEs. On the one hand, the linear MF-BSDEs over an infinite horizon is really interesting, by which we can investigate the infinite-horizon mean-field LQ problems. On the other hand, the nonlinear MF-BSDEs are also interesting. Furthermore, the discrete-time MF-BSDE studied in this paper is different from that in [16] and [17], where the BSDEs are of finite state. Lastly, Problem (MF-LQI) is a zero-endpoint problem as any admissible control stabilizes $[A, \bar{A}, B, \bar{B}; C, \bar{C}, D, \bar{D}]$. It is reasonable to consider the free-endpoint problem [27], [39] and to study the set of the solutions of the GAREs.

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*. Basel, Switzerland: Birkhäuser Verlag, 2003.

[2] D. Ahdersson and B. Djehiche, “A maximum principle for SDEs of mean-field type,” *Appl. Math. Optim.*, vol. 63, pp. 341–356, 2011.

[3] N. U. Ahmed and X. Ding, “Controlled McKean-Vlasov equations,” *Commun. Pur. Appl. Anal.*, vol. 5, pp. 183–206, 2001.

[4] M. Ait Rami, X. Chen, and X. Y Zhou, “Discrete-time indefinite LQ control with state and control dependent noises,” *J. Global Optimiz.*, vol. 23, pp. 245–265, 2002.

[5] M. Ait Rami, J. B. Moore, and X. Y. Zhou, "Sovability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls," *IEEE Trans. Autom, Control*, vol. 46, no. 3, pp. 428–440, Mar. 2001.

[6] M. Ait Rami, J. B. Moore, and X. Y. Zhou, "Indefinite stochastic linear quadratic control and generalized differential Riccati equation," *SIAM J. Control Optimiz.*, vol. 40, no. 4, pp. 1296–1311, 2001.

[7] M. Ait Rami and X. Y. Zhou, "Linear matrix inequalities, Riccati equations, indefinite stochastic linear quadratic controls," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1131–1143, 2000.

[8] A. Albert, "Conditions for positive and nonnegative definiteness in term of pseudoinverses," *SIAM J. Appl. Mathemat.*, vol. 17, pp. 434–440, 1969.

[9] K. J. Astrom, *Introduction to Stochastic Control Theory*. New York: Academic Press, 1970, vol. 70, Mathematics in Science and Engineering.

[10] M. Bardi and F. S. Priuli, "Linear-quadratic N-person and mean-field games with ergodic cost," *SIAM J. Control Optimiz.*, vol. 52, pp. 3022–3052, 2014.

[11] M. Bardi, "Explicit solutions of some linear-quadratic mean field games," *Networks and Heterog. Media*, vol. 7, pp. 243–261, 2012.

[12] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*. New York: Springer Briefs in Mathematics, 2013.

[13] A. Bensoussan, K. Sung, P. Yam, and S. Yung, Linear-Quadratic Mean-Field Games, arXiv:1404.5741.

[14] R. Carmona and F. Delarue, "Probability analysis of mean-field games," *SIAM J. Control Optim.*, vol. 51, pp. 2705–2734, 2013.

[15] R. Carmona, F. Delarue, and A. Lachapelle, "Control of McKean-Vlasov dynamics versus mean field games," *Math. Finan. Econ.*, vol. 7, pp. 131–166, 2013.

[16] S. N. Cohen and R. J. Elliott, "Solutions of backward stochastic differential equations on Markov chains," *Commun. Stochast. Analy.*, vol. 2, pp. 251–262, 2008.

[17] S. N. Cohen and R. J. Elliott, "A general theory of finite state backward stochastic difference equations," *Stochast. Process. Their Applic.*, vol. 120, pp. 442–466, 2010.

[18] T. Damm, *Rational Matrix Equations in Stochastic Control*. New York: Springer-Verlag, 2004, vol. 297, Lecture Notes in Control and Information Sciences.

[19] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*. Philadelphia, PA: SIAM Classics in Applied Mathematics, 1999.

[20] R. J. Elliott, X. Li, and Y. H. Ni, "Discrete time mean-field stochastic inear-quadratic optimal control problems," *Automatica*, vol. 49, no. 11, pp. 3222–3233, 2013.

[21] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. New York: Springer Science and Business Media, Inc., 2006.

[22] M. Hafayed, "A mean-field necessary and sufficient conditions for optimal singular stochastic control," *Commun. Math. Stat.*, vol. 1, no. 4, pp. 417–435, 2014.

[23] J. Huang, X. Li, and J. M. Yong, "A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite hrizon," *Mathemat. Control Related Fields*, vol. 5, pp. 97–139, 2015.

[24] M. Huang, P. E. Caines, and R. P. Malhame, "Individual and mass behaviour in large population stochastic wireless power controlproblems: Centralized and Nash equilibrium solutions," in *Proc. 42nd IEEE Conf. Decision Control*, Maui, HI, Dec. 2003, pp. 98–103.

[25] M. Huang, P. E. Caines, and R. P. Malhame, "Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1560–1571, Sep. 2007.

[26] M. Huang, P. E. Caines, and R. P. Malhame, "Social optima in mean field LQG control: Centralized and decentralized strategies," *IEEE Trans. Autom. Control*, vol. 57, pp. 1736–1751, 2010.

[27] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*. New York: Oxford University Press Inc., 1995.

[28] J. M. Lasry and P. L. Lions, "Mean-field games," *Jap. J. Mathemat.*, vol. 2, pp. 229–260, 2007.

[29] D. Li and W. L. Ng, "Optimal dynamic portfolio selection: Multi-period mean-variance formulation," *Mathemat. Finance*, vol. 10, pp. 387–406, 2000.

[30] J. Li, "Stochastic maximum principle in the mean-field controls," *Automatica*, vol. 48, pp. 366–373, 2012.

[31] T. Li and J. F. Zhang, "Asymptotically optimal decentralized control for large population stochastic multi-agent systems," *IEEE Trans. Autom. Control*, vol. 53, pp. 1643–1660, 2008.

[32] H. P. McKean, "A class of Markov processes associated with nonlinear parabolic equations," in *Proc. Nat. Acad. Sci.* USA, 1966, vol. 56, pp. 1907–1911.

[33] Y. H. Ni, R. Elliott, and X. Li, "Discrete-time mean-field stochastic linear-quadratic optimal controls. II—Infinite horizon case," *Automatica*, vol. 57, pp. 65–77, 2015.

[34] Y. H. Ni, J. F. Zhang, and X. Li, "Indefinite mean-field stochastic linear-quadratic optimal control," *IEEE Trans. Autom. Control*, vol. 60, pp. 1786–1800, 2015.

[35] S. Peng and Y. Shi, "Infinite horizon forward-backward stochastic differential equations," *Stochast. Process. Their Applic.*, vol. 85, pp. 75–92, 2000.

[36] J. R. Sun and J. M. Yong, "Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points," *SIAM J. Control Optimiz.*, vol. 52, no. 6, pp. 4082–4121, 2014.

[37] J. R. Sun, J. M. Yong, and S. G. Zhang, Linear Quadratic Stochastic Two-Persion Zero-Sum Differential Games in an Infinite Horizon, ESAIM: Control, Optimisation and Calculus of Variations, doi: 10.1051/cocv/2015024.

[38] H. Tembine, Q. Zhu, and T. Basar, "Risk-sensitive mean-field games," *IEEE Trans. Autom. Control*, vol. 59, pp. 835–850, 2014.

[39] J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation ," *IEEE Trans. Autom. Control*, vol. 16, no. 6, pp. 621–634, Dec. 1971.

[40] B. C. Wang and J. F. Zhang, "Mean field games for large-popilation multiagent systems with Markov jump parameters," *SIAM J. Control Optimiz.*, vol. 50, pp. 2308–2334, 2012.

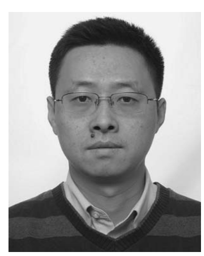
[41] G. Wang, C. Zhang, and W. Zhang, "Stochastic maximum principle for mean-field type optimal control under partial information," *IEEE Trans. Autom. Control*, vol. 59, pp. 522–528, 2014.

[42] D. Yao, S. Z. Zhang, and X. Y. Zhou, "Stochastic LQ control via primal-dual semidefinite programming," *SIAM Rev.*, vol. 46, pp. 87–111, 2004.

[43] J. M. Yong, "A linear-quadratic optimal control problem for mean-field stochastic differential equations," *SIAM J. Control Optimiz.*, vol. 51, pp. 2809–2838, 2013.

[44] J. M. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer, 1999.

[45] X. Y. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: A stochastic LQ framework," *Appl. Mathemat. Optimiz.*, vol. 42, pp. 19–33, 2000.



Yuan-Hua Ni received the Ph.D. degree in operation research and cybernetics from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2010.

Since April 2011, he has been with the Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin, China, as an Assistant Professor. Since June 2013, he has been being a postdoctoral researcher with the Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences. From April 2014 to May 2015, he was a visiting scholar at the University of California, San Diego. His current research interests are stochastic systems and stochastic optimal control.

Xun Li received the B.S. degree in 1992 from the Department of Mathematics, Shanghai University of Science and Technology, Shanghai, China, and the M.S. degrees in 1995 from the Department of Mathematics, Shanghai University. He received the Ph.D. degree in 2000 from the Department of Systems Engineering and Engineering Management, Chinese University of Hong Kong.

He was a postdoctoral Research Fellow until 2001 with the Department of Systems Engineering and Engineering Management, Chinese University of Hong Kong. From 2001 to 2003, he was a postdoctoral Fellow in the Mathematical and Computational Finance Laboratory, University of Calgary. From 2003 to 2007, he was a Visiting Fellow in the Department of Mathematics, National University of Singapore. He joined the Department of Applied Mathematics at the Hong Kong Polytechnic University as an Assistant Professor in 2007 and is currently an Associate Professor. His main research areas are applied probability and stochastic control with financial applications.

Dr. Li has published in journals such as *SIAM Journal on Control and Optimization*, *Annals of Applied Probability*, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, *Automatica*, *Mathematical Finance*, and *Annals of Finance*.

Ji-Feng Zhang (M'92–SM'97–F'14) received the B.S. degree in mathematics from Shandong University, Jinan, China, in 1985 and the Ph.D. degree from the Institute of Systems Science (ISS), Chinese Academy of Sciences (CAS), Beijing, China, in 1991.

Since 1985, he has been with the ISS, CAS, and now is a Guan Zhao-Zhi Chair Professor of AMSS and the Director of ISS. His current research interests include system modeling, adaptive control, stochastic systems, and multi-agent systems.

Dr. Zhang is Editor-in-Chief of the *Journal of Systems Science and Mathematical Sciences* and *All About Systems and Control*, Deputy Editor-in-Chief of *Science China: Information Sciences* and *Systems Engineering—Theory and Practice*. He was a Managing Editor of the *Journal of Systems Science and Complexity*, Deputy Editor-in-Chief of *Acta Automatica Sinica* and *Control Theory and Applications*, Associate Editor of several other journals, including IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and *SIAM Journal on Control and Optimization*. He received the Second Prize of the State Natural Science Award of China in 2010 and 2015, respectively, the Distinguished Young Scholar Fund from National Natural Science Foundation of China in 1997, the First Prize of the Young Scientist Award of CAS in 1995, the Outstanding Advisor Award of CAS in 2007, 2008, and 2009, respectively. He has served as a Vice Chair of the IFAC Technical Board, member of the Board of Governors, IEEE Control Systems Society, Vice President of the Systems Engineering Society of China, and the Chinese Association of Automation. He was the General Co-Chair of the 32nd and 33rd Chinese Control Conference, and the IPC Chair or Vice-Chair of the 2012 IEEE Conference on Control Applications, the 9th World Congress on Intelligent Control and Automation, the 17th IFAC Symposium on System Identification, and the 20th World Congress of the International Federation of Automatic Control.