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# STOCHASTIC APPROXIMATION BASED CONSENSUS DYNAMICS OVER MARKOVIAN NETWORKS*

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**Abstract.** This paper considers consensus problems with random networks. A key object of our analysis is a sequence of stochastic matrices which involve Markovian switches and decreasing step sizes. We establish ergodicity of the backward products of these stochastic matrices. The basic technique is to consider the second moment dynamics of an associated Markovian jump linear system and exploit its two-scale interaction property resulting from the decreasing step sizes. The mean square convergence rate of the backward products is also obtained. The ergodicity results are used to prove mean square consensus of stochastic approximation algorithms where agents collect noisy information. The approach is further applied to a token scheduled averaging model.

**Key words.** backward product, consensus, ergodicity, Markovian switch, mean square convergence, stochastic approximation

**AMS subject classifications.** 93E03, 93E15, 94C15, 68R10, 60J10

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## 1. Introduction.

Consensus algorithms with imperfect information exchange or randomly perturbed state evolution have been systematically investigated, addressing many important issues including measurement noise, exogenous perturbations entering system state dynamics, and the quantization effect [1, 12, 13, 15, 24, 30]. The work [28] made an early effort introducing stochastic gradient based consensus algorithms. For noisy modeling of collective motion in multiagent systems, see, e.g., [6, 22].

When noisy measurements of neighboring agents' states are available, stochastic approximation with decreasing step sizes may be applied to reduce long-term fluctuation of the iteration [12, 13, 18, 19, 23, 26]. A popular tool for proving convergence is to use quadratic Lyapunov functions. For fixed network topologies containing a spanning tree, the existence of such functions is guaranteed. This is provable by the constructive method in [11, 31]. For time-varying topologies, the use of Lyapunov functions typically depends on assuming balanced graphs or restrictive eigenvalue

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conditions [1, 10, 19]. For time-varying directed graphs, the assumption of balanced weights is very restrictive.

To overcome the limitation of the Lyaponov approach, a new technique is introduced in [9]. Consider the stochastic consensus algorithm

$$X_{t+1} = (I + a_t B_t^o)X_t + a_t D_t^o W_t, \quad t \geq 0,$$

for $n$ agents with randomly varying network topologies. Each matrix $B_t^o$ is determined by the random network topology at time $t$ modeled by a directed graph $G_t^o$. The random matrix sequence $\{D_t^o, t \geq 0\}$ is bounded. The sequence $\{W_t, t \geq 0\}$ consists of independent vector random variables and is independent of $\{(B_t^o, D_t^o), t \geq 0\}$. We take the step size $a_t$ satisfying the standard conditions in stochastic approximation. It is shown that mean square consensus is ensured if and only if $\{A_t := I + a_t B_t^o, t \geq 0\}$ has ergodic backward products with probability one. By studying the trajectory behavior of a switching linear system, it is further shown that such ergodicity holds, and so the balanced graph assumption is removed. This idea is very different from using paracontractions [7] or Wolfowitz's theorem [29]. A key condition used in [9] is that for a zero probability set $N_0$ and each $\omega \in \Omega \backslash N_0$, there exists a sequence $0 = T_0(\omega) < T_1(\omega) < T_2(\omega) < \ldots$ such that the graph union is strongly connected on each discrete time interval $[T_l(\omega), T_{l+1}(\omega))$, $l \geq 0$, and

$$\sup_l [T_{l+1}(\omega) - T_l(\omega)] < \infty. \tag{1.1}$$

In this paper, we are interested in an important class of networks, where the switches are governed by a finite state Markov chain $\{\theta_t, t \geq 0\}$ and condition (1.1) does not hold in general. The Markovian switches can model communication failure [10, 33] and randomized scheduling for signal transmission. Our analysis starts by considering the matrix sequence $\{I + a_t B_{\theta_t}, t \geq 0\}$ which naturally arises in stochastic approximation algorithms in order to attenuate noise and is also of interest in its own right. In some other situations, general stochastic matrices of the form $\{I + a_t B_t^o, t \geq 0\}$, converging to an identity matrix and so referred to as degenerating stochastic matrices [9], can be used to model hardening positions in consensus models [2, 4]. In relation to [9], the route of analyzing stochastic approximation through studying ergodic backward products of $\{I + a_t B_{\theta_t}, t \geq 0\}$ is still valid in this Markovian switching model. However, we need to develop very different techniques to establish ergodicity. We introduce an auxiliary noiseless Markovian jump linear system associated with degenerating stochastic matrices and next examine the dynamics of its second moment matrix, which is similar to [5, 21]. Based on the second moment dynamics, we further identify a class of time-varying linear systems with two-scale interactions, on which we will develop the main machinery for eventually proving ergodicity of backward products. We also obtain the mean square convergence rate of the backward products.

The approach of this paper will be further applied to study a noisy averaging model where a token is used to schedule the broadcast of the state information of a node. A well-known randomized scheduling rule for broadcast is to employ independent Poisson clocks [8, 32]. Our scheduling mechanism has certain advantages since the nodes have more autonomy in their operation. In contrast, Poisson clocks implicitly demand more coordination since all agents should refer to a common time scale.

We mention some recent literature on ergodicity of stochastic matrices over random networks. The work [21] considers backward products of $\{A_{\theta_t}, t \geq 0\}$ and establishes their almost sure convergence by the second moment dynamics. Average consensus is proved when each matrix $A_{\theta_t}$ is further assumed to be doubly stochastic [20]. The approach of [21] is to tackle a time-invariant linear difference equation and the main condition is that the Markov chain is irreducible and that the graph union contains a spanning tree. Our model gives rise to a time-varying difference equation for the second moment dynamics and for this reason the associated asymptotic analysis is very different from [21]. For a sequence of independent stochastic matrices $\{A_t, t \geq 0\}$, ergodicity is proved by an infinite flow approach in [27].

The key idea in our two-scale analysis of the second moment dynamics is to construct a lower dimensional model which is able to reflect certain connectivity properties ensured by the graph union. In a different context, two-scale consensus modeling with Markovian regime switching is introduced in [16, 34] and weak convergence analysis is developed. The model in [16] includes a faster Markov chain to be tracked by multiple sensors. The work [34] treats different relative values of the regime switching rate and the step size used in the state update. For multiagent parameter estimation problems, [14] uses step sizes of different scales for averaging states and incorporating local parameter estimation.

We make some notes on notation. We use $1_k$ to denote a column vector consisting of $k$ ones, and $J_k = \frac{1}{k}1_k 1_k^T$. The indicator function of an event $A$ is denoted by $1_A$. We use $I$ to denote an identity matrix with its dimension clear from the context. For clarity, we sometimes indicate the dimension by adding a subscript (such as $k$ in $I_k$). The number $M(i,j)$ denotes the $(i,j)$th entry of a matrix $M$. For a vector or matrix $M$, denote the Frobenius norm $|M| = [\mathrm{Tr}(M^T M)]^{1/2}$. For column vectors $Z_1, \dots, Z_k$, $[Z_1; \dots; Z_k]$ denotes the column vector obtained by vertical concatenation of the $k$ vectors. Let $\{g_t, t \geq 0\}$ and $\{h_t, t \geq 0\}$ be two sequences where the latter is a nonnegative sequence. Then $g_t = O(h_t)$ means that there exist constants $C$ and $T$ such that $|g_t| \leq Ch_t$ for all $t \geq T$, and $g_t = o(h_t)$ means that for any $\epsilon > 0$, there exists $T$ such that $|g_t| \leq \epsilon h_t$ for all $t \geq T$. The agent or node index is often used as a superscript ($x_t^i$, $\zeta_t^j$, $\kappa_t^i$, etc.) and should not be understood as an exponent. We also write some vectors ($\phi_t^k, z_t^k \in \mathbb{R}^N$) with superscript $k$, which is obviously seen not to be an exponent. The identification of these widely used superscripts should be clear from the context.

The paper is organized as follows. Section 2 introduces the stochastic matrix model with Markovian switches and decreasing step sizes, and section 3 presents the main results on ergodicity and stochastic approximation. Section 4 analyzes the second moment dynamics, and a two-scale model is obtained in section 5. Section 6 develops its convergence analysis. Section 7 analyzes the mean square convergence rate of the backward products. An application of the main result in section 3 is presented in section 8, which deals with a token scheduled averaging model. Section 9 concludes the paper.

## 2. The Markovian switching model.

### 2.1. Graph theoretic preliminaries.

We introduce some standard preliminaries on graph modeling of the network topology. A directed graph (digraph) $G = (\mathcal{N}, \mathcal{E})$ consists of a set of nodes $\mathcal{N} = \{1, \dots, n\}$ and a set of directed edges $\mathcal{E}$. A directed edge (simply called an edge) is denoted by an ordered pair $(i,j) \in \mathcal{N} \times \mathcal{N}$, where $i \neq j$. A directed path (from node $i_1$ to node $i_l$) consists of a sequence of nodes $i_1, \dots, i_l$, $l \geq 2$, such that $(i_k, i_{k+1}) \in \mathcal{E}$. The digraph $G$ is strongly connected if from any node to any other node, there exists a directed path. A directed tree is a digraph where each node $i$, except the root, has exactly one parent node $j$ so that $(j,i) \in \mathcal{E}$.

We call $G' = (\mathcal{N}', \mathcal{E}')$ a subgraph of $G$ if $\mathcal{N}' \subset \mathcal{N}$ and $\mathcal{E}' \subset \mathcal{E}$. The digraph $G$ is said to contain a spanning tree if there exists a directed tree $G_{\mathrm{tr}} = (\mathcal{N}, \mathcal{E}_{\mathrm{tr}})$ as a subgraph of $G$. If $(j,i) \in \mathcal{E}$, $j$ is called an in-neighbor (or neighbor) of $i$, and $i$ is called an out-neighbor of $j$. Denote $\mathcal{N}_i = \{j | (j,i) \in \mathcal{E}\}$. If $G$ is an undirected graph, each edge is denoted as an unordered pair $(i,j)$, where $i \neq j$.

For a matrix $M = (m_{ij})_{i,j \le k} \in \mathbb{R}^{k \times k}$, if it either is a stochastic matrix or has zero row sums and nonnegative off-diagonal entries, we define its interaction graph as a digraph denoted by $\mathrm{graph}(M) = (\mathcal{N}_M, \mathcal{E}_M)$, where $\mathcal{N}_M = \{1, \ldots, k\}$ and $(j,i) \in \mathcal{E}_M$ if and only if $m_{ij} > 0$.

**2.2. The Markovian model.** Let the underlying probability space be denoted by $(\Omega, \mathcal{F}, P)$. Suppose that $\{\theta_t, t = 0, 1, 2, \ldots\}$ is a Markov chain with state space $\{1, \ldots, N\}$ and transition probability matrix

$$P_\theta = (p_{lm})_{1 \le l,m \le N}.$$

Let $\{B_k, k = 1, \ldots, N\}$ be $n \times n$ matrices. Each $B_k$ has zero row sums and nonnegative off-diagonal entries and can be interpreted as the generator of an $n$ state continuous time Markov chain. Each $B_k$ is associated with its interaction digraph $G_k = (\mathcal{N}, \mathcal{E}_k)$, where $\mathcal{N} = \{1, \ldots, n\}$ and $(j,i) \in \mathcal{E}_k$ if and only if $b_{ij} > 0$.

Consider the sequence of matrices

$$\{I + a_t B_{\theta_t}, t \ge 0\}.$$

As $t \to \infty$, $I + a_t B_{\theta_t}$ tends to the identity matrix describing a trivial Markov chain without transitions. Following [9], we call it a sequence of degenerating stochastic matrices. Denote the backward product $\Psi_{t,s} = (I + a_{t-1} B_{\theta_{t-1}}) \cdots (I + a_s B_{\theta_s})$ for $t > s$ and $\Psi_{s,s} = I$. Our first task is to examine the asymptotic property of $\Psi_{t,s}$ for any fixed $s$ when $t \to \infty$.

*Remark* 1. Throughout the paper we assume

$$\inf_{t \ge 0, l, i} (1 + a_t B_l(i,i)) \ge 0$$

and otherwise may start with a large fixed initial time $t_0$ instead of time 0 and consider $t \ge t_0$.

We make the following assumptions:

(A1) $\{a_t, t = 0, 1, 2, \ldots\}$ is a nonnegative sequence satisfying (i) $\sum_{t=0}^{\infty} a_t = \infty$, (ii) $\sum_{t=0}^{\infty} a_t^2 < \infty$.

(A2) The Markov chain $\{\theta_t, t \ge 0\}$ with state space $\{1, \ldots, N\}$ is ergodic (i.e., irreducible and aperiodic).

The initial distribution of $\{\theta_t, t \ge 0\}$ is fixed and denoted by $\mu_{\theta_0}$. By (A2), the Markov chain has a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$ consisting of $N$ positive entries [25].

(A3) The union graph $\cup_{k=1}^N G_k$ contains a spanning tree $G_{\cup,\mathrm{tr}}$.

**3. Ergodicity and stochastic approximation.** A sequence of stochastic matrices $\{A_t, t \ge 0\}$ has ergodic backward products if for any given $s$, $\lim_{t \to \infty} A_t \ldots A_{s+1} A_s$ exists and is a matrix of identical rows.

THEOREM 3.1. *Assume* (A1)–(A3)*. The sequence of stochastic matrices* $\{I + a_t B_{\theta_t}, t \ge 0\}$ *has ergodic backward products with probability one.*

Before being able to prove this basic result, we need to develop the analytical tools in sections 4–6. The proof of Theorem 3.1 is postponed to Appendix B.

The ergodicity analysis for $\{I + a_t B_{\theta_t}, t \geq 0\}$ on one hand is important for establishing the mean square consensus result in Theorem 3.4 and on the other hand is interesting in its own right.

**3.1. Stochastic approximation.** Denote $X_t = [x_t^1, \dots, x_t^n]^T$. Consider the stochastic approximation based consensus algorithm

$$X_{t+1} = (I + a_t B_{\theta_t})X_t + a_t D_{\theta_t} W_t, \qquad t \geq 0, \tag{3.1}$$

where the Markov chain $\{\theta_t, t \geq 0\}$ determines the underlying network topology for information exchange between the agents. The dimension of the constant matrices $\{D_1, \dots, D_N\}$ is compatible with the noise vector $W_t$. This conceptually simple modeling can characterize the temporal correlation in the evolution of the network.

A similar Markovian switching noisy consensus model has been studied in [10]. However, that work assumed either balanced graphs or, more restrictively, the existence of a common Lyapunov function. The present work does not depend on such assumptions.

(A4) $\{W_t, t \geq 0\}$ is a sequence of independent vector random variables of zero mean, which is independent of $\{\theta_t, t \geq 0\}$. In addition, $\sup_t E|W_t|^2 < \infty$ and $E|X_0|^2 < \infty$.

To study the convergence of (3.1), we introduce the definition.

DEFINITION 3.2. *The $n$ nodes are said to achieve mean square consensus if $E|x_t^i|^2 < \infty$, $t \geq 0$, $1 \leq i \leq n$, and there exists a random variable $x^*$ such that $\lim_{t\to\infty} E|x_t^i - x^*|^2 = 0$ for $1 \leq i \leq n$.*

The next lemma is an immediate consequence of [9, Theorem 3] by running (3.1) with a general initial time-state pair $(t_0, X_{t_0})$, $t_0 \geq 0$.

LEMMA 3.3. *Under* (A1)–(A4)*,* (3.1) *ensures mean square consensus for any given initial time-state pair $(t_0, X_{t_0})$ with $E|X_{t_0}|^2 < \infty$ if and only if $\{I + a_t B_{\theta_t}\}$ has ergodic backward products with probability one.*

THEOREM 3.4. *Assume* (A1)–(A4)*. The algorithm* (3.1) *ensures mean square consensus.*

*Proof.* This theorem follows from Lemma 3.3 and Theorem 3.1. ☐

**4. The second moment dynamics.** Throughout this section, (A1)–(A2) are assumed. The backward products of $\{I + a_t B_{\theta_t}, t \geq 0\}$ will be studied by use of the difference equation

$$X_{t+1} = (I + a_t B_{\theta_t})X_t. \tag{4.1}$$

For this linear system, we run it with any initial time-state pair $(t_0, X_{t_0})$, where $X_{t_0}$ is deterministic. The process $\{\theta_t, t \geq t_0\}$ is the restriction of the original Markov chain $\{\theta_t, t \geq 0\}$ on the discrete time interval $[t_0, \infty)$. For $t \geq 0$, let $\mu_{\theta_t}$ be the distribution of $\theta_t$.

Denote

$$V_l(t) = E\left[X_t X_t^T 1_{\{\theta_t = l\}}\right], \quad t \geq t_0, \tag{4.2}$$

$$V(t) = \sum_{l=1}^{N} V_l(t). \tag{4.3}$$

The expectation in (4.2) is evaluated using $(X_{t_0}, \mu_{\theta_{t_0}})$, where $\mu_{\theta_{t_0}}$ in turn is determined from $\mu_{\theta_0}$. The object $V_l(t)$ was also used in [21] for a Markovian switching linear consensus model $X_{t+1} = A_{\theta_t} X_t$ which does not have a step size $a_t$ as in (4.1). The approach of [21] is to obtain a time-invariant linear system for $\{V_l, 1 \le l \le N\}$ and check its asymptotic property, which is very different from our approach to be developed below.

Recall that $\pi = (\pi_1, \dots, \pi_N)$ is the stationary distribution of $\{\theta_t, t \ge 0\}$. For $t \ge t_0$, we have the second moment dynamics

$$\begin{aligned} V_l(t+1) &= E\left[X_{t+1}X_{t+1}^T 1_{\{\theta_{t+1}=l\}}\right] \\ &= \sum_{m=1}^N E\left[(I + a_t B_m) X_t X_t^T (I + a_t B_m)^T 1_{\{\theta_{t+1}=l, \theta_t = m\}}\right] \\ &= \sum_{m=1}^N p_{ml} E\left[(I + a_t B_m) X_t X_t^T (I + a_t B_m)^T 1_{\{\theta_t = m\}}\right] \\ &= \sum_{m=1}^N p_{ml} (I + a_t B_m) V_m(t) \left(I + a_t B_m^T\right). \end{aligned}$$

For an $m \times n$ matrix $M$, $\mathrm{vec}(M)$ is an $mn$ dimensional column vector obtained by stacking its $n$ columns in order with the first column on top. Let $\xi_t^l = \mathrm{vec}(V_l(t))$ and $\xi_t = [\xi_t^1; \dots; \xi_t^N]$ as vertical concatenation of the $N$ components. Denote the Kronecker sum $A \oplus B = A \otimes I_n + I_n \otimes B$ for $n \times n$ matrices $A$ and $B$. We have

$$\begin{aligned} \xi_{t+1} &= \begin{pmatrix} p_{11} I_{n^2} & p_{21} I_{n^2} & \dots & p_{N1} I_{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N} I_{n^2} & p_{2N} I_{n^2} & \dots & p_{NN} I_{n^2} \end{pmatrix} \xi_t \\ &\quad + a_t \begin{pmatrix} p_{11}(B_1 \oplus B_1) & \dots & p_{N1}(B_N \oplus B_N) \\ \vdots & \ddots & \vdots \\ p_{1N}(B_1 \oplus B_1) & \dots & p_{NN}(B_N \oplus B_N) \end{pmatrix} \xi_t \\ &\quad + a_t^2 \begin{pmatrix} p_{11}(B_1 \otimes B_1) & \dots & p_{N1}(B_N \otimes B_N) \\ \vdots & \ddots & \vdots \\ p_{1N}(B_1 \otimes B_1) & \dots & p_{NN}(B_N \otimes B_N) \end{pmatrix} \xi_t \\ &=: (M_{1,0} + a_t M_{2,0} + a_t^2 M_{3,0}) \xi_t. \end{aligned} \tag{4.4}$$

A matrix is said to be nonnegative if all its entries are nonnegative.

PROPOSITION 4.1. (i) *Both* $M_{2,0}$ *and* $M_{3,0}$ *have zero row sums.* (ii) $M_{1,0} + a_t M_{2,0} + a_t^2 M_{3,0}$ *is a nonnegative matrix.*

*Proof*. Part (i) can be verified directly. We check (ii). By Remark 1, the only possible entries within $a_t M_{2,0} + a_t^2 M_{3,0}$ to have negative values are the $((l-1)n^2 + i, (m-1)n^2 + i)$th entries, $l, m = 1, \dots, N$, $i = 1, \dots, n^2$. We take $l = 1, m = 1, i = 1$, and all other cases can be checked similarly. The $(1,1)$th entry of the matrix $M_{1,0} + a_t M_{2,0} + a_t^2 M_{3,0}$ is

$$p_{11}[1 + 2b_1(1,1)a_t + (b_1(1,1))^2 a_t^2] \ge 0.$$

This proves (ii). ∎

To facilitate further analysis, we will modify (4.4) into a new form. Denote the matrix $\Pi = \mathrm{diag}(\pi_1 I_{n^2}, \dots, \pi_N I_{n^2}) \in \mathbb{R}^{Nn^2 \times Nn^2}$ and introduce the linear transformation

$$\bar{\xi}_t = \Pi^{-1} \xi_t, \quad t \ge t_0.$$

Denote

$$M_1 = \Pi^{-1} M_{1,0} \Pi = \begin{pmatrix} \frac{\pi_1 p_{11}}{\pi_1} I_{n^2} & \frac{\pi_2 p_{21}}{\pi_1} I_{n^2} & \dots & \frac{\pi_N p_{N1}}{\pi_1} I_{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\pi_1 p_{1N}}{\pi_N} I_{n^2} & \frac{\pi_2 p_{2N}}{\pi_N} I_{n^2} & \dots & \frac{\pi_N p_{NN}}{\pi_N} I_{n^2} \end{pmatrix},$$

$$M_2 = \Pi^{-1} M_{2,0} \Pi = \begin{pmatrix} \frac{\pi_1 p_{11}}{\pi_1} (B_1 \oplus B_1) & \dots & \frac{\pi_N p_{N1}}{\pi_1} (B_N \oplus B_N) \\ \vdots & \ddots & \vdots \\ \frac{\pi_1 p_{1N}}{\pi_N} (B_1 \oplus B_1) & \dots & \frac{\pi_N p_{NN}}{\pi_N} (B_N \oplus B_N) \end{pmatrix},$$

and $M_3 = \Pi^{-1} M_{3,0} \Pi$. Then

$$\bar{\xi}_{t+1} = (M_1 + a_t M_2 + a_t^2 M_3) \bar{\xi}_t. \tag{4.5}$$

Although $\bar{\xi}_t$ has been defined in terms of $\{V_l(t), 1 \leq l \leq N\}$ for $t \geq t_0$, the linear system (4.5) can be studied in terms of any initial pair $(t_1, \bar{\xi}_{t_1}) \in \mathbb{Z}_+ \times \mathbb{R}^{Nn^2}$ for $t_1 \geq 0$.

PROPOSITION 4.2. (i) *$M_2$ and $M_3$ have zero row sums.* (ii) *For each $t \geq 0$, $M_1 + a_t M_2 + a_t^2 M_3$ is a stochastic matrix.* (iii) *$M_1 + a_t M_2$ is a stochastic matrix for all $t \geq t_0^*$ provided that $\inf_{t \geq t_0^*, l, i} (1 + 2a_t B_l(i,i)) \geq 0$.*

*Proof.* Analogous to the proof of Proposition 4.1, we can show that (i) holds. Furthermore, $M_1 + a_t M_2 + a_t^2 M_3$ is a nonnegative matrix. Now it suffices to show that $M_1$ has unit row sums. For each $l$, the stationary distribution $(\pi_1, \dots, \pi_N)$ satisfies $\sum_{k=1}^N \pi_k p_{kl} = \pi_l$. So the $l$th row sum of $M_1$ equals 1. Part (ii) follows. We check the $((l-1)n^2 + i, (m-1)n^2 + i)$th entry of $M_1 + a_t M_2$, $l, m = 1, \dots, N$, $i = 1, \dots, n^2$. For instance,

$$\begin{aligned} &[M_1 + a_t M_2](1,1) = p_{11}(1 + 2a_t B_1(1,1)) \geq 0, \\ &[M_1 + a_t M_2](2,2) = p_{11}(1 + a_t B_1(1,1) + a_t B_1(2,2)) \geq 0 \end{aligned}$$

for $t \geq t_0^*$. In this manner, the $N^2 n^2$ entries are verified to be nonnegative. All remaining entries of $M_1 + a_t M_2$ are clearly nonnegative. Part (iii) follows. ☐

Since $a_t \to 0$, there exists $t_0^*$ satisfying the condition in Proposition 4.2. We consider the new linear system

$$\zeta_{t+1} = (M_1 + a_t M_2) \zeta_t, \qquad t \geq t_0^*. \tag{4.6}$$

We denote two statements: $S1$ (resp., $S2$)—Algorithm (4.5) (resp., (4.6)) ensures consensus with any given initial pair $(t_1, \bar{\xi}_{t_1})$ (resp., $(t_1, \zeta_{t_1}), t_1 \geq t_0^*$).

LEMMA 4.3. *$S1$ is equivalent to $S2$.*

*Proof.* For given initial pairs $(t_1, \bar{\xi}_{t_1})$ and $(t_1, \zeta_{t_1})$, we have $\sum_{t=t_1}^\infty a_t^2 |M_3 \bar{\xi}_t| < \infty$ and $\sum_{t=t_1}^\infty a_t^2 |M_3 \zeta_t| < \infty$. Thus one algorithm may be viewed as another subject to small perturbation. The method is similar to the proof of [9, Lemma B.2]. ☐

**5. The averaging model with two-scale interactions.** Throughout this section, (A1)–(A3) are assumed. We view (4.6) as a consensus problem with $Nn^2$ agents indexed by $\{1, 2, \dots, Nn^2\}$. To identify the interaction relation of these agents, we introduce a small parameter $\epsilon > 0$ and define the matrix

$$M_\epsilon = M_1 + \epsilon M_2.$$

Denote $\beta = \max_{k,i} |B_k(i,i)| > 0$. For each fixed

$$\epsilon \in (0, (4\beta)^{-1}],$$

$M_\epsilon$ is a stochastic matrix and can be associated with a Markov chain $\{\Upsilon_t, t \geq 0\}$ of $Nn^2$ states $\{1, 2, \dots, Nn^2\}$. Denote the list

$$\begin{aligned} S_1 &= \{1, n^2+1, \dots, (N-1)n^2+1\}, \\ S_2 &= \{2, n^2+2, \dots, (N-1)n^2+2\}, \\ &\vdots \\ S_{n^2} &= \{n^2, 2n^2, \dots, Nn^2\}. \end{aligned}$$

This list will be used as a partition of the states of $\{\Upsilon_t, t \geq 0\}$ and later on for classifying the $Nn^2$ agents of (4.6) into $n^2$ groups.

Denote the matrix

$$P_\pi = (q_{lm})_{1\leq l,m\leq N} = \left(\frac{\pi_m p_{ml}}{\pi_l}\right)_{1\leq l,m\leq N},$$

which can be verified to be a stochastic matrix.

LEMMA 5.1. *The stochastic matrix $(q_{lm})_{l,m\leq N}$ is ergodic and its stationary distribution is $\pi$.*

*Proof.* See Appendix A. ∎

THEOREM 5.2. *Suppose that* (A3) *holds with $i_0$ being the root of $G_{\cup,\mathrm{tr}}$. Then the state $i_0$ of the $Nn^2$ state Markov chain $\{\Upsilon_t, t \geq 0\}$ is reachable from any other state with positive probability; equivalently,* graph$(M_\epsilon)$ *contains a spanning tree $G_{M_\epsilon,\mathrm{tr}}$ with $i_0$ being its root.*

*Proof.* See Appendix A. ∎

For the $N$ states in $S_i$, $i \leq n^2$, denote the transition probability

$$p^{(i)}_{lm} = P\left(\Upsilon_{t+1} = (m-1)n^2 + i | \Upsilon_t = (l-1)n^2 + i\right)$$

and $P^{(i)} = (p^{(i)}_{lm})_{l,m\leq N}$. It is straightforward to show

$$P^{(i)} = (q_{lm})_{l,m\leq N} + \epsilon Q^{(i)},$$

which is a substochastic matrix and where $Q^{(i)}$ does not depend on $\epsilon$.

*Remark* 2. When $\epsilon$ becomes very small, the transition probabilities among the states within $S_i$ are mainly determined by the ergodic matrix $(q_{lm})_{l,m\leq N}$. By the structure of $M_\epsilon$, the transition probability from one state in $S_i$ to another in $S_j$, $i \neq j$ (if nonzero) is on the order of $\epsilon$.

We visualize $S_1 \cup \cdots \cup S_{n^2}$ as a decomposition of the state space of $\{\Upsilon_t, t \geq 0\}$ where strong interactions exist within each set $S_i$ and no strong interactions exist between any $S_i$ and $S_j$, $i \neq j$. Below we will exploit this structure to transform (4.6) into an equivalent form, which appears to be simpler. This will be done using $a_t$ in place of $\epsilon$.

Recall that $\zeta_t$ in (4.6) is viewed as the state vector of $Nn^2$ agents. Denote $\zeta_t = [\zeta_t^1, \zeta_t^2, \dots, \zeta_t^{Nn^2}]^T$, where each superscript $j \leq Nn^2$ is used as an agent index. Now we rewrite (4.6) by reordering the position of the $Nn^2$ agents. The collection $S_1, \dots, S_{n^2}$

will be used to denote different groups of the $Nn^2$ agents of (4.6). Let $\phi_t^k \in \mathbb{R}^N$ be the states of the agents with indices in $S_k$,

$$\phi_t^k = \left[\zeta_t^k, \zeta_t^{n^2+k}, \dots, \zeta_t^{(N-1)n^2+k}\right]^T, \quad 1 \le k \le n^2. \tag{5.1}$$

We take a permutation of the components of $\zeta_t$ to get the new vector

$$\phi_t := \left[\phi_t^1; \phi_t^2; \dots; \phi_t^{n^2}\right].$$

In fact, there exists a unique nonsingular matrix $\Gamma$ such that

$$\phi_t = \Gamma \zeta_t.$$

By (4.6), the new state vector $\phi_t$ satisfies

$$\phi_{t+1} = \Gamma(M_1 + a_t M_2)\Gamma^{-1}\phi_t =: \hat{M}(a_t)\phi_t. \tag{5.2}$$

It is clear that $\hat{M}(a_t)$ is a stochastic matrix if $M_1 + a_t M_2$ is.

*Remark* 3. By Proposition 4.2, $\hat{M}(a_t)$ is a stochastic matrix for all large $t$.

THEOREM 5.3. *$\hat{M}(a_t)$ has the representation*

$$\hat{M}(a_t) = \begin{bmatrix} \hat{M}_{11}(a_t) & a_t\hat{M}_{12} & \cdots & a_t\hat{M}_{1n^2} \\ a_t\hat{M}_{21} & \hat{M}_{22}(a_t) & \cdots & a_t\hat{M}_{2n^2} \\ \vdots & & \ddots & \\ a_t\hat{M}_{n^21} & a_t\hat{M}_{n^22} & \cdots & \hat{M}_{n^2n^2}(a_t) \end{bmatrix}, \tag{5.3}$$

*where*

(i) *$\hat{M}_{ij} \in \mathbb{R}^{N\times N}$ is a constant nonnegative matrix for any $i \neq j$, and so independent of the value of $a_t$.*

(ii) *$\hat{M}_{ii}(a_t) + a_t \sum_{j\neq i} \hat{M}_{ij} = (q_{lm})_{l,m\le N}$ for all $i \le n^2$.*

*Proof.* Consider (4.6) and any agent $i' \in S_i$ with state of the form $\zeta_t^{(j-1)n^2+i}$ for some $1 \le j \le N$. If this agent updates its state using the state of an agent $j' \in S_j$, the weight assigned to $j'$ can only originate as an entry of $a_t M_2$; see Remark 2. This implies that all off-diagonal blocks in (5.3) must take the form $a_t\hat{M}_{ij}$. By Proposition 4.2, whenever $a_t$ is sufficiently small, $M_1 + a_t M_2$ and so $\hat{M}(a_t)$ are nonnegative matrices. So $\hat{M}_{ij}$ is nonnegative for $i \neq j$. This proves (i).

To show (ii), we check $A_1 := \hat{M}_{11}(a_t) + a_t \sum_{j\neq 1} \hat{M}_{1j}$. First, $\hat{M}_{11}(a_t)(1,1) = q_{11} + 2a_t q_{11} B_1(1,1)$. Next, $\sum_{j=2}^{n^2} a_t \hat{M}_{1j}(1,1) = 2a_t q_{11} \sum_{j=2}^{n} B_1(1,j)$. Therefore, $A_1(1,1) = q_{11}$. We continue to check $A_1(1,l)$, $1 < l \le N$. Then

$$\begin{aligned} \hat{M}_{11}(1,l) &= M_1(1,(l-1)n^2+1) + a_t M_2(1,(l-1)n^2+1) \\ &= q_{1l} + a_t q_{1l}(2B_l(1,1)), \\ a_t\hat{M}_{1j}(1,l) &= a_t M_2(1,(l-1)n^2+j) \\ &= a_t q_{1l}(B_l \otimes I_n + I_n \otimes B_l)(1,j), \quad j \ge 2, \end{aligned}$$

which is the weight agent 1 in (4.6) assigns to the agent as the $l$th member of the $j$th group $S_j$. It can be checked that

$$\sum_{j=2}^{n^2} (B_l \otimes I_n + I_n \otimes B_l)(1,j) = 2\sum_{k=2}^{n} B_l(1,k).$$

It follows that

$$A_1(1,l) = \hat{M}_{11}(1,l) + \sum_{j=2}^{n^2} a_t \hat{M}_{1j}(1,l) = q_{1l}.$$

In the same manner we can check the remaining entries of $A_1$ and also other cases of $\hat{M}_{ii}$, $2 \le i \le n^2 - 1$. The theorem follows. ∎

By Theorem 5.3, we may write (5.2) as

$$\begin{bmatrix} \phi_{t+1}^1 \\ \phi_{t+1}^2 \\ \vdots \\ \phi_{t+1}^{n^2} \end{bmatrix} = \begin{bmatrix} \hat{M}_{11}(a_t) & a_t\hat{M}_{12} & \cdots & a_t\hat{M}_{1n^2} \\ a_t\hat{M}_{21} & \hat{M}_{22}(a_t) & \cdots & a_t\hat{M}_{2n^2} \\ \vdots & & \ddots & \\ a_t\hat{M}_{n^21} & a_t\hat{M}_{n^22} & \cdots & \hat{M}_{n^2n^2}(a_t) \end{bmatrix} \begin{bmatrix} \phi_t^1 \\ \phi_t^2 \\ \vdots \\ \phi_t^{n^2} \end{bmatrix}, \tag{5.4}$$

which will be called a canonical form of (4.6).

We may view (5.4) as a two-scale averaging model. To avoid confusion, when a consensus model is examined with a corresponding number of agents, the index of an agent is specified according to the position of its state within the state vector. For instance, $\phi_t^1$ denotes the states of agents with indices $\{1,\ldots,N\}$. Denote $\hat{S}_k = \{(k-1)N+1,\ldots,kN\}$, $k=1,\ldots,n^2$. By (5.1), it is evident that the agent indices $\hat{S}_k$ in (5.4) and $S_k$ in (4.6) refer to the same group of agents physically.

The canonical form makes it convenient to identify the interaction structure of the $Nn^2$ agents. Within each group $\hat{S}_k$, averaging takes place rapidly when (5.4) is iterated. The interconnection between the groups is controlled by the step size $a_t$. By Theorem 5.3, once $a_t$ is fixed, the matrix $\hat{M}(a_t)$ is completely determined by the set of off-diagonal blocks. We will continue to check whether they will be able to generate adequate interactions among the groups $\{\hat{S}_1,\ldots,\hat{S}_{n^2}\}$ in some sense.

We define a new graph which has fewer nodes than graph$(\hat{M}(\epsilon))$. Its purpose is to indicate the information flow among different agent groups $\hat{S}_1,\ldots,\hat{S}_{n^2}$ of (5.4).

Let $\hat{G}_{\mathrm{q}}$ be a digraph with nodes $\mathcal{N}_{\mathrm{q}} = \{1,2,\ldots,n^2\}$ and the set of edges $\mathcal{E}_{\mathrm{q}}$. An edge $(j,i) \in \mathcal{E}_{\mathrm{q}}$ if and only if $\hat{M}_{ij} \neq 0$. If we identify all nodes of each $S_i$ as an equivalent class, $\hat{G}_{\mathrm{q}}$ defined above may be called a quotient graph of graph$(M_\epsilon)$. The graph $\hat{G}_{\mathrm{q}}$ does not depend on the particular value of the small parameter $\epsilon$.

LEMMA 5.4. For $\hat{G}_{\mathrm{q}}$, $(j,i) \in \mathcal{E}_{\mathrm{q}}$ if and only if there is an edge on graph$(M_\epsilon)$ from a node in $S_j$ to a node in $S_i$.

*Proof.* There is an edge on graph$(M_\epsilon)$ from a node in $S_j$ to a node in $S_i$ if and only if $\hat{M}_{ij} \neq 0$. ∎

THEOREM 5.5. *$\hat{G}_{\mathrm{q}}$ contains a spanning tree.*

*Proof.* By Theorem 5.2, graph$(M_\epsilon)$ contains a spanning tree $G_{M_\epsilon,\mathrm{tr}}$. Without loss of generality, assume that the root of $G_{M_\epsilon,\mathrm{tr}}$ is node 1. It suffices to show that node 1 of $\hat{G}_{\mathrm{q}}$ can reach any other node $j \in \{2,\ldots,n^2\}$ by a directed path. Select such a node $j$.

Consider graph$(M_\epsilon)$. There exists a directed path from node $1 \in S_1$ to node $j \in S_j$. Denote this directed path by $1, k_2, k_3, \ldots, k_r, j$. Suppose that $k_i \in S_{d_i}$. We list $S_1, S_{d_2}, \ldots, S_{d_r}, S_j$. For this list, if $S_k$ appears successively in a segment, we list $S_k$ only once corresponding to that segment. By Lemma 5.4, the resulting list identifies a directed path from node 1 to node $j$ in $\hat{G}_{\mathrm{q}}$. ∎

*Remark* 4. Theorems 5.2, 5.3, 5.5, and Lemma 5.4 still hold if (A2) is replaced by the weaker assumption that $\{\theta_t, t \ge 0\}$ is irreducible while all other assumptions remain the same.

**6. Convergence of algorithm (5.4).** Assume (A1)–(A3) for this section. For each $\phi_t^k$, denote $\phi_t^k = [\phi_t^{k,1}, \dots, \phi_t^{k,N}]^T \in \mathbb{R}^N$. In this section the integer $k \le n^2$ will frequently be used as a superscript but not an exponent for various vectors. Consider (5.4) with any given initial pair $(t_1, \phi_{t_1})$. Our method is to derive a lower dimensional model. Each component $\phi_t^k$ corresponds to $N$ equations within (5.4) for which we attempt to only retain the equation for $\phi_t^{k,1}$.

Recall that $P_\pi = (q_{lm})_{l,m \le N}$ is an ergodic stochastic matrix. Denote its $N$ eigenvalues by $\lambda_1 = 1, \lambda_2, \dots, \lambda_N$. Then $\max_{2 \le l \le N} |\lambda_l| < 1$. Fix any $\delta \in (\max_{2 \le l \le N} |\lambda_l|, 1)$. Define

$$a_t^* = \sum_{s=0}^{t} \delta^{t-s} a_s, \quad t \ge 0.$$

The next lemma provides some prior estimate of the difference between different entries in $\phi_t^k$.

LEMMA 6.1. *We have*

$$\max_k \max_{l,m} |\phi_t^{k,l} - \phi_t^{k,m}| = O(a_t^*), \quad t \ge t_1.$$

*Proof.* First, there exists a constant $C$, depending on the initial pair $(t_1, \phi_{t_1})$ of (5.4), such that $\sup_{t,k} |\phi_t^k| \le C$; see Remark 3. Denote $H_k(a_t) = a_t \sum_{j \ne k} \hat{M}_{kj}(\phi_t^j - \phi_t^k)$. Hence $|H_k(a_t)| = O(a_t)$. Next, we check $\phi_t^k$ and by Theorem 5.3 have the relation

$$\phi_{t+1}^k = P_\pi \phi_t^k + H_k(a_t). \tag{6.1}$$

Note that $P_\pi - I$ has rank $N-1$. Let $\Phi_{N-1}$ be an $n \times (n-1)$ matrix such that $\mathrm{span}(\Phi_{N-1}) = \mathrm{span}(P_\pi - I)$. Denote $\Phi = [1_N, \Phi_{N-1}] \in \mathbb{R}^{N \times N}$. By the method in [12] we can show that $\Phi$ is nonsingular and

$$\Phi^{-1} P_\pi \Phi = \begin{bmatrix} 1 & 0 \\ 0 & A_\pi \end{bmatrix},$$

where $A_\pi$ is an $(N-1) \times (N-1)$ matrix having all eigenvalues with absolute value less than $\delta$. In fact the first row of $\Phi^{-1}$ is equal to $\pi$. There exists a constant $C$ such that the power of $A_\pi$ satisfies

$$|A_\pi^t| \le C \delta^t, \quad t \ge 0.$$

Take a change of coordinates $z_t^k = \Phi^{-1} \phi_t^k \in \mathbb{R}^N$, and denote $z_t^k = [z_t^{k,1}, \dots, z_t^{k,N}]^T = [z_t^{k,1}; z_t^{k,-1}]$. Thus, $z_t^{k,1} = \pi \phi_t^k$. We obtain

$$\begin{aligned} z_{t+1}^{k,1} &= z_t^{k,1} + O(a_t), \\ z_{t+1}^{k,-1} &= A_\pi z_t^{k,-1} + H_{k,-1}(a_t), \end{aligned} \tag{6.2}$$

where $H_{k,-1}(a_t)$ is determined from $H_k(a_t)$ and so $|H_{k,-1}(a_t)| = O(a_t)$. The second equation leads to

$$\begin{aligned} |z_t^{k,-1}| &= \left| A_\pi^{t-t_1} z_{t_1}^{k,-1} + \sum_{s=t_1}^{t-1} A_\pi^{t-1-s} H_{k,-1}(a_s) \right| \\ &= O\left(\delta^{t-t_1} + a_{t-1}^*\right) = O(a_t^*). \end{aligned} \tag{6.3}$$

Now for $t \geq t_1$,

$$\phi_t^k = \Phi z_t^k = [1_N, \Phi_{N-1}]z_t = z_t^{k,1}1_N + \Phi_{N-1}z_t^{k,-1}. \tag{6.4}$$

The lemma follows. ∎

For a matrix $M$, we use $\mathrm{rsum}_l(M)$ to denote the sum of its $l$th row. With a slight abuse of notation, we will sometimes use $O(a_t)$ (or $o(a_t)$, $O(a_t^*)$, etc.) to denote a vector or matrix of compatible dimension. It means that each entry of the vector or matrix is of the form $O(a_t)$ (or $o(a_t)$, $O(a_t^*)$).

THEOREM 6.2. *For $k = 1, 2, \ldots, n^2$, we have*

$$z_{t+1}^{k,1} = \left(1 + a_t\hat{b}_{kk}\right) z_t^{k,1} + a_t \sum_{j=1, j\neq k}^{n^2} \hat{b}_{kj} z_t^{j,1} + O((a_t^*)^2), \quad t \geq t_1, \tag{6.5}$$

*where $\hat{b}_{kj} = \sum_{l=1}^N \pi_l \mathrm{rsum}_l(\hat{M}_{kj})$ for $j \neq k$, and $\hat{b}_{kk} = -\sum_{j=1,j\neq k}^{n^2} \hat{b}_{kj}$.*

*Proof.* By (6.1), we have

$$\begin{aligned} z_{t+1}^{k,1} &= \pi\phi_{t+1}^k \\ &= \pi P_\pi \phi_t^k - a_t\pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^k + a_t\pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^j \\ &= z_t^{k,1} - a_t\pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^k + a_t\pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^j. \end{aligned} \tag{6.6}$$

Since $z_t^{k,1} = \pi\phi_t^k$, it follows from Lemma 6.1 that for each $l$,

$$|\phi_t^{k,l} - z_t^{k,1}| = \left| \sum_{m=1}^N \pi_m(\phi_t^{k,l} - \phi_t^{k,m}) \right| = O(a_t^*).$$

Therefore,

$$\begin{aligned} \pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^k &= \pi \sum_{j\neq k} \hat{M}_{kj} \left[ z_t^{k,1}1_N + O(a_t^*) \right] = z_t^{k,1} \sum_{j\neq k} \sum_{l=1}^N \pi_l \mathrm{rsum}_l(\hat{M}_{kj}) + O(a_t^*) \\ &= \Big( \sum_{j\neq k} \hat{b}_{kj} \Big) z_t^{k,1} + O(a_t^*). \end{aligned}$$

Similarly,

$$\begin{aligned} \pi \sum_{j\neq k} \hat{M}_{kj}\phi_t^j &= \pi \sum_{j\neq k} \hat{M}_{kj} \left[ z_t^{j,1}1_N + O(a_t^*) \right] = \sum_{j\neq k} z_t^{j,1} \sum_{l=1}^N \pi_l \mathrm{rsum}_l(\hat{M}_{kj}) + O(a_t^*) \\ &= \sum_{j\neq k} \hat{b}_{kj} z_t^{j,1} + O(a_t^*). \end{aligned}$$

The theorem follows by combining (6.6) with the above estimates and the fact that $a_t = O(a_t^*)$. ∎

*Remark* 5. Lemma 6.1 and Theorem 6.2 hold under a much weaker condition on $a_t, t \geq 0$. We only need $0 \leq a_t \to 0$ and $\sup_t a_t > 0$; the new condition does not affect Theorem 5.3 and it ensures that (6.3) holds and that $M_1 + a_tM_2$ is a stochastic matrix for all large $t$.

Define

$$\hat{B} = \left(\hat{b}_{kj}\right)_{k,j\le n^2}, \tag{6.7}$$

which has zero row sums and nonnegative off-diagonal entries. Denote $y_t = [z_t^{1,1}, \ldots, z_t^{n^2,1}]^T$. Let (6.5) be written in the vector form

$$y_{t+1} = \left(I_{n^2} + a_t\hat{B}\right) y_t + O\left((a_t^*)^2\right), \quad t \ge t_1. \tag{6.8}$$

LEMMA 6.3. graph$(\hat{B}) = \hat{G}_{\rm q}$.

*Proof.* Both digraphs have the set of nodes $\{1,\ldots,n^2\}$. Note that $\hat{M}_{kj}$ is a nonnegative matrix for any $(j,k)$. Also, the stationary distribution $\pi$ has $N$ positive entries. So $\hat{b}_{kj} > 0$ if and only if $\hat{M}_{kj} \ne 0$. On the other hand, $(j,k)$ is an edge of graph$(\hat{B})$ if and only if $\hat{b}_{kj} > 0$; $(j,k)$ is an edge of $\hat{G}_{\rm q}$ if and only if $\hat{M}_{kj} \ne 0$. We conclude that both graph$(\hat{B})$ and $\hat{G}_{\rm q}$ have the same set of edges. ∎

THEOREM 6.4. *The algorithm* (5.4) *ensures consensus for any give initial pair* $(t_1, \phi_{t_1})$.

*Proof.* Consider the algorithm

$$y'_{t+1} = \left(I_{n^2} + a_t\hat{B}\right) y'_t. \tag{6.9}$$

This is a special case of the stochastic approximation algorithm in [12] by setting the noise as zero. By Theorem 5.5, Lemma 6.3, and the step size condition (A1), (6.9) ensures consensus with any initial pair $(t_0, y'_{t_0})$.

Given any initial pair $(t_1, \phi_{t_1})$, we accordingly determine the initial pair $(t_1, y_{t_1})$ in (6.8). Denote $r_t = \frac{1-\delta^{t+1}}{1-\delta}$. We observe that

$$(a_t^*)^2 = r_t^2 \left(\sum_{s=0}^{t} \frac{\delta^{t-s}}{r_t} a_s\right)^2 \le r_t^2 \sum_{s=0}^{t} \frac{\delta^{t-s}}{r_t} a_s^2 \le \frac{1}{1-\delta} \sum_{s=0}^{t} \delta^{t-s} a_s^2.$$

This implies that

$$\sum_{t=0}^{\infty} (a_t^*)^2 \le \frac{1}{1-\delta} \left(\sum_{k=0}^{\infty} \delta^k\right) \sum_{s=0}^{\infty} a_s^2 = \frac{1}{(1-\delta)^2} \sum_{s=0}^{\infty} a_s^2 < \infty. \tag{6.10}$$

By the convergence of (6.9), it follows from (6.10) and [9, Lemmas B.1, B.2] that for (6.8) with any given initial pair $(t_1, y_{t_1})$, $y_t$ converges to a limit vector in span$\{1_{n^2}\}$. In other words, there exists a common constant $c$ such that

$$\lim_{t\to\infty} z_t^{k,1} = c \qquad \text{for all } k = 1, \ldots, n^2.$$

Subsequently, $\lim_{t\to\infty} \phi_t^k = \lim_{t\to\infty} \Phi z_t^k = c1_N$ since $\lim_{t\to\infty} |z_t^{k,-1}| = 0$. This gives $\lim_{t\to\infty} \phi_t = c1_{Nn^2}$. The theorem follows. ∎

**7. Convergence rate.** Ergodicity of the backward products of $\{I + a_t B_{\theta_t}, t \ge 0\}$ has a central role in analyzing the stochastic approximation algorithm (3.1). Theorem 3.1 only characterizes a qualitative property of the sequence of backward products. Here we aim to obtain more information on its asymptotic behavior by establishing its mean square convergence rate.

With some regularity on $\{a_t, t \geq 0\}$, we may simplify the estimates in section 6. We prove the lemma below without requiring (A1).

LEMMA 7.1. *If* $\{a_t, t \geq 0\}$ *satisfies* $0 < a_t \to 0$ *and* $\lim_{t\to\infty} \frac{a_t}{a_{t+1}} = 1$, *then for any initial pair* $(t_1, y_{t_1})$,

$$y_{t+1} = \left(I_{n^2} + a_t \hat{B}\right) y_t + O(a_t^2), \quad t \geq t_1, \tag{7.1}$$

*where* $y_t = [z_t^{1,1}, \ldots, z_t^{n^2,1}]^T$ *and* $\hat{B}$ *is defined by* (6.7).

*Proof.* We follow the notation in section 6 and recall Remark 5. Rewrite (6.2) in the form

$$a_{t+1}^{-1} z_{t+1}^{k,-1} = A_\pi \left(a_t^{-1} z_t^{k,-1}\right) \frac{a_t}{a_{t+1}} + a_{t+1}^{-1}\, H_{k,-1}(a_t), \quad t \geq t_1.$$

Denote $v_t = a_t^{-1} z_t^{k,-1}$. This gives

$$v_{t+1} = (1 + o(1)) A_\pi v_t + O(1). \tag{7.2}$$

Since $A_\pi$ is stable (i.e., all its eigenvalues are inside the unit circle), we may specify any $Q_0 > 0$ and solve a unique $P_0 > 0$ from the Lyapunov equation $A_\pi^T P_0 A_\pi - P_0 + Q_0 = 0$. By use of (7.2), we may find a small constant $0 < c_0 < 1$ such that

$$v_{t+1}^T P_0 v_{t+1} \leq (1 - c_0) v_t^T P_0 v_t + O(1).$$

Hence $\sup_{t \geq t_1} |v_t| < \infty$ and

$$|z_t^{k,-1}| = O(a_t). \tag{7.3}$$

By adapting the proofs of Lemma 6.1 and Theorem 6.2, we see that under the current assumption, Lemma 6.1 and consequently (6.5) still hold when $a_t^*$ is replaced by $a_t$. This completes the proof. ∎

Taking $\gamma \in (1/2, 1]$, we choose

$$a_t = \frac{1}{t^\gamma}, \qquad t \geq 1, \tag{7.4}$$

and $a_0 > 0$. The more general case of $a_t = \frac{c}{t^\gamma}$ for $t \geq 1$, $c > 0$ can be reduced to (7.4) by replacing $\{B_1, \ldots, B_N\}$ by a new set of matrices. It is clear that (7.4) satisfies the assumption on $\{a_t, t \geq 0\}$ in Lemma 7.1.

Denote the backward product

$$\Psi_{t+1,t_0} = (I + a_t B_{\theta_t}) \ldots (I + a_{t_0} B_{\theta_{t_0}}), \quad t \geq t_0,$$

$\Psi_{t_0,t_0} = I$, where $\{a_t, t \geq 0\}$ is given by (7.4). According to Remark 1, we still assume that $I + a_t B_{\theta_t}$ is a stochastic matrix for all $t$. Under (A1)–(A3), Theorem 3.1 shows that $\Psi_{t+1,t_0}$ converges with probability one to a random matrix denoted by $\Psi_{\infty,t_0}$ which has identical rows. Since graph$(\hat{B})$ contains a spanning tree by Theorem 5.5 and Lemma 6.3, $\hat{B}$ has 1 eigenvalue equal to zero and $n^2 - 1$ eigenvalues having strictly negative real parts [12]. Suppose that $\sigma_0 > 0$ is a constant such that all nonzero eigenvalues of $\hat{B}$ have a real part strictly less than $-\sigma_0$.

THEOREM 7.2. *Let the step sizes be given by* (7.4) *and assume* (A2)–(A3).

(i) *If* $1/2 < \gamma < 1$, *we have*

$$E|\Psi_{t+1,t_0} - \Psi_{\infty,t_0}|^2 = O\left(\frac{1}{t^{2\gamma-1}}\right).$$

(ii) *If* $\gamma = 1$,

$$E|\Psi_{t+1,t_0} - \Psi_{\infty,t_0}|^2 = O\left(\frac{1}{t^{\eta}}\right).$$

*where* $\eta = \min\{1, \sigma_0\}$.

*Proof. Step* 1. Consider the linear system

$$X_{t+1} = (I + a_t B_{\theta_t})X_t, \quad t \geq t_0.$$

As in (B.2), set the initial condition $X_{t_0}^{(i)} = e_i$ and denote the corresponding solution $X_t^{(i)} = \Psi_{t,t_0} X_{t_0}^{(i)}$ for $t \geq t_0$. Then

$$\Psi_{t+1,t_0} = \left[X_{t+1}^{(1)}, \ldots, X_{t+1}^{(n)}\right], \quad t \geq t_0.$$

It follows that with probability one $X_t^{(i)}$ converges to $\eta_i 1_n$ which is equal to the $i$th column of $\Psi_{\infty,t_0}$ and where $\eta_i$ is a random variable. We have

$$|\Psi_{t+1,t_0} - \Psi_{\infty,t_0}|^2 = \sum_{i=1}^{n} \left|X_{t+1}^{(i)} - \eta_i 1_n\right|^2.$$

Below we check $X_t^{(1)}$ and simply write it as $X_t = [X_{t,1}, \ldots, X_{t,n}]^T$. Since $\eta_1 1_n$ is obtained as the limit state vector of a consensus model, we necessarily have

$$\min_k X_{t,k} \leq \eta_1 \leq \max_k X_{t,k}, \quad t \geq t_0.$$

Consequently, $|X_{t,k} - \eta_1| \leq \max_j |X_{t,k} - X_{t,j}| \leq \sum_{j=1}^{n} |X_{t,k} - X_{t,j}|$ almost surely. We need to estimate $E|X_{t,k} - X_{t,j}|^2$. For the initial condition $X_{t_0} = e_1$, we accordingly define $V_l(t)$ by (4.2) and $\bar{\xi}_t$ by (4.5) for $t \geq t_0$. The cases of $X^{(i)}$, $i \geq 2$, can be handled in exactly the same manner.

*Step* 2. Recalling (7.1), we write

$$y_{t+1} = \left(I_{n^2} + a_t \hat{B}\right) y_t + O(a_t^2),$$

for which we set the initial time $t_0$. By an appropriate change of coordinates $y_t = \hat{\Phi} p_t$ [12], we have

$$\begin{aligned} p_{t+1}^{(1)} &= p_t^{(1)} + O(a_t^2), \\ p_{t+1}^{(-1)} &= \left(I_{n^2-1} + a_t \hat{B}_0\right) p_t^{(-1)} + O(a_t^2), \end{aligned}$$

where $p_t = [p_t^{(1)}; p_t^{(-1)}]$, $p_t^{(-1)} \in \mathbb{R}^{n^2-1}$ and $\hat{B}_0$ is an $(n^2-1)\times(n^2-1)$ Hurwitz matrix. We have the limits $p_t^{(1)} \to p_\infty^{(1)}$ and $p_t^{(-1)} \to 0$ as $t \to \infty$. For $\{a_t, t \geq 0\}$ given by (7.4), denote $\epsilon_t = \sum_{s=t}^{\infty} a_s^2$. Then

$$\left|p_t^{(1)} - p_\infty^{(1)}\right| = O(\epsilon_t) = O\left(t^{1-2\gamma}\right).$$

Denote $\delta_t = |p_t^{(-1)}|$. There exists a constant $c$ such that $\lim_{t\to\infty} y_t = c1_{n^2}$, and

$$|y_t - c1_{n^2}| = O(\epsilon_t + \delta_t).$$

In other words,

$$\left|\left[z_t^{1,1}, \dots, z_t^{n^2,1}\right]^T - c1_{n^2}\right| = O(\epsilon_t + \delta_t).$$

By (6.4) and (7.3),

$$|\phi_t^k - c1_N| = O(a_t + \epsilon_t + \delta_t).$$

Thus,

$$|\zeta_t - c1_{Nn^2}| = O(a_t + \epsilon_t + \delta_t).$$

The above estimate is valid for any given $(t_0, \zeta_{t_0})$ and it allows us to have $t_0 \le t_0^*$ in (4.6).

*Step* 3. For $X_t$ in Step 1 with initial pair $(t_0, e_1)$, we determine $V_l(t_0)$ and accordingly $\bar{\xi}_{t_0}$ for (4.5). Denote the limit of $\bar{\xi}_t$ by $c_1 1_{Nn^2}$ which exists. By setting $\zeta_{t_0} = \bar{\xi}_{t_0}$ in (4.5)–(4.6) and comparing the two solutions, we further obtain

$$|\bar{\xi}_t - c_1 1_{Nn^2}| = O(a_t + \epsilon_t + \delta_t).$$

Let $\Pi$ be defined as in section 4. It follows that

$$|\xi_t - c_1 \Pi 1_{Nn^2}| = |\Pi\bar{\xi}_t - c_1 \Pi 1_{Nn^2}| = O(a_t + \epsilon_t + \delta_t).$$

On the other hand,

$$\left|V_l(t) - c_1 \pi_l 1_n 1_n^T\right| = \left|\xi_t^l - c_1 \pi_l 1_{n^2}\right| \le |\xi_t - c_1 \Pi 1_{Nn^2}|.$$

Let $V(t)$ be defined by (4.3) and recall $J_n = \frac{1}{n} 1_n 1_n^T$. It follows that

$$|V(t) - c_1 n J_n| = O(a_t + \epsilon_t + \delta_t).$$

Therefore,

$$\begin{aligned} E|X_{t,i} - X_{t,j}|^2 &= (e_i - e_j)^T [V(t) - c_1 n J_n](e_i - e_j) \\ &\le (e_i - e_j)^T |V(t) - c_1 n J_n| (e_i - e_j) \\ &= O(a_t + \epsilon_t + \delta_t). \end{aligned}$$

*Step* 4. If $1/2 < \gamma < 1$, $\delta_t = O(t^{-\gamma})$ by Lemma A.2. Hence

$$E|X_{t,i} - X_{t,j}|^2 = O(t^{-\gamma} + t^{1-2\gamma}) = O(t^{1-2\gamma}).$$

If $\gamma = 1$, $\delta_t = O(t^{-\eta})$ by Lemma A.2. This gives

$$E|X_{t,i} - X_{t,j}|^2 = O(t^{-1} + t^{-\eta}) = O(t^{-\eta}).$$

By Step 1, the theorem follows. $\square$

**8. Application to token scheduled averaging.** Let $G = (\mathcal{N}, \mathcal{E})$ be a strongly connected digraph, where $\mathcal{N} = \{1, \ldots, n\}$. The token process $\{T_t, t = 0, 1, \ldots\}$ is a random walk on $G$, and so is a Markov chain with state space $\{1, \ldots, n\}$. Let $\hat{\mu}_0$ be the distribution of $T_0$. The transition probability is $P(T_{t+1} = j | T_t = i) = \hat{p}_{ij}$, where $\hat{p}_{ij} > 0$ if and only if $i \in \mathcal{N}_j$. Denote $P_T = (\hat{p}_{ij})_{i,j \le n}$. It is evident that $G$ is strongly connected if and only if $\{T_t, t \ge 0\}$ is irreducible.

Each node has a counter $\kappa_t^i$, $i \in \mathcal{N}$, $t \ge 0$. The initial value $\kappa_0^i \ge 0$ is a deterministic integer. The counter is updated by the rule

$$\kappa_{t+1}^i = \kappa_t^i + 1_{\{T_{t+1}=i\}}, \quad t \ge 0,$$

where $1_A$ stands for the indictor function of an event $A$. This means that the counter is incremented by one upon each new possession of the token.

If $T_t = i$, node $i$ broadcasts its state $x_t^i$ which is received with additive noise by its out-neighbors. If $i \in \mathcal{N}_j$, node $j$ receives the measurement

$$y_t^{ji} = x_t^i + w_t^{ji}, \quad t \ge 0.$$

For convenience of modeling, we define $w_t^{ji}$ for all $(i, j) \in \mathcal{E}$. At time $t$ if no measurement occurs along the edge $(i, j)$, $w_t^{ji}$ is simply included as a dummy random variable. Let $\{a_t, t \ge 0\}$ be a nonnegative step size sequence. When $T_t = i$, the state of node $j$ evolves by the rule

$$x_{t+1}^j = \begin{cases} (1 - a_{\kappa_t^i}) x_t^j + a_{\kappa_t^i} y_t^{ji}, & i \in \mathcal{N}_j, \\ x_t^j, & i \notin \mathcal{N}_j, \end{cases} \quad t \ge 0. \tag{8.1}$$

The above modeling uses $t$ to mark the transitions of the token. There is no need for the nodes to share slotted time. When a node is during a period neither possessing the token nor collecting measurements, it remains in an idle status. Neither its counter nor its state is changed.

For each $i \in \mathcal{N}$, define the matrix $B_i = (B_i(j,k))_{j,k \le n}$ by the following rule. If $i \notin \mathcal{N}_j$, then $B_i(j,k) = 0$ for all $k$. If $i \in \mathcal{N}_j$,

$$B_i(j,k) = \begin{cases} -1, & k = j, \\ 1, & k = i, \\ 0, & \text{all other } k. \end{cases}$$

For a given $t \ge 0$, we list all random variables $\{w_t^{ji}, (i,j) \in \mathcal{E}\}$ into a vector $W_t$. The position of $w_t^{ji}$ within $W_t$ is determined only by $(i, j)$. Denote $X_t = [x_t^1, \ldots, x_t^n]^T$. Define $\mathbf{a}_{\kappa_t} = \mathrm{diag}(a_{\kappa_t^1}, \ldots, a_{\kappa_t^n})$. We write (8.1) in the vector form

$$X_{t+1} = (I + \mathbf{a}_{\kappa_t} B_{T_t}) X_t + \mathbf{a}_{\kappa_t} D_{T_t} W_t, \quad t \ge 0, \tag{8.2}$$

where the collection of matrices $\{D_1, \ldots, D_n\}$ can be defined accordingly and we omit the details.

We take $\gamma \in (1/2, 1]$ and

$$a_t = \frac{1}{t^\gamma}, \quad t \ge t_a,$$

for some $t_a \ge 1$ and $a_t \in [0, 1]$ for $t < t_a$. Then for each $t$, $I + \mathbf{a}_{\kappa_t} B_{T_t}$ is a stochastic matrix.

We introduce the following assumptions for the rest of this section.

(H1) $\{T_t, t \geq 0\}$ is ergodic with stationary distribution $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)$.

(H2) $\{W_t, t \geq 0\}$ is a sequence of independent vector random variables of zero mean and $\sup_t E|W_t|^2 < \infty$.

(H3) $\{T_t, t \geq 0\}$ and $\{W_t, t \geq 0\}$ are independent, and $E|X_0|^2 < \infty$.

LEMMA 8.1. *Under* (H1)*, there exists a deterministic constant $C$ such that for each $i$,*

$$\limsup_{t\to\infty} \frac{|\kappa_t^i - \hat{\pi}_i t|}{\sqrt{t \log\log t}} \leq C.$$

*Proof.* Consider a fixed $i$. We write $\kappa_t^i = \kappa_0^i + \sum_{s=1}^t 1_{\{T_s=i\}}$, $t \geq 1$. Following [3, section I.14], let

$$\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$$

be an increasing sequence of all values of $t \geq 1$ for which $T_t = i$. Denote $\rho_k = \tau_{k+1} - \tau_k$, which is called the $k$th return time. The random variables $\{\rho_k, k \geq 1\}$ are independent and identically distributed [3]. Since $\{T_t, t \geq 0\}$ has finite states and is ergodic,

$$P(\rho_k > s) = O(e^{-\alpha s}) \tag{8.3}$$

for some $\alpha > 0$. Therefore, the finite moment assumptions in [3, Theorem 5, p. 101] hold and the proving argument by the dissection formula implies that there exists $C$ such that

$$\limsup_{t\to\infty} \frac{\left|\sum_{s=1}^t 1_{\{T_s=i\}} - m_i t\right|}{\sqrt{t \log\log t}} \leq C,$$

where for this ergodic Markov chain we use [3, Theorem 4, p. 90] to determine $m_i = \hat{\pi}_i$. The lemma follows easily. ∎

THEOREM 8.2. *Under* (H1)*, the sequence $\{I + \mathbf{a}_{\kappa_t} B_{T_t}, t \geq 0\}$ has ergodic backward products with probability one.*

*Proof.* Consider the consensus algorithm

$$Y_{t+1} = (I + \mathbf{a}_{\kappa_t} B_{T_t}) Y_t$$

with the deterministic initial pair $(t_0, Y_{t_0})$. Denote $\Lambda = \mathrm{diag}(\hat{\pi}_1^{-1}, \ldots, \hat{\pi}_n^{-1})$. We have

$$Y_{t+1} = (I + a_t \Lambda B_{T_t}) Y_t + (\mathbf{a}_{\kappa_t} - a_t \Lambda) B_{T_t} Y_t.$$

Select $t_1 \geq t_0$ such that $I + a_t \Lambda B_i$ is nonnegative for all $i \leq n$ and $t \geq t_1$. By Theorem 3.1, there exists a set $N_1$ with $P(N_1) = 0$ such that for all $\omega \in \Omega \backslash N_1$, $\{I + a_t \Lambda B_{T_t(\omega)}, t \geq t_1\}$ has ergodic backward products since $\cup_{i=1}^n \mathrm{graph}(\Lambda B_i) = G$ is strongly connected. Denote $Y_t = [Y_{t,1}, \ldots, Y_{t,n}]^T$. For some $C > 0$, we have a prior upper bound

$$|B_{T_t} Y_t| \leq C \max_j |Y_{t_0,j}|.$$

By Lemma 8.1, there exists a set $N_2$ with $P(N_2) = 0$ such that for all $\omega \in \Omega \backslash N_2$,

$$\left|a_{\kappa_t^i(\omega)} - a_t\hat{\pi}_i^{-\gamma}\right| = O\left(\frac{1}{\left[t + O(\sqrt{t\log\log t})\right]^\gamma} - \frac{1}{t^\gamma}\right)$$
$$= O\left(\frac{\sqrt{\log\log t}}{t^{\gamma+\frac{1}{2}}}\right).$$

Note that both $N_1$ and $N_2$ are determined by $\{T_t, t \geq 0\}$. Denote $\Delta_t = (\mathbf{a}_{\kappa_t} - a_t\Lambda)B_{T_t}Y_t$. For each $\omega \in \Omega\backslash(N_1 \cup N_2)$, $\{I + a_t\Lambda B_{T_t(\omega)}, t \geq t_1\}$ has ergodic backward products and $\sum_{t=t_0}^\infty |\Delta_t(\omega)| < \infty$ since $\gamma \in (1/2, 1]$. By [9, Lemma B.2], there exists $y$ such that $\lim_{t\to\infty} Y_t(\omega) = y1_n$ for $\omega \in \Omega\backslash(N_1 \cup N_2)$. Since $(t_0, Y_{t_0})$ can be arbitrarily selected, by [9, Lemma B.1], $\{I + \mathbf{a}_{\kappa_t(\omega)}B_{T_t(\omega)}, t \geq 0\}$ has ergodic backward products for all $\omega \in \Omega\backslash(N_1 \cup N_2)$. The theorem follows. ∎

THEOREM 8.3. *Under* (H1)–(H3)*, the algorithm* (8.2) *ensures mean square consensus.*

*Proof.* Since $\{T_t, t \geq 0\}$ is independent of $\{W_t, t \geq 0\}$, we have

$$E|\mathbf{a}_{\kappa_t}D_{T_t}W_t|^2 \leq CE|\mathbf{a}_{\kappa_t}|^2 = C\sum_{i=1}^n Ea_{\kappa_t^i}^2.$$

Fix $i$ and as in the proof of Lemma 8.1, define the sequence $\{\tau_k, k \geq 1\}$. Set $\tau_0 = 0$.

Take a large $l_0 > 1$ so that $\{a_t, t \geq \tau_{l_0}\}$ satisfies $a_t = \frac{1}{t^\gamma}$. We have

$$\sum_{t=0}^\infty Ea_{\kappa_t^i}^2 = E\sum_{l=0}^\infty \sum_{t=\tau_l}^{\tau_{l+1}-1} a_{\kappa_t^i}^2.$$

Then for $l \geq l_0$, by (8.3)

$$E\sum_{t=\tau_l}^{\tau_{l+1}-1} a_{\kappa_t^i}^2 \leq \frac{E(\tau_{l+1} - \tau_l)}{(k_0^i + l)^{2\gamma}} \leq \frac{C}{(k_0^i + l)^{2\gamma}},$$

where $C > 0$ does not depend on $l$. By (8.3), it is easy to show

$$E\sum_{l=0}^{l_0-1}\sum_{t=\tau_l}^{\tau_{l+1}-1} a_{\kappa_t^i}^2 < \infty.$$

Consequently, $\sum_{t=0}^\infty Ea_{\kappa_t^i}^2 < \infty$ for each $i$, which implies that

$$\sum_{t=0}^\infty E|\mathbf{a}_{\kappa_t}D_{T_t}W_t|^2 < \infty. \tag{8.4}$$

By Theorem 8.2, (8.4), and (H1)–(H3), we apply [9, Theorem 3] to conclude that (8.2) ensures mean square consensus. ∎

**9. Concluding remarks.** We have studied ergodicity of backward products of a class of stochastic matrices with Markovian switches and decreasing step sizes. The ergodicity theorem is used to prove mean square consensus of stochastic approximation algorithms. Our proof of the ergodicity theorem assumes that the Markov chain is ergodic. An interesting question is what happens if the Markov chain is irreducible but periodic. This scenario seems to be more challenging. The dimension reduction technique via the canonical form in section 6 cannot be applied since the matrix $P_\pi$ in this case has several eigenvalues with absolute value equal to one. To handle this scenario, a promising method is to explore the stochastic averaging approach [17] by identifying a limiting ordinary differential equation governing the stochastic approximation algorithm since the irreducible and periodic case still offers good long-run average properties for the model. We hope to pursue this idea in our future studies.

**Appendix A.**

**Proof of Lemma 5.1.** Associate $P_\pi = (q_{lm})_{l,m\le N}$ with a Markov chain $\{\theta'_t, t \ge 0\}$, whose irreducibility follows from that of $\{\theta_t, t \ge 0\}$. Since $P_\theta$ is ergodic, there exists $k_0 \ge 1$ such that for all $k \ge k_0$, the $k$-step transition probability $p_{11}^{[k]} > 0$. It implies that there exists a transition path $1, l_1, l_2, \dots, l_{k-1}, 1$ such that $p_{1l_1}p_{l_1l_2}\dots p_{l_{k-1}1} > 0$. For the Markov chain $\{\theta'_t, t \ge 0\}$, the probability of the path $1, l_{k-1}, \dots, l_2, l_1, 1$ is

$$q_{1l_{k-1}}\dots q_{l_2l_1}q_{l_11} = p_{l_{k-1}1}(\pi_{l_{k-1}}/\pi_1)\dots p_{l_1l_2}(\pi_{l_1}/\pi_{l_2})p_{1l_1}(\pi_1/\pi_{l_1}) > 0.$$

The $k$-step transition probability $q_{11}^{[k]} \ge q_{1l_{k-1}}\dots q_{l_2l_1}q_{l_11} > 0$ for all $k \ge k_0$ and so $\{\theta'_t, t \ge 0\}$ is aperiodic.

Since $P_\pi$ is ergodic, it has a unique stationary distribution. For any $m \le N$,

$$\sum_{l=1}^N \pi_l q_{lm} = \sum_{l=1}^N \pi_l\pi_m\pi_l^{-1}p_{ml} = \pi_m.$$

This verifies that $\pi = (\pi_1, \dots, \pi_N)$ is its stationary distribution. The lemma follows. ∎

LEMMA A.1. *Suppose that $B$ is a $k\times k$ matrix having zero row sums and nonnegative off-diagonal entries. Denote $Q = I_k \otimes B + B \otimes I_k$. If* graph$(B)$ *contains a spanning tree with root $k_0 \in \{1,\dots,k\}$, then* graph$(Q)$ *contains a spanning tree with root $k_0 \in \{1,\dots,k^2\}$.*

*Proof.* Without loss of generality, we take $k_0 = 1$. We introduce a sufficiently small $\tau > 0$ to define a stochastic matrix $I + \tau Q$ corresponding to a discrete time Markov chain with state space $\{1, 2, \dots, k^2\}$. Denote $B = (b_{ij})_{i,j\le k}$. We have the blockwise representation

$$I + \tau Q = (\delta_{ij}(I + \tau B) + b_{ij}\tau I)_{i,j\le k}\,.$$

Partition the states of the Markov chain into the sets $S'_i = \{(i-1)k+1, \dots, ik\}$, $i = 1, \dots, k$. The $i$th diagonal block of $I + \tau Q$ is $I + \tau B + b_{ii}\tau I$. Since graph$(B)$ contains a spanning tree with root 1, each state of $S'_i$ other than $(i-1)k+1$ can reach $(i-1)k+1$ by a sequence of transitions staying within $S'_i$.

Now it suffices to show that $(i-1)k+1$ can reach state $k_0 = 1$ with positive probability for $i > 1$. Consider $i = 2$, and all other cases are similar. Since graph$(B)$ contains a spanning tree, there exists a product of the form

$$b_{2i_1}b_{i_1i_2}\dots b_{i_l1} > 0$$

and we can ensure that $2, i_1, i_2, \dots, i_l, 1$ are different integers from $\{1, 2, \dots, k\}$. Then we can show that there is a positive probability for the $k^2$ state Markov chain to make the sequence of transitions

$$(2-1)k+1 \to (i_1-1)k+1 \to (i_2-1)k+1 \to \dots \to (i_l-1)k+1 \to 1,$$

and the corresponding probability is obtained from $I + \tau Q$ as $\tau^{l+1}(b_{2i_1}b_{i_1i_2}\dots b_{i_l1})$. ∎

**Proof of Theorem 5.2.** Without loss of generality, suppose that node $i_0 = 1$ is the root of $G_{\cup,\mathrm{tr}}$. Due to the particular structure of $M_1$ and Lemma 5.1, for each $S_j$, any two states can reach one another by a transition path within $S_j$. Denote the stochastic matrix $M_\epsilon = (\bar{p}_{ij})_{i,j\le Nn^2}$ for $0 < \epsilon \le \frac{1}{4\beta}$. It suffices to show that from each state $i \in \{2, \dots, n^2\}$, there exists a transition path of $\{\Upsilon_t, t \ge 0\}$ to give

$$\bar{p}_{ii_1}\bar{p}_{i_1i_2}\dots\bar{p}_{i_r1} > 0. \tag{A.1}$$

Denote $Q_l = I_n \otimes B_l + B_l \otimes I_n$. Then $M_\epsilon = M_1 + \epsilon M_2 = (q_{lm}(I_{n^2} + \epsilon Q_m))_{l,m\le N}$.

*Step* 1. Let $Q = \sum_{l=1}^N Q_l$. Since $\cup_{k=1}^N G_k$ has a spanning tree $G_{\cup,\mathrm{tr}}$ with node 1 being the root, graph$(\sum_{l=1}^N B_l)$ contains a spanning tree with node $1 \in \{1, \dots, n\}$ being its root. So graph$(Q)$ contains a spanning tree with node $1 \in \{1, \dots, n^2\}$ as its root by Lemma A.1. Therefore, $I + \frac{\epsilon}{N}Q$ is a stochastic matrix of positive diagonal entries where state 1 is reachable from any state in $\{2, \dots, n^2\}$ with positive probability. Thus, the first column of $(I + \frac{\epsilon}{N}Q)^{n^2-1}$ has only positive entries since each state can transit to state 1 with at most $n^2 - 1$ steps.

*Step* 2. Consider the product

$$D := q_{1j_1}(I + \epsilon Q_{j_1})q_{j_1j_2}(I + \epsilon Q_{j_2})\dots q_{j_s1}(I + \epsilon Q_1).$$

Since $(q_{lm})_{l,m\le N}$ is irreducible by Lemma 5.1, there exists an integer $K_0$ depending on $(q_{lm})_{l,m\le N}$ such that the above product has at most $K_0$ matrix terms, i.e., $s+1 \le K_0$, including each matrix in $\{Q_l, l \le N\}$ at least once and satisfying $q_{1j_1}q_{j_1j_2}\dots q_{j_s1} > 0$. For two nonnegative matrices, $A_1 \ge A_2$ means that the inequality holds componentwise. Note that $I + \epsilon Q_j \ge I/2$ since $0 < \epsilon \le \frac{1}{4\beta}$. Then $I + \epsilon Q_j \ge I/4 + I/2 + (\epsilon/2)Q_j$. For some constants $C_1, C_2$, we have the estimate

$$\begin{aligned} D &\ge C_1(I + \epsilon Q_1)\dots(I + \epsilon Q_N) \\ &\ge C_1\left[I/4 + I/2 + (\epsilon/2)Q_1\right]\dots\left[I/4 + I/2 + (\epsilon/2)Q_N\right] \\ &\ge C_1\left[4^{-N}I + 4^{-N+1}\sum_{l=1}^N (I/2 + (\epsilon/2)Q_l)\right] \\ &\ge C_2\left(I + \frac{\epsilon}{N}Q\right). \end{aligned}$$

So

$$D^{n^2-1} \ge C_2^{n^2-1}\left(I + \frac{\epsilon}{N}Q\right)^{n^2-1},$$

where the first column of $(I + \frac{\epsilon}{N}Q)^{n^2-1}$ has $n^2$ positive entries by Step 1. Thus, we may find a product of the form

$$D' := q_{1j_1}(I + \epsilon Q_{j_1})q_{j_1j_2}(I + \epsilon Q_{j_2})\dots q_{j_{s'}1}(I + \epsilon Q_1)$$

so that the first column has all positive entries.

*Step* 3. Take any $1 \le j \le n^2$. We check the $(j, 1)$th entry of $D'$. By Step 2,

$$\begin{aligned} D'(j,1) = \sum_{t_1,t_2,\dots,t_{s'}} & q_{1j_1}(I + \epsilon Q_{j_1})(j, t_1)q_{j_1j_2}(I + \epsilon Q_{j_2})(t_1, t_2) \\ & \times \dots q_{j_{s'}1}(I + \epsilon Q_1)(t_{s'}, 1) > 0. \end{aligned} \tag{A.2}$$

Recall that $M(i,j)$ denotes the $(i,j)$th entry of a matrix $M$. By (A.2), there exists a particular choice $(\hat{t}_1, \hat{t}_2, \dots, \hat{t}_{s'})$ such that

$$q_{1j_1}(I+\epsilon Q_{j_1})(j,\hat{t}_1)q_{j_1j_2}(I+\epsilon Q_{j_2})(\hat{t}_1,\hat{t}_2)\dots q_{j_{s'}1}(I+\epsilon Q_1)(\hat{t}_{s'},1) > 0,$$

which implies that $\{\Upsilon_t, t \geq 0\}$ has the transition path

$$j \to (j_1-1)n^2 + \hat{t}_1 \to (j_2-1)n^2 + \hat{t}_2 \to \cdots \to (j_{s'}-1)n^2 + \hat{t}_{s'} \to 1$$

with positive probability. Since $j \leq n^2$ is arbitrary, (A.1) holds. This completes the proof. $\square$

Lemma A.2. *Let $\{a_t, t \geq 1\}$ be a nonnegative sequence converging to zero (not necessarily satisfying* (A1)*). Suppose*

$$v_{t+1} = (I + a_t M)v_t + O(a_t^2), \quad t \geq 1,$$

*where $M$ is a Hurwitz matrix with all its eigenvalues having a real part strictly less than $-\sigma_0$ for some $\sigma_0 > 0$. Suppose that the sequence $\{b_t, t \geq 1\}$ satisfies $0 < b_t \to 0$, $(I + a_t M)\frac{b_t}{b_{t+1}} = I + a_t M_0 + o(a_t)$ for some Hurwitz matrix $M_0$, $\frac{a_t}{b_t} = O(1)$. Then the following assertions hold:*

(i) $|v_t| = O(b_t)$.

(ii) *If $a_t = t^{-\gamma}$, $0 < \gamma < 1$, we have $|v_t| = O(t^{-\gamma})$. If $a_t = t^{-1}$, $|v_t| = O(t^{-\eta})$, where $\eta = \min\{1, \sigma_0\}$.*

*Proof.* We have

$$b_{t+1}^{-1} v_{t+1} = (I + a_t M_0 + o(a_t))\left(b_t^{-1} v_t\right) + O(a_t), \quad t \geq 1.$$

Denote $r_t = b_t^{-1} v_t$. Taking any $Q > 0$, we solve a unique $P > 0$ from $PM_0 + M_0^T P = -Q$. Then

$$\begin{aligned} r_{t+1}^T P r_{t+1} = {} & r_t^T (I + a_t M_0 + o(a_t))^T P (I + a_t M_0 + o(a_t)) r_t + O(a_t^2) \\ & + 2 r_t^T (I + a_t M_0 + o(a_t))^T P O(a_t), \end{aligned} \tag{A.3}$$

where $o(a_t)$ and $O(a_t)$ on the right-hand side of (A.3) are a matrix and a vector, respectively. Denote $d_t = r_t^T P r_t$. By taking a large $t_0$, we can find $\delta_0 > 0$ and $C_0 > 0$ to ensure

$$d_{t+1} \leq (1 - \delta_0 a_t) d_t + C_0 a_t^2 + C_0 a_t |r_t|, \quad t \geq t_0,$$

where $1 - \delta_0 a_t > 0$. Next, we can find a large $C_1$ to ensure

$$C_0 |r_t| \leq \frac{\delta_0}{2} d_t + C_1.$$

Hence for some $C_2 > 0$,

$$d_{t+1} \leq \left(1 - \frac{\delta_0}{2} a_t\right) d_t + C_2 a_t, \quad t \geq t_0.$$

Consider

$$h_{t+1} = \left(1 - \frac{\delta_0}{2} a_t\right) h_t + C_2 a_t, \quad h_{t_0} = d_{t_0}.$$

By induction we can show $0 \le d_t \le h_t$. On the other hand, it is easy to show that $h_t - \frac{2C_2}{\delta_0}$ converges to a finite limit $(h_{t_0} - \frac{2C_2}{\delta_0})\prod_{s=t_0}^{\infty}(1 - \frac{\delta_0}{2}a_s)$. Hence $d_t = O(1)$. Part (i) follows.

*Case* 1: $a_t = t^{-\gamma}$, $0 < \gamma < 1$. We take $b_t = a_t$. It can be checked that

$$\frac{a_t}{a_{t+1}} = \left(1 + t^{-1}\right)^{\gamma} = 1 + \gamma t^{-1} + o\left(t^{-1}\right) = 1 + o(a_t).$$

For this case $M_0 = M$.

*Case* 2: $a_t = t^{-1}$. We take $b_t = t^{-\eta}$. Then

$$(I + a_t M)\frac{b_t}{b_{t+1}} = (I + t^{-1}M)\left(1 + \eta t^{-1} + O(t^{-2})\right) = I + t^{-1}(M + \eta I) + o(a_t).$$

The matrix $M_0 = M + \eta I$ is Hurwitz. Moreover, $\frac{a_t}{b_t} = O(1)$. This completes the proof of part (ii). ∎

**Appendix B.**

**Proof of Theorem 3.1.** Note that (5.4) is obtained from (4.6) by reordering the $Nn^2$ agents. By Theorem 6.4, S2 holds and hence S1 holds by Lemma 4.3.

*Step* 1. Consider any given deterministic value $X_{t_0}$ for (4.1). There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \bar{\xi}_t = \alpha 1_{Nn^2}.$$

Hence

$$\lim_{t\to\infty} \xi_t = \alpha \mathrm{diag}(\pi_1 I_{n^2}, \ldots, \pi_N I_{n^2}) 1_{Nn^2}. \tag{B.1}$$

For $V(t) = \sum_{l=1}^{N} V_l(t)$ and $J_n = \frac{1}{n}1_n 1_n^T$, (B.1) implies that $\lim_{t\to\infty} V(t) = \alpha 1_n 1_n^T = \alpha n J_n$. Next,

$$\begin{aligned} E|(I_n - J_n)X_t|^2 &= E\left[X_t^T (I_n - J_n)^2 X_t\right] \\ &= \mathrm{Tr}[(I_n - J_n)V(t)]. \end{aligned}$$

It is clear that

$$\lim_{t\to\infty} E|X_t - J_n X_t|^2 = 0,$$

which implies that the difference between the states of any two agents converges to zero in mean square. By the proving argument in [10, Theorem 9], we can further show mean square consensus of (4.1).

*Step* 2. Let $\{e_i, 1 \le i \le n\}$ be the canonical basis of $\mathbb{R}^n$. We set $X_{t_0} = X_{t_0}^{(i)} = e_i$, respectively, and by Step 1 we can show that

$$\begin{aligned} \Psi_{t+1,t_0} &:= (I + a_t B_{\theta_t}) \ldots (I + a_{t_0} B_{\theta_{t_0}}) \\ &= (I + a_t B_{\theta_t}) \ldots (I + a_{t_0} B_{\theta_{t_0}}) \left[X_{t_0}^{(1)}, \ldots, X_{t_0}^{(n)}\right] \\ &= \left[X_{t+1}^{(1)}, \ldots, X_{t+1}^{(n)}\right] \end{aligned} \tag{B.2}$$

converges in mean square to a stochastic matrix of identical rows. By the method in [9, Theorem 3, necessity proof], we may further obtain that $\Psi_{t,t_0}$ converges with probability one to a stochastic matrix of identical rows for the given $t_0$. This completes the proof of Theorem 3.1. ∎

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