Indefinite Mean-Field Stochastic Linear-Quadratic Optimal Control

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***Abstract*—This paper is concerned with the discrete-time indefinite *mean-field* linear-quadratic optimal control problem. The so-called mean-field type stochastic control problems refer to the problem of incorporating the means of the state variables into the state equations and cost functionals, such as the mean-variance portfolio selection problems. A dynamic optimization problem is called to be nonseparable in the sense of dynamic programming if it is not decomposable by a stage-wise backward recursion. The classical dynamic-programming-based optimal stochastic control methods would fail in such nonseparable situations as the principle of optimality no longer applies. In this paper, we show that both the well-posedness and the solvability of the indefinite mean-field linear-quadratic problem are equivalent to the solvability of two coupled constrained generalized difference Riccati equations and a constrained linear recursive equation. We characterize the optimal control set completely, and obtain a set of necessary and sufficient conditions on the mean-variance portfolio selection problem. The results established in this paper offer a more accurate solution scheme in tackling directly the issue of nonseparability and deriving the optimal policies analytically for the mean-variance-type portfolio selection problems.**

***Index Terms*—Indefinite stochastic linear-quadratic optimal control, mean-field theory, multi-period mean-variance portfolio selection.**

I. Introduction

In this paper, a kind of discrete-time stochastic linear-quadratic (LQ) optimal control problem of mean-field type is investigated. Comparing with the classical stochastic optimal LQ control, a new feature of the problems is that both the objective functional and the system dynamics involve the states and the controls as well as their expected values. In this case, the system dynamics is a discrete-time stochastic difference equation (SDE) of McKean-Vlasov type, which is also referred as mean-field SDE (MF-SDE). As a feature of such a class of SDEs, the dynamics depend on the statistical distribution of the solution, which provides simple but effective techniques for studying large systems by reducing the dimension and the complexity. This new feature roots itself in the category of mean-field theory, which is developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamical systems. According to mean-field theory, the interactions among agents are modeled by a mean-field term. When the number of individuals goes to infinity, the mean-field term can approach the expected value. The past few years have witnessed many successful applications of the mean-field formulation in various fields of engineering, games, finance and economics.

Mean-field stochastic LQ optimal control with indefinite cost weighting matrices is studied in this paper, which will be referred as the indefinite mean-field stochastic LQ optimal control. This problem is a natural generalization of those in [13], [31], where the definite and homogeneous versions of mean-field LQ problem are dealt with for a finite horizon and an infinite horizon, respectively. In fact, there is an increasing interest in mean-field control theory in the mathematics and control communities during recent years. The investigation of continuous-time mean-field stochastic differential equations may be traced back to the 1960s [29]; see also [34] for early developments. In [4], to cope with the possible time-inconsistency of optimal control, an extended version of the dynamic programming principle was derived using the Nisio nonlinear operator semigroup. Subsequently, stochastic maximum principles were studied in several works [3], [7], [24], which specify the necessary conditions for optimality. The results range from the case of a convex action space to the case of a general action space. As applications, the Markowitz mean-variance portfolio selection and a class of mean-field LQ problems are studied in [3], [24] using the derived stochastic maximum principle. In [36], the definite mean-field LQ control with a finite time horizon is systemically studied using a variational method and a decoupling technique. It is shown that the optimal control is of linear feedback form and that the gains are represented using solutions of two coupled differential Riccati equations. In [13], the discrete-time definite mean-field LQ problem is formulated as an operator stochastic LQ optimal control problem. By the kernel-range decomposition representation of the expectation operator and its pseudo-inverse, an optimal control is obtained based on the solutions of two Riccati difference equations. Later, [18] generalizes results obtained in [36] to the case with an infinite time horizon. Reference [35] studies the stochastic maximum principle under partial information. Reference [16] presents the maximum principle principle of mean-field type for the controlled mean-field forward-backward stochastic differential equations with Poisson jump. References [15], [17] give

Manuscript received April 15, 2014; revised August 16, 2014 and December 1, 2014; accepted December 15, 2014. Date of publication December 23, 2014; date of current version June 24, 2015. This work was supported by the National Natural Science Foundation of China under grant 11471242, the China Postdoctoral Science Foundation and the China Scholarship Council, by the National Natural Science Foundation of China under grant 612279002, the National Key Basic Research Program of China (973 Program) under grant 2014CB845301, and by the Hong Kong RGC under grants 520412 and 15209614. Recommended by Associate Editor S. Anderson.

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Digital Object Identifier 10.1109/TAC.2014.2385253

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necessary and sufficient conditions for mean-field stochastic optimal and near-optimal singular control problems, respectively. For other interesting aspects of mean-field optimal control problems, readers may refer to, for example, [30] and other related works. It is worth mentioning that the study of controlled mean-field stochastic differential or difference equations is also partially motivated by a recent surge of interest in mean-field games [6], [9], [19]–[22], [25]. Compared with the topic of this paper, mean-field games use decentralized controls, that is, the controls are selected to achieve each individual's own goal by using local information.

Indefinite stochastic LQ optimal control without mean-field terms is first studied in [10]. It is found in [10] that a stochastic LQ problem with indefinite cost weighting matrices may still be well-posed, which challenges the standard belief about LQ problems. For more about such class of LQ problems, readers may refer to, for example, [1], [2], [38]. It is further shown that indefinite stochastic LQ problems are closely related to Markowitz's mean-variance portfolio selection problems [11], [14], [23], [26], [27], [30] in financial investment. The important feature of Markowitz's model is that the risk is quantified by using variance of the wealth. This Noble-Prize-winning approach becomes the foundation of modern finance theory and has inspired hundreds of extensions and applications. In the multi-period case, the difficulty in dealing with such problem is the loss of the smoothing property of the variance term. This is due to the nonlinear term $(\mathbb{E}x)^2$ in the variance operator $\mathrm{Var}(x) = \mathbb{E}x^2 - (\mathbb{E}x)^2$ and leads to a nonseparable problem, to which the dynamic-programming-based methodology often used in stochastic optimal control theory fails to work. Moreover, nonseparability may result in the time-inconsistency of the optimal control [37]. To overcome the difficulty of nonseparability, [23] and [39] introduce an embedding scheme to tackle the original problem. It is worth mentioning that the auxiliary problem is an indefinite stochastic LQ problem, which can be solved explicitly by the indefinite LQ theory [39]. On the one hand, the embedding scheme has promising feature, which provides a beautiful framework to deal with more complicated situations [14]. On the other hand, the mean-variance portfolio selection problems provide a soil to blossom out for the indefinite stochastic LQ theory.

However, to our knowledge, the indefinite stochastic LQ optimal control problems and the mean-variance portfolio selection problems have been not yet completely combined in a unified framework. In [23] and [39], the return rates of the risky securities [23] and the volatility of the stocks [39] are required to be nondegenerate. This may come from the embedding scheme, in which the inverses of the corresponding matrices is required to ensure the existence of a unique optimal portfolio strategy. To make the formulation more practical, it is natural to consider, at least in theory, how to generalize these results to the case where degeneracy is allowed. In fact, mean-variance portfolio selection problems with degenerate covariance matrices may date back to 1970s. In [8] or the "corrected" version [33], Buser *et al.* propose the single-period version with possibly singular covariance matrix. Clearly, such class of problems are more general than the classical ones [28], and more consistent with the reality.

Note that $\mathrm{Var}(x) = \mathbb{E}x^2 - (\mathbb{E}x)^2$. Instead of using embedding scheme to solve the case with degenerate variance matrices, we view the multi-period mean-variance portfolio selection problem as a special example of indefinite mean-field stochastic LQ problem. By using a method developed for general indefinite mean-field stochastic LQ problem, we can promote existing theory on multi-period mean-variance portfolio selection problems to the case with degenerate variance matrices. This is the second motivation to consider indefinite mean-field LQ problem in addition to the mean-field formulation. Therefore, the contribution of this paper is two-fold. On the one hand, this paper develops general theory for indefinite mean-field LQ optimal control, which generalizes existing results about definite mean-field LQ optimal control problems. On the other hand, by using theory developed, we can promote the old problems—multi-period mean-variance portfolio selection—to the case with degenerate variance matrices.

In this paper, we first show that the solvability of the two constrained generalized difference Riccati equations (GDREs) and a constrained linear recursive equation (LRE) are necessary to the solvability of Problem (MF-LQ) defined in Section II by using a modified backward recursive technique. Then, we show that the sufficiency also holds by completing the square. Different from the definite one, the indefinite mean-field LQ optimal control problem is with five degrees of freedom. We characterize the set of all the optimal controls, and apply the results to solve a kind of multi-period mean-variance portfolio selection problem, such as Problem (MV). For the solvability of Problem (MV), necessary and sufficient conditions, which are completely characterized by the returns of the risky and riskless assets, are obtained. Moreover, when the returns of risky assets are nondegenerate, the results of this paper can be reduced to those obtained in existing literature.

The rest of the paper is organized as follows. Section II gives some preliminaries. Sections III and IV show that the solvability of the GDREs and LRE is equivalent to that of Problem (MF-LQ). Section V deals with the dynamic multi-period mean-variance portfolio selection problem. Sections VI and VII give some examples and concluding remarks, respectively.

II. Preliminaries

Consider the following controlled MF-SDE:

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k + f_k) \\ \qquad\quad +(C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k + d_k) w_k \\ x_l = \zeta,\ k = l, \cdots, N-1,\ l \in \{0, 1, .., N-1\}. \end{cases} \tag{1}$$

Here, $A_k, \bar{A}_k, C_k, \bar{C}_k \in \mathbb{R}^{n\times n}$, $B_k, \bar{B}_k, D_k, \bar{D}_k \in \mathbb{R}^{n\times m}$ and $f_k, d_k \in \mathbb{R}^n$ are given deterministic matrices and vectors, respectively; $\mathbb{E}$ is the expectation operator; $\{x_k, k = 0, 1, \cdots, N\}$ and $\{u_k, k = 0, \cdots, N-1\}$ are the state process and control process, respectively. $\{w_k, k = 0, 1, \cdots, N-1\}$ is a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ in the sense that

$$\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0,\ \mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1,\ k \geq 0 \tag{2}$$

where $\mathbb{E}[\cdot|\mathcal{F}_k]$ is the conditional mathematical expectation with respect to $\mathcal{F}_k = \{x_0, w_l, l = 0, 1, \cdots, k\}$ and $\mathcal{F}_{-1} = \sigma(x_0)$. The initial value ζ is square integrable and measurable with respect to $\mathcal{F}_{l-1}$.

The cost functional is described as

$$J(l,\zeta,u) = \sum_{k=l}^{N-1} \mathbb{E}\left[x_k^\tau Q_k x_k + \mathbb{E}x_k^\tau \bar{Q}_k \mathbb{E}x_k + u_k^\tau R_k u_k + \mathbb{E}u_k^\tau \bar{R}_k \mathbb{E}u_k + 2q_k^\tau x_k + 2\rho_k^\tau u_k\right] + \mathbb{E}\left[x_N^\tau G x_N\right] + \mathbb{E}x_N^\tau \bar{G}\mathbb{E}x_N + 2\mathbb{E}g^\tau x_N \tag{3}$$

where x^τ denotes the transpose of x; $Q_k, \bar{Q}_k, R_k, \bar{R}_k, G, \bar{G}$ are deterministic symmetric matrices and q_k, ρ_k, g are deterministic vectors. Here and in what follows, we shall use the following notation $(\mathbb{E}x_k)^\tau$ and $\mathbb{E}x_k^\tau$ interchangeably. Introduce the following admissible control set of $u = (u_0, u_1, \cdots, u_{N-1}) : \mathbb{N} \times \Omega \to \mathbb{R}^m$

$$\mathcal{U}_{ad} \equiv \left\{ u \,\middle|\, u_k \text{ is } \mathcal{F}'_k \text{ measurable}, \sum_{k=0}^{N-1} \mathbb{E}|u_k|^2 < \infty \right\}.$$

Hereafter, $\mathcal{F}'_k = \sigma(x_l, l = 0, \cdots, l)$; $\mathbb{N}_l$ denotes the set $\{l, l+1, \cdots, N-1\}$ for $l \in \{0, 1, \cdots, N-1\}$; $\bar{\mathbb{N}}_l$ denotes $\{l, l+1, \cdots, N\}$; $\mathbb{N}_0$ and $\bar{\mathbb{N}}_0$ will be simply written as $\mathbb{N}$ and $\bar{\mathbb{N}}$, respectively. From (1), we know that $\mathcal{F}'_k \subset \mathcal{F}_{k-1}$ if u is selected from $\mathcal{U}_{ad}$.

In this paper, for (3), when we select a u in $\mathcal{U}_{ad}$, we mean selecting the part $(u_l, \cdots, u_{N-1})$ of u. The optimal control problem considered in this paper is stated as follows.

Problem (MF-LQ). For any given initial pair (l, ζ) with ζ being square integrable and measurable with respect to $\mathcal{F}_{l-1}$, find $u^* \in \mathcal{U}_{ad}$ such that

$$J(l,\zeta,u^*) = \inf_{u \in \mathcal{U}_{ad}} J(l,\zeta,u). \tag{4}$$

We then call u^* an optimal control for Problem (MF-LQ).

Define now the value function

$$V(l,\zeta) = \inf_{u \in \mathcal{U}_{ad}} J(l,\zeta,u).$$

Since the weighting matrices $Q_k, \bar{Q}_k, R_k, \bar{R}_k, k \in \mathbb{N}, G, \bar{G}$ are possibly negative, Problem (MF-LQ) may be ill-posed, i.e., $V(l,\zeta)$ may be $-\infty$ for some l and ζ. We then give the following notions.

Definition 2.1: (i) Problem (MF-LQ) is said to be finite or well-posed at (l,ζ) if

$$V(l,\zeta) > -\infty.$$

Problem (MF-LQ) is said to be finite or well-posed if it is finite or well-posed at any (l,ζ).

(ii) Problem (MF-LQ) is said to be (uniquely) solvable or attainable at (l,ζ) if there exists a (unique) $u^* \in \mathcal{U}_{ad}$ such that (4) holds at (l,ζ). Problem (MF-LQ) is said to be (uniquely) solvable or attainable if it is solvable at any (l,ζ).

We recall the pseudo-inverse of a matrix. By [32], for a given matrix $M \in \mathbb{R}^{n\times m}$, there exists a unique matrix in $\mathbb{R}^{m\times n}$ denoted by $M^\dagger$ such that

$$\begin{cases} MM^\dagger M = M, \quad M^\dagger M M^\dagger = M^\dagger \\ (MM^\dagger)^\tau = MM^\dagger, \quad (M^\dagger M)^\tau = M^\dagger M. \end{cases} \tag{5}$$

This $M^\dagger$ is called the Moore-Penrose inverse of M. If M is symmetric, we have the result [1].

Lemma 2.1: If M is symmetric, then

(i) $M^\dagger = (M^\dagger)^\tau$;
(ii) $M \geq 0$ if and only if $M^\dagger \geq 0$;
(iii) $MM^\dagger = M^\dagger M$.

Hereafter, $M \geq 0$ means that M is a positive semi-definite square matrix.

Lemma 2.2: (Extended Schur's Lemma [5]). Let matrices $M = M^T, N, R = R^T$ be given with appropriate sizes. Then the following conditions are equivalent:

(i) $\begin{bmatrix} M & N \\ N^\tau & R \end{bmatrix} \geq 0$;
(ii) $\begin{bmatrix} R & N^\tau \\ N & M \end{bmatrix} \geq 0$;
(iii) $R \geq 0, M - NR^\dagger N^\tau \geq 0$, and $N(I - RR^\dagger) = 0$.

Now, we introduce a set of GDREs and LRE.

Definition 2.2: The following constrained difference equations:

$$\begin{cases} P_k = Q_k + A_k^\tau P_{k+1} A_k + C_k^\tau P_{k+1} C_k - H_k^\tau W_k^\dagger H_k, \\ T_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^\tau P_{k+1}(C_k + \bar{C}_k) \\ \qquad + (A_k + \bar{A}_k)^\tau T_{k+1}(A_k + \bar{A}_k) - \bar{H}_k^\tau \bar{W}_k^\dagger \bar{H}_k, \\ P_N = G_N, \quad T_N = G_N + \bar{G}_N, \\ W_k, \bar{W}_k \geq 0, \quad W_k W_k^\dagger H_k - H_k = 0, \\ \bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0, \quad k \in \mathbb{N} \end{cases} \tag{6}$$

are called two coupled generalized difference Riccati equations, where $\{(P_k, T_k), k \in \mathbb{N}\}$ is the solution, and

$$\begin{cases} W_k = R_k + B_k^\tau P_{k+1} B_k + D_k^\tau P_{k+1} D_k, \\ H_k = B_k^\tau P_{k+1} A_k + D_k^\tau P_{k+1} C_k, \\ \bar{W}_k = R_k + \bar{R}_k + (B_k + \bar{B}_k)^\tau T_{k+1}(B_k + \bar{B}_k) \\ \qquad + (D_k + \bar{D}_k)^\tau P_{k+1}(D_k + \bar{D}_k), \\ \bar{H}_k = (B_k + \bar{B}_k)^\tau T_{k+1}(A_k + \bar{A}_k) \\ \qquad + (D_k + \bar{D}_k)^\tau P_{k+1}(C_k + \bar{C}_k), \\ k \in \mathbb{N}. \end{cases} \tag{7}$$

Furthermore, introduce the following constrained linear recursive equation:

$$\begin{cases} \varphi_k = (C_k + \bar{C}_k)^\tau P_{k+1} d_k + (A_k + \bar{A}_k)^\tau (T_{k+1} f_k + \varphi_{k+1}) \\ \qquad - \bar{H}_k^T \bar{W}_k^\dagger \mu_k + q_k, \\ \varphi_N = g, \\ \bar{W}_k \bar{W}_k^\dagger \mu_k - \mu_k = 0, \quad k \in \mathbb{N}, \end{cases} \tag{8}$$

where $\{\varphi_k, k \in \mathbb{N}\}$ is the solution and

$$\mu_k = (D_k + \bar{D}_k)^\tau P_{k+1} d_k + (B_k + \bar{B}_k)^\tau (T_{k+1} f_k + \varphi_{k+1}) + \rho_k. \tag{9}$$

Note that (6) and (8) are constrained equations, since $W_k W_k^\dagger H_k - H_k = 0$, $\bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0$, $\bar{W}_k \bar{W}_k^\dagger \mu_k - \mu_k = 0$, are needed for any k. In what follows, by GDREs and LRE we mean (6) and (8), respectively.

III. NECESSITY OF SOLVABILITY OF THE GDREs AND LRE

In this section we shall show that the solvability of the GDREs (6) and LRE (8) is necessary for the well-posedness of Problem (MF-LQ). We first present two lemmas.

Lemma 3.1: Let $F = F^\tau$, $G = G^\tau$ and H be given deterministic matrices and q, ρ be given deterministic vectors with appropriate size. Consider the following quadratic form:

$$h(x,u) = \mathbb{E}[x^\tau Fx + 2x^\tau Hu + u^\tau Gu + 2q^\tau x + 2\rho^\tau u] \quad (10)$$

where x, u are square integrable random variables defined on a probability space. Then the following statements are equivalent:

(i) $\inf_u h(x,u) > -\infty$ for any square integrable$x$;
(ii) $G \geq 0, \mathrm{Ker}(G) \subset (H), \rho \in \mathrm{Ran}(G)$;
(iii) $G \geq 0, H(I - GG^\dagger) = 0, \rho^T(I - GG^\dagger) = 0$.

Here, $\mathrm{Ker}(G) = \{x \mid Gx = 0\}$ is the kernel space of G, and $\mathrm{Ran}(G) = \{Gx \mid x \in \mathbb{R}^n\}$ with n being the dimension of x is the range of G.

Proof: $(i) \Rightarrow (ii)$. We prove this by contradiction. Suppose that $G < 0$, i.e., there exists a u_1 such that $u_1^\tau Gu_1 < 0$. Then

$$h(x,\alpha u_1) = \mathbb{E}\left[x^\tau Fx + 2\alpha x^\tau Hu_1 + \alpha^2 u_1^\tau Gu_1 + 2q^\tau x + 2\alpha\rho^\tau u_1\right]. \quad (11)$$

This implies $\lim_{\alpha\to+\infty} h(x,\alpha u_1) = -\infty$, which contradicts (i). Thus, $G \geq 0$.

Furthermore, assume that there exists a u_2 such that $Gu_2 = 0$ and $Hu_2 \neq 0$. Then

$$h(\beta Hu_2, -\alpha u_2) = \mathbb{E}\left[\beta^2 (Hu_2)^\tau FHu_2 + 2\beta q^\tau Hu_2 - 2\alpha\left(\beta|Hu_2|^2 + \rho^\tau u_2\right)\right]. \quad (12)$$

By selecting $\beta > 0$ such that $\beta|Hu_2|^2 + \rho^\tau u_2 > 0$, we have $\lim_{\alpha\to+\infty} h(\beta Hu_2, -\alpha u_2) = -\infty$. Thus, $Ker(G) \subset Ker(H)$.

Finally, suppose $\rho \notin Ran(G)$. Then $\rho = \rho_1 + \rho_2$ with $0 \neq \rho_1 \in Ker(G), \rho_2 \in Ran(G)$. Hence

$$h(\beta x, -\alpha\rho_1) = \mathbb{E}\left[\beta^2 x^\tau Fx + 2\beta q^\tau x - 2\alpha\left(\beta x^\tau H\rho_1 + 2|\rho_1|^2\right)\right]. \quad (13)$$

By selecting β sufficiently small such that $\beta x^\tau H\rho_1 + 2|\rho_1|^2 > 0$, we have $\lim_{\alpha\to+\infty} h(\beta x, -\alpha\rho_1) = -\infty$. This contradicts (i). Therefore, $\rho \in Ran(G)$ follows.

$(ii) \Leftrightarrow (iii)$. The equivalence between $Ker(G) \subset Ker(H)$ and $H(I - GG^\dagger) = 0$ is from Lemma 4.2 of [1]. We now show that $\rho \in Ran(G)$ is equivalent to $\rho^\tau(I - GG^\dagger) = 0$. On the one hand, if $\rho \in Ran(G)$, then there exist $\bar\rho$ such that $\rho = G\bar\rho$. Thus, $\rho^\tau(I - GG^\dagger) = \bar\rho^\tau G(I - GG^\dagger) = 0$. On the other hand, if $\rho^\tau(I - GG^\dagger) = 0$, then for any $z \in Ker(G)$

$$\rho^\tau z = \rho^\tau GG^\dagger z = \rho^\tau G^\dagger Gz = 0.$$

Therefore, $\rho \in Ran(G)$.

$(iii) \Rightarrow (i)$. Under (ii), $h(x,u)$ can be rewritten as

$$h(x,u) = \mathbb{E}\left[(u + G^\dagger H^\tau x + G^\dagger\rho)^\tau G(u + G^\dagger H^\tau x + G^\dagger\rho)\right] + \mathbb{E}\left[x^\tau(F - HG^\dagger H^\tau)x + 2q^\tau x - \rho^\tau G^\dagger\rho - 2x^\tau HG^\dagger\rho\right].$$

From this, (i) follows. This completes the proof. □

Remark 3.1: This result is an extension of Lemma 4.3 of [1]. In [1], the homogeneous version (i.e., $q = 0, \rho = 0$) is presented. In addition, if G is nonsingular, we must have $\rho \in \mathrm{Ran}(G)$ as $\rho = G(G^{-1}\rho)$.

Lemma 3.2: Let $G_1 = G_1^\tau$, $G_2 = G_2^\tau$ and H_1, H_2 be given deterministic matrices and ρ be given deterministic vectors with appropriate size. Consider the following quadratic form

$$\bar h(x,u) = \mathbb{E}\,[2(x - \mathbb{E}x)^\tau H_1(u - \mathbb{E}u) + (u - \mathbb{E}u)^\tau G_1(u - \mathbb{E}u) + 2(\mathbb{E}x)^\tau H_2\mathbb{E}u + (\mathbb{E}u)^\tau G_2\mathbb{E}u + 2\rho^\tau\mathbb{E}u] \quad (14)$$

where x, u are square integrable random variables defined on a probability space. Then, the following statements are equivalent:

(i) $\inf_u \bar h(x,u) > -\infty$ for any square integrable x;
(ii) $G_1 \geq 0$, $\mathrm{Ker}(G_1) \subset \mathrm{Ker}(H_1)$, $G_2 \geq 0$, $\mathrm{Ker}(G_2) \subset \mathrm{Ker}(H_2), \rho \in \mathrm{Ran}(G_2)$;
(iii) $G_1 \geq 0$, $H_1(I - G_1G_1^\dagger) = 0$, $G_2 \geq 0$, $H_2(I - G_2G_2^\dagger) = 0, \rho^\tau(I - G_2G_2^\dagger) = 0$.

Proof: It is easy to see $(ii) \Leftrightarrow (iii)$. By completing the square, we have that $(iii) \Rightarrow (i)$. We now prove $(i) \Rightarrow (ii)$ by the method of contradiction. First, suppose that $G_1 < 0$, i.e., there exists a (deterministic) $\bar u_1$ such that $\bar u_1^\tau G_1\bar u_1 < 0$. Let ε be a random variable such that $P(\varepsilon = 1) = P(\varepsilon = -1) = (1/2)$. Then, we have

$$\bar h(x, \alpha\bar u_1\varepsilon) = \mathbb{E}\left[2\alpha\varepsilon(x - \mathbb{E}x)^\tau H_1\bar u_1 + \alpha^2\bar u_1^\tau G_1\bar u_1\right].$$

This implies that $\lim_{\alpha\to+\infty}\bar h(x, \alpha\varepsilon\bar u_1) = -\infty$, which contradicts (i). Thus, $G_1 \geq 0$.

Secondly, assume that there exists an $\bar u_2$ such that $G_1\bar u_2 = 0$ and $H_1\bar u_2 \neq 0$. Then

$$\bar h(\varepsilon H\bar u_2, -\alpha\varepsilon\bar u_2) = -2\alpha|H\bar u_2|^2.$$

This gives $\lim_{\alpha\to+\infty}\bar h(\varepsilon H\bar u_2, -\alpha\varepsilon\bar u_2) = -\infty$. Thus, we have $\mathrm{Ker}(G_1) \subset \mathrm{Ker}(H_1)$.

Finally, by selecting u_1, u_2 and ρ_1 as those in Lemma 3.1, we can prove that $G_2 \geq 0, \mathrm{Ker}(G_2) \subset \mathrm{Ker}(H_2), \rho \in \mathrm{Ran}(G_2)$ via equations similar to (11)–(13). The details are omitted here. This completes the proof. □

The following result shows that the solvability of the GDREs (6) and LRE (8) is necessary to the well-posedness of Problem (MF-LQ).

Theorem 3.1: If Problem (MF-LQ) is well-posed, then the GDREs (6) and LRE (8) are solvable.

Proof: We prove this by induction. For $l = N$, note $P_N = G, T_N = G + \bar G, \varphi_N = g$. For $l = N - 1$, by substituting x_N with representation of form (1), we have

$$\begin{aligned}
&V(N-1,\zeta)\\
&= \inf_{u_{N-1}}\mathbb{E}\Big[x_{N-1}^\tau Q_{N-1}x_{N-1} + \mathbb{E}x_{N-1}^\tau\bar Q_{N-1}\mathbb{E}x_{N-1}\\
&\quad + u_{N-1}^\tau R_{N-1}u_{N-1} + \mathbb{E}u_{N-1}^\tau\bar R_{N-1}\mathbb{E}u_{N-1}\\
&\quad + 2q_{N-1}^\tau x_{N-1} + 2\rho_{N-1}^\tau u_{N-1}\\
&\quad + \mathbb{E}x_N^\tau Gx_N + \mathbb{E}x_N^\tau\bar G\mathbb{E}x_N + \mathbb{E}g^\tau x_N\Big]\\
&= \inf_{u_{N-1}}\mathbb{E}[(x_{N-1} - \mathbb{E}x_{N-1})^\tau\\
&\quad \cdot\left(Q_{N-1} + A_{N-1}^\tau P_NA_{N-1} + C_{N-1}^\tau P_NC_{N-1}\right)\\
&\quad \cdot(x_{N-1} - \mathbb{E}x_{N-1})\\
&\quad + 2(x_{N-1} - \mathbb{E}x_{N-1})^\tau
\end{aligned}$$

$$
\begin{aligned}
&\cdot \big(A_{N-1}^{\tau} P_N B_{N-1} + C_{N-1}^{\tau} P_N D_{N-1}\big)\\
&\cdot (u_{N-1} - \mathbb{E}u_{N-1})\\
&+ (u_{N-1} - \mathbb{E}u_{N-1})^{\tau}\\
&\cdot \big(R_{N-1} + B_{N-1}^{\tau} P_N B_{N-1} + D_{N-1}^{\tau} P_N D_{N-1}\big)\\
&\cdot (u_{N-1} - \mathbb{E}u_{N-1})\\
&+ (\mathbb{E}x_{N-1})^{\tau} \big(Q_{N-1} + \bar{Q}_{N-1}\\
&+ (C_{N-1} + \bar{C}_{N-1})^{\tau} P_N (C_{N-1} + \bar{C}_{N-1})\\
&+ (A_{N-1} + \bar{A}_{N-1})^{\tau} T_N (A_{N-1} + \bar{A}_{N-1})\big)\, \mathbb{E}x_{N-1}\\
&+ 2(\mathbb{E}x_{N-1})^{\tau} \big((C_{N-1} + \bar{C}_{N-1})^{\tau}\\
&\cdot P_N (D_{N-1} + \bar{D}_{N-1})\\
&+ (A_{N-1} + \bar{A}_{N-1}) T_N (B_{N-1} + \bar{B}_{N-1})\big)\, \mathbb{E}u_{N-1}\\
&+ (\mathbb{E}u_{N-1})^{\tau} \big(R_{N-1} + \bar{R}_{N-1} + (D_{N-1} + \bar{D}_{N-1})^{\tau}\\
&\cdot P_N (D_{N-1} + \bar{D}_{N-1})\\
&+ (B_{N-1} + \bar{B}_{N-1})^{\tau} T_N (B_{N-1} + \bar{B}_{N-1})\big)\, \mathbb{E}u_{N-1}\\
&+ 2(\mathbb{E}u_{N-1})^{\tau} \big((D_{N-1} + \bar{D}_{N-1}) P_N d_{N-1}\\
&+ (B_{N-1} + \bar{B}_{N-1})(T_N f_{N-1} + \varphi_N) + \rho_{N-1}\big)\\
&+ 2(\mathbb{E}x_{N-1})^{\tau} \big((C_{N-1} + \bar{C}_{N-1})^{\tau} P_N d_{N-1}\\
&+ (A_{N-1} + \bar{A}_{N-1})^{\tau} (T_N f_{N-1} + \varphi_N) + q_{N-1}\big)\\
&+ f_{N-1}^{\tau} T_N f_{N-1} + d_{N-1}^{\tau} P_N d_{N-1} + 2\varphi_N^{\tau} f_{N-1}\big]\\
= &\inf_{u_{N-1}} \mathbb{E}\big[2(u_{N-1} - \mathbb{E}u_{N-1})^{\tau} H_{N-1} (x_{N-1} - \mathbb{E}x_{N-1})\\
&+ (u_{N-1} - \mathbb{E}u_{N-1})^{\tau} W_{N-1} (u_{N-1} - \mathbb{E}u_{N-1})\\
&+ 2(\mathbb{E}u_{N-1})^{\tau} \bar{H}_{N-1} \mathbb{E}x_{N-1}\\
&+ (\mathbb{E}u_{N-1})^{\tau} \bar{W}_{N-1} \mathbb{E}u_{N-1} + 2(\mathbb{E}u_{N-1})^{\tau} \mu_{N-1}\\
&+ (x_{N-1} - \mathbb{E}x_{N-1})^{\tau} (Q_{N-1} + A_{N-1}^{\tau} P_N A_{N-1}\\
&+ C_{N-1}^{\tau} P_N C_{N-1})(x_{N-1} - \mathbb{E}x_{N-1})\\
&+ (\mathbb{E}x_{N-1})^{\tau} \big(Q_{N-1} + \bar{Q}_{N-1}\\
&+ (C_{N-1} + \bar{C}_{N-1})^{\tau} P_N (C_{N-1} + \bar{C}_{N-1})\\
&+ (A_{N-1} + \bar{A}_{N-1})^{\tau} T_N (A_{N-1} + \bar{A}_{N-1})\big)\, \mathbb{E}x_{N-1}\\
&+ 2(\mathbb{E}x_{N-1})^{\tau} \big((C_{N-1} + \bar{C}_{N-1})^{\tau} P_N d_{N-1}\\
&+ (A_{N-1} + \bar{A}_{N-1})^{\tau} (T_N f_{N-1} + \varphi_N) + q_{N-1}\big)\\
&+ f_{N-1}^{\tau} T_N f_{N-1} + d_{N-1}^{\tau} P_N d_{N-1} + 2\varphi_N^{\tau} f_{N-1}\big], \quad (15)
\end{aligned}
$$

where $x_{N-1} = \zeta$. To simplify the notation, hereafter we will replace the expression

$$\inf_{\{u_{N-1} \text{ is } \mathcal{F}'_{N-1}\text{-measurable and } \mathbb{E}|u_{N-1}|^2 < \infty\}}$$

by $\inf_{u_{N-1}}$. As $V(N-1,\zeta) > -\infty$, by Lemma 3.2 we have

$$
\begin{cases}
W_{N-1} \geq 0, \;\; \bar{W}_{N-1} \geq 0,\\
W_{N-1} W_{N-1}^{\dagger} H_{N-1} - H_{N-1} = 0,\\
\bar{W}_{N-1} \bar{W}_{N-1}^{\dagger} \bar{H}_{N-1} - \bar{H}_{N-1} = 0,\\
\bar{W}_{N-1} \bar{W}_{N-1}^{\dagger} \mu_{N-1} - \mu_{N-1} = 0,
\end{cases}
$$

which implies that the GDREs (6) and LRE (8) are solvable when $k = N-1$. Furthermore, the optimal control u_{N-1}^* can be selected as

$$
\begin{aligned}
u_{N-1}^* = -W_{N-1}^{\dagger} H_{N-1} (x_{N-1} - \mathbb{E}x_{N-1})&\\
- \bar{W}_{N-1}^{\dagger} \bar{H}_{N-1} \mathbb{E}x_{N-1} - \bar{W}_{N-1}^{\dagger} \mu_{N-1}&
\end{aligned}
$$

and

$$
\begin{aligned}
&V(N-1,\zeta)\\
&= \Big[2\varphi_N^{\tau} f_{N-1} + f_{N-1}^{\tau} T_N f_{N-1} + d_{N-1}^{\tau} P_N d_{N-1}\\
&\quad - \mu_{N-1}^{\tau} \bar{W}_{N-1}^{\dagger} \mu_{N-1}\Big]\\
&\quad + \mathbb{E}\left[(x_{N-1} - \mathbb{E}x_{N-1})^{\tau} P_{N-1} (x_{N-1} - \mathbb{E}x_{N-1})\right]\\
&\quad + (\mathbb{E}x_{N-1})^{\tau} T_{N-1} \mathbb{E}x_{N-1} + 2\varphi_{N-1}^{\tau} \mathbb{E}x_{N-1}. \quad (16)
\end{aligned}
$$

For $l = N-2$, we have

$$
\begin{aligned}
&V(N-2,\zeta)\\
&= \inf_{u_{N-2},u_{N-1}} \Bigg[\sum_{k=N-2}^{N-1} \mathbb{E}\big(x_k^{\tau} Q_k x_k + \mathbb{E}x_k^{\tau} \bar{Q}_k \mathbb{E}x_k\\
&\qquad + u_k^{\tau} R_k u_k + \mathbb{E}u_k^{\tau} \bar{R}_k \mathbb{E}u_k + 2q_k^{\tau} x_k + 2\rho_k^{\tau} u_k\big)\\
&\qquad + \mathbb{E}x_N^{\tau} G x_N + \mathbb{E}x_N^{\tau} \bar{G} \mathbb{E}x_N + \mathbb{E}g^{\tau} x_N\Bigg]\\
&\leq \inf_{u_{N-2}} \mathbb{E}\left[x_{N-2}^{\tau} Q_{N-2} x_{N-2} + \mathbb{E}x_{N-2}^{\tau} \bar{Q}_{N-2} \mathbb{E}x_{N-2}\right.\\
&\qquad + u_{N-2}^{\tau} R_{N-2} u_{N-2} + \mathbb{E}u_{N-2}^{\tau} \bar{R}_{N-2} \mathbb{E}u_{N-2}\\
&\qquad + 2q_{N-2}^{\tau} x_{N-2} + 2\rho_{N-2}^{\tau} u_{N-2}\\
&\qquad \left. + V(N-1, x_{N-1})\right], \quad (17)
\end{aligned}
$$

where $\zeta = x_{N-2}$. Noting that $V(N-1, x_{N-1}) = J(N-1, x_{N-1}, u_{N-1}^*)$ and the nonseparability of cost functional, "$\leq$" in (17) is due to the *possible* time-inconsistency of optimal control. Roughly speaking, the time-inconsistency of optimal control means that, a control is optimal now and will be not optimal in the future [37].

From (16) we have

$$
\begin{aligned}
&V(N-2,\zeta)\\
&\leq \inf_{u_{N-2}} \mathbb{E}\Big[x_{N-2}^{\tau} Q_{N-2} x_{N-2} + \mathbb{E}x_{N-2}^{\tau} \bar{Q}_{N-2} \mathbb{E}x_{N-2}\\
&\qquad + u_{N-2}^{\tau} R_{N-2} u_{N-2} + \mathbb{E}u_{N-2}^{\tau} \bar{R}_{N-2} \mathbb{E}u_{N-2}\\
&\qquad + 2q_{N-2}^{\tau} x_{N-2} + 2\rho_{N-2}^{\tau} u_{N-2}\\
&\qquad + \mathbb{E}\big((x_{N-1} - \mathbb{E}x_{N-1})^{\tau} P_{N-1} (x_{N-1} - \mathbb{E}x_{N-1})\big)\\
&\qquad + (\mathbb{E}x_{N-1})^{\tau} T_{N-1} \mathbb{E}x_{N-1} + 2\varphi_{N-1}^{\tau} \mathbb{E}x_{N-1}\\
&\qquad + 2\varphi_N^{\tau} f_{N-1} + f_{N-1}^{\tau} T_N f_{N-1} + d_{N-1}^{\tau} P_N d_{N-1}\\
&\qquad - \mu_{N-1}^{\tau} \bar{W}_{N-1}^{\dagger} \mu_{N-1}\Big]\\
&\equiv \widetilde{V}(N-2,\zeta).
\end{aligned}
$$

By substituting x_{N-1}, we have

$$
\begin{aligned}
&\widetilde{V}(N-2,\zeta)\\
&= \inf_{u_{N-2}} \mathbb{E}\Big[2(u_{N-2} - \mathbb{E}u_{N-2})^{\tau} H_{N-2} (x_{N-2} - \mathbb{E}x_{N-2})\\
&\quad + (u_{N-2} - \mathbb{E}u_{N-2})^{\tau} W_{N-2} (u_{N-2} - \mathbb{E}u_{N-2})\\
&\quad + 2(\mathbb{E}u_{N-2})^{\tau} \bar{H}_{N-2} \mathbb{E}x_{N-2}\\
&\quad + (\mathbb{E}u_{N-2})^{\tau} \bar{W}_{N-2} \mathbb{E}u_{N-2} + 2(\mathbb{E}u_{N-2})^{\tau} \mu_{N-2}\\
&\quad + (x_{N-2} - \mathbb{E}x_{N-2})^{\tau}\\
&\quad \cdot \big(Q_{N-2} + A_{N-2}^{\tau} P_{N-1} A_{N-2} + C_{N-2}^{\tau} P_{N-1} C_{N-2}\big)\\
&\quad \cdot (x_{N-2} - \mathbb{E}x_{N-2})\\
&\quad + (\mathbb{E}x_{N-2})^{\tau} \big(Q_{N-2} + \bar{Q}_{N-2}\\
&\quad + (C_{N-2} + \bar{C}_{N-2})^{\tau} P_{N-1} (C_{N-2} + \bar{C}_{N-2})\\
&\quad + (A_{N-2} + \bar{A}_{N-2})^{\tau} T_{N-1} (A_{N-2} + \bar{A}_{N-2})\big) \mathbb{E}x_{N-2}\\
&\quad + 2(\mathbb{E}x_{N-2})^{\tau} \big((C_{N-2} + \bar{C}_{N-2})^{\tau} P_{N-1} d_{N-2}\\
&\quad + (A_{N-2} + \bar{A}_{N-2})^{\tau} (T_{N-1} f_{N-2} + \varphi_{N-1}) + q_{N-2}\big)\\
&\quad + 2\varphi_{N-1}^{\tau} f_{N-2} + 2\varphi_N^{\tau} f_{N-1} + f_{N-1}^{\tau} T_N f_{N-1}\\
&\quad + d_{N-1}^{\tau} P_N d_{N-1} - \mu_{N-1}^{\tau} \bar{W}_{N-1}^{\dagger} \mu_{N-1}\Big].
\end{aligned}
$$

As $V(N-2,\zeta) > -\infty$, we have $\widetilde{V}(N-2,\zeta) > -\infty$. From Lemma 3.2 it follows that:

$$
\begin{cases}
W_{N-2} \geq 0, \; \bar{W}_{N-2} \geq 0,\\
W_{N-2} W_{N-2}^{\dagger} H_{N-2} - H_{N-2} = 0,\\
\bar{W}_{N-2} \bar{W}_{N-2}^{\dagger} \bar{H}_{N-2} - \bar{H}_{N-2} = 0,\\
\bar{W}_{N-2} \bar{W}_{N-2}^{\dagger} \mu_{N-2} - \mu_{N-2} = 0,
\end{cases}
$$

which implies that GDREs (6) and LRE (8) are solvable at $k = N-2$. Furthermore, we have

$$\begin{aligned} &V(N-2,\zeta) \le \widetilde{V}(N-2,\zeta)\\ &= \sum_{k=N-2}^{N-1}\left[2\varphi_{k+1}^\tau f_k + f_k^\tau T_{k+1} f_k + d_k^\tau P_{k+1} d_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k\right]\\ &\quad + \mathbb{E}\left[(x_{N-2}-\mathbb{E}x_{N-2})^\tau P_{N-2}(x_{N-2}-\mathbb{E}x_{N-2})\right]\\ &\quad + (\mathbb{E}x_{N-2})^\tau T_{N-2}\mathbb{E}x_{N-2} + 2\varphi_{N-2}^\tau \mathbb{E}x_{N-2}. \end{aligned}$$

Now, suppose that for $l = l_1, l_1+1, \cdots, N-1, N$, the GDREs (6) and LRE (8) are solvable, and

$$\begin{aligned} &V(l,\zeta) \le \widetilde{V}(l,\zeta)\\ &= \sum_{k=l}^{N-1}\left[2\varphi_{k+1}^\tau f_k + f_k^\tau T_{k+1} f_k + d_k^\tau P_{k+1} d_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k\right]\\ &\quad + \mathbb{E}\left[(x_l-\mathbb{E}x_l)^\tau P_l(x_l-\mathbb{E}x_l)\right]\\ &\quad + (\mathbb{E}x_l)^\tau T_l\mathbb{E}x_l + 2\varphi_l^\tau \mathbb{E}x_l. \end{aligned}$$

Then, for $l = l_1 - 1$

$$\begin{aligned} &V(l_1-1,\zeta)\\ &= \inf_u \Bigg[\sum_{k=l_1-1}^{N-1}\mathbb{E}\big(x_k^\tau Q_k x_k + \mathbb{E}x_k^\tau \bar{Q}_k \mathbb{E}x_k + u_k^\tau R_k u_k\\ &\qquad + \mathbb{E}u_k^\tau \bar{R}_k \mathbb{E}u_k + 2q_k^\tau x_k + 2\rho_k^\tau u_k\big)\\ &\qquad + \mathbb{E}x_N^\tau G x_N + \mathbb{E}x_N^\tau \bar{G}\mathbb{E}x_N + \mathbb{E}g^\tau x_N\Bigg]\\ &\le \inf_{u_{l_1-1}} \mathbb{E}\Big[x_{l_1-1}^\tau Q_{l_1-1}x_{l_1-1} + \mathbb{E}x_{l_1-1}^\tau \bar{Q}_{l_1-1}\mathbb{E}x_{l_1-1}\\ &\qquad + u_{l_1-1}^\tau R_{l_1-1}u_{l_1-1} + \mathbb{E}u_{l_1-1}^\tau \bar{R}_{l_1-1}\mathbb{E}u_{l_1-1}\\ &\qquad + 2q_{l_1-1}^\tau x_{l_1-1} + 2\rho_{l_1-1}^\tau u_{l_1-1} + V(l_1, x_{l_1})\Big]\\ &\le \inf_{u_{l_1-1}} \mathbb{E}\Big[x_{l_1-1}^\tau Q_{l_1-1}x_{l_1-1} + \mathbb{E}x_{l_1-1}^\tau \bar{Q}_{l_1-1}\mathbb{E}x_{l_1-1}\\ &\qquad + u_{l_1-1}^\tau R_{l_1-1}u_{l_1-1} + \mathbb{E}u_{l_1-1}^\tau \bar{R}_{l_1-1}\mathbb{E}u_{l_1-1}\\ &\qquad + 2q_{l_1-1}^\tau x_{l_1-1} + 2\rho_{l_1-1}^\tau u_{l_1-1}\\ &\qquad + \sum_{k=l_1}^{N-1}\big(2\varphi_{k+1}^\tau f_k + f_k^\tau T_{k+1}f_k + d_k^\tau P_{k+1}d_k\big)\\ &\qquad - \sum_{k=l_1}^{N-1}\mu_k^\tau \bar{W}_k^\dagger \mu_k\\ &\qquad + \mathbb{E}\big((x_{l_1}-\mathbb{E}x_{l_1})^\tau P_{l_1}(x_{l_1}-\mathbb{E}x_{l_1})\big)\\ &\qquad + (\mathbb{E}x_{l_1})^\tau T_{l_1}\mathbb{E}x_{l_1} + 2\varphi_{l_1}^\tau \mathbb{E}x_{l_1}\Big]. \end{aligned}$$

Similarly, we have

$$\begin{cases} W_{l_1-1} \ge 0,\ \bar{W}_{l_1-1} \ge 0,\\ W_{l_1-1}W_{l_1-1}^\dagger H_{l_1-1} - H_{l_1-1} = 0,\\ \bar{W}_{l_1-1}\bar{W}_{l_1-1}^\dagger \bar{H}_{l_1-1} - \bar{H}_{l_1-1} = 0,\\ \bar{W}_{l_1-1}\bar{W}_{l_1-1}^\dagger \mu_{l_1-1} - \mu_{l_1-1} = 0. \end{cases}$$

Therefore, the GDREs (6) and LRE (8) are solvable at $k = l_1 - 1$, and

$$\begin{aligned} &V(l_1-1,\zeta) \le \widetilde{V}(l_1-1,\zeta)\\ &= \sum_{k=l_1-1}^{N-1}\left[2\varphi_{k+1}^\tau f_k + f_k^\tau T_{k+1} f_k + d_k^\tau P_{k+1} d_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k\right]\\ &\quad + \mathbb{E}\left[(x_{l_1-1}-\mathbb{E}x_{l_1-1})^\tau P_{l_1-1}(x_{l_1-1}-\mathbb{E}x_{l_1-1})\right]\\ &\quad + (\mathbb{E}x_{l_1-1})^\tau T_{l_1-1}\mathbb{E}x_{l_1-1} + 2\varphi_{l_1-1}^\tau \mathbb{E}x_{l_1-1}. \end{aligned} \tag{18}$$

By induction, this completes the proof. □

Remark 3.2: To prove Theorem 3.1, we use a backward recursive method, i.e., proving the solvability of the GEREs (6) and LRE (8) backwards from N, $N-1$ to 0. While this backward recursive method is different from the dynamic-programming-based one. Fortunately, it works and is referred in this paper as the modified backward recursive method.

IV. Sufficiency of the GDREs and LRE

In the previous section, we prove that the solvability of the GDREs (6) and LRE (8) is necessary for the well-posedness of Problem (MF-LQ). In this section, we shall show that the solvability of the GDREs (6) and LRE (8) is also sufficient for the well-posedness of Problem (MF-LQ), and the optimal control can be constructed via the solutions of the GDREs and LRE. To proceed, we first have

$$\begin{cases} \mathbb{E}x_{k+1} = (A_k+\bar{A}_k)\mathbb{E}x_k + (B_k+\bar{B}_k)\mathbb{E}u_k + f_k,\\ \mathbb{E}x_l = \mathbb{E}\zeta. \end{cases} \tag{19}$$

Therefore

$$\begin{cases} x_{k+1} - \mathbb{E}x_{k+1}\\ \quad = [A_k(x_k-\mathbb{E}x_k) + B_k(u_k-\mathbb{E}u_k)]\\ \quad\quad + \big[C_k(x_k-\mathbb{E}x_k) + (C_k+\bar{C}_k)\mathbb{E}x_k\\ \quad\quad + D_k(u_k-\mathbb{E}u_k) + (D_k+\bar{D}_k)\mathbb{E}u_k + d_k\big]w_k,\\ x_l - \mathbb{E}x_l = \zeta - \mathbb{E}\zeta. \end{cases} \tag{20}$$

Theorem 4.1: If the GDREs (6) and LRE (8) admit a solution, then Problem (MF-LQ) is well-posed, solvable and

$$u_k = -W_k^\dagger H_k(x_k-\mathbb{E}x_k) - \bar{W}_k^\dagger \bar{H}_k\mathbb{E}x_k - \bar{W}_k^\dagger \mu_k \tag{21}$$

is an optimal control. Moreover, the value function with respect to (l,ζ) is

$$\begin{aligned} &V(l,\zeta)\\ &= \sum_{k=l}^{N-1}\left[2\varphi_{k+1}^\tau f_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k + f_k^\tau T_{k+1} f_k + d_k^\tau P_{k+1} d_k\right]\\ &\quad + \mathbb{E}[(x_l-\mathbb{E}x_l)^\tau P_l(x_l-\mathbb{E}x_l)] + (\mathbb{E}x_l)^\tau T_l\mathbb{E}x_l + 2\varphi_l^\tau \mathbb{E}x_l. \end{aligned} \tag{22}$$

Proof: Let $(P_\cdot, T_\cdot)$ be the solution of the GDREs (6). Then, we have

$$\begin{aligned} &\mathbb{E}\Big[(x_N-\mathbb{E}x_N)^\tau P_N(x_N-\mathbb{E}x_N)\\ &\quad - (x_l-\mathbb{E}x_l)^\tau P_l(x_l-\mathbb{E}x_l)\\ &\quad + (\mathbb{E}x_N)^\tau T_N\mathbb{E}x_N - (\mathbb{E}x_l)^\tau T_l\mathbb{E}x_l\Big] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=l}^{N} \mathbb{E}\left[(x_{k+1}-\mathbb{E}x_{k+1})^\tau P_{k+1}(x_{k+1}-\mathbb{E}x_{k+1})\right.\\
&\quad \left. -(x_k-\mathbb{E}x_k)^\tau P_k(x_k-\mathbb{E}x_k)\right]\\
&\quad + \sum_{k=l}^{N} \mathbb{E}\left[(\mathbb{E}x_{k+1})^\tau T_{k+1}\mathbb{E}x_{k+1} - (\mathbb{E}x_k)^\tau T_k \mathbb{E}x_k\right]\\
&= \sum_{k=l}^{N} \mathbb{E}\Big[(x_k-\mathbb{E}x_k)^\tau (A_k^\tau P_{k+1}A_k\\
&\quad + C_k^\tau P_{k+1}C_k - P_k)(x_k-\mathbb{E}x_k)\\
&\quad + 2(x_k-\mathbb{E}x_k)^\tau (A_k^\tau P_{k+1}B_k + C_k^\tau P_{k+1}D_k)(u_k-\mathbb{E}u_k)\\
&\quad + (u_k-\mathbb{E}u_k)^\tau (B_k^\tau P_{k+1}B_k + D_k^\tau P_{k+1}D_k)(u_k-\mathbb{E}u_k)\\
&\quad + (\mathbb{E}x_k)^\tau \big((C_k+\bar{C}_k)^\tau P_{k+1}(C_k+\bar{C}_k)\\
&\quad + (A_k+\bar{A}_k)^\tau T_{k+1}(A_k+\bar{A}_k) - T_k\big)\mathbb{E}x_k\\
&\quad + 2(\mathbb{E}x_k)^\tau \big((C_k+\bar{C}_k)^\tau P_{k+1}(D_k+\bar{D}_k)\\
&\quad +(A_k+\bar{A}_k)^\tau T_{k+1}(B_k+\bar{B}_k)\big)(\mathbb{E}u_k)\\
&\quad + (\mathbb{E}u_k)^\tau \big((D_k+\bar{D}_k)^\tau P_{k+1}(D_k+\bar{D}_k)\\
&\quad +(B_k+\bar{B}_k)^\tau T_{k+1}(B_k+\bar{B}_k)\big)(\mathbb{E}u_k)\\
&\quad + 2(\mathbb{E}x_k)^\tau (C_k+\bar{C}_k)^\tau P_{k+1}d_k\\
&\quad + 2(\mathbb{E}u_k)^\tau (D_k+\bar{D}_k)^\tau P_{k+1}d_k\\
&\quad + 2(\mathbb{E}x_k)^\tau (A_k+\bar{A}_k)^\tau T_{k+1}f_k\\
&\quad + 2(\mathbb{E}u_k)^\tau (B_k+\bar{B}_k)^\tau T_{k+1}f_k\\
&\quad + f_k^\tau T_{k+1}f_k + d_k^\tau P_{k+1}d_k\Big].
\end{aligned} \tag{23}
$$

For φ_k $(k=l,\cdots,N)$ in (8), it follows that:

$$
\begin{aligned}
\mathbb{E}(\varphi_N^\tau x_N) - \mathbb{E}(\varphi_l^\tau x_l)
&= \sum_{k=l}^{N-1}\left[\mathbb{E}(\varphi_{k+1}^\tau x_{k+1}) - \mathbb{E}(\varphi_k^\tau x_k)\right]\\
&= \sum_{k=l}^{N}\left[\varphi_{k+1}^\tau (A_k+\bar{A}_k)\mathbb{E}x_k - \varphi_k^\tau \mathbb{E}x_k\right]\\
&\quad + \sum_{k=l}^{N}\left[\varphi_{k+1}^\tau (B_k+\bar{B}_k)\mathbb{E}u_k + \varphi_{k+1}^\tau f_k\right].
\end{aligned} \tag{24}
$$

Adding (23) and (24) into the cost functional (3), we have

$$
\begin{aligned}
&J(l,\zeta,u) + \mathbb{E}\left[(x_N-\mathbb{E}x_N)^\tau P_N(x_N-\mathbb{E}x_N)\right]\\
&\quad - \mathbb{E}\left[(x_l-\mathbb{E}x_l)^\tau P_l(x_l-\mathbb{E}x_l)\right]\\
&\quad + (\mathbb{E}x_N)^\tau T_N \mathbb{E}x_N - (\mathbb{E}x_l)^\tau T_l\mathbb{E}x_l + 2\varphi_N^\tau \mathbb{E}x_N - 2\varphi_l^\tau \mathbb{E}x_l\\
&\quad = \sum_{k=l}^{N-1}\mathbb{E}\Big[(x_k-\mathbb{E}x_k)^\tau\\
&\quad \cdot (Q_k + A_k^\tau P_{k+1}A_k + C_k^\tau P_{k+1}C_k - P_k)(x_k-\mathbb{E}x_k)\\
&\quad +2(x_k-\mathbb{E}x_k)^\tau (A_k^\tau P_{k+1}B_k + C_k^\tau P_{k+1}D_k)(u_k-\mathbb{E}u_k)\\
&\quad +(u_k-\mathbb{E}u_k)^\tau (B_k^\tau P_{k+1}B_k + D_k^\tau P_{k+1}D_k)(u_k-\mathbb{E}u_k)\\
&\quad + (u_k-\mathbb{E}u_k)^\tau R_k(u_k-\mathbb{E}u_k)\\
&\quad + (\mathbb{E}x_k)^\tau \big(Q_k+\bar{Q}_k + (C_k+\bar{C}_k)^\tau P_{k+1}(C_k+\bar{C}_k)\\
&\quad + (A_k+\bar{A}_k)^\tau T_{k+1}(A_k+\bar{A}_k) - T_k\big)\mathbb{E}x_k\\
&\quad + 2(\mathbb{E}x_k)^\tau \big((C_k+\bar{C}_k)^\tau P_{k+1}(D_k+\bar{D}_k)\\
&\quad + (A_k+\bar{A}_k)T_{k+1}(B_k+\bar{B}_k)\big)\mathbb{E}u_k\\
&\quad + (\mathbb{E}u_k)^\tau \big(R_k+\bar{R}_k + (D_k+\bar{D}_k)^\tau P_{k+1}(D_k+\bar{D}_k)\\
&\quad + (B_k+\bar{B}_k)^\tau T_{k+1}(B_k+\bar{B}_k)\big)\mathbb{E}u_k\\
&\quad + 2(\mathbb{E}x_k)^\tau (C_k+\bar{C}_k)^\tau P_{k+1}d_k\\
&\quad + 2(\mathbb{E}u_k)^\tau (D_k+\bar{D}_k)^\tau P_{k+1}d_k\\
&\quad + 2(\mathbb{E}x_k)^\tau (A_k+\bar{A}_k)^\tau T_{k+1}f_k\\
&\quad + 2(\mathbb{E}u_k)^\tau (B_k+\bar{B}_k)^\tau T_{k+1}f_k\\
&\quad + f_k^\tau T_{k+1}f_k + d_k^\tau P_{k+1}d_k + 2\varphi_{k+1}^\tau (A_k+\bar{A}_k)\mathbb{E}x_k\\
&\quad - 2\varphi_k^\tau \mathbb{E}x_k + 2\varphi_{k+1}^\tau (B_k+\bar{B}_k)\mathbb{E}u_k\\
&\quad + 2\varphi_{k+1}^\tau f_k + 2q_k^\tau x_k + 2\rho_k^\tau u_k\Big]\\
&\quad + \mathbb{E}\left[(x_N-\mathbb{E}x_N)^\tau G_N(x_N-\mathbb{E}x_N)\right]\\
&\quad + (\mathbb{E}x_N)^\tau (G_N+\bar{G}_N)\mathbb{E}x_N + 2g^\tau \mathbb{E}x_N.
\end{aligned} \tag{25}
$$

As (6) and (8) are solvable, by completing the squares we have

$$
\begin{aligned}
&J(l,\zeta,u)\\
&\quad = \sum_{k=l}^{N-1}\mathbb{E}\Big[\Big(u_k-\mathbb{E}u_k + W_k^\dagger H_k(x_k-\mathbb{E}x_k)\Big)^\tau W_k\\
&\qquad \cdot \Big(u_k-\mathbb{E}u_k + W_k^\dagger H_k(x_k-\mathbb{E}x_k)\Big)\\
&\qquad + \Big(\mathbb{E}u_k + \bar{W}_k^\dagger \bar{H}_k\mathbb{E}x_k + \bar{W}_k^\dagger \mu_k\Big)^\tau \bar{W}_k\\
&\qquad \cdot \Big(\mathbb{E}u_k + \bar{W}_k^\dagger \bar{H}_k\mathbb{E}x_k + \bar{W}_k^\dagger \mu_k\Big)\\
&\qquad + 2\varphi_{k+1}^\tau f_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k + f_k^\tau T_{k+1}f_k + d_k^\tau P_{k+1}d_k\Big]\\
&\qquad + \mathbb{E}\Big[(x_l-\mathbb{E}x_l)^\tau P_l(x_l-\mathbb{E}x_l)\Big]\\
&\qquad + (\mathbb{E}x_l)^\tau T_l\mathbb{E}x_l + 2\varphi_l^\tau \mathbb{E}x_l,
\end{aligned} \tag{26}
$$

where μ_k is defined in (9). By selecting

$$
\begin{cases}
u_k - \mathbb{E}u_k = -W_k^\dagger H_k(x_k-\mathbb{E}x_k),\\
\mathbb{E}u_k = -\bar{W}_k^\dagger \bar{H}_k\mathbb{E}x_k - \bar{W}_k^\dagger \mu_k,
\end{cases}
$$

and thus

$$
u_k = -W_k^\dagger H_k(x_k-\mathbb{E}x_k) - \bar{W}_k^\dagger \bar{H}_k\mathbb{E}x_k - \bar{W}_k^\dagger \mu_k,
$$

we can get (22). □

Corollary 4.1: If $Q_k, Q_k+\bar{Q}, G_N, G_N+\bar{G}_N \geq 0, R_k, R+\bar{R}_k > 0$, $k\in\mathbb{N}$, then the optimal control is unique and is expressed as

$$
u_k = -W_k^{-1}H_k(x_k-\mathbb{E}x_k) - \bar{W}_k^{-1}\bar{H}_k\mathbb{E}x_k - \bar{W}_k^{-1}\mu_k, k\in\mathbb{N}.
$$

Proof: Under the given condition, we have $W_k, \bar{W}_k > 0$, and the uniqueness of the solutions of the GDREs (6) and LRE (8). Thus, from Theorem 4.1 the result follows. □

From Theorem 3.1 and Theorem 4.1, we arrive at the main result of this paper.

Theorem 4.2: The following statements are equivalent

(i) Problem (MF-LQ) is well-posed;
(ii) Problem (MF-LQ) is attainable;
(iii) the GDREs (6) and LRE (8) are solvable.

Moreover, when any of the above statements is true, the LQ problem is attainable by

$$u_k = -W_k^{\dagger} H_k(x_k - \mathbb{E}x_k) - \bar{W}_k^{\dagger}\bar{H}_k\mathbb{E}x_k - \bar{W}_k^{\dagger}\mu_k, k \in \mathbb{N}. \tag{27}$$

Moreover, the value function with respect to (l, ζ) is

$$\begin{aligned} V(l,\zeta) = &\sum_{k=l}^{N-1}\left[2\varphi_{k+1}^{\tau}f_k + f_k^{\tau}T_{k+1}f_k + d_k^{\tau}P_{k+1}d_k\right] \\ &- \sum_{k=l}^{N-1}\mu_k^{\tau}\bar{W}_k^{\dagger}\mu_k + \mathbb{E}\left[(x_l - \mathbb{E}x_l)^{\tau}P_l(x_l-\mathbb{E}x_l)\right] \\ &+ (\mathbb{E}x_l)^{\tau}T_l\mathbb{E}x_l + 2\varphi_l^{\tau}\mathbb{E}x_l. \end{aligned} \tag{28}$$

Remark 4.1: From (28), we know that "≤" in (18) should read as "=". Hence, the modified backward recursive method mentioned in Remark 3.2 reduces to the classical backward recursive method, and the time consistency of optimal controls holds. Note that the expectations $\mathbb{E}x_k, \mathbb{E}u_k$ enter nonlinearly into the cost functional (3). It now seems a common view that the dynamic programming principle may not be applicable to such a class of problems. To better understand such phenomena and investigate the possible time inconsistency of optimal control, we may focus in the future on the dynamic version of Problem (MF-LQ), i.e., replace the expectation operator in the dynamics and cost functional by the conditional expectation operator. The modified backward recursive method might be also applicable to such class of problems.

In the following, we show that the GDREs (6) are solvable for a special case.

Theorem 4.3: Under the condition that $Q_k, Q_k + \bar{Q}_k, R_k, R_k + \bar{R}_k, k \in \mathbb{N}, G_N, G_N + \bar{G}_N \geq 0$, the GDREs (6) are solvable.

Proof: Introduce the homogeneous versions of controlled system (1) and cost functional (3)

$$\begin{cases} x_{k+1} = (A_kx_k + \bar{A}_k\mathbb{E}x_k + B_ku_k + \bar{B}_k\mathbb{E}u_k) \\ \qquad\quad +(C_kx_k + \bar{C}_k\mathbb{E}x_k + D_ku_k + \bar{D}_k\mathbb{E}u_k)w_k, \\ x_l = \zeta,\ k \in \mathbb{N} \end{cases}$$

and

$$\begin{aligned} J^H(l,\zeta,u) = &\sum_{k=l}^{N-1}\mathbb{E}\left[x_k^{\tau}Q_kx_k + \mathbb{E}x_k^{\tau}\bar{Q}_k\mathbb{E}x_k + u_k^{\tau}R_ku_k\right] \\ &+ \sum_{k=l}^{N-1}\mathbb{E}u_k^{\tau}\bar{R}_k\mathbb{E}u_k + \mathbb{E}x_N^{\tau}Gx_N + \mathbb{E}x_N^{\tau}\bar{G}\mathbb{E}x_N. \end{aligned}$$

Clearly, $J^H(l,\zeta,u) \geq 0 > -\infty$ for any (l,ζ,u) under the condition that $Q_k, Q_k + \bar{Q}_k, R_k, R_k + \bar{R}_k, k \in \mathbb{N}, G_N, G_N + \bar{G}_N \geq 0$. Hence, by the proof of Theorem 3.1, we can assert that the GDREs are solvable. □

If the GDREs (6) and LRE (8) are solvable, then an optimal control of Problem (MF-LQ) can be expressed in the form of (27). To see this, we will give a complete characterization of all the optimal controls of Problem (MF-LQ) with five degrees of freedom.

Theorem 4.4: Let the GDREs (6) and LRE (8) be solvable. Then, the set of all optimal controls is determined by the following (parameterized by $Y_k, \bar{Y}_k, y_k, Z_k, z_k$)

$$\begin{aligned} u_k = &-\left[\bar{W}_k^{\dagger}\bar{H}_k + \left(I - \bar{W}_k^{\dagger}\bar{W}_k\right)Y_k\right]\mathbb{E}x_k \\ &+ \left(I - \bar{W}_k^{\dagger}\bar{W}_k\right)y_k + \left[\bar{W}_k^{\dagger} + \left(I - \bar{W}_k^{\dagger}\bar{W}_k\right)\bar{Y}_k\right]\mu_k \\ &- \left[W_k^{\dagger}H_k + \left(I - W_k^{\dagger}W_k\right)Z_k\right](x_k - \mathbb{E}x_k) \\ &+ \left(I - W_k^{\dagger}W_k\right)z_k,\ \ k \in \mathbb{N}. \end{aligned} \tag{29}$$

Here, $Y_k, \bar{Y}_k \in \mathbb{R}^{n\times n}, y_k \in \mathbb{R}^n$ are deterministic and $Z_k \in \mathbb{R}^{n\times n}, z_k \in \mathbb{R}^n$ are arbitrary square integrable random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and adapted to $\mathcal{F}_k'$ such that

$$\begin{aligned} \mathbb{E}\Big\{ &- \left[W_k^{\dagger}H_k + \left(I - W_k^{\dagger}W_k\right)Z_k\right](x_k - \mathbb{E}x_k) \\ &+ \left(I - W_k^{\dagger}W_k\right)z_k\Big\} = 0, \quad k \in \mathbb{N}. \end{aligned}$$

Proof: We first show that (29) is optimal. By the proof of Theorem 4.1, we have

$$\begin{aligned} &J(l,\zeta,u) \\ &= \sum_{k=l}^{N-1}\mathbb{E}\Big[\left(u_k - \mathbb{E}u_k + W_k^{\dagger}H_k(x_k - \mathbb{E}x_k)\right)^{\tau}W_k \\ &\quad \cdot\left(u_k - \mathbb{E}u_k + W_k^{\dagger}H_k(x_k-\mathbb{E}x_k)\right) \\ &\quad + \left(\mathbb{E}u_k + \bar{W}_k^{\dagger}\bar{H}_k\mathbb{E}x_k + \bar{W}_k^{\dagger}\mu_k\right)^{\tau}\bar{W}_k \\ &\quad \cdot\left(\mathbb{E}u_k + \bar{W}_k^{\dagger}\bar{H}_k\mathbb{E}x_k + \bar{W}_k^{\dagger}\mu_k\right) \\ &\quad + 2\varphi_{k+1}^{\tau}f_k + f_k^{\tau}T_{k+1}f_k + d_k^{\tau}P_{k+1}d_k - \mu_k^{\tau}\bar{W}_k^{\dagger}\mu_k\Big] \\ &\quad + (x_l - \mathbb{E}x_l)^{\tau}P_l(x_l - \mathbb{E}x_l) \\ &\quad + (\mathbb{E}x_l)^{\tau}T_l\mathbb{E}x_l + 2\varphi_l^{\tau}\mathbb{E}x_l. \end{aligned} \tag{30}$$

By the properties of pseudo-inverse

$$\begin{cases} \bar{W}_k(Y_k - \bar{W}_k^{\dagger}\bar{W}_kY_k) = 0,\ \bar{W}_k(Y_k - \bar{W}_k^{\dagger}\bar{W}_kY_k) = 0, \\ \bar{W}_k(y_k - \bar{W}_k^{\dagger}\bar{W}_ky_k) = 0,\ W_k(Z_k - W_k^{\dagger}W_kZ_k) = 0, \\ W_k(z_k - W_k^{\dagger}W_kz_k) = 0, \end{cases}$$

we have

$$\begin{aligned} &J(\zeta,u) \\ &= \sum_{k=l}^{N-1}\mathbb{E}\Big\{\Big[u_k - \mathbb{E}u_k - z_k + W_k^{\dagger}W_kz_k \\ &\quad + \left(W_k^{\dagger}H_k + Z_k - W_k^{\dagger}W_kZ_k\right)(x_k - \mathbb{E}x_k)\Big]^{\tau}W_k \end{aligned}$$

$$
\begin{aligned}
&\cdot \Big[u_k - \mathbb{E}u_k - z_k + W_k^\dagger W_k z_k \\
&+ \Big(W_k^\dagger H_k + Z_k - W_k^\dagger W_k Z_k\Big)(x_k - \mathbb{E}x_k)\Big] \\
&+ \Big[\mathbb{E}u_k - y_k + \bar{W}_k^\dagger \bar{W}_k y_k \\
&+ \Big(\bar{W}_k^\dagger \bar{H}_k \mathbb{E} + Y_k - \bar{W}_k^\dagger \bar{W}_k Y_k\Big)\mathbb{E}x_k + \Big(\bar{W}_k^\dagger + \bar{Y}_k \\
&- \bar{W}_k^\dagger \bar{W}_k \bar{Y}_k\Big)\mu_k\Big]^\tau \bar{W}_k \\
&\cdot \Big[\mathbb{E}u_k - y_k + \bar{W}_k^\dagger \bar{W}_k y_k \\
&+ \Big(\bar{W}_k^\dagger \bar{H}_k \mathbb{E} + Y_k - \bar{W}_k^\dagger \bar{W}_k Y_k\Big)\mathbb{E}x_k + (\bar{W}_k^\dagger + \bar{Y}_k \\
&- \bar{W}_k^\dagger \bar{W}_k \bar{Y}_k)\mu_k\Big] \\
&+ 2\varphi_{k+1}^\tau f_k + f_k^\tau T_{k+1} f_k + d_k^\tau P_{k+1} d_k - \mu_k^\tau \bar{W}_k^\dagger \mu_k\Big\} \\
&+ \mathbb{E}\Big[(x_l - \mathbb{E}x_l)^\tau P_l (x_l - \mathbb{E}x_l)\Big] \\
&+ (\mathbb{E}x_l)^\tau T_l \mathbb{E}x_l + 2\varphi_l^\tau \mathbb{E}x_l.
\end{aligned}
$$

As $W_k, \bar{W}_k \geq 0, k \in \mathbb{N}$, (29) is the optimal control.

The remaining is to show that any optimal control $\bar{u}$ can be expressed as the form of (29). From (30) and (28), we have

$$
\begin{aligned}
\sum_{k=l}^{N-1} \mathbb{E}\Big[&\Big(\bar{u}_k - \mathbb{E}\bar{u}_k + W_k^\dagger H_k (x_k - \mathbb{E}x_k)\Big)^\tau W_k \\
&\cdot \Big(\bar{u}_k - \mathbb{E}\bar{u}_k + W_k^\dagger H_k (x_k - \mathbb{E}x_k)\Big) \\
&+ \Big(\mathbb{E}\bar{u}_k + \bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k + \bar{W}_k^\dagger \mu_k\Big)^\tau \bar{W}_k \\
&\cdot \Big(\mathbb{E}\bar{u}_k + \bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k + \bar{W}_k^\dagger \mu_k\Big)\Big] = 0.
\end{aligned}
$$

As $W_k, \bar{W}_k \geq 0, k \in \mathbb{N}$, we have

$$
\begin{cases}
W_k\Big(\bar{u}_k - \mathbb{E}\bar{u}_k + W_k^\dagger H_k (x_k - \mathbb{E}x_k)\Big) = 0, \\
\bar{W}_k\Big(\mathbb{E}\bar{u}_k + \bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k + \bar{W}_k^\dagger \mu_k\Big) = 0, k \in \mathbb{N}.
\end{cases}
$$

By Lemma 2.7 in [2], (29) follows. This completes the proof. □

Remark 4.2: In (29), $Y_k, \bar{Y}_k, y_k, k \in \mathbb{N}$, may be relaxed to be random under the constraint that $(I - \bar{W}_k^\dagger \bar{W}_k)\bar{Y}_k, (I - \bar{W}_k^\dagger \bar{W}_k)Y_k$ and $(I - \bar{W}_k^\dagger \bar{W}_k)y_k$ are deterministic. Here, the constraint is to ensure that $\mathbb{E}u_k = -[\bar{W}_k^\dagger \bar{H}_k + (I - \bar{W}_k^\dagger \bar{W}_k)Y_k]\mathbb{E}x_k + (I - \bar{W}_k^\dagger \bar{W}_k)y_k + [\bar{W}_k^\dagger + (I - \bar{W}_k^\dagger \bar{W}_k)\bar{Y}_k]\mu_k, k \in \mathbb{N}$, are deterministic.

In fact, we can extend Theorem 4.2 to the case with multidimensional noises. Specifically, consider the following controlled MF-SDE

$$
\begin{cases}
x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k + f_k) \\
\qquad + \sum_{i=1}^t (C_{i,k} x_k + \bar{C}_{i,k} \mathbb{E}x_k) w_{i,k} \\
\qquad + \sum_{i=1}^t (D_{i,k} u_k + \bar{D}_{i,k} \mathbb{E}u_k + d_{i,k}) w_{i,k} \\
x_l = \zeta,\ k = l, \cdots, N-1,\ l \in \{0, 1, \cdots, N-1\}.
\end{cases} \tag{31}
$$

Here, $\{w_k = (w_{1,k}, \cdots, w_{t,k})^\tau, k = 0, 1, \cdots, N-1\}$ is a vector-valued martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ in the sense that

$$
\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0,\ \mathbb{E}\left[w_{k+1} w_{k+1}^\tau | \mathcal{F}_k\right] = \Gamma_{k+1},\ k \geq 0, \tag{32}
$$

where $\mathcal{F}_k = \{x_0, w_l, l = 0, 1, \cdots, k\}$, $\mathcal{F}_{-1} = \sigma(x_0)$ and $\Gamma_{k+1} = (\gamma_{k+1}^{ij})_{t \times t}$ is a deterministic nonnegative definite matrix of $t \times t$. Taking expectations in both sides of (31) gives

$$
\begin{cases}
\mathbb{E}x_{k+1} = (A_k + \bar{A}_k)\mathbb{E}x_k + (B_k + \bar{B}_k)\mathbb{E}u_k + f_k, \\
\mathbb{E}x_l = \mathbb{E}\zeta.
\end{cases}
$$

Therefore,

$$
\begin{cases}
x_{k+1} - \mathbb{E}x_{k+1} \\
= [A_k(x_k - \mathbb{E}x_k) + B_k(u_k - \mathbb{E}u_k)] \\
\quad + \sum_{i=1}^t \left[C_{i,k}(x_k - \mathbb{E}x_k) + (C_{i,k} + \bar{C}_{i,k})\mathbb{E}x_k\right] w_{i,k} \\
\quad + \sum_{i=1}^t \left[D_{i,k}(u_k - \mathbb{E}u_k) + (D_{i,k} + \bar{D}_{i,k})\mathbb{E}u_k + d_{i,k}\right] w_{i,k}, \\
x_l - \mathbb{E}x_l = \zeta - \mathbb{E}\zeta.
\end{cases}
$$

Noting (32), we can get equations similar to (15) and (23). Hence, the GDREs and LRE corresponding to (31) should be

$$
\begin{cases}
P_k = Q_k + A_k^\tau P_{k+1} A_k + \sum_{i,j=1}^t \gamma_k^{ij} C_{i,k}^\tau P_{k+1} C_{j,k} \\
\qquad - H_k^\tau W_k^\dagger H_k, \\
T_k = Q_k + \bar{Q}_k \\
\qquad + \sum_{i,j=1}^t \gamma_k^{ij} (C_{i,k} + \bar{C}_{i,k})^\tau P_{k+1} (C_{j,k} + \bar{C}_{j,k}) \\
\qquad + (A_k + \bar{A}_k)^\tau T_{k+1} (A_k + \bar{A}_k) - \bar{H}_k^\tau \bar{W}_k^\dagger \bar{H}_k, \\
P_N = G_N,\ T_N = G_N + \bar{G}_N, \\
W_k, \bar{W}_k \geq 0,\ W_k W_k^\dagger H_k - H_k = 0, \\
\bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0,\ \ k \in \mathbb{N},
\end{cases} \tag{33}
$$

and

$$
\begin{cases}
\varphi_k = \sum_{i,j=1}^t \gamma_k^{ij} (C_{i,k} + \bar{C}_{i,k})^\tau P_{k+1} d_{j,k} \\
\qquad + (A_k + \bar{A}_k)^\tau (T_{k+1} f_k + \varphi_{k+1}) \\
\qquad - \bar{H}_k^T \bar{W}_k^\dagger \mu_k + q_k, \\
\varphi_N = g, \\
\bar{W}_k \bar{W}_k^\dagger \mu_k - \mu_k = 0, \quad k \in \mathbb{N},
\end{cases} \tag{34}
$$

where

$$
\begin{cases}
W_k = R_k + B_k^\tau P_{k+1} B_k \\
\qquad + \sum_{i,j=1}^t \gamma_k^{ij} D_{i,k}^\tau P_{k+1} D_{j,k}, \\
H_k = B_k^\tau P_{k+1} A_k + \sum_{i,j=1}^t \gamma_k^{ij} D_{i,k}^\tau P_{k+1} C_{j,k}, \\
\bar{W}_k = R_k + \bar{R}_k + (B_k + \bar{B}_k)^\tau T_{k+1} (B_k + \bar{B}_k) \\
\qquad + \sum_{i,j=1}^t \gamma_k^{ij} (D_{i,k} + \bar{D}_{i,k})^\tau P_{k+1} (D_{j,k} + \bar{D}_{j,k}), \\
\bar{H}_k = (B_k + \bar{B}_k)^\tau T_{k+1} (A_k + \bar{A}_k) \\
\qquad + \sum_{i,j=1}^t \gamma_k^{ij} (D_{i,k} + \bar{D}_{i,k})^\tau P_{k+1} (C_{j,k} + \bar{C}_{j,k}), \\
k \in \mathbb{N}
\end{cases} \tag{35}
$$

and

$$
\mu_k = \sum_{i,j=1}^t \gamma_k^{ij} (D_{i,k} + \bar{D}_{i,k})^\tau P_{k+1} d_{j,k} + (B_k + \bar{B}_k)^\tau (T_{k+1} f_k + \varphi_{k+1}) + \rho_k. \tag{36}
$$

Denote the version of Problem (MF-LQ) corresponding to (31) by Problem (MMF-LQ). Then, we can have the following result, whose proof is a straightforward extension of that of Theorem 4.2 and hence is omitted here.

Theorem 4.5: The following statements are equivalent:

(i) Problem (MMF-LQ) is well-posed;
(ii) Problem (MMF-LQ) is attainable;
(iii) the GDREs (33) and LRE (34) are solvable

Moreover, when any of the above statements is true, the LQ problem is attainable by

$$u_k = -W_k^{\dagger} H_k(x_k - \mathbb{E}x_k) - \bar{W}_k^{\dagger}\bar{H}_k\mathbb{E}x_k - \bar{W}_k^{\dagger}\mu_k, k \in \mathbb{N}, \tag{37}$$

where $W_k, \bar{W}_k, H_k, \bar{H}_k, \mu_k$ are defined in (35) and (36), respectively. Moreover, the value function with respect to (l, ζ) is

$$\begin{aligned} V(l,\zeta) = &\sum_{k=l}^{N-1}\Big[2\varphi_{k+1}^{\tau} f_k + f_k^{\tau} T_{k+1} f_k\Big] \\ &- \sum_{k=l}^{N-1}\mu_k^{\tau}\bar{W}_k^{\dagger}\mu_k + \mathbb{E}\Big[(x_l - \mathbb{E}x_l)^{\tau} P_l (x_l - \mathbb{E}x_l)\Big] \\ &+ \sum_{k=l}^{N-1}\sum_{i,j=1}^{t}\gamma_k^{ij} d_{i,k}^{\tau} P_{k+1} d_{j,k} \\ &+ (\mathbb{E}x_l)^{\tau} T_l \mathbb{E}x_l + 2\varphi_l^{\tau}\mathbb{E}x_l. \end{aligned} \tag{38}$$

V. Multi-Period Mean-Variance Portfolio Selection

A. Problem Formulation

Basing upon the general theory above, in this section, we shall study a particular example of Problem (MF-LQ)—the multi-period portfolio selection [23], [39]. Consider a capital market consisting of one riskless asset and n risky assets within a time horizon N. Let $s_k(>1)$ be a given deterministic return of the riskless asset at time period k and $e_k = (e_k^1, \cdots, e_k^n)^{\tau}$ be the vector of random returns of the n risky assets at period k. We assume that vectors $e_k, k = 0, 1, \cdots, N-1$, are statistically independent and the only information known about the random return vector e_k is its first two moments: its mean $\mathbb{E}(e_k) = (\mathbb{E}e_k^1, \mathbb{E}e_k^2, \cdots, \mathbb{E}e_k^n)^{\tau}$ and its covariance $\mathrm{Cov}(e_k) = \mathbb{E}[(e_k - \mathbb{E}e_k)(e_k - \mathbb{E}e_k)^{\tau}]$.

Clearly, $\mathrm{Cov}(e_k)$ is nonnegative definite, i.e., $\mathrm{Cov}(e_k) \geq 0$. Different from [23], here we do not suppose that $\mathrm{Cov}(e_k)$ is positive definite. In fact, mean-variance portfolio selection problems of such class can be traced back to 1970s. In [8] or the "corrected" version [33], Buser *et al.* propose the single-period version with possibly singular covariance matrix. Such class of problems is more general than classical mean-variance portfolio selection problems [28], and is more consistent with the reality. In this section, we shall derive necessary and sufficient conditions for the solvability of multi-period mean-variance portfolio selection upon the general theory of indefinite mean-field LQ problems above.

To proceed, let x_k be the wealth of the investor at the beginning of the k-th period, and let $u_k^i, i = 1, 2, \cdots, n$, be the amount invested in the i-th risky asset at period k. Then $x_k - \sum_{i=1}^{n} u_k^i$ is the amount invested in the riskless asset at period k, and the wealth at the beginning of the $(k+1)$-th period [23] is given by

$$x_{k+1} = \sum_{i=1}^{n} e_k^i u_k^i + \left(x_k - \sum_{i=1}^{n} u_k^i\right) s_k = s_k x_k + O_k^{\tau} u_k, \tag{39}$$

where O_k is the excess return vector of risky assets [23] defined as

$$\begin{aligned} O_k &= (O_k^1, O_k^2, \cdots, O_k^n)^{\tau} \\ &= (e_k^1 - s_k, e_k^2 - s_k, \cdots, e_k^n - s_k)^{\tau}. \end{aligned} \tag{40}$$

Clearly, $x_k \in \mathbb{R}, k \in \bar{\mathbb{N}}$. Define the information set at the beginning of period k as

$$\mathcal{F}_k' = \sigma(s_l, x_l, l = 0, 1, \cdots, k).$$

In this paper, we consider the case where short-selling of stocks is allowed, i.e., $u_k^i, i = 1, \ldots, k$, could take values in $\mathbb{R}$, which leads to an unconstrained mean-variance portfolio selection formulation. Hence, the admissible policy set of $u = (u_0, u_1, \cdots, u_{N-1}) : \mathbb{N} \times \Omega \to \mathbb{R}^n$ in this section is

$$\mathcal{U}_{ad} \equiv \left\{ u \,\middle|\, u_k \text{ is } \mathcal{F}_{k-1}'\text{-measurable}, \sum_{k=l}^{N-1}\mathbb{E}|u_k|^2 < \infty \right\}.$$

The conventional multi-period mean-variance problem [23] can be formulated as follows:

Problem (MV). For any given x_0, find $u^* \in \mathcal{U}_{ad}$ such that

$$J_1(x_0, u^*) = \inf_{u \in \mathcal{U}_{ad}} J_1(x_0, u),$$

where

$$J_1(x_0, u) = \lambda \mathrm{Var}(x_N) - \mathbb{E}x_N \tag{41}$$

with $\lambda > 0$ being the trade-off parameter between the mean and the variance of the terminal wealth.

Before stating the main result of this subsection, we review a recent progress in mean-variance problem related to the topic of this paper. Recently, [11] proposes an mean-field formulation to deal with multi-period mean-variance portfolio selection problems in a unified way and formulate the considered problems as mean-field LQ ones, which motivates the study of indefinite mean-field LQ optimal control theory of this paper. By decomposing of x_k into two orthogonal parts $\mathbb{E}x_k$ and $x_k - \mathbb{E}x_k$, the authors recursively construct the optimal strategy by elegant analysis. The difference between this paper and [11] lies in the following two points. The first point is that in [11] the return rates of the risky securities are nondegenerate, but here the degeneracy is allowed. The second point of the difference is that we regard the multi-period mean-variance portfolio selection as an application of the general indefinite mean-field LQ optimal control theory.

B. Optimal Strategies to Problem (MV)

To proceed, we shall transform (39) into a linear controlled system of form (1), by which the general theory in above sections will work. Precisely, we define

$$\begin{cases} w_{i,k} = e_k^i - s_k - \mathbb{E}(e_k^i - s_k), \\ D_{i,k} = (0, \cdots, 0, 1, 0, \cdots, 0), \\ i = 1, \cdots, n, k = 0, 1, \cdots, N-1. \end{cases}$$

Then, $\{w_k = (w_{1,k}, \ldots, w_{n,k})^{\tau}, k = 0, 1, \cdots, N-1\}$ is a martingale difference sequence as $e_k, k = 1, \ldots, N-1$, are statistically independent. Furthermore, $\mathbb{E}[w_{k+1}w_{k+1}^\tau|\sigma(x_0, e_l, l = 0, \ldots, k)] = \mathrm{Cov}(e_{k+1})$. This leads to

$$x_{k+1} = (s_k x_k + (\mathbb{E}O_k)^\tau u_k) + \sum_{i=1}^{n} D_{i,k} u_k w_{i,k}. \tag{42}$$

Before proceeding, we recall the following lemma [12].

Lemma 5.1: Let $M \in \mathbb{R}^{n\times n}, c \in \mathbb{R}^n$. If $c \in \mathrm{Ran}(M)$, then

$$(M \pm cc^\tau)^\dagger = M^\dagger - \frac{M^\dagger cc^\tau M^\dagger}{c^\tau M^\dagger c \pm 1}.$$

Theorem 5.1: Problem (MV) is solvable in the sense of Definition 2.1 if and only if for any $k \in \mathbb{N}, \mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k))$. Furthermore, if this condition holds, any optimal portfolio selection strategy can be expressed as

$$\begin{aligned} u_k = &-\left[W_k^\dagger H_k + (I - W_k^\dagger W_k)Z_k\right](x_k - \mathbb{E}x_k) \\ &+ (I - W_k^\dagger W_k)z_k - \left[\bar{W}_k^\dagger + (I - \bar{W}_k^\dagger \bar{W}_k)\bar{Y}_k\right]\mu_k \\ &+ (I - \bar{W}_k^\dagger \bar{W}_k)y_k, \quad k \in \mathbb{N}. \end{aligned} \tag{43}$$

Here

$$\begin{cases} P_k = s_k^2 P_{k+1}\left[1 - \mathbb{E}O_k^\tau[\mathbb{E}(O_k O_k^\tau)]^\dagger \mathbb{E}O_k\right], \\ P_N = \lambda, \\ W_k = P_{k+1}\mathbb{E}(O_k O_k^\tau), \\ H_k = s_k \mathbb{E}(O_k)P_{k+1}, \\ \bar{W}_k = P_{k+1}\mathrm{Cov}(O_k), \\ \mu_k = -\frac{1}{2}s_{k+1}\cdots s_{N-1}\mathbb{E}O_k, \end{cases} \tag{44}$$

$Y_k \in \mathbb{R}^{n\times n}, y_k \in \mathbb{R}^n$ are deterministic, and $Z_k \in \mathbb{R}^{n\times n}, z_k \in \mathbb{R}^n$ are arbitrary square integrable random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and adapted to $\mathcal{F}_k'$ such that

$$\mathbb{E}\left\{-\left[W_k^\dagger H_k + (I - W_k^\dagger W_k)Z_k\right](x_k - \mathbb{E}x_k) + (I - W_k^\dagger W_k)z_k\right\} = 0, \quad k \in \mathbb{N}.$$

Moreover, the value function is given by

$$V(x_0) = -\sum_{k=0}^{N-1} \mu_k^\tau \bar{W}_k^\dagger \mu_k + P_0\mathbb{E}\left[(x_0 - \mathbb{E}x_0)^2\right] + 2\varphi_0 \mathbb{E}x_0, \tag{45}$$

where $\varphi_0 = -(1/2)s_0 s_1 \cdots s_N - 1$.

Proof: Due to the general theory of above sections, the solvability of Problem (MV), i.e., the existence of optimal strategy, in the sense of Definition 2.1, is equivalent to the solvability of the following GDREs and LRE:

$$\begin{cases} P_k = s_k^2 P_{k+1} - H_k^\tau W_k^\dagger H_k, \\ T_k = s_k^2 T_{k+1} - \bar{H}_k^\tau \bar{W}_k^\dagger \bar{H}_k, \\ P_N = \lambda, T_N = 0, k \in \mathbb{N}, \\ W_k, \bar{W}_k \geq 0, W_k W_k^\dagger H_k - H_k = 0, \\ \bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0, \end{cases} \tag{46}$$

and

$$\begin{cases} \varphi_k = s_k \varphi_{k+1} - \bar{H}_k^\tau \bar{W}_k^\dagger \mu_k, \\ \varphi_N = -\frac{1}{2}, k \in \mathbb{N}, \\ \bar{W}_k \bar{W}_k^\dagger \mu_k - \mu_k = 0. \end{cases} \tag{47}$$

Here

$$\begin{cases} W_k = \mathbb{E}O_k P_{k+1}\mathbb{E}O_k^\tau \\ \qquad + \sum_{i=1}^n \sum_{j=1}^n v_k^{ij} D_{i,k}^\tau P_{k+1} D_{j,k} \\ \quad = P_{k+1}\mathbb{E}(O_k O_k^\tau), \\ H_k = s_k \mathbb{E}O_k P_{k+1}, \\ \bar{W}_k = \mathbb{E}O_k T_{k+1}\mathbb{E}O_k^\tau \\ \qquad + \sum_{i=1}^n \sum_{j=1}^n v_k^{ij} D_{i,k}^\tau P_{k+1} D_{j,k} \\ \quad = \mathbb{E}O_k T_{k+1}\mathbb{E}O_k^\tau + P_{k+1}\mathrm{Cov}(O_k), \\ \bar{H}_k = s_k \mathbb{E}O_k T_{k+1}, \\ \mu_k = \mathbb{E}O_k \varphi_{k+1}, \\ k \in \mathbb{N}, \end{cases} \tag{48}$$

where $v_k^{ij} = \mathbb{E}[w_{i,k}w_{j,k}]$. From Theorem 4.3, we know that the GDREs (46) are solvable. Furthermore, we have $T_k = 0, \bar{H}_k = 0, k \in \mathbb{N}$. Thus (47)–(48) can be reduced to

$$E\left\{\left[p_1(O_{k_0}^1 - \mathbb{E}O_{k_0}^1) + \cdots + p_n(O_{k_0}^n - \mathbb{E}O_{k_0}^n)\right]^2\right\} = 0$$

with the constraint $\bar{W}_k \bar{W}_k^\dagger \mu_k - \mu_k = 0$.

From Lemma 3.2, we have that the solvability of the LRE (47), and hence the solvability of Problem (MV) is equivalent to $\mathbb{E}O_k \in \mathrm{Ran}(P_{k+1}\mathrm{Cov}(O_k)), k \in \mathbb{N}$. We now show that $\mathbb{E}O_k \in \mathrm{Ran}(P_{k+1}\mathrm{Cov}(O_k)), k \in \mathbb{N}$, is equivalent to $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{N}$. By (46), we know

$$P_k = s_k^2 P_{k+1}\left[1 - \mathbb{E}O_k^\tau \left[\mathbb{E}(O_k O_k^\tau)\right]^\dagger \mathbb{E}O_k\right]. \tag{49}$$

If $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{N}$, hold, by Lemma 5.1 we have that for any $k \in \mathbb{N}$

$$\begin{aligned} 1 &- \mathbb{E}O_k^\tau \left[\mathbb{E}(O_k O_k^\tau)\right]^\dagger \mathbb{E}O_k \\ &= 1 - \mathbb{E}O_k^\tau \left[\mathrm{Cov}(O_k) + \mathbb{E}O_k\mathbb{E}O_k^\tau)\right]^\dagger \mathbb{E}O_k \\ &= \frac{1}{1 + \mathbb{E}O_k^\tau\left(\mathrm{Cov}(O_k)\right)^\dagger \mathbb{E}O_k} > 0. \end{aligned} \tag{50}$$

This, together with (49), implies that $P_k > 0$ for any $k \in \mathbb{N}$. Therefore, $\mathbb{E}O_k \in \mathrm{Ran}(P_{k+1}\mathrm{Cov}(O_k)), k \in \mathbb{N}$. On the other hand, if $\mathbb{E}O_k \in \mathrm{Ran}(P_{k+1}\mathrm{Cov}(O_k)), k \in \mathbb{N}$, we have $\mathbb{E}O_{N-1} \in \mathrm{Ran}(\mathrm{Cov}(O_{N-1}))$ as $P_N = \lambda > 0$. Combining this with (50), we then have $P_{N-1} > 0$, which implies $\mathbb{E}O_{N-2} \in \mathrm{Ran}(\mathrm{Cov}(O_{N-2}))$. By induction, we derive that $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{N}$. Hence, we arrive at the following result: Problem (MV) is solvable in the sense of Definition 2.1 if and only if for any $k \in \mathbb{N}, \mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k))$. We then complete the proof by Theorem 4.2 and Theorem 4.4. □

Remark 5.1: We now give an intuitive explanation of the condition $\mathbb{E}O_k \in \mathrm{Ran}(\mathrm{Cov}(O_k)), k \in \mathbb{N}$, to the well-posedness of Problem (MV). Suppose now $\mathbb{E}O_{k_0} \notin \mathrm{Ran}(\mathrm{Cov}(O_{k_0}))$ for some k_0. Hence, $\mathrm{Cov}(O_{k_0})$ is just nonnegative definite and the dimension of $\text{Ker}(\text{Cov}(O_{k_0}))$ is big than 0. For any $0 \neq p = (p_1, \ldots, p_n)^\tau \in \text{Ker}(\text{Cov}(O_{k_0})), \text{Cov}(O_{k_0})p = 0$, equivalently

$$\begin{pmatrix} \mathbb{E}\left[(O^1_{k_0} - \mathbb{E}O^1_{k_0}) \sum_{i=1}^n p_i(O^i_{k_0} - \mathbb{E}O^i_{k_0})\right] \\ \vdots \\ \mathbb{E}\left[(O^n_{k_0} - \mathbb{E}O^n_{k_0}) \sum_{i=1}^n p_i(O^i_{k_0} - \mathbb{E}O^i_{k_0})\right] \end{pmatrix} = 0. \quad (51)$$

From (51), it follows that:

$$q^\tau_{k_0}\mathbb{E}O_{k_0} = q^\tau_{k_0}(q^\tau_{k_0} + \hat{o}_{k_0}) = \|q^\tau_{k_0}\|^2 > 0.$$

Hence, $\sum_{i=1}^n p_i e^i_{k_0}$ is deterministic, whose variance is zero. Conversely, if $\sum_{i=1}^n p_i e^i_{k_0}$ is deterministic, so is $\sum_{i=1}^n p_i O^i_{k_0}$, and (51) is satisfied. Therefore, $p \in \text{Ker}(\text{Cov}(O_{k_0}))$ if and only if $\sum_{i=1}^n p_i e^i_{k_0}$ is deterministic.

Furthermore, as $\mathbb{E}O_{k_0} \notin \text{Ran}(\text{Cov}(O_{k_0}))$, there exist $0 \neq q_{k_0} = (q^1_{k_0}, \ldots, q^n_{k_0}) \in \text{Ker}(\text{Cov}(O_{k_0}))$ and $\hat{o}_{k_0} \in \text{Ran}(\text{Cov}(O_{k_0}))$ such that $\mathbb{E}O_{k_0} = q_{k_0} + \hat{o}_{k_0}$. It follows that:

$$\begin{cases} \varphi_k = -\frac{1}{2}s_k s_{k+1} \cdots s_{N-1}, \\ W_k = P_{k+1}\mathbb{E}(O_k O^\tau_k), \\ H_k = s_k \mathbb{E}O_k P_{k+1}, \\ \bar{W}_k = P_{k+1}\text{Cov}(O_k), \\ \bar{H}_k = 0, \\ \mu_k = -\frac{1}{2}\mathbb{E}O_k s_{k+1} \cdots s_{N-1}, \\ k \in \mathbb{N}. \end{cases}$$

As $q^\tau_{k_0}\mathbb{E}O_k = \sum_{i=1}^n q^i_{k_0} e^i_k - s_k \sum_{i=1}^n q^i_{k_0}$, we have

$$\sum_{i=1}^n q^i_{k_0} e^i_k > s_k \sum_{i=1}^n q^i_{k_0}. \quad (52)$$

Now, take $u^i_{k_0} = q^i_{k_0}\mu$ in (39) with μ being a control parameter at time k_0. Hence

$$x_{k_0+1} = s_{k_0}x_{k_0} + \mu\left(\sum_{i=1}^n q^i k_0 e^i_{k_0} - s_{k_0}\sum_{i=1}^n q^i_{k_0}\right).$$

Therefore

$$\mathbb{E}x_{k_0+1} = s_{k_0}\mathbb{E}x_{k_0} + \mu\left(\sum_{i=1}^n q^i k_0 e^i_{k_0} - s_{k_0}\sum_{i=1}^n q^i_{k_0}\right) \to +\infty$$

as $\mu \to +\infty$. Further investigations may show that Problem (MV) is not well-posed under the above condition that $\mathbb{E}O_{k_0} \notin \text{Ran}(\text{Cov}(O_{k_0}))$ for some k_0

From (43), we know that the optimal control is expressed in terms of the solutions of the GDREs (6) and LRE (8) with four degrees of freedom. That is to say that we have infinite optimal strategies in hands. This fact is different from the case where the return rates of the risky securities are nondegenerate [23]. The questions arise naturally: Is the efficient frontier, i.e., $\text{Var}(x_N)$ under (43), unique? When efficient frontier is not unique, how to get the minimal one? Fortunately, we can show in the following that the efficient frontier is unique.

Theorem 5.2: Under the optimal strategy (43), the optimal expected wealth and efficient frontier are, respectively,

$$\mathbb{E}x_T = \frac{1}{2}\sum_{k=0}^{N-1}\left[s^2_{k+1}\cdots s^2_{N-1}P^{-1}_{k+1}\mathbb{E}O^\tau_k\left(\text{Cov}(O_k)\right)^\dagger\mathbb{E}O_k\right] + s_0\cdots s_{N-1}\mathbb{E}x_0,$$

$$\begin{aligned}\text{Var}(x_N) &= \frac{1}{4\lambda}\sum_{k=0}^{N-1}\left[s^2_{k+1}\cdots s^2_{N-1}P^{-1}_{k+1}\mathbb{E}O^\tau_k\left(\text{Cov}(O_k)\right)^\dagger\mathbb{E}O_k\right] \\ &\quad + \frac{1}{\lambda}P_0\mathbb{E}\left[(x_0 - \mathbb{E}x_0)^2\right].\end{aligned}$$

Proof: Similarly to the proof of Theorem 4.1, we have for any u

$$\begin{aligned}\lambda\text{Var}(x_N) &= \sum_{k=0}^{N-1}\mathbb{E}\Big[\Big(u_k - \mathbb{E}u_k + W^\dagger_k H_k(x_k - \mathbb{E}x_k)\Big)^\tau W_k \\ &\quad \cdot\Big(u_k - \mathbb{E}u_k + W^\dagger_k H_k(x_k - \mathbb{E}x_k)\Big) \\ &\quad + \Big(\mathbb{E}u_k + \bar{W}^\dagger_k\bar{H}_k\mathbb{E}x_k\Big)^\tau\bar{W}_k\Big(\mathbb{E}u_k + \bar{W}^\dagger_k\bar{H}_k\mathbb{E}x_k\Big)\Big] \\ &\quad + \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^\tau P_0(x_0 - \mathbb{E}x_0)\right] + (\mathbb{E}x_0)^\tau T_0\mathbb{E}x_0 \\ &= \sum_{k=0}^{N-1}\mathbb{E}\Big[\Big(u_k - \mathbb{E}u_k + W^\dagger_k H_k(x_k - \mathbb{E}x_k)\Big)^\tau W_k \\ &\quad \cdot\Big(u_k - \mathbb{E}u_k + W^\dagger_k H_k(x_k - \mathbb{E}x_k)\Big) \\ &\quad + (\mathbb{E}u_k)^\tau\bar{W}_k\mathbb{E}u_k\Big] + \mathbb{E}(x_0 - \mathbb{E}x_0)^\tau P_0(x_0 - \mathbb{E}x_0),\end{aligned}$$

where the last equality is from $\bar{H}_k = 0, T_k = 0$ for all $k \in \bar{\mathbb{N}}$. Therefore, under the optimal strategy (43) (parameterized by $y_k, \bar{Y}_k, z_k, Z_k$)

$$\lambda\text{Var}(x_N) = \sum_{k=0}^{N-1}\mu^\tau_k\bar{W}^\dagger_k\mu_k + \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^\tau P_0(x_0 - \mathbb{E}x_0)\right].$$

Thus, we have

$$\text{Var}(x_N) = \frac{1}{\lambda}\left[\sum_{k=0}^{N-1}\mu^\tau_k\bar{W}^\dagger_k\mu_k + \mathbb{E}\left[(x_0 - \mathbb{E}x_0)^\tau P_0(x_0 - \mathbb{E}x_0)\right]\right].$$

By (45), under the optimal control (43), the optimal expected wealth is

$$\begin{aligned}\mathbb{E}x_N &= \lambda\text{Var}(x_N) - V(x_0) \\ &= \sum_{k=0}^{N-1}\mu^\tau_k\bar{W}^\dagger_k\mu_k - \left[-\sum_{k=0}^{N-1}\mu^\tau_k\bar{W}^\dagger_k\mu_k + 2\varphi^\tau_0\mathbb{E}x_0\right] \\ &= 2\sum_{k=0}^{N-1}\mu^\tau_k\bar{W}^\dagger_k\mu_k - 2\varphi_0\mathbb{E}x_0.\end{aligned}$$

This completes the proof by substituting μ_k, φ_0 by $-(1/2)s_{k+1}\cdots s_{N-1}\mathbb{E}O_k$ and $-(1/2)s_0\cdots s_{N-1}$, respectively. □

Remark 5.2: Different from the method of direct computation of $\mathbb{E}x_N$ and $\text{Var}(x_N)$ in [11], here, we derive the optimal expected wealth and efficient frontier by making use of the value function. It is also worth mentioning that the expected wealth and efficient frontier are unique though we have infinite optimal strategies ((43) has four degrees of freedom).

We now consider the case where $\mathbb{E}(O_k O^\tau_k), \text{Cov}(O_k), k \in \mathbb{N}$ are positive definite [11], [23]. In this case, the conditions of Theorem 5.1 are satisfied, and the optimal strategy is unique.

TABLE I
DATA FOR THE ASSET ALLOCATION EXAMPLE

	Asset 1	Asset 2	Asset 3	Asset 4
Expected Return	11.84%	21.65%	15.56%	18.75%
Standard Deviation	18%	20%	24%	20%
		Correlation		
Asset 1	1	0.250	0	0.625
Asset 2	0.250	1	0.250	0.750
Asset 3	0	0.250	1	0.625
Asset 4	0.625	0.750	0.625	1

Since $\mathrm{Cov}(O_k)=\mathbb{E}(O_kO_k^\tau)-\mathbb{E}O_k(\mathbb{E}O_k)^\tau$, by the Sherman-Morrison formular (a particular form of Lemma 5.1), we have

$$\begin{aligned}(\mathrm{Cov}(O_k))^{-1}\,\mathbb{E}O_k &= \left[\mathbb{E}(O_kO_k^\tau)-\mathbb{E}O_k(\mathbb{E}O_k)^\tau\right]^{-1}\mathbb{E}O_k\\ &=\frac{[\mathbb{E}(O_kO_k^\tau)]^{-1}}{1-\mathbb{E}O_k^\tau\,[\mathbb{E}(O_kO_k^\tau)]^{-1}\,\mathbb{E}O_k},\end{aligned}$$

which is Lemma 2 of [11]. Therefore, the optimal strategy is given by

$$\begin{aligned}u_k=&-s_k\,[\mathbb{E}(O_kO_k^\tau)]^{-1}\,\mathbb{E}O_k(x_k-\mathbb{E}x_k)\\ &+\frac{1}{2}s_{k+1}\cdots s_{N-1}P_{k+1}^{-1}\frac{[\mathbb{E}(O_kO_k^\tau)]^{-1}\,\mathbb{E}O_k}{1-\mathbb{E}O_k^\tau\,[\mathbb{E}(O_kO_k^\tau)]^{-1}\,\mathbb{E}O_k},\end{aligned}\quad (53)$$

where

$$\begin{aligned}P_k&=s_k^2P_{k+1}\left[1-\mathbb{E}O_k^\tau\,[\mathbb{E}(O_kO_k^\tau)]^{-1}\,\mathbb{E}O_k\right]\\ &=\lambda\prod_{i=k}^{N-1}s_i^2\left[1-\mathbb{E}O_i^\tau\,[\mathbb{E}(O_iO_i^\tau)]^{-1}\,\mathbb{E}O_i\right].\end{aligned}$$

Clearly, (53) coincides with result in [11], [23]. This subsection is new in two folds. First, it extends the results of [23]. Secondly, it generalizes the results of [8], [33] to the multi-period case.

VI. EXAMPLES

Consider an example of constructing a pension fund consisting of four risky assets and a bank account for 4-period. The expected values, variances and correlations of the annual return rates of these assets are given in Table I.

We also assume that the annual risk free rate is 5% ($s_k=1.05$). Then, $\mathbb{E}(O_k)$, $\mathrm{Cov}(O_k)$ and $\mathbb{E}(O_kO_k^\tau)$ can be computed as follows, for $k=0,1,2,3$:

$$\mathbb{E}(O_k)=\begin{bmatrix}0.0684\\0.1665\\0.1056\\0.1375\end{bmatrix},$$

$$\mathrm{Cov}(O_k)=\begin{bmatrix}0.0324&0.0135&0&0.0225\\0.0135&0.0900&0.0180&0.0450\\0&0.0180&0.0576&0.0300\\0.0225&0.0450&0.0300&0.0400\end{bmatrix},$$

$$\mathbb{E}\left(O_kO_k^\tau\right)=\begin{bmatrix}0.03707856&0.02488860&0.00722304&0.03190500\\0.02488860&0.11772225&0.03558240&0.06789375\\0.00722304&0.03558240&0.06875136&0.04452000\\0.03190500&0.06789375&0.04452000&0.05890625\end{bmatrix},$$

where the ranks of $\mathrm{Cov}(O_k)$ and $\mathbb{E}(O_kO_k^\tau)$ equal to 3, i.e., $\mathrm{Cov}(O_k)\geq 0$ and $\mathbb{E}(O_kO_k^\tau)\geq 0$. For $k=0,1,2,3$, we compute $1-\mathbb{E}O_k^\tau[\mathbb{E}(O_kO_k^\tau)]^\dagger\mathbb{E}O_k=0.67658999$. Here, we use the same values of $\mathbb{E}O_k$ and $\mathrm{Cov}(O_k)$, $k=1,2,3,4$, just for simplicity.

According to Theorem 5.1, since $\mathbb{E}O_k\in\mathrm{Ran}(\mathrm{Cov}(O_k))$ for $k=0,1,2,3$ holds, then the optimal strategy can be expressed as

$$\begin{aligned}u_k=&-\left[W_k^\dagger H_k+(I-W_k^\dagger W_kZ_k\right](x_k-\mathbb{E}x_k)\\ &+z_k-W_k^\dagger W_kz_k-\left[\bar{W}_k^\dagger+\left(I-\bar{W}_k^\dagger\bar{W}_k\right)\bar{Y}_k\right]\mu_k\\ &+\left(I-\bar{W}_k^\dagger\bar{W}_k\right)y_k\end{aligned}$$

with $Y_k\in\mathbb{R}^{n\times n}, y_k\in\mathbb{R}^n, Z_k\in\mathbb{R}^{n\times n}, z_k\in\mathbb{R}^n$ being parameters in Theorem 5.1, and

$$W_k^\dagger H_k=\begin{bmatrix}0.6347\\0.6650\\0.6537\\0.8467\end{bmatrix},$$

$$W_k^\dagger W_k=\begin{bmatrix}0.8063&-0.1162&-0.1453&0.3487\\-0.1162&0.9303&-0.0872&0.2092\\-0.1453&-0.0872&0.8910&0.2615\\0.3487&0.2092&0.2615&0.3724\end{bmatrix},$$

$$\bar{W}_k^\dagger\bar{W}_k=\begin{bmatrix}0.8063&-0.1162&-0.1453&0.3487\\-0.1162&0.9303&-0.0872&0.2092\\-0.1453&-0.0872&0.8910&0.2615\\0.3487&0.2092&0.2615&0.3724\end{bmatrix},$$

for $k=0,1,2,3$

$$\bar{W}_0^\dagger=\frac{1}{\lambda}\begin{bmatrix}72.7778&-25.4142&-13.8458&26.1916\\-25.4142&43.2747&-15.2590&-6.0520\\-13.8458&-15.2590&50.4416&8.2388\\26.1916&-6.0520&8.2388&15.9664\end{bmatrix},$$

$$\mu_0=\begin{bmatrix}-0.0416\\-0.1012\\-0.0642\\-0.0836\end{bmatrix},$$

$$\bar{W}_1^\dagger=\frac{1}{\lambda}\begin{bmatrix}54.2879&-18.9575&-10.3282&19.5374\\-18.9575&32.2804&-11.3823&-4.5145\\-10.3282&-11.3823&37.6264&6.1457\\19.5374&-4.5145&6.1457&11.9100\end{bmatrix},$$

$$\mu_1=\begin{bmatrix}-0.0396\\-0.0964\\-0.0611\\-0.0796\end{bmatrix},$$

$$\bar{W}_2^\dagger=\frac{1}{\lambda}\begin{bmatrix}40.4955&-14.1412&-7.7042&14.5737\\-14.1412&24.0792&-8.4905&-3.3675\\-7.7042&-8.4905&28.0671&4.5843\\14.5737&-3.3675&4.5843&8.8841\end{bmatrix},$$

$$\mu_2=\begin{bmatrix}-0.0377\\-0.0918\\-0.0582\\-0.0758\end{bmatrix},$$

$$\bar{W}_3^\dagger=\frac{1}{\lambda}\begin{bmatrix}30.2073&-10.5485&-5.7469&10.8711\\-10.5485&17.9617&-6.3334&-2.5120\\-5.7469&-6.3334&20.9364&3.4196\\10.8711&-2.5120&3.4196&6.6270\end{bmatrix},$$

$$\mu_3=\begin{bmatrix}-0.0359\\-0.0874\\-0.0554\\-0.0722\end{bmatrix},$$

where λ is given in (41).

The optimal expected wealth level is then given by

$$\mathbb{E}(x_4) = \frac{2.7875}{\lambda} + 1.1025\mathbb{E}x_0,$$

and the variances of the optimal wealth levels are given as

$$\mathrm{Var}(x_4) = \frac{1.3937}{\lambda} + \frac{0.3096}{\lambda}\mathbb{E}\left[(x_0 - \mathbb{E}x_0)^2\right].$$

VII. Conclusion

In this paper, we studied the indefinite mean-field LQ problem, for which a set of the necessary and sufficient conditions are given. As application, the multi-period mean-variance portfolio selection issues were investigated. For future research, we may study the indefinite mean-field LQ problems with positive controls, which can be applied to the mean-variance one under no-shorting constraint. Another interesting question is the (definite, or indefinite) mean-field LQ problem with random coefficients. For this, we may begin with the case with regime switching (or Markov jump parameters).

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