Time-Inconsistent Discrete-Time Stochastic Linear-Quadratic Optimal Control: Time-consistent Solutions

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Abstract—In this paper, the time-consistent solutions of a time-inconsistent discrete-time stochastic linear-quadratic optimal control are investigated. Different from the existing literature, the definiteness constraint is not posed on the state and the control weight matrices of the cost functional. Necessary and sufficient conditions are, respectively, obtained to the existence of the open-loop time-consistent equilibrium control and the closed-loop time-consistent equilibrium strategy, which contain the solvability of certain system of forward-backward stochastic difference equations, the stationary conditions and the convexity conditions. Furthermore, for the case that the system matrices are independent of the initial time, the existence of the open-loop equilibrium pair for all the initial pairs is shown to be equivalent to the solvability of a system of generalized difference Riccati equations and a system of linear difference equations. Moreover, the existence of the closed-loop equilibrium strategy is shown to be equivalent to the solvability of another system of generalized difference Riccati equations. Interestingly, if solvable, the system of generalized difference Riccati equations for the open-loop formulation does not admit symmetric solution while the one for the closed-loop formulation has symmetric solution.

Index Terms—Time-inconsistency, indefinite stochastic linear-quadratic optimal control, forward-backward stochastic difference equation

I. INTRODUCTION

We begin with a stochastic optimal control problem for discrete-time dynamic system. There are several situations when it is necessary or natural to describe a system by a discrete-time model. A typical case is that the signal values are only available for measurement or manipulation at certain times; for example, a continuous time system is sampled at certain times. Generally speaking, the standard discrete-time controlled stochastic system is described by

\[
\begin{align*}
X_{k+1} &= f_k(X_k, u_k, w_k), \\
X_t &= x \in \mathbb{R}^n, \quad k \in \mathbb{T}_t, \quad t \in T.
\end{align*}
\]

(1)

Here, \( N \) is a positive integer, \( \mathbb{T} = \{0, 1, \cdots, N-1\} \), and \( \mathbb{T}_t = \{t, \cdots, N-1\} \) for \( t \in \mathbb{T} \); \( \{X_k, k \in \mathbb{T}_t\} \) and \( \{u_k, k \in \mathbb{T}_t\} \) with \( \mathbb{T}_t = \{t, t+1, \cdots, N\} \) are the state process and the control process, respectively; \( \{w_k, k \in \mathbb{T}_t\} \) represents the stochastic disturbance. The cost functional associated with (1) is

\[
J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}[L_k(X_k, u_k)] + \mathbb{E}[h(X_N)].
\]

(2)

Let \( U[t, N-1] \) be some set of admissible controls that is available to the controller. Then the standard discrete-time stochastic optimal control problem is stated as follows.

**Problem (C).** Concerned with (1), (2) and the initial pair \((t, x) \in T \times \mathbb{R}^n\), find a \( \bar{u} \in U[t, N-1] \) such that

\[
J(t, x; \bar{u}) = \inf_{u \in U[t, N-1]} J(t, x; u).
\]

Any \( \bar{u} \in U[t, N-1] \) satisfying the above is called an optimal control for the initial pair \((t, x)\); \( \bar{X} = \{\bar{X}_k = X(k; t, x, \bar{u}), k \in \mathbb{T}_t\} \) is called the optimal trajectory and \( (\bar{X}, \bar{u}) \) is referred to as an optimal pair.

Dynamic programming is a fundamental technique to deal with the above stochastic optimal control problem. The basic idea is to consider the behavior of the optimal cost-to-go function of Problem (C) over all candidate controls. The optimal cost-to-go function is also called the value function, which is defined as

\[
V(t, x) = \inf_{u \in U[t, N-1]} J(t, x; u).
\]

Then by studying the incremental behavior of \( V \) as one works backward in time, the corresponding difference equation derived is called the Bellman dynamic programming equation:

\[
V(t, x) = \inf_{u \in U[t]} \mathbb{E}[L_t(x, u_t) + V(t + 1, X(t + 1; t, x, u_t))],
\]

\( \forall (t, x) \in T \times \mathbb{R}^n \)

with the terminal condition

\[
V(N - 1, x) = \inf_{u_{N-1} \in U[N-1]} \mathbb{E}[L_{N-1}(x, u_{N-1}) + h(X(N; N - 1, x, u_{N-1}))].
\]
In the above, $U(t) = U[t, t]$ represents the admissible control set at $t$, and similar meaning holds for $U[N - 1]$. A key property derived from the Bellman dynamic programming equation is the Bellman optimality principle. Specifically, let $\hat{u}$ be an optimal control of Problem (C) for the initial pair $(t, x)$, then for any $\tau \in T_{t+1}$, $\hat{u}_{\tau}$ (the restriction of $\hat{u}$ on $T_{\tau}$) is an optimal control of Problem (C) for the initial pair $(\tau, \hat{X}(\tau; t, \hat{u}_{\tau(t, \ldots, \tau-1)}))$. This phenomenon is also referred to as the time-consistency of the optimal control. The time-consistency ensures that one needs only to solve an optimal control problem for a given initial pair, and the obtained optimal control is also optimal along the optimal trajectory.

However, in reality, the time-consistency of optimal control fails quite often. For this, let us see the following interesting example.

**Example 1.1:** Consider a one-dimensional controlled system

$$
\begin{align*}
X_{k+1} &= u_k + X_k w_k, \\
X_t &= x, \quad k \in T_t
\end{align*}
$$

with the cost functional

$$
J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}u_k^2 + \mathbb{E}[h_t X_N^2].
$$

(3)

Here, $w_1, \ldots, w_{N-1}$ are mutually independent of each other with properties $\mathbb{E}[w_k] = 0, \mathbb{E}[w_k^2] = 1, k \in T_t$, and $h : T \to [h_0, \infty)$ is continuous for some $h_0 > 0$, and the control takes value in $\mathbb{R}$. We intend to find a control that minimizes the cost functional (4).

**Solution.** First of all, we have the following result

$$
\frac{h_t X_{k+1}^2}{1 + h_t(N - k - 1)} - \frac{h_t X_k^2}{1 + h_t(N - k)} = \frac{h_t(2X_k u_k w_k + u_k^2)}{1 + h_t(N - k - 1)} + \frac{h_t X_k^2(w_k^2 - 1)[1 + h_t(N - k - 1)] + h_t^2 X_k^2 w_k^2}{[1 + h_t(N - k - 1)][1 + h_t(N - k)]}.
$$

Taking summation for both sides of the above equation and doing some calculations, we have

$$
\begin{align*}
\frac{h_t X_N^2}{1 + h_t(N - t)} + \sum_{k=t}^{N-1} \mathbb{E} & \left\{ \frac{h_t X_k^2}{1 + h_t(N - k - 1)} \right\} + \frac{h_t X_k^2(w_k^2 - 1)[1 + h_t(N - k - 1)] + h_t^2 X_k^2 w_k^2}{[1 + h_t(N - k - 1)][1 + h_t(N - k)]}
\end{align*}
$$

Then the cost functional is expressed as

$$
J(t, x; u) = \mathbb{E}\left\{ \frac{h_t X_t^2}{1 + h_t(N - t)} \right\} + \sum_{k=t}^{N-1} \mathbb{E} \left\{ \frac{2h_t X_k u_k w_k + [1 + h_t(N - k)]u_k^2}{1 + h_t(N - k - 1)} \right\} + \frac{h_t^2 X_k^2}{[1 + h_t(N - k - 1)][1 + h_t(N - k)]}.
$$

Thus, we have a unique optimal control

$$
\hat{u}_k \equiv \hat{u}(k; t, x) = -\frac{h_t X_k w_k}{1 + h_t(N - k)}.
$$

(5)

where $(t, x)$ is specialized in $u(k; t, x)$ to indicate the dependence on the initial pair $(t, x)$. Substituting it into (3) yields

$$
\begin{align*}
\hat{X}_{k+1} &= \frac{1 + h_t(N - k - 1) \hat{X}_k w_k}{1 + h_t(N - k)} \\
\hat{X}_t &= x, \quad k \in T_t.
\end{align*}
$$

(6)

Then the corresponding optimal trajectory is given by

$$
\hat{X}_{k(t, x, u)} = \hat{X}_k = x + \frac{1 + h_t(N - k) - 1}{1 + h_t(N - t)} \prod_{j=t}^{k-1} w_j, \quad k \in T_t.
$$

(7)

Hence,

$$
\hat{u}_{k(t, x)} = \frac{xh_t}{1 + h_t(N - t)} \prod_{j=t}^{k-1} w_j, \quad k \in T_t.
$$

(8)

Now, for any $\tau \in T_{t+1}$, we consider the optimal control problem on $T_\tau$, with the initial state $y = \hat{X}_{k(t, x, u)}$. We now derive the expression of $J(\tau, y; \hat{u}_{\tau})$. Using (8), we have

$$
J(\tau, y; \hat{u}_{\tau}) = \mathbb{E}\left\{ \sum_{k=\tau}^{N-1} \left\{ \frac{xh_t}{1 + h_t(N - t)} \prod_{j=\tau}^{k-1} w_j \right\}^2 \right\} + h_t \left\{ \frac{y^2 h_t^2(N - \tau) + h_t}{[1 + h_t(N - \tau)]^2} \right\}.
$$

Furthermore, similar to (5) we have the optimal control for the initial pair $(\tau, y)$

$$
\hat{u}_k \equiv \hat{u}(k; \tau, y) = -\frac{yh_t}{1 + h_t(N - \tau)} \prod_{j=\tau}^{k-1} w_j, \quad k \in T_\tau,
$$

(9)

and the corresponding optimal trajectory

$$
\hat{X}_{k(\tau, y, u)} = \hat{X}_\tau = y + \frac{1 + h_t(N - k) - 1}{1 + h_t(N - \tau)} \prod_{j=\tau}^{k-1} w_j, \quad k \in T_\tau.
$$

Hence,

$$
J(\tau, y; \hat{u}) = \mathbb{E}\left\{ \sum_{k=\tau}^{N-1} \hat{u}_k^2 + h_t \hat{X}_{k(\tau, y, u)} \right\} = \frac{y^2 h_t}{1 + h_t(N - \tau)}.
$$

Therefore, we have

$$
J(\tau, y; \hat{u}) = \left\{ \sum_{k=\tau}^{N-1} \hat{u}_k^2 + h_t \hat{X}_{k(\tau, y, u)} \right\} = \frac{y^2 h_t}{[1 + h_t(N - t)]^2[1 + h_t(N - \tau)]}.
$$

(10)

Thus, when $x \neq 0$ and $h_t \neq h_\tau$, the restriction of $\hat{u}_{k(t, x)}$ on $T_\tau$ is not optimal for the initial pair $(\tau, \hat{X}_{k(t, x, u)})$. Such a phenomenon is referred to as the time-inconsistency of the optimal control problem. □
Generally speaking, the time-inconsistent optimal control problem can be formulated in the following way. Consider the controlled system
\[
\begin{align*}
X_{k+1} &= f_k(t, x, X_k, u_k, w_k), \\
X_t &= x \in \mathbb{R}^n, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T}
\end{align*}
\]  
(11)
with its cost functional
\[
J(t, x; u) = \sum_{k=1}^{N-1} \mathbb{E} \left[ L_k(t, x, X_k, u_k) \right] + \mathbb{E} \left[ h(t, x, X_N) \right].
\]  
(12)
To mimic Problem (C), the “optimal control” problem corresponding to (11) and (12) is naturally stated as follows.

**Problem (N).** Concerned with (11), (12) and the initial pair \((t, x) \in \mathbb{T} \times \mathbb{R}^n\), find a \(\bar{u} \in \mathcal{U}[t, N-1]\) such that
\[
J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}[t, N-1]} J(t, x; u),
\]
where \(\mathcal{U}[t, N-1]\) is some set of admissible controls.

Clearly, Example 1.1 is a special case of Problem (N). Therefore, Problem (N) is time-inconsistent, in general. In contrast to the standard stochastic optimal control, the initial pair enters into (11) and (12) explicitly. This is really the reason that Problem (N) is time-inconsistent. The explicit dependence on the initial pair of the quantities in (11) and (12) reflects some situations of people modifying the controlled system and/or the cost functional at different initial pairs. Hyperbolic discounting and quasi-geometric discounting, mean variance utility, and endogenous habit formation are several representative examples of such situations [10] and [4].

Due to its time-inconsistency, there are two different ways to handle Problem (N). The first one is static formulation or pre-commitment formulation. If one is able to commit to his/her initial strategy and does not revisit the problem in the future, then this strategy can be implemented as planned. This approach neglects the time-inconsistency and the optimal control is optimal only when viewed at the initial time. Though the static formulation is of practical and theoretical value, it has not really addressed the time-inconsistency nor provided solutions in a dynamic sense. Note that Problem (N) is a dynamic optimization problem. Relative to the naive static formulation, another approach addresses the time-inconsistency in a dynamic manner. Instead of seeking an “optimal control”, some kind of equilibrium controls are concerned with. This is mainly motivated by practical applications such as in mathematical finance and economics, and has recently attracted considerable interest and efforts.

The mathematical formulation of the time-inconsistency was first reported by [16], while its qualitative analysis may be traced back to the work of [15]. Following [16], the works [8], [10], [11] and [13] are for discrete dynamic systems or simple ordinary differential equations (ODEs). Though the phenomena of time-inconsistency above are mostly studied by researchers from the community of economics, this type of decision problems are essentially optimal control problems and also attract increasing attention from the control community.

Recently, [6] and [7] studied the non-exponential discounting problems both for simple ODEs and stochastic differential equations and introduced the notion of time-consistent control, while [4] started to discuss the problems of general Markovian time-inconsistent stochastic optimal control. Concerned with the deterministic continuous-time linear-quadratic (LQ) optimal control, [18] and [19] addressed it by an essentially cooperative game approach. Specifically, take a partition of the concerned time interval and consider a multi-person differential game with hierarchical structure: the initial pair of \(k\)-th player is the terminal pair of the \((k-1)\)-th player, and each player knows that the later players will do their best and modify their control systems as well as their cost functionals. Then when the mesh size of the partition approaches to zero, the desired time-consistent strategy is achieved. This line is followed by a part of [20] to consider the stochastic LQ problem of mean-field type, which is called the closed-loop formulation there. Different from [18], [19], in [9] the authors studied another kind of time-consistent equilibrium control, which is an infinitesimally open-loop optimal control. Roughly speaking, a controller ([9]) can commit to his equilibrium control in an infinitesimal manner. By investigating the maximal principle, the existence of the equilibrium control is ensured via the solvability of a flow of forward-backward stochastic differential equations. As an application, a mean-variance portfolio selection problem is considered, and the explicit equilibrium strategy is obtained. In [20], the author thoroughly investigated both the open-loop and the closed-loop time-consistent solutions for general mean-field stochastic LQ problems. It is shown in [20] that the existence of the open-loop equilibrium control and the closed-loop equilibrium strategy is ensured via the solvability of certain systems of Riccati-type equations. To end this paragraph, it is valuable to mention that all the relevant existing results are for the definite LQ problems. Here, by the definiteness, we mean that in the cost functionals the state weight matrices are nonnegative definite and the control weight matrices are positive definite.

Interestingly, it is found in [5] that a stochastic LQ problem with indefinite state and control weight matrices may still be well-posed. This challenges the standard belief, and such stochastic LQ problems are termed as indefinite stochastic LQ problems. For more about such class of problems, readers may refer to, for example, [1], [2], [3] and [17]. Furthermore, indefinite stochastic LQ problems are shown to be closely related to Markowitz’s mean-variance portfolio selection [21], which motivates lots of merged investigations on indefinite stochastic LQ optimal control and mean-variance portfolio selection. Therefore, indefinite stochastic LQ optimal control not only has theoretic value to expand standard LQ theory, but also has practical merit to handle problems such as mean-variance portfolio selection. Note that all the existing results about indefinite stochastic LQ optimal control are for the cases with time-consistency.

In this paper, we shall investigate the general time-inconsistent discrete-time stochastic LQ optimal control. Different from [9], [18], [19] and [20], we do not pose the definiteness constraint on the state and the control weight matrices. Such a problem is then termed as time-inconsistent indefinite stochastic LQ optimal control. Clearly, the indefinite setting provides a maximal capacity to model and deal with problems of LQ type. Therefore, the study of time-inconsistent
indefinite stochastic LQ optimal control not only intends to generalize existing results to the indefinite setting, but also provides possible methodology to deal with problems such as the time-inconsistent version of mean-variance portfolio selection. In Section 2 of this paper, the open-loop time-consistent equilibrium control of Problem (LQ) is introduced, and is shown in (18)-(20) to be the Nash equilibrium of a multi-person game with hierarchical structure. In Theorem 2.1, we show that the existence of the open-loop equilibrium pair is equivalent to the solvability of a system of forward-backward stochastic difference equations (FBSΔEs) with stationary conditions and convexity conditions. For a special case of Problem (LQ) (the system dynamics is independent of the initial time), the existence of the open-loop equilibrium pair for all the initial pairs is shown to be equivalent to the solvability of a system of generalized difference Riccati equations (GDREs) and a system of linear difference equations (LDEs). Interestingly, the system of GDREs that we have derived does not have symmetric structure, i.e., the solution to the system of GDREs are not symmetric. Furthermore, when the state and the control weight matrices are also independent of the initial time, the corresponding GEREs and LDEs are different from the standard GEREs [1].

In Section 3, the closed-loop time-consistent equilibrium strategy for Problem (LQ) is investigated. Different from the backward recursive method in [18] and [20], using a variational method we derive the equivalent characterization of the existence of the closed-loop equilibrium strategy by the solvability of a system of FBSΔEs with stationary conditions and convexity conditions. We then show the relationship between the existence of the closed-loop equilibrium strategy and the solvability of a system of GDREs. Interestingly, the GDREs here have symmetric structure and hence the solution are symmetric. This is different from the case of open-loop formulation. In Section 4, several examples are presented and the paper ends with the concluding remarks in Section 5.

II. OPEN-LOOP TIME-CONSISTENT EQUILIBRIUM CONTROL

Consider the following controlled stochastic difference equation (SDE)\footnote{Throughout this paper, (\(L_X\)) denotes the linear operators acting on \(\mathbb{E}X\).}

\[
\begin{aligned}
X_{k+1} &= A_{t,k}X_k + B_{t,k}u_k + (C_{t,k}X_k + D_{t,k}u_k)w_k, \\
X_t^x &= x, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T},
\end{aligned}
\]  

(13)

where \(A_{t,k}, C_{t,k} \in \mathbb{R}^{n \times n}, B_{t,k}, D_{t,k} \in \mathbb{R}^{n \times m}\) are deterministic matrices; \(\{X_k^x, k \in \mathbb{T}\} \triangleq \{X^x\} \) and \(\{u_k, k \in \mathbb{T}\} \triangleq u\) are the state and the control process, respectively. In (13), the initial time \(t\) is specialized in these matrices and the state to emphasize the property that the matrices and thus the state may change according to \(t\). The noise \(\{w_k, k \in \mathbb{T}\}\) is assumed to be a martingale difference sequence defined on a probability space \((\Omega, \mathcal{F}, P)\) in the sense that

\[
\mathbb{E}[w_{k+1} | \mathcal{F}_k] = 0, \quad \mathbb{E}[(w_{k+1})^2 | \mathcal{F}_k] = 1, \quad k \geq 0.
\]  

(14)

Here, \(\mathbb{E}[\cdot | \mathcal{F}_k]\) is the conditional mathematical expectation with respect to \(\mathcal{F}_k = \sigma\{x_0, w_l | l = 0, 1, \ldots, k\}\) and \(\mathcal{F}_-\) is understood as \(\{\emptyset, \Omega\}\). The cost functional associated with system (13) is

\[
J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}\left[\left(X_t^\text{r}T Q_{t,k}X_k^\text{r} + u_k^T R_{t,k}u_k\right)\right] + \mathbb{E}\left[\left(X_N^\text{r}T G_{t}X_N^\text{r}\right)\right],
\]  

(15)

where \(Q_{t,k}, R_{t,k}, k \in \mathbb{T}_t\), and \(G_{t}\) are deterministic symmetric matrices of appropriate dimensions. Different from [9], [18], [19] and [20], we do not pose the definiteness constraints on the state and the control weight matrices.

Let \(L^2_{\mathbb{T}}(\mathbb{H}_t; \mathcal{H})\) be the set of \(\mathbb{H}\)-valued processes \(\nu = \{\nu_k, k \in \mathbb{T}_t\}\) such that \(\nu_k\) is \(\mathcal{F}_{k-1}\)-measurable and \(\sum_{k=t}^{N-1} \mathbb{E}|\nu_k|^2 < \infty\). In addition, \(L^2_{\mathbb{T}}(\mathbb{H}; \mathcal{H})\) is the set of random variables \(\xi\) such that \(\xi \in \mathcal{H}\) is \(\mathcal{F}_{t-1}\)-measurable and \(\mathbb{E}|\xi|^2 < \infty\). Throughout this paper, \((t, x)\) is said to be an admissible initial time-state pair or simply an initial pair for (13) if \(t \in \mathbb{T} \) and \(x \in L^2_{\mathbb{T}}(\mathbb{T}; \mathbb{R}^n)\).

Corresponding to Problem (N), we have the following problem.

**Problem (LQ).** Concerned with (13), (15) and the initial pair \((t, x)\), find a \(u^* \in L^2_{\mathbb{T}}(\mathbb{T}; \mathbb{R}^m)\), such that

\[
J(t, x; u^*) = \inf_{u \in L^2_{\mathbb{T}}(\mathbb{T}; \mathbb{R}^m)} J(t, x; u).
\]  

(16)

Instead of solving Problem (LQ) for the pre-committed optimal control, we adopt the concept of dynamic equilibrium control, which is optimal in an infinitesimal manner and is consistent with the dynamic nature of Problem (LQ).

**Definition 2.1.** A state-control pair denoted as \((X^t, x, u^t, x, u^t)\) is called an open-loop equilibrium pair of Problem (LQ) for the initial pair \((t, x)\) if \(X^t_t = x\) and the following holds for any \(k \in \mathbb{T}_t\) and any \(u_k \in L^2_{\mathbb{T}}(k; \mathbb{R}^m)\)

\[
J(k, X^t_k; u^t_k, u^t_k | \mathcal{T}_k) \leq J(k, X^t_k; (u_k, u^t_k | \mathcal{T}_{k+1})).
\]  

(17)

Here, \(u^t_k | \mathcal{T}_k\) and \(u^t_k | \mathcal{T}_{k+1}\) are the restrictions of \(u^t_k\) on \(\mathcal{T}_k\) and \(\mathcal{T}_{k+1}\), respectively. Furthermore, such a \(u^t_k\) is called an open-loop equilibrium control for the initial pair \((t, x)\).

**Remark 2.1.** By its definition, an open-loop equilibrium control \(u^t, x, u^t\) is time-consistent along the equilibrium trajectory \(\{(k, X^t_k, x, u^t_k), k \in \mathbb{T}_t\}\) in the sense that for any \(k \in \mathbb{T}_t\), \(u^t_k\) is an open-loop equilibrium control for the initial pair \((k, X^t_k, x, u^t_k)\). Furthermore, \(17\) is a local optimality condition, as the control \((u_k, u^t_k | \mathcal{T}_{k+1})\) differs from \(u^t_k | \mathcal{T}_k\) only at time point \(k\).

**Remark 2.2.** To understand the so-called “equilibrium” in Definition 2.1, we introduce a game, termed as Problem (HG), with a hierarchical structure. The cost functional of Player \(k = t, \ldots, N - 1\) is

\[
J_k(u_k, u_{-k}) = J(k, X^t_k; u_k | \mathcal{T}_k) = \sum_{k=t}^{N-1} \mathbb{E}\left[\left((X_{\ell,k}^t)^T Q_{\ell,k}X_{\ell,k} + u_{\ell,k}^T R_{\ell,k}u_{\ell,k}\right)\right] + \mathbb{E}\left[\left(X_{N,k}^t)^T G_{\ell}X_{N,k}^t\right)\right],
\]  

(18)

with \(u_{-k} \triangleq \{u_t, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{N-1}\}\). In (18), \(u_k\) is the action of Player \(k\), and \(\{X_{\ell,k}^t, \ell \in \mathbb{T}_k\} \triangleq X^t_{\ell,k}\) is the internal
state of Player $k$ driven by all the actions $\{u_{i}, \ell \in T_{i}\}$. Indeed, $\{u_{1}, \cdots, u_{k-1}\}$ enters into $X^{k}$ via its initial state
\begin{equation}
\begin{aligned}
X_{k}^{\ell} &= A_{k}\ell X_{k}^{\ell} + B_{k}\ell u_{\ell} + (C_{k}\ell X_{k}^{\ell} + D_{k}\ell u_{\ell}) u_{\ell} , \\
X_{k} &= X_{k}^{T_{k}}, \quad \ell \in T_{k} 
\end{aligned}
\end{equation}
with $X_{k}^{T_{k}}$ being the internal state of Player $k-1$ at time point $k$, as $X_{k-1}^{T_{k}}$ is essentially a function of $\{u_{1}, \cdots, u_{k-1}\}$. This is why we denote $J_{k}(X_{k}, u_{k}; u_{\mid_{T_{k}\ell}})$ by $J_{k}(u_{k}, u_{\mid_{T_{k}\ell}})$. Furthermore, in (19), $X_{k}^{T_{k}} = X_{k}^{T_{k}}$ indicates a forward hierarchical structure of Problem (HG): Player $k-1$ could be viewed as the leader of Player $k$. From (15), (17), and (18), we have that for any $k \in T_{i}$ and any $u_{k} \in L^{2}_{T_{i}}(k; \mathbb{R}^{m})$ the following inequality
\begin{equation}
J_{k}(u_{k}, u_{k}^{t}, u_{k}^{t}, u_{k}^{t}) \leq J_{k}(u_{k}, u_{k}^{t}, u_{k}^{t}) 
\end{equation}
holds. Therefore, $u_{k}^{t}, x_{k}^{t}$ is a Nash equilibrium of Problem (HG). Hence, in Definition 2.1, we call $u_{k}^{t}, x_{k}^{t}$ the equilibrium control.

The following theorem is concerned with the existence of the open-loop equilibrium pair.

**Theorem 2.1:** Given an initial pair $(t, x)$, the following statements are equivalent.

(i) There exists an open-loop equilibrium pair of Problem (LQ) for the initial pair $(t, x)$.

(ii) There exists a control $u_{t}, x_{t}$ such that for any $k \in T_{i}$, the following FBSDE admits a solution $(X^{k,t}, Z^{k,t})$
\begin{equation}
\begin{aligned}
X_{x}^{k} &= A_{k} t X_{x}^{k} + B_{k} t u_{t}^{k} + (C_{k} t X_{x}^{k} + D_{k} t u_{t}^{k}) t u_{t}^{k} , \\
Z_{k}^{k,t} &= A_{k} t E(Z_{x}^{k,t}) + C_{k} t E(Z_{x}^{k,t}) + Q_{k}, t X_{k}^{k,t} , \\
X_{k}^{k,t} &= X_{k}^{k,t}, \quad Z_{N}^{k,t} = G_{k} X_{N}^{k,t}, \quad t \in T_{k} 
\end{aligned}
\end{equation}
with the stationary condition
\begin{equation}
0 = R_{k, k} t X_{x}^{k,t} + B_{k, k} t E(Z^{k,t}_{x} t \mid \mathcal{F}_{k-1}) + D_{k, k} t E(Z^{k,t}_{x} t w_{k} \mid \mathcal{F}_{k-1}),
\end{equation}
and the convexity condition
\begin{equation}
\inf_{u_{k} \in L^{2}_{T_{i}}(k; \mathbb{R}^{m})} \left\{ \sum_{n=1}^{N-1} \left\{ E\left[ (Y_{k}^{t})^{T} Q_{k} t X_{k}^{t} + E\left[ (\bar{u}_{k} R_{k,k} \bar{u}_{k}) + E\left[ (Y_{k}^{t})^{T} G_{k} X_{N}^{t} \right] \right) \right] \geq 0. \right. \right. \)
\end{equation}
Here, $Y^{k}$ is given by
\begin{equation}
\begin{aligned}
Y_{k}^{t+1} &= A_{k} t Y_{k}^{t} + C_{k} t Y_{k}^{t} t u_{t}^{k} , \\
Y_{k}^{t} &= B_{k} t \bar{u}_{k} + D_{k} t \bar{u}_{k} w_{k} , \\
Y_{k}^{t} &= 0.
\end{aligned}
\end{equation}
and $X^{t,t}$ is given by
\begin{equation}
\begin{aligned}
X_{t}^{t,t} &= A_{k, k} t X_{t}^{t,t} + B_{k, k} t u_{t}^{t} + (C_{k, k} t X_{t}^{t,t} + D_{k, k} t u_{t}^{t}) t u_{t}^{t} , \\
X_{t}^{t,t} &= x, \quad k \in T_{k} 
\end{aligned}
\end{equation}
Furthermore, $(X^{t,t}, u^{t,t})$ given in (ii) is an open-loop equilibrium pair.

**Proof.** See Appendix A. □

To proceed, we now recall the pseudo-inverse of a matrix. By [14], for a given matrix $M \in \mathbb{R}^{m \times n}$, there exists a unique matrix in $\mathbb{R}^{n \times m}$ denoted by $M^{\dagger}$ such that
\begin{equation}
\begin{aligned}
M M^{\dagger} M &= M, \quad M^{\dagger} M M^{\dagger} = M^{\dagger}, \\
(M M^{\dagger})^{T} &= M M^{\dagger}, \quad (M^{\dagger} M)^{T} = M^{\dagger} M.
\end{aligned}
\end{equation}
This $M^{\dagger}$ is called the Moore-Penrose inverse of $M$. The following lemma is from [1].

**Lemma 2.1.** Let matrices $L$, $M$, and $N$ be given with appropriate size. Then, $L X M = N$ has a solution $X$ if and only if $L L^{T} N M M^{\dagger} = N$. Moreover, the solution of $L X M = N$ can be expressed as $X = L^{T} N M^{\dagger} + Y - L^{T} L Y M M^{\dagger}$, where $Y$ is a matrix with appropriate size.

From (21) and Lemma 2.1, an open-loop equilibrium control is given by
\begin{equation}
\begin{aligned}
u_{k}^{t,t} &= -R_{k,k}^{t} E(Z_{k+1}^{t,t} | \mathcal{F}_{k-1}) + D_{k,k}^{t} E(Z_{k+1}^{t,t} w_{k} | \mathcal{F}_{k-1}), \quad k \in T_{i},
\end{aligned}
\end{equation}
where the backward state $Z_{k+1}^{t,t}$ is involved. Substituting (26) into (20), we get a system of FBSDEs (when $k$ ranges in $T_{i}$)
\begin{equation}
\begin{aligned}
X_{k+1}^{t,t} &= -B_{k,k} t R_{k,k}^{t} E(Z_{k+1}^{t,t} | \mathcal{F}_{k-1}) + D_{k,k}^{t} E(Z_{k+1}^{t,t} w_{k} | \mathcal{F}_{k-1}), \\
+ C_{k,k} t E(Z_{k+1}^{t,t} | \mathcal{F}_{k-1}) \end{aligned}
\end{equation}
with coupled with
\begin{equation}
\begin{aligned}
X_{k+1}^{t,t} &= A_{k,k} t X_{k}^{t,t} + B_{k,k} t R_{k,k}^{t} E(Z_{k+1}^{t,t} | \mathcal{F}_{k-1}) + D_{k,k}^{t} E(Z_{k+1}^{t,t} w_{k} | \mathcal{F}_{k-1}), \\
+ C_{k,k} t E(Z_{k+1}^{t,t} | \mathcal{F}_{k-1}) \end{aligned}
\end{equation}
and $X_{k+1}^{t,t} = x, \quad k \in T_{i}$.

Note that the backward state $Z_{k+1}^{t,t}$ is involved in the forward SDE of (27). Therefore, (27) is a system of coupled FBSDEs. To get a more convenient form of the open-loop equilibrium control and inspired by [12] and known results about forward-backward stochastic differential equations, we should decouple the forward state and backward state of (27) to obtain Riccati-like equations. However, (27) is not a single FBSDE but a system of FBSDEs coupled with the system of the open-loop equilibrium state $X^{t,t}$. Until now, it is hard for us to obtain neat results for this general case. Alternatively, we confine ourselves in a more specific situation to see what is going on. Exactly, we consider the following system
\begin{equation}
\begin{aligned}
X_{k+1} &= (A_{k,k} X_{k} + B_{k,k} u_{k}) + (C_{k,k} X_{k} + D_{k,k} u_{k}) w_{k}, \quad k \in T_{i},
X_{1} &= x, \quad k \in T_{i}.
\end{aligned}
\end{equation}
Different from (13), in (28) the system matrices are assumed to be independent of the initial time $t$.

The following result via the solvability of a system of GDREs and a system of LDEs gives the equivalent characterization on the existence of the open-loop equilibrium pair of Problem (LQ) corresponding to (28) and (15). Here, Problem (LQ) corresponding to (28) and (15) is understood as a version of Problem (LQ) whose system dynamics and cost functional are, respectively, (28) and (15).

**Theorem 2.2:** The following statements are equivalent.

(i) For any initial pair $(t, x)$, there exists an open-loop equilibrium pair of Problem (LQ) corresponding to (28) and (15).

(ii) The system of GDREs

\[
\begin{align*}
\begin{cases}
P_{t,k} &= Q_{t,k} + A_k^TP_{t,k+1}A_k + C_k^TP_{t,k+1}C_k \quad - H_k^TW_{k,k}H_k, \\
S_{t,N} &= G_t, \\
P_{t,N} &= G_t, \\
W_{t,t} &= 0,
\end{cases}
\end{align*}
\]

where the last equality follows from the solvability of (29).

By backward deduction, we have

\[
Z_{N-1}^k = [A_{N-1}^TP_{N-1}N + C_{N-1}^TP_{N-1}N + Q_{N-1}N],
\]

admits a solution $(\tilde{X}^k, \tilde{Z}^k)$, as the backward state $\tilde{Z}^k$ does not appear in the forward SDE. Comparing the forward SDE in (34) (by substituting $\tilde{u}_t^t$) with (33), we have

\[
\tilde{X}_t^k = X_{t}^t, \quad \ell \in T_k.
\]

To apply Theorem 2.1, we now validate the stationary condition. Noting $Z_{N-1} = G_NX_{N-1},$ for $t = N - 1$, we have

\[
Z_{N-1}^k = \begin{cases}
A_{N-1}^TP_{N-1}N + C_{N-1}^TP_{N-1}N + Q_{N-1}N, \\
0,
\end{cases}
\]

and the system of LDEs

\[
\begin{align*}
\begin{cases}
S_{t,t} &= Q_{t,t} + A_k^TP_{t,t+1}A_k + C_k^TP_{t,t+1}C_k, \\
S_{t,N} &= G_t, \\
P_{t,N} &= G_t, \\
R_{t,t} &= B_k^TP_{t,t+1}B_k + D_k^TP_{t,t+1}D_k, \\
W_{t,t} &= 0,
\end{cases}
\end{align*}
\]

are solvable in the sense that $W_{t,t}H_{t,t} - H_{t,t} = 0, R_{t,t} + B_k^TP_{t,t+1}B_k + D_k^TP_{t,t+1}D_k, t \in T_k$. Hold. In the above,

\[
\begin{align*}
W_{k,k} &= R_k + B_k^TP_{k+1}B_k + D_k^TP_{k+1}D_k, \\
H_{k,k} &= B_k^TP_{k+1}A_k + C_k^TP_{k+1}C_k, \\
H_{t,k} &= B_k^TP_{k+1}A_k + C_k^TP_{k+1}C_k, \\
k \in T_k, \quad \ell \in T_k.
\end{align*}
\]

Proof. (ii)⇒(i). For any given initial pair $(t, x)$, let

\[
\tilde{u}_t^t = -W_{k,k}H_{k,k}X_{k}^t, \quad k \in T_k,
\]

where

\[
\begin{align*}
\begin{cases}
X_{k+1}^t &= (A_k - B_k^TP_{k}H_{k})X_{k}^t + (C_k - D_k^TP_{k}H_{k})X_{k}^t w_k, \\
X_{t}^t &= x, \quad k \in T_k,
\end{cases}
\end{align*}
\]

Under this $\tilde{u}_t^t$, the following FBSDE

\[
\begin{align*}
\begin{cases}
\bar{X}_{t+1}^k &= A_k\bar{X}_t^k + B_t\tilde{u}_t^t, \\
\bar{X}_t^k &= C_k\bar{X}_t^k + D_t\tilde{u}_t^t \quad w_t, \\
\bar{Z}_t^k &= A_k^T\bar{Z}_{t+1}^k, \\
\bar{Z}_t^k &= A_k^T\bar{Z}_{t+1}^k \quad + Q_{k,t}\bar{X}_t^k,
\end{cases}
\end{align*}
\]

admits a solution $(\bar{X}_t^k, \bar{Z}_t^k)$. Therefore, the stationary condition holds. Furthermore, corresponding to (23) and (28), let

\[
\begin{align*}
Y_{t+1} &= A_tY_t + C_tY_t u_t, \quad \ell \in T_k+1, \\
Y_{k+1} &= B_k\tilde{u}_k + D_k\tilde{u}_k w_k, \\
Y_{k} &= 0,
\end{align*}
\]

and by simple calculations we have for any $\tilde{u}_k \in L^2(\mathbb{T}; k; \mathbb{R}_m)$

\[
\begin{align*}
&\langle Q_kY_t, Y_t \rangle_{\mathbb{T}_k} + \langle R_k\tilde{u}_k, \tilde{u}_k \rangle + \langle G_kY_t, Y_t \rangle_N \\
&= E[\tilde{u}_t^T R_k\tilde{u}_k] + E[Y_t^T G_kY_t] \\
&+ \sum_{\ell=k}^{N-1} E[(Y_t)^T Q_{k,\ell} Y_t] \\
&= \sum_{\ell=k}^{N-1} E[(Y_{t+1})^T S_{k,\ell+1}Y_{t+1} + (Y_{t})^T (S_{k,\ell} - Q_{k,\ell}) Y_t] \\
&+ E[\tilde{u}_t^T R_k\tilde{u}_k] \\
&= E[\tilde{u}_t^T (R_k + B_k^T S_{k,t+1}B_k + D_k^T S_{k,t+1}D_k)\tilde{u}_k] \\
&\geq 0,
\end{align*}
\]

where the inequality is from the solvability of (30). Therefore, the convexity condition holds. By Theorem 2.1, the pair $(X_t^t, \bar{u}_t^t)$ given in (32)-(33) is the open-loop equilibrium pair of Problem (LQ) for (28), (15) and $(t, x)$.  

(i)⇒(ii). For any given initial pair \((t, x)\), let \((X^{t,x,*}, u^{t,x,*})\) be the equilibrium pair of Problem (LQ) corresponding to (28) and (15). Then (20) becomes

\[
\begin{align*}
X^{k,*}_{t+1} &= A_tX^{k,*}_t + B_tu^{t,x,*}_t + \left\{C_tX^{k,*}_t + D_tu^{t,x,*}_t\right\}w_t, \\
Z^{k,*}_t &= A_t^T(\mathcal{Z}^{k,*}_t|F_{t-1}) + C_t^T(\mathcal{Z}^{k,*}_t|u^{t,x,*}_t) + Q_ku^{t,x,*}_t, \\
X^{k,*}_k &= X^{k,*}_0, \quad Z^{k,*}_0 = G_kX^{k,*}_0,
\end{align*}
\]

which is solvable from Theorem 2.1. Combining (37) and the stationary condition, we have for \(k = N - 1\)

\[
0 = R_{N-1,N-1}u^{t,x,*}_{N-1} + B_{N-1}^T\mathbb{E}(Z^{N-1}|F_{N-2}) + D_{N-1}^T\mathbb{E}(Z^{N-1}|u^{t,x,*}_{N-1}) = [R_{N-1,N-1} + B_{N-1}^TG_{N-1}B_{N-1} + D_{N-1}^TG_{N-1}D_{N-1}]u^{t,x,*}_{N-1} + [B_{N-1}^TG_{N-1}A_{N-1} + D_{N-1}^TG_{N-1}C_{N-1}]X^{t,x,*}_{N-1} + W_{N-1,N-1}u^{t,x,*}_{N-1} + H_{N-1,N-1}X^{t,x,*}_{N-1},
\]

(38)

where \(W_{N-1,N-1}, H_{N-1,N-1}\) are defined in (31) with \(P_{N-1,N} = G_{N-1}\). Letting \(t = N - 1\) in (38) and by the arbitrariness of \(x\), we have

\[
W_{N-1,N-1} = -H_{N-1,N-1}X^{t,x,*}_{N-1}
\]

By this property, it holds from (38) that

\[
u^{t,x,*}_{N-1} = -W_{N-1,N-1}H_{N-1,N-1}X^{t,x,*}_{N-1}.
\]

For \(k = N - 2\), similar to (35), we have that

\[
Z^{N-2,*}_N = \left\{A_{N-2}^T\left[N_{N-2}G_{N-2}A_{N-1} + C_{N-1}^TG_{N-2}C_{N-1} + Q_{k,N-1} - H_{N-2,N-1}W_{N-1,N-1}H_{N-1,N-1}\right]\right\}X^{N-2,*}_N \\
\Rightarrow P_{N-2,N-1} = Q_{k,N-1} + A_{N-2}^TP_{N-2,N}A_{N-1} + C_{N-1}^TP_{N-2,N}C_{N-1} - H_{N-2,N-1}W_{N-1,N-1}H_{N-1,N-1} = G_{N-2}.
\]

Similar to (38), we have

\[
0 = [R_{N-2,N-2} + B_{N-2}^TG_{N-2}B_{N-2} + D_{N-2}^TG_{N-2}D_{N-2}]u^{t,x,*}_{N-2} + [B_{N-2}^TG_{N-2}A_{N-2} + D_{N-2}^TG_{N-2}C_{N-2}]X^{t,x,*}_{N-2} + W_{N-2,N-2}u^{t,x,*}_{N-2} + H_{N-2,N-2}X^{t,x,*}_{N-2}.
\]

(40)

Therefore, it holds from (40) that

\[
u^{t,x,*}_{N-2} = -W_{N-2,N-2}H_{N-2,N-2}X^{t,x,*}_{N-2}.
\]

Continuing above procedure backward, we have the solvability of (29) and the equilibrium control. To conclude the proof, we need to show the solvability of (30). From (22) and similar to (36), we have for any \(k \in \mathbb{T}\)

\[
0 \leq \inf_{\tilde{u}_k \in L^2_T(k;\mathbb{R}^m)} \mathbb{E}\left[\tilde{u}_k^T(R_k + B_k^TS_k+1B_k + D_k^TS_k+1D_k)\tilde{u}_k\right].
\]

Therefore, it holds that

\[
R_{k,k} + B_k^TS_k+1B_k + D_k^TS_k+1D_k \geq 0, \quad k \in \mathbb{T},
\]

and (30) is solvable.

If (ii) of Theorem 2.2 holds, the open-loop equilibrium control is given by

\[
u^{t,x,*}_k = -W_k^kH_kX^{t,x,*}_k, \quad k \in \mathbb{T},
\]

where

\[
X^{t,x,*}_k = (A_k - B_kW_k^kH_k)X^{t,x,*}_{k+1} + (C_k - D_kW_k^kH_k)X^{t,x,*}_{k+1}w_k,
\]

(42)

\[
X^{t,x,*}_t = x, \quad k \in \mathbb{T}.
\]

In (29), the GDREs are coupled via \(W_k^kH_k, k \in \mathbb{T}\). Generally speaking, \(H_k\) is not equal to \(H_{t,k}\), as \(P_{k,k+1}\) is different from \(P_{k,k+1}\). Therefore, \(P_{k,k}, k \in \mathbb{T}, t \in \mathbb{T}\), are generally nonsymmetric. In contrast to this, the LDEs in (30) are decoupled. Hence, \(S_{t,k}, k \in \mathbb{T}, t \in \mathbb{T}\), are all symmetric as \(G_t, t \in \mathbb{T}\), are symmetric. Interestingly, there is no definite constraint on matrices associated with (29), while the definite constraint is posed through (30). This is different from the standard indefinite stochastic LQ optimal control. Furthermore, note that just \(R_k, k \in \mathbb{T}\), are involved in (29) and (30). Hence, if some \(R_{t,k}, k \in \mathbb{T}, t \in \mathbb{T}\), are modified, the existence of the open-loop equilibrium control will not change!

Let us further assume that \(Q_{k}, k \in \mathbb{T}, t \in \mathbb{T}\), are all independent of \(t\). Then, the systems (29) and (30) become

\[
\begin{align*}
P_k &= Q_k + A_k^TP_{k+1}A_k + C_k^TP_{k+1}C_k - H_k^TW_k^kH_k, \\
P_N &= G, \\
W_k^kH_k - H_k = 0, \\
k \in \mathbb{T},
\end{align*}
\]

(43)

and

\[
\begin{align*}
S_k &= Q_k + A_k^TS_{k+1}A_k + C_k^TS_{k+1}C_k, \\
S_N &= G, \\
R_{k,k} + B_k^TS_{k+1}B_k + D_k^TS_{k+1}D_k &\geq 0, \\
k \in \mathbb{T},
\end{align*}
\]

(44)

where

\[
\begin{align*}
W_k &= R_{k,k} + B_k^TP_{k+1}B_k + D_k^TP_{k+1}D_k, \\
H_k &= B_k^TP_{k+1}A_k + D_k^TP_{k+1}C_k, \\
k \in \mathbb{T}.
\end{align*}
\]
In this case, \( P_k, k \in \mathbb{T} \), are all symmetric. Review the standard GDRE [1]

\[
P_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - H_k^T W_k H_k,
\]

\[
P_N = G,
\]

\[
W_k W_k^T H_k - H_k = 0,
\]

\[
W_k \geq 0,
\]

\[
k \in \mathbb{T}.
\]

(45)

Though the GDRE (43) and the LDE (44) are different from the GDRE (45), we claim that (43) and (44) are both solvable for the case \( Q_k \geq 0, G \geq 0 \) and \( R_k, k \geq 0, k \in \mathbb{T} \). In fact, we easily have the solvability of (44). Under the conditions, we have that the GDRE (45) is solvable by standard results. Hence, (43) is also solvable as (45) has one more constraint \((W_k \geq 0, k \in \mathbb{T})\). However, the conditions that ensure the solvability of the system of GDREs (29) are hard to derive up to now; even for the definite case, the solvability of (29) can not be ensured due to its nonsymmetric structure. At the present time, we therefore need to validate the solvability of (29) case by case. In the future, we shall study the conditions that ensure the solvability of (29), and further focus on more general situations other than the case for (28).

III. CLOSED-LOOP TIME-CONSISTENT EQUILIBRIUM STRATEGY

In the last section, a notion of open-loop time-consistent equilibrium control of Problem (LQ) is introduced. In this section, we shall focus on the time-consistency of the strategy instead of on the control itself.

**Definition 3.1:** \( \Phi = \{\Phi_0, \cdots, \Phi_{N-1}\} \) with \( \Phi_t \in \mathbb{R}^{m \times n}, t \in \mathbb{T} \), is called a closed-loop equilibrium strategy of Problem (LQ) if for any initial pair \((t, x)\) and \( u_k \in L_2^X(k; \mathbb{R}^m) \) with \( k \in \mathbb{T}_t \)

\[
J(\{k, X_{t,x}^{k,u}\}; (\Phi X_k^{k,u,\Phi})_{|T_k+1}) \leq J(\{k, X_{t,x}^{k,u}\}; (u_k, (\Phi X_k^{k,u,\Phi})_{|T_k+1}))
\]

holds. Above, \( X_{t,x}^{k,u} = \{X_k^{k,u,\Phi} \in \mathbb{T}_k\} \) and \( X_k^{k,u,\Phi} = \{X_k^{k,u,\Phi} \in \mathbb{T}_k\} \) are given by

\[
X_{t,x}^{k+1} = (A_{k+1} + B_{k+1} \Phi_{k+1}) X_{t,x}^k + C_{k+1} u_{k+1} + D_{k+1} Y_{k+1},
\]

\[
X_k^{k,u,\Phi} = (A_{k+1} + B_{k+1} \Phi_{k+1}) X_k^{k,u,\Phi} + C_{k+1} u_{k+1} + D_{k+1} Y_{k+1}.
\]

In (50), \( u(t,x) \in \mathbb{T}_{k+1} \) is not influenced by \( u_k \). On the contrary, in (49), the control \((\Phi X_k^{k,u,\Phi})_{|T_k+1}) \) is influenced by \( u_k \) via the term \( X_k^{k,u,\Phi} \), though the strategy \( \Phi \) is fixed. This is the essential difference between the open-loop equilibrium control and the closed-loop equilibrium strategy.

The following theorem is concerned with the existence of the closed-loop equilibrium strategy, which is parallel to Theorem 2.1.

**Theorem 3.1:** The following statements are equivalent.

(i) There exists a closed-loop equilibrium strategy of Problem (LQ).

(ii) There exists a \( \Phi = \{\Phi_0, \cdots, \Phi_{N-1}\} \) with \( \Phi_t \in \mathbb{R}^{m \times n}, t \in \mathbb{T} \) such that for any initial pair \((t, x)\) and \( k \in \mathbb{T}_t \), the following FBS\( \Delta \)E admits a solution \((X_k^{k,u,\Phi}, Z_k^{k,u,\Phi})\)

\[
X_k^{k,u,\Phi} = (A_{k+1} + B_{k+1} \Phi_{k+1}) X_k^{k,u,\Phi} + C_{k+1} u_{k+1} + D_{k+1} Y_{k+1} + \phi_k u_k + \gamma_k u_k
\]

\[
Z_k^{k,u,\Phi} = (A_{k+1} + B_{k+1} \Phi_{k+1}) Z_k^{k,u,\Phi} + C_{k+1} u_{k+1} + D_{k+1} Y_{k+1} + \phi_k u_k + \gamma_k u_k
\]

with the stationary condition

\[
0 = R_k \Phi_k X_k^{k,u,\Phi} + B_k^T \mathcal{E}(Z_k^{k,u,\Phi} | F_{k-1}) + D_{k}^T \mathcal{E}(Z_k^{k,u,\Phi} | F_{k-1})
\]

and the convexity condition

\[
\inf_{\Phi_k \in \mathbb{R}^{m \times n}} \left[ (Q_k X_k^{k,u,\Phi} + Y_k u_k)_{|T_k} \right]
\]
In (53), \( Y^k,\bar{u}_k,\Phi \) is given by
\[
\begin{aligned}
Y^k,\bar{u}_k,\Phi_{t+1} &= (A_{k,t} + B_{k,t}\Phi_t)Y^k,\bar{u}_k,\Phi_t + (C_{k,t} + D_{k,t}\Phi_t)w_t, \quad \ell \in \mathbb{T}_{k+1},
Y^k,\bar{u}_k,\Phi_{t+1} &= B_{k,k,\bar{u}_k} + D_{k,k,\bar{u}_k}w_t,
Y^k,\bar{u}_k,\Phi_{t+1} &= 0.
\end{aligned}
\]

Proof. See Appendix B. \( \Box \)

Based on Theorem 3.1, the relationship between the existence of the closed-loop equilibrium strategy and the solvability of a system of difference equations is established, which is stated in the following theorem.

**Theorem 3.2.** The following statements hold.

(i) Problem (LQ) admits a closed-loop equilibrium strategy.

(ii) The following system of equations
\[
\begin{aligned}
P_{t,k} &= Q_{t,k} + \Phi_t^T R_{t,k} \Phi_t,
p_{t,k} + (A_{t,k} + B_{t,k} \Phi_t)P_{t,k+1} (A_{t,k} + B_{t,k} \Phi_t)^T &+ (C_{t,k} + D_{t,k} \Phi_t)P_{t,k+1} (C_{t,k} + D_{t,k} \Phi_t)^T = 0,
N &\geq k \in \mathbb{T}_t,
W_{t,t}W_{t,t}^T H_{t,t} - H_{t,t} = 0,
W_{t,t} \geq 0,
\end{aligned}
\]
is solvable in the sense that \( W_{t,t}W_{t,t}^T H_{t,t} - H_{t,t} = 0 \) and \( W_{t,t} \geq 0 \) hold for any \( t \in \mathbb{T} \). In the above,
\[
\begin{aligned}
W_{t,t} &= R_{t,t} + B_{t,t}^T P_{t,t+1} A_{t,t} + D_{t,t}^T P_{t,t+1} C_{t,t} \Phi_t,
\Phi_t &= -W_{t,t}^T H_{t,t},
\end{aligned}
\]
\( t \in \mathbb{T} \).

Proof. (i)⇒(ii). Let \( \Phi \) be a closed-loop equilibrium strategy. By the condition and Theorem 3.1, we have that for any initial pair \((t,x)\) (51) admits a solution. Noting that \( Z_{N}^{t,\Phi} = G_{t}X_{N}^{t,\Phi} \) in (51) and inspired by [12] and the known results about forward-backward stochastic differential equations, we let
\[
Z_{N}^{t,\Phi} = P_{t,k} X_{N}^{t,\Phi}, \quad k \in \mathbb{T}_t,
\]
where \( P_{t,k} \), \( k \in \mathbb{T}_t \), are deterministic matrices and determined below. From (52) and Lemma 2.1, we have
\[
\begin{aligned}
0 &= R_{t,t} \Phi_t X_{t}^{t,\Phi} + B_{t,t}^T \mathbb{E}(P_{t,t+1} X_{t+1}^{t,\Phi}|\mathcal{F}_{t-1})
+ D_{t,t}^T \mathbb{E}(P_{t,t+1} X_{t+1}^{t,\Phi}|\mathcal{F}_{t-1})
= [R_{t,t} \Phi_t + B_{t,t}^T P_{t,t+1}(A_{t,t} + B_{t,t} \Phi_t)
+ D_{t,t}^T P_{t,t+1}(C_{t,t} + D_{t,t} \Phi_t)] X_{t}^{t,\Phi}.
\end{aligned}
\]

As \( x = X_{t}^{t,\Phi} \) can be arbitrarily selected, we have
\[
0 = R_{t,t} \Phi_t + B_{t,t}^T P_{t,t+1}(A_{t,t} + B_{t,t} \Phi_t)
+ D_{t,t}^T P_{t,t+1}(C_{t,t} + D_{t,t} \Phi_t)
+ D_{t,t}^T P_{t,t+1}(C_{t,t} + D_{t,t} \Phi_t).
\]

From Lemma 2.1, it holds that
\[
\Phi_t = -W_{t,t}^T H_{t,t},
\]
and
\[
W_{t,t}W_{t,t}^T H_{t,t} - H_{t,t} = 0.
\]

Note that
\[
Z_{N}^{t,\Phi} = \left\{ Q_{t,k} + \Phi_t^T R_{t,k} \Phi_k,
+ (A_{t,k} + B_{t,k} \Phi_k)^T P_{t,k+1} (A_{t,k} + B_{t,k} \Phi_k)
+ (C_{t,k} + D_{t,k} \Phi_k)^T P_{t,k+1} (C_{t,k} + D_{t,k} \Phi_k) \right\} X_{t}^{t,\Phi}.
\]

For \( k \in \mathbb{T}_t \), let
\[
P_{t,k} = Q_{t,k} + \Phi_t^T R_{t,k} \Phi_k,
+ (A_{t,k} + B_{t,k} \Phi_k)^T P_{t,k+1} (A_{t,k} + B_{t,k} \Phi_k)
+ (C_{t,k} + D_{t,k} \Phi_k)^T P_{t,k+1} (C_{t,k} + D_{t,k} \Phi_k).
\]

Clearly, (57) holds. Furthermore, from the convexity condition (53), we have for any \( u_t \in L^2([t; \mathbb{R}^m]) \)
\[
0 \leq \langle Q_{t} Y_{t}^{t,\bar{u}_t,\Phi}, Y_{t}^{t,\bar{u}_t,\Phi} \rangle_{\mathbb{T}_t} + 2 \langle R_{t} \Phi_t Y_{t}^{t,\bar{u}_t,\Phi}, \bar{u}_t \rangle_t
+ \langle R_{t} \Phi_t Y_{t}^{t,\bar{u}_t,\Phi}, \Phi_t \rangle_{\mathbb{T}_t}
+ \langle G_{t} Y_{N}^{t,\bar{u}_t,\Phi}, Y_{N}^{t,\bar{u}_t,\Phi} \rangle_{N}
= \mathbb{E} [\bar{u}_t^T R_{t} \Phi_t \bar{u}_t] + \mathbb{E} [Y_{N}^{t,\bar{u}_t,\Phi}^T P_{t} N Y_{N}^{t,\bar{u}_t,\Phi}]
+ \sum_{t} \mathbb{E} [Y_{k}^{t,\bar{u}_t,\Phi}^T (Q_{t,k} + \Phi_k^T R_{t,k} \Phi_k) Y_{k}^{t,\bar{u}_t,\Phi}]
= \mathbb{E} [\bar{u}_t^T (R_{t} + B_{t} P_{t+1} B_{t} + D_{t}^T P_{t+1} D_{t}) \bar{u}_t].
\]

Hence, it holds that
\[
W_{t,t} = R_{t,t} + B_{t,t}^T P_{t,t+1} B_{t,t} + D_{t,t}^T P_{t,t+1} D_{t,t} \geq 0.
\]

When \( t \) ranges from \( N - 1 \) to 0, we have the solvability of (55).

(ii)⇒(i). Note that \( \Phi = \{\Phi_0, ..., \Phi_{N-1}\} \) with \( \Phi_t = -W_{t,t}^T H_{t,t}, t \in \mathbb{T} \). As (55) is solvable, we have that (51) is solvable with property (57). Furthermore, by reversing some presentations of “(i)⇒(ii)”, the stationary condition (52) and the convexity condition (53) are both satisfied. Therefore, \( \Phi \) is a closed-loop equilibrium strategy. \( \Box \)

By substituting \( \Phi \), (55) is equivalent to
\[
\begin{aligned}
P_{t,k} &= Q_{t,k} + A_{t,k}^T P_{t,k+1} A_{t,k}
+ C_{t,k}^T P_{t,k+1} C_{t,k} - H_{t,k}^T W_{t,k} H_{t,k},
- H_{t,k}^T W_{t,k}^T H_{t,k} + H_{t,k}^T W_{t,k} H_{t,k} W_{t,k}^T H_{t,k},
P_{t,N} &= G_{t},
W_{t,t} W_{t,t}^T H_{t,t} - H_{t,t} = 0,
W_{t,t} \geq 0,
\end{aligned}
\]
\( t \in \mathbb{T} \).
where
\[
\begin{align*}
W_{t,k} &= R_{t,k} + B_{t,k}^T P_{t,k+1} B_{t,k} + D_{t,k}^T P_{t,k+1} D_{t,k}, \\
H_{t,k} &= B_{t,k}^T P_{t,k+1} A_{t,k} + D_{t,k}^T P_{t,k+1} C_{t,k},
\end{align*}
\]
\[
\begin{cases}
W_{t,k} = R_{t,k} + B_{t,k}^T P_{t,k+1} B_{t,k} + D_{t,k}^T P_{t,k+1} D_{t,k}, \\
H_{t,k} = B_{t,k}^T P_{t,k+1} A_{t,k} + D_{t,k}^T P_{t,k+1} C_{t,k},
\end{cases}
\]
\[
k \in T_t, \quad t \in T.
\]
Clearly, for any \( t \in T, k \in T_t, P_{t,k} \) is symmetric. Furthermore, if \( Q_{t,k} \geq 0, R_{t,k} \geq 0, G_{t,k} \geq 0, k \in T_t, t \in T \), we can prove backward that \( W_{t,k} > 0 \) and \( P_{t,k} \geq 0, k \in T_t, t \in T \). Therefore, (58) is solvable. We then have the following corollary.

**Corollary 3.1:** If \( Q_{t,k} \geq 0, R_{t,k} > 0, G_{t} \geq 0, k \in T_t, t \in T \), then Problem (LQ) admits a closed-loop equilibrium strategy \( \Phi \).

Interestingly, all the \( R_{t,k}, k \in T_t, t \in T \) are involved in (58). This is different from that for (29). Furthermore, for the case that \( A_{t,k}, B_{t,k}, C_{t,k}, D_{t,k}, Q_{t,k}, k \in T_t, G_{t} \), are all independent of \( t \), (58) reads as
\[
\begin{align*}
P_{t,k} &= Q_k + A_k^T P_{t,k+1} A_k + C_k^T P_{t,k+1} C_k \\
&- H_k^T W_{k,k} H_{k,k} - H_k^T W_{k,k} H_{k,k} \\
&+ H_k^T W_{k,k} W_{k,k}^T H_{k,k}
\end{align*}
\]
\[
P_{t,N} = G,
\]
\[
k \in T_t, \\
W_{t,k} H_{t,k} - H_{t,k} = 0,
\]
\[
t \in T,
\]
\[
(59)
\]
(59) is different from (43) and (44), which relate to the open-loop equilibrium control of the corresponding situation. Moreover, when all the system matrices in the dynamics and cost functional are independent of the initial time, (58) will reduce to the standard GDRE [1]. In [20], a backward recursive method to deal with continuous-time definite time-inconsistent LQ optimal control of mean-field type is introduced. The difference between [20] and this paper lies in the following two points. Firstly, apart from the mean-field terms, this paper deals with the indefinite case while the definite one is studied in [20]. In this paper, the necessary and sufficient conditions are presented to the existence of the closed-loop equilibrium strategy. Secondly, the techniques to derive the main results are different. The method of this paper is potentially applicable to the continuous-time case.

### IV. Examples

In this section, we shall present several examples to illustrate the theory derived above.

**Example 4.1:** Let
\[
A_0 = \begin{bmatrix} 1.12 & 0.21 \\ -0.13 & 0.98 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2.12 & -0.35 \\ -0.21 & 3.43 \end{bmatrix},
\]
\[
A_2 = \begin{bmatrix} 5.46 & 1.21 \\ -0.98 & 4.21 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.45 & -0.23 \\ -0.2 & 4 \end{bmatrix},
\]
\[
B_1 = \begin{bmatrix} 1.5 & 0.3 \\ -0.2 & 3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -4.36 & 0.82 \\ 1.21 & 4.21 \end{bmatrix},
\]
\[
C_0 = \begin{bmatrix} 1 & 0.32 \\ 0.25 & 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1.65 & -0.13 \\ -0.42 & 6 \end{bmatrix},
\]
\[
C_2 = \begin{bmatrix} -3 & 1.53 \\ -0.62 & 4.78 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 5 & 1 \\ -0.85 & 8 \end{bmatrix},
\]
\[
D_1 = \begin{bmatrix} 4 & 0.53 \\ -0.42 & 5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 9.21 & -2.03 \\ -1.52 & 6.98 \end{bmatrix},
\]
\[
Q_0 = \begin{bmatrix} -2 & 0.8 \\ 0.8 & 1.6 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 1.56 & -0.23 \\ -0.23 & 2.54 \end{bmatrix}, \quad R_{0,0} = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix},
\]
\[
R_{0,1} = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \quad R_{0,2} = \begin{bmatrix} -9 & 0 \\ 0 & 10 \end{bmatrix},
\]
\[
R_{1,1} = \begin{bmatrix} 4 & -0.3 \\ -0.3 & 7 \end{bmatrix}, \quad R_{1,2} = \begin{bmatrix} 2.24 & -5.67 \\ -5.67 & -1.27 \end{bmatrix},
\]
\[
R_{2,2} = \begin{bmatrix} 6.29 & -1.67 \\ -1.67 & 8.38 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.
\]
Here, we have assumed that the system matrices except for \( R_{c} \) are independent of the initial time. Clearly, \( Q_1 \geq 0, Q_2 > 0, G > 0, R_{1,1,2} > 0, \) and \( Q_0, R_{0,0}, R_{1,2} \) are indefinite, and \( R_{0,1} \) is negative definite.

We have from (43) and (44) that
\[
P_2 = \begin{bmatrix} 16.6571 & 5.8520 \\ 5.8520 & 11.5436 \end{bmatrix},
\]
\[
P_3 = \begin{bmatrix} 6.9700 & -1.3882 \\ -1.3882 & 9.1396 \end{bmatrix},
\]
\[
P_0 = \begin{bmatrix} 7.8991 & 4.2276 \\ 4.2276 & 6.4336 \end{bmatrix},
\]
\[
S_2 = \begin{bmatrix} 43.0612 & -12.3922 \\ -12.3922 & 87.4900 \end{bmatrix},
\]
\[
S_1 = \begin{bmatrix} 11.6579 & -7.4371 \\ -7.4371 & 95.6692 \end{bmatrix},
\]
\[
S_0 = \begin{bmatrix} 34.3248 & 35.9699 \\ 35.9699 & 938.8710 \end{bmatrix}
\]
with properties
\[
W_2 = \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0,
\]
\[
W_1 = \begin{bmatrix} 287.3160 & 153.0623 \\ 153.0623 & 447.2115 \end{bmatrix} > 0,
\]
\[
W_0 = \begin{bmatrix} 1138.3 & -410.9 \\ -410.9 & 915.8 \end{bmatrix} > 0,
\]
\[
R_{2,2} + B_2^T S_3 B_2 + D_2^T S_3 D_2
\]
\[
= \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0,
\]
\[
R_{1,1} + B_1^T S_2 B_1 + D_1^T S_2 D_1
\]
\[
= \begin{bmatrix} 857.9 & -426.0 \\ -426.0 & 2909.6 \end{bmatrix} > 0,
\]
\[
R_{0,0} + B_0^T S_1 B_0 + D_0^T S_1 D_0
\]
\[
= \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0.
\]
From (60)-(65), the corresponding (43) and (44) are solvable, and thus the open-loop equilibrium pair exists. Furthermore, an open-loop equilibrium control is given by

\[ u_k^{0,x,*} = -W_k^{\dagger}H_k x_k^{0,x,*}, \quad k = 0, 1, 2 \]

with

\[ -W_0^{\dagger}H_0 = \begin{bmatrix} -0.2183 & 0.0031 \\ 0.0023 & -0.3286 \end{bmatrix}, \]
\[ -W_1^{\dagger}H_1 = \begin{bmatrix} -0.5138 & 0.1973 \\ 0.0026 & -1.1339 \end{bmatrix}, \]
\[ -W_2^{\dagger}H_2 = \begin{bmatrix} 0.4889 & -0.2601 \\ 0.1605 & -0.7474 \end{bmatrix}, \]

and

\[
\begin{align*}
X_k^{0,x,*} &= (A_k x_k^{0,x,*} + B_k u_k^{0,x,*}) + (C_k x_k^{0,x,*} + D_k u_k^{0,x,*})w_k, \\
X_0^{0,x,*} &= x, \quad k = 0, 1, 2.
\end{align*}
\]

On the other hand, from (59) we can compute the solution. However, we have

\[ W_{1,1} = \begin{bmatrix} 239.0218 & 247.7565 \\ 247.7565 & 224.5117 \end{bmatrix}, \]

whose eigenvalues are

\[ \lambda_1 = -16.096, \quad \lambda_2 = 479.6294. \]

Clearly, \( W_{1,1} \) is indefinite, and thus the corresponding (59) is not solvable. This means that the closed-loop equilibrium strategy does not exist.

**Example 4.2:** The system matrices and the weight matrices are the same as those of Example 4.1 except for \( R_{0,1}, R_{0,2}, R_{1,2} \), which are now

\[ R_{0,1} = \begin{bmatrix} -1 & 0 \\ 0 & -0.6 \end{bmatrix}, \quad R_{0,2} = \begin{bmatrix} 9.45 & 1.32 \\ 1.32 & 10.78 \end{bmatrix}, \]
\[ R_{1,2} = \begin{bmatrix} 5.24 & -1.67 \\ -1.67 & 7.27 \end{bmatrix}. \]

Note that \( R_{0,1} \) is negative definite.

In this case, the corresponding (43) and (44) are the same as those of Example 4.1, because \( R_{0,1}, R_{0,2}, R_{1,2} \) do not enter the GDRE and LDE. Hence, the open-loop equilibrium pair exists for any initial pair. Let us check the existence of the closed-loop equilibrium strategy. By (59), we have

\[ P_{0,0} = \begin{bmatrix} 6.1615 & 4.3853 \\ 4.3853 & 3.2889 \end{bmatrix}, \]

with properties

\[ W_{2,2} = \begin{bmatrix} 117.6727 & -34.9725 \\ -34.9725 & 146.0623 \end{bmatrix} > 0, \]
\[ W_{1,1} = \begin{bmatrix} 281.0078 & 160.6675 \\ 160.6675 & 425.5062 \end{bmatrix} > 0, \]
\[ W_{0,0} = \begin{bmatrix} 1178.0 & -334.5 \\ -334.5 & 143.3 \end{bmatrix} > 0. \]

From (66)-(68), we have that (59) is solvable. Furthermore, a closed-loop equilibrium strategy \( (\Phi_0, \Phi_1, \Phi_2) \) is given by

\[ \Phi_0 = \begin{bmatrix} -0.0368 & 0.0884 \\ 0.6555 & -0.0192 \end{bmatrix}, \]
\[ \Phi_1 = \begin{bmatrix} -0.5094 & 0.1935 \\ -0.0021 & -1.1301 \end{bmatrix}, \]
\[ \Phi_2 = \begin{bmatrix} 0.4889 & -0.2601 \\ 0.1605 & -0.7474 \end{bmatrix}. \]

**Example 4.3:** Let

\[ A_{0,0} = \begin{bmatrix} 2.3 & 0.41 \\ -0.3 & 1.9 \end{bmatrix}, \quad A_{0,1} = \begin{bmatrix} 4.12 & -0.35 \\ 0.31 & 3.03 \end{bmatrix}, \]
\[ B_{0,0} = \begin{bmatrix} 2.45 & -0.3 \\ 0.2 & 4 \end{bmatrix}, \quad B_{0,1} = \begin{bmatrix} 2.5 & 0.6 \\ -0.2 & 3 \end{bmatrix}, \]
\[ C_{0,0} = \begin{bmatrix} 2.2 & 0.32 \\ 0.5 & 3 \end{bmatrix}, \quad C_{0,1} = \begin{bmatrix} 3.65 & -0.3 \\ -0.42 & 5.6 \end{bmatrix}, \]
\[ D_{0,0} = \begin{bmatrix} 5.6 & 1 \\ 0.73 & 7.8 \end{bmatrix}, \quad D_{0,1} = \begin{bmatrix} 5 & 0.73 \\ -0.47 & 5.2 \end{bmatrix}, \]
\[ A_{1,1} = \begin{bmatrix} 6 & 1.63 \\ -1.37 & 7 \end{bmatrix}, \quad B_{1,1} = \begin{bmatrix} 4 & 0.93 \\ 1.07 & 3 \end{bmatrix}, \]
\[ C_{1,1} = \begin{bmatrix} 8 & 2.03 \\ -1.23 & 10 \end{bmatrix}, \quad D_{1,1} = \begin{bmatrix} 5 & -0.93 \\ 1.016 & 4.65 \end{bmatrix}, \]
\[ Q_{0,0} = \begin{bmatrix} 2 & 0.8 \\ 0.8 & 1.6 \end{bmatrix}, \quad Q_{0,1} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ R_{0,0} = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_{0,1} = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \]
\[ Q_{1,1} = \begin{bmatrix} 2 & 0.1 \\ 0.1 & 5 \end{bmatrix}, \quad R_{1,1} = \begin{bmatrix} 4 & -0.3 \\ -0.3 & 7 \end{bmatrix}, \]
\[ G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 2 & -0.3 \\ -0.3 & 3 \end{bmatrix}. \]

Note that \( R_{0,0} \) is indefinite and \( R_{0,1} \) is negative definite. By (58), we have

\[ P_{1,1} = \begin{bmatrix} 18.8304 & -11.9513 \\ -11.9513 & 46.5418 \end{bmatrix}, \]
\[ P_{0,1} = \begin{bmatrix} 40.6027 & -28.7266 \\ -28.7266 & 50.9647 \end{bmatrix}, \]
\[ P_{0,0} = \begin{bmatrix} 99.6787 & 14.1112 \\ 14.1112 & 8.3265 \end{bmatrix}. \]
with properties

\[ W_{1,1} = \begin{bmatrix} 86.9155 & 11.0531 \\ 11.0531 & 103.2478 \end{bmatrix} > 0, \]

\[ W_{0,0} = \begin{bmatrix} 1282.7 & -1027.0 \\ -1027.0 & 3582.2 \end{bmatrix} > 0. \]

From (69) (70), we have that (58) is solvable. Hence, the closed-loop equilibrium strategy does exist. Furthermore, a closed-loop equilibrium strategy \( (\Phi_0, \Phi_1) \) is given by

\[ \Phi_0 = \begin{bmatrix} -0.4665 & -0.0206 \\ 0.0206 & -0.3965 \end{bmatrix}, \]

\[ \Phi_1 = \begin{bmatrix} -1.4499 & -0.4726 \\ 0.6369 & -1.8700 \end{bmatrix}. \]

V. Conclusion

In this paper, we have carefully investigated the open-loop equilibrium control and the closed-loop equilibrium strategy of the time-inconsistent indefinite stochastic LQ optimal control. Necessary and sufficient conditions are, respectively, presented for these both cases. Furthermore, the GDREs and LDEs are introduced to characterize the closed-loop form of the open-loop equilibrium control and the closed-loop equilibrium strategy. For future researches, we would like to study time-inconsistent problems with random coefficients, or more generally, extend the methodology developed in this paper to other types of time-inconsistency.

References


Appendix A

Note that \( u^l,x^*|_{k+1} \) appears in the both sides of (17). For any given \( k \in \mathbb{T}_k \), we then denote \( \bar{J}(k,X^l,k^*;u,k) \) by \( \bar{J}(k,X^l,k^*;u) \) as \( u^l,x^*|_{k+1} \) is fixed, i.e.,

\[ \bar{J}(k,X^l,k^*;u) = J(k,X^l,k^*;u,k) \] (71)

Hence, (17) implies that

\[ \bar{J}(k,X^l,k^*;u) \leq \bar{J}(k,X^l,k^*;u,k), \] (72)

holds for any \( u \in L^2_\mathbb{P}(k;\mathbb{R}^m) \). This means that \( u^l,x^* \) is an optimal control of the following nonstandard optimal control problem (denoted as Problem (LQ)_k):

\[
\begin{align*}
\text{Minimize } & J(k,X^l,k^*;u,k) \text{ over } L^2_\mathbb{P}(k;\mathbb{R}^m), \\
\text{subject to } & X^l_{k+1} = A_kX^l_k + B_ku^l,x, \\
& X^l_{k+1} = (A_kX^l_k + B_ku^l,x)_{k+1} + (C_kX^l_k + D_ku^l,x)w_k, \\
& X^l_k = (A_kX^l_k + B_ku^l,x)_{k+1} + (C_kX^l_k + D_ku^l,x)w_k, \\
& X^l_k = X^l_k, \quad \ell \in \mathbb{T}_{k+1}.
\end{align*}
\]

Here, we call Problem (LQ)_k a nonstandard optimal control problem as the control \( u^l,x^*|_{k+1} \) in the dynamics of \( X^l_k \) on \( \mathbb{T}_{k+1} \) is fixed, and we just select \( u_k \) to minimize \( J(k,X^l,k^*;u,k) \).

To proceed, we introduce the inner product in \( L^2_\mathbb{P}(\mathbb{T}_k;\mathbb{R}^p) \) \( \langle y,z \rangle_{\mathbb{T}_k} = \sum_{\ell=k}^{N-1} \mathbb{E}(y^\ell z_\ell), \) for \( y,z \in L^2_\mathbb{P}(\mathbb{T}_k;\mathbb{R}^p), \)

and use the convention

\[
\begin{align*}
\langle Q_kx, x \rangle &= Q_kx, x, \quad \forall x \in L^2_\mathbb{P}(\mathbb{T}_k;\mathbb{R}^n), \\
\langle R_ku, u \rangle &= R_ku, u, \quad \forall u \in L^2_\mathbb{P}(\mathbb{T}_{k+1};\mathbb{R}^m).
\end{align*}
\] (73)

For \( L^2_\mathbb{P}(k;\mathbb{R}^p) \) \( \langle y,z \rangle_k = \mathbb{E}(y^Tk z), \) for \( y,z \in L^2_\mathbb{P}(k;\mathbb{R}^p), \)

then the cost functional \( \bar{J}(k,X^l,k^*;u) \) can be written as

\[
\bar{J}(k,X^l,k^*;u) = \langle Q_kX^l,k^*,X^l \rangle_{\mathbb{T}_k} + \langle R_ku_k, u_k \rangle_k
\]
We now calculate the first order and second order directional derivatives of $J(k, X^{k,x,*}; u_k)$ at $u_k^{t,x,*}$ (when they exist). With the initial state $X^{k,x,*}$, denote solutions to (19) corresponding to controls $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ and $(u^{t,x,*}_k + \lambda \bar{u}_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})$ by $X^{k,*}$ and $X^{k,\lambda}$, respectively, where $\bar{u}_k \in L_2^2(k; \mathbb{R}^m)$. We then have

$$
\begin{aligned}
&\begin{cases}
\frac{X^{k,\lambda}-X^{k,x,*}}{\lambda} = A_{k,\ell} \frac{X^{k,\lambda}-X^{k,x,*}}{\lambda} + C_{k,\ell} \frac{X^{k,\lambda}-X^{k,x,*}}{\lambda} w_{t,\ell}, \\
X^{k,\lambda}_{k+1} = (A_{k,\ell} X^{k,\lambda}_{k}+B_{k,\ell} \bar{u}_k)w_{k,\ell},
\end{cases} \\
&X^{k,\lambda}_{k+1} = 0, \quad \ell \in \mathbb{T}_{k+1}.
\end{aligned}
$$

Denote $\frac{X^{k,\lambda}-X^{k,x,*}}{\lambda}$ by $Y^{k}_\ell$. Therefore, it holds that

$$
\begin{aligned}
\begin{cases}
Y^{k}_\ell = A_{k,\ell} Y^{k}_\ell + C_{k,\ell} Y^{k}_{t,\ell} w_{t,\ell}, \quad \ell \in \mathbb{T}_{k+1}, \\
Y^{k}_{k+1} = B_{k,\ell} \bar{u}_k + D_{k,\ell} \bar{u}_k w_{k,\ell}, \\
Y^{k}_k = 0.
\end{cases}
\end{aligned}
$$

Clearly, for any $\ell \in \mathbb{T}_k$, we have $X^{k,\lambda}_\ell = X^{k,*}_\ell + \lambda Y^{k}_\ell$. To get the first order directional derivative, we need some calculations. It is clear that

$$
\begin{aligned}
&\lim_{\lambda \to 0} \frac{(R_{k,k}(u_k^{t,x,*} + \lambda \bar{u}_k), u_k^{t,x,*} + \lambda \bar{u}_k) - (R_{k,k}(u_k^{t,x,*}, u_k^{t,x,*}))_k}{\lambda} \\
= &2(R_{k,k}(u_k^{t,x,*}, \bar{u}_k))_k + \lim_{\lambda \to 0} \lambda (R_{k,k}(\bar{u}_k, \bar{u}_k))_t \\
= &2(R_{k,k}(u_k^{t,x,*}, \bar{u}_k))_k.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\lim_{\lambda \to 0} \frac{(Q_k X^{k,\lambda}, X^{k,\lambda})_{\mathbb{T}_k} - (Q_k X^{k,*}, X^{k,*})_{\mathbb{T}_k}}{\lambda} \\
= &2(Q_k(\bar{u}_k), Y^{k})_{\mathbb{T}_k}, \\
&\lim_{\lambda \to 0} \frac{(G_k X^{k,\lambda}_N, X^{k,\lambda}_N)_{N} - (G_k X^{k,*}_N, X^{k,*}_N)_{N}}{\lambda} \\
= &2(G_k X^{k,*}_N, Y^{k}_N)_{N}.
\end{aligned}
$$

Therefore, from (74) and the above equalities, we have the first order directional derivative of $\tilde{J}(k, X^{k,x,*}; u_k)$ at $u_k^{t,x,*}$ with the direction $\bar{u}_k$

$$
\begin{aligned}
d\tilde{J}(k, X^{k,x,*}; u_k^{t,x,*}; \bar{u}_k) = &\lim_{\lambda \to 0} \frac{\tilde{J}(k, X^{k,x,*}; u_k^{t,x,*} + \lambda \bar{u}_k) - \tilde{J}(k, X^{k,x,*}; u_k^{t,x,*})}{\lambda} \\
= &2(Q_k \hat{Y}^{k}, Y^{k})_{\mathbb{T}_k} + 2(R_{k,k}\tilde{u}_k, \tilde{u}_k) + 2(G_k \hat{Y}^{k}_{N}, Y^{k}_N)_{N},
\end{aligned}
$$

where

$$
\begin{aligned}
\begin{cases}
\hat{Y}^{k}_\ell &\triangleq \frac{X^{k,\beta}_{k,\ell}-X^{k,x,*}_{k,\ell}}{\beta}, \\
\hat{Y}^{k}_{k+1} &\triangleq A_{k,\ell} \hat{Y}^{k}_\ell + C_{k,\ell} \hat{Y}^{k}_{t,\ell} w_{t,\ell}, \quad \ell \in \mathbb{T}_{k+1}, \\
\hat{Y}^{k}_{k+1} &\triangleq B_{k,\ell} \tilde{u}_k + D_{k,\ell} \bar{u}_k w_{k,\ell}, \\
\hat{Y}^{k}_k &\triangleq 0.
\end{cases}
\end{aligned}
$$

If $\bar{u}_k = \tilde{u}_k$, then it holds that

$$
\begin{aligned}
d^2 \tilde{J}(k, X^{k,x,*}; u_k^{t,x,*}; \bar{u}_k) = &2(Q_k Y^{k}, Y^{k})_{\mathbb{T}_k} + 2(R_{k,k}(\bar{u}_k, \bar{u}_k))_k + 2(G_k Y^{k}_N, Y^{k}_N)_N. \quad (77)
\end{aligned}
$$

Note that the right hand side of (77) is independent of $u_k^{t,x,*}$. Hence, for any $u_k \in L_2^2(k; \mathbb{R}^m)$, it holds that

$$
\begin{aligned}
d^2 \tilde{J}(k, X^{k,x,*}; u_k; \bar{u}_k) = &2(Q_k Y^{k}, Y^{k})_{\mathbb{T}_k} + 2(R_{k,k}(\bar{u}_k, \bar{u}_k))_k + 2(G_k Y^{k}_N, Y^{k}_N)_N. \quad (78)
\end{aligned}
$$

Furthermore, we can show that $\tilde{J}(k, X^{k,x,*}; u_k)$ is infinitely differentiable in the sense that the directional derivatives of all orders exist. By the classical results on convex analysis [6], we have the following result.

**Lemma A.1:** The following statements are equivalent.

(i) The map $u_k \mapsto \tilde{J}(k, X^{k,x,*}; u_k)$ is convex.

(ii) The following holds

$$
\inf_{\bar{u}_k \in L_2^2(k; \mathbb{R}^m)} \left[ (Q_k Y^{k}, Y^{k})_{\mathbb{T}_k} + (R_{k,k}(\bar{u}_k, \bar{u}_k))_k + (G_k Y^{k}_N, Y^{k}_N)_N \right] \geq 0.
$$

**Proof of Theorem 2.1.** (i) $\Rightarrow$ (ii). Let $(X^{t,x,*}, u^{t,x,*})$ be an open-loop equilibrium pair of Problem (LQ) for the initial pair $(t, x)$. Since $\tilde{J}(k, X^{k,x,*}; u_k)$ is infinitely differentiable with respect to $u_k$ and (78) is independent of $u_k$, the minimum point $u_k^{t,x,*}$ of $\tilde{J}(k, X^{k,x,*}; u_k)$ is characterized by the first and second order derivatives: $d\tilde{J}(k, X^{k,x,*}; u_k^{t,x,*}; \bar{u}_k) = 0$ and $d^2 \tilde{J}(k, X^{k,x,*}; u_k^{t,x,*}; u_k; \bar{u}_k) \geq 0$ for any $\bar{u}_k \in L_2^2(k; \mathbb{R}^m)$. By Lemma A.1, (22) follows. The forward SAE of $X^{k,*}$ is clearly solvable as $Z^{k,*}$ does not appear in this SAE. After obtaining $X^{k,*}$ and substituting $X^{k,*}$ into the backward SAE, we then have $Z^{k,*}$. This means that the FBSDE (20) admits a solution $(X^{k,*}, Z^{k,*})$. Furthermore, by (76), it holds that

$$
\begin{aligned}
&\frac{1}{2}d\tilde{J}(k, X^{k,x,*}; u_k^{t,x,*}; \bar{u}_k) \\
= &\sum_{\ell=k}^{N-1} \mathbb{E}\left[ (X^{k,*}_\ell)^T Q_k, t Y^{k}_\ell \right] + \mathbb{E}\left[ (u^{t,x,*}_\ell)^T R_{k,k} \bar{u}_k \right] \\
&+ \mathbb{E}\left[ (X^{k,*}_N)^T G_k Y^{k}_N \right] \\
= &\sum_{\ell=k}^{N-1} \mathbb{E}\left[ (X^{k,*}_\ell)^T Q_k, t Y^{k}_\ell \right] + \mathbb{E}\left[ (u^{t,x,*}_\ell)^T R_{k,k} \bar{u}_k \right] \\
&+ \sum_{\ell=k}^{N-1} \mathbb{E}\left[ (Z^{k,*}_\ell)^T Y^{k}_\ell \right] \\
= &\sum_{\ell=k}^{N-1} \mathbb{E}\left[ (A_{k,\ell})^T \mathbb{E} (Z^{k,*}_{\ell+1})_{F_{\ell-1}} + Q_k, t X^{k,*}_{\ell} + C_{k,\ell} \mathbb{E} (Z^{k,*}_{\ell+1})_{F_{\ell-1}} - Z^{k,*}_\ell \right] \\
&+ \mathbb{E}\left[ (R_{k,k} u^{t,x,*}_\ell + B_{k,k} \mathbb{E} Z^{k,*}_\ell)_{F_{\ell-1}} \right] \\
+ \mathbb{E}\left[ (R_{k,k} u^{t,x,*}_\ell + B_{k,k} \mathbb{E} Z^{k,*}_\ell)_{F_{\ell-1}} \right].
\end{aligned}
$$
Continuing the above procedure of obtaining (80)-(83), we have
\[
\begin{align*}
\bar{u}_k &= 0, \quad \forall \bar{u}_k \in L_2^2(k; \mathbb{R}^m).
\end{align*}
\]
Combining the above result and (22), we know that \( u_{k,t}^{\ast,x} \) is the minimizer of \( J(k, X_{k,t}^{x,x}; u_k) \) over \( L_2^2(k; \mathbb{R}^m) \). This means that (72), (equivalent to (17)), holds for \( k \in \mathbb{T}_t \). To avoid notation complexity, we now calculate the open-loop equilibrium state. Let us begin with \( k = t \). We have from (13) that
\[
\begin{align*}
X_{t+1}^{x,x} &= A_{t+1}X_{t+1}^{x,x} + B_{t+1}u_{t+1}^{x,x} + (C_{t+1}X_{t+1}^{x,x} + D_{t+1}u_{t+1}^{x,x}) w_t, \\
X_t^{x,x} &= x,
\end{align*}
\]
and for any \( u_t \in L_2^2(t; \mathbb{R}^m) \)
\[
J(t, x; u_{t}^{x,x}) \leq J(t, x; (u_t, u_{t}^{x,x} | t_{t+1})).
\]
We then move to \( k = t + 1 \). In this case, the starting point of the state is \( X_{t+1}^{x,x} \). Hence, we have
\[
\begin{align*}
X_{t+1}^{x,x} &= A_{t+1}X_{t+1}^{x,x} + B_{t+1}u_{t+1}^{x,x} + (C_{t+1}X_{t+1}^{x,x} + D_{t+1}u_{t+1}^{x,x}) w_{t+1}, \\
X_t^{x,x} &= X_{t+1}^{x,x},
\end{align*}
\]
and for any \( u_{t+1} \in L_2^2(t+1; \mathbb{R}^m) \)
\[
J(t+1, X_{t+1}^{x,x}; u_{t+1}^{x,x} | t_{t+1}) \leq J(t+1, X_{t+1}^{x,x}; (u_{t+1}, u_{t+1}^{x,x} | t_{t+2})).
\]
Continuing the above procedure of obtaining (80)-(83), we have for any \( k \in \mathbb{T}_t \)
\[
\begin{align*}
X_{k+1}^{x,x} &= A_kX_{k+1}^{x,x} + B_ku_k^{x,x} + (C_kX_{k+1}^{x,x} + D_ku_k^{x,x}) w_k, \\
X_k^{x,x} &= X_{k+1}^{x,x},
\end{align*}
\]
and for any \( u_k \in L_2^2(k; \mathbb{R}^m) \)
\[
J(k, X_{k}^{x,x}; u_k^{x,x} | t_k) \leq J(k, X_{k}^{x,x}; (u_k, u_k^{x,x} | t_{k+1})).
\]
Denote \( \{x, X_{t+1}^{x,x}, X_{t+2}^{x,x}, \ldots, X_{N-1}^{x,x}, X_N^{x,x}\} \) by \( X_{t}^{x,x} \) which satisfies (24) and is an open-loop equilibrium state. This proves the conclusion. \( \square \)

**APPENDIX B**

Let
\[
\begin{align*}
\hat{J}(k, X_k^{x,x}; v | t_k) &= \mathbb{E}\left[ (X_N^{x})^T G_k X_N^{x} \right] + \sum_{\ell=k}^{N-1} \mathbb{E}\left[ (X_{\ell}^{x})^T Q_{k,\ell} X_{\ell}^{x} \right] + (u_{\ell} + \Phi_k X_{\ell}^{x})^T R_{k,\ell} (u_{\ell} + \Phi_k X_{\ell}^{x}), \quad k \in \mathbb{T}_t,
\end{align*}
\]
and
\[
\begin{align*}
X_{\ell+1}^{x} &= (A_{k,\ell} + B_k,\ell \Phi_k) X_{\ell}^{x} + B_k,\ell u_{\ell},
\end{align*}
\]
Clearly, it holds that for any \( k \in \mathbb{T}_t \)
\[
\hat{J}(k, X_k^{x,x}; 0 | t_k) = J(k, X_k^{x,x}; (\Phi_k X_k^{x,x} | t_k)),
\]
\[
\hat{J}(k, X_k^{x,x}; (u_k - \Phi_k X_k^{x,x}; 0 | t_{k+1})),
\]
\[
J(k, X_k^{x,x}; (u_k, (\Phi_k X_k^{x,x} u_k, 0 | t_{k+1}))).
\]
Here, \( 0 | t_k \) is understood as \( u_{t_k} = \{u_k, \ldots, u_{N-1} \} \) with \( u_{t_k} = 0, \ell \in \mathbb{T}_k \). Hence, (46) reads as
\[
\begin{align*}
\hat{J}(k, X_k^{x,x}; 0 | t_k) &\leq J(k, X_k^{x,x}; (u_k - \Phi_k X_k^{x,x}; 0 | t_{k+1})),
\end{align*}
\]
Noting that \( u_k \in L_2^2(k; \mathbb{R}^m) \) in (87) is arbitrary, we then have that \( u_k - \Phi_k X_k^{x,x} \) ranges over all the elements in \( L_2^2(k; \mathbb{R}^m) \). Therefore, if \( \Phi_k \) is a closed-loop equilibrium strategy of Problem (LQ), then \( 0 | t_k \) is the open-loop equilibrium control of the time-inconsistent stochastic LQ problem corresponding to (86) and (87). Furthermore, letting \( u_k - \Phi_k X_k^{x,x} = v_k \), we then denote \( \hat{J}(k, X_k^{x,x}; (u_k - \Phi_k X_k^{x,x}; 0 | t_{k+1})), \) and \( J(k, X_k^{x,x}; 0 | t_k) \), respectively, by \( \hat{J}(k, X_k^{x,x}; v_k) \) and \( J(k, X_k^{x,x}; 0) \).

To avoid notation complexity, we now calculate the first order and second order direction derivatives of \( \hat{J}(k, X_k^{x,x}; u_k) \) instead of \( \hat{J}(k, X_k^{x,x}; v_k) \). Here, \( u_k \) is a generic element in \( L_2^2(k; \mathbb{R}^m) \). Note that
\[
\begin{align*}
\hat{J}(k, X_k^{x,x}; u_k + \lambda \delta u_k) &= J(k, X_k^{x,x}; (u_k + \lambda \delta u_k, (\Phi_k X_k^{x,x} u_k, \delta \Phi_k, \delta u_k) | t_{k+1})),
\end{align*}
\]
and
\[
\begin{align*}
\hat{J}(k, X_k^{x,x}; u_k + \lambda \delta u_k) &= J(k, X_k^{x,x}; (u_k + \lambda \delta u_k, (\Phi_k X_k^{x,x} u_k, \delta \Phi_k, \delta u_k) | t_{k+1})),
\end{align*}
\]
where \( Q_{k,\ell}, R_{k,\ell} \) are similarly defined as those in (73), and
\[
\begin{align*}
\begin{cases}
X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} &= (A, k, \ell + B_k, \ell \Phi_k) X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} + (C_k, k, \ell + D_k, \ell \Phi_k) w_k, \ell \in \mathbb{T}_{k+1}, \\
X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} &= (A, k, \ell + B_k, \ell \Phi_k) X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} + (C_k, k, \ell + D_k, \ell \Phi_k) w_k, \ell \in \mathbb{T}_{k+1}, \\
X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} &= (A, k, \ell + B_k, \ell \Phi_k) X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} + (C_k, k, \ell + D_k, \ell \Phi_k) w_k, \ell \in \mathbb{T}_{k+1}, \\
X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} &= (A, k, \ell + B_k, \ell \Phi_k) X_{k+1}^{x,x,\delta u_k, \Phi, \lambda} + (C_k, k, \ell + D_k, \ell \Phi_k) w_k, \ell \in \mathbb{T}_{k+1},
\end{cases}
\end{align*}
\]
Noting that $X^{t,x,r}_{k,u_k,\Phi} = \tilde{X}^{t,x,r}_{k,u_k,\Phi,\lambda} = X^{t,x,\ast}_{k}$, we then have
\[
\begin{align*}
\begin{cases}
\lambda^{X^{t,x,r}_{k,u_k,\Phi},\lambda} X^{t,x,r}_{k,u_k,\Phi} \\
= (A_{k,\ell} + B_{k,\ell} \Phi) X^{t,x,r}_{k,u_k,\Phi} + (C_{k,\ell} + D_{k,\ell} \Phi) X^{t,x,r}_{k,u_k,\Phi} \lambda^{-2} X^{t,x,r}_{k,u_k,\Phi} \\
= \left( A_{k,\ell} X^{t,x,r}_{k,u_k,\Phi} + B_{k,\ell} \tilde{u}_k \right) \\
+ \left( C_{k,\ell} X^{t,x,r}_{k,u_k,\Phi} + D_{k,\ell} \tilde{u}_k \right) \lambda^{-2} X^{t,x,r}_{k,u_k,\Phi} + \{ X^{t,x,r}_{k,u_k,\Phi} \}_{T_k} \\
\lambda^{-2} X^{t,x,r}_{k,u_k,\Phi} = 0,
\end{cases}
\end{align*}
\]
for all $\ell \in T_{k+1}$.

Denote $X^{t,x,r}_{k,u_k,\Phi,\lambda} = Y^{t,x,r}_{k,u_k,\Phi,\lambda}$, which is independent of $u_k$ and $\lambda$. Therefore, it holds that
\[
\begin{align*}
\begin{cases}
Y^{k,\bar{u}_k,\Phi} = (A_{k,\ell} + B_{k,\ell} \Phi) Y^{k,\bar{u}_k,\Phi} + (C_{k,\ell} + D_{k,\ell} \Phi) Y^{k,\bar{u}_k,\Phi} \lambda^{-2} X^{k,\bar{u}_k,\Phi} \lambda^{-2} X^{k,\bar{u}_k,\Phi} \\
= \left( A_{k,\ell} Y^{k,\bar{u}_k,\Phi} + B_{k,\ell} \tilde{u}_k \right) \\
+ \left( C_{k,\ell} Y^{k,\bar{u}_k,\Phi} + D_{k,\ell} \tilde{u}_k \right) X^{k,\bar{u}_k,\Phi} \lambda^{-2} X^{k,\bar{u}_k,\Phi} + \{ Y^{k,\bar{u}_k,\Phi} \}_{T_k} \\
\lambda^{-2} X^{k,\bar{u}_k,\Phi} = 0,
\end{cases}
\end{align*}
\]
for all $\ell \in T_{k+1}$.

Clearly, for any $\ell \in T_k$, it holds that
\[
X^{t,x,r}_{k,u_k,\Phi,\lambda} = X^{t,x,r}_{k,u_k,\Phi} + \lambda X^{k,\bar{u}_k,\Phi}.
\]

Similar to the deviation of (76), we have
\[
d\tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = \lim_{\lambda \downarrow 0} \tilde{J}(k, X^{t,x,r}_{k,u_k,\lambda}) - \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = 2(Q_k Y^{k,\bar{u}_k,\Phi}, Y^{k,\bar{u}_k,\Phi})_{T_k} + 4(R_{k,k} \Phi Y^{k,\bar{u}_k,\Phi})_{T_k}
\]
and for any $\bar{u}_k \in L^2_T(k; \mathbb{R}^m)$
\[
d^2 \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = \tilde{J}(k, X^{t,x,r}_{k,u_k,\lambda}) - \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k})
\]
where
\[
\begin{align*}
Y^{k,\bar{u}_k,\Phi} = (A_{k,\ell} + B_{k,\ell} \Phi) Y^{k,\bar{u}_k,\Phi} + (C_{k,\ell} + D_{k,\ell} \Phi) Y^{k,\bar{u}_k,\Phi} \lambda^{-2} X^{k,\bar{u}_k,\Phi} \\
= \left( A_{k,\ell} Y^{k,\bar{u}_k,\Phi} + B_{k,\ell} \tilde{u}_k \right) \\
+ \left( C_{k,\ell} Y^{k,\bar{u}_k,\Phi} + D_{k,\ell} \tilde{u}_k \right) X^{k,\bar{u}_k,\Phi} \lambda^{-2} X^{k,\bar{u}_k,\Phi} + \{ Y^{k,\bar{u}_k,\Phi} \}_{T_k} \\
\lambda^{-2} X^{k,\bar{u}_k,\Phi} = 0.
\end{align*}
\]
If $\tilde{u}_k = \bar{u}_k$, then it holds that
\[
d^2 \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = 2(Q_k Y^{k,\bar{u}_k,\Phi}, Y^{k,\bar{u}_k,\Phi})_{T_k} + 4(R_{k,k} \Phi Y^{k,\bar{u}_k,\Phi})_{T_k}
\]
and for any $\bar{u}_k \in L^2_T(k; \mathbb{R}^m)$
\[
d^2 \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = \tilde{J}(k, X^{t,x,r}_{k,u_k,\lambda}) - \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = 2(Q_k Y^{k,\bar{u}_k,\Phi}, Y^{k,\bar{u}_k,\Phi})_{T_k} + 4(R_{k,k} \Phi Y^{k,\bar{u}_k,\Phi})_{T_k}
\]
which is independent of $u_k$.

Proof of Theorem 3.1. (i)⇒(ii). Let $\Phi$ be a closed-loop equilibrium strategy of Problem (LQ). From the derivation between (86) and (87), we know that 0 is the open-loop equilibrium control of the stochastic LQ problem corresponding to (86) and (87). Combining the fact that (91) is independent of $u_k$, we then have
\[
\begin{align*}
\begin{cases}
d\tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = 0, \\
d^2 \tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = 0,
\end{cases}
\end{align*}
\]
hold for all $u_k, \bar{u}_k \in L^2_T(k; \mathbb{R}^m)$. Noting (92), (90) and (91), we have (53) and
\[
0 = \frac{1}{2} d\tilde{J}(k, X^{t,x,r}_{k,u_k,\bar{u}_k}) = \sum_{\ell=k}^{N-1} \mathbb{E}\left[\left( (A_{k,\ell} + B_{k,\ell} \Phi) \right)^T \mathbb{E}(Z^{k,\Phi}_{\ell+1} | \mathcal{F}_{\ell-1}) + (C_{k,\ell} + D_{k,\ell} \Phi) \right] Y^{k,\Phi}_{\ell} - Z^{k,\Phi}_{\ell} \right] T Y^{k,\Phi}_{\ell} + \mathbb{E}\left[ (R_{k,k} \Phi Y^{k,\Phi}_{\ell} + B_{k,k} \Phi) \right] \mathbb{E}(Z^{k,\Phi}_{\ell+1} | \mathcal{F}_{\ell-1}) + \mathbb{E}(Z^{k,\Phi}_{\ell+1} \mathbb{E}(Z^{k,\Phi}_{\ell+1} | \mathcal{F}_{\ell-1}) - Z^{k,\Phi}_{\ell+1}) \mathbb{E}(Z^{k,\Phi}_{\ell+1} | \mathcal{F}_{\ell-1})
\]