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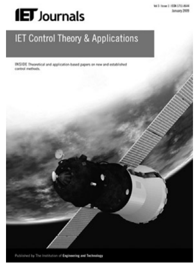

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Brief Paper

Continuous-time multi-agent averaging with relative-state-dependent measurement noises: matrix intensity functions

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Abstract: In this study, the distributed averaging of high-dimensional first-order agents is investigated with relative-state-dependent measurement noises. Each agent can measure or receive its neighbours' state information with random noises, whose intensity is a non-linear matrix function of agents' relative states. By the tools of stochastic differential equations and algebraic graph theory, the authors give sufficient conditions to ensure mean square and almost sure average consensus and the convergence rate and the steady-state error for average consensus are quantified. Especially, if the noise intensity function depends linearly on the relative distance of agents' states, then a sufficient condition is given in terms of the control gain, the noise intensity coefficient constant, the number of agents and the dimension of agents' dynamics.

1 Introduction

In recent years, the distributed coordination of multi-agent systems with environmental uncertainties has been a hot topic of the systems and control community [1–16]. There are various kinds of uncertainties in multi-agent networks, such as the communication and measurement noises involved by the information exchange between adjacent agents, the communication delay, the encoding–decoding error, the packet dropouts for digital communications and so on. Fruitful results have been obtained for distributed consensus with random noises [1–4]. For continuous-time stochastic approximation-type protocols, Li and Zhang [3] gave the necessary and sufficient conditions on the control gains to ensure mean square average consensus. More extended results of continuous-time stochastic approximation-type multi-agent consensus can be found in [14] for the case with time delays, in [15] for the case of leader following and in [16, 17] for the second-order and general linear dynamics with local state feedback.

Most of the above literature assume that the intensity of noises is time invariant and independent of agents' states. However, this assumption does not always hold for some important measurement or communication schemes. For consensus with logarithmic quantised measurements, the uncertainties introduced by the quantisation can be modelled by relative-state-dependent white noises in a stochastic framework [8]. For the case with analogue Gaussian fading measurements, the uncertainties of the measurement are also relative-state-dependent noises [6, 18]. In [19], we considered the distributed averaging corrupted by relative-state-dependent measurement noises. Each agent can measure or receive its neighbours' state information with relative-state-dependent measurement random noises. By investigating the structure of this interaction and the tools of stochastic differential equations, we developed several small consensus gain theorems to give sufficient conditions in terms of the control gain, the number of agents and the noise intensity function to ensure mean square and almost sure consensus. Especially, for the case with homogeneous communication and control channels, a necessary and sufficient condition to ensure mean square consensus on the control gain is given and it is shown that the control gain is independent of the specific network topology, but only depends on the number of nodes and the noise coefficient constant. For the measurement model of [19], the measurement noises of different state components are the same and the noise intensity is a vector function. This model cannot cover the case where the measurement noises for different state components are mutually different and coupled together. For this case, the noise intensity function is more suitable to be modelled as a matrix function of the relative state, but not a vector. For some communication models, the relative distances of nodes are important factors of the statistical properties of channels [20, 21]. For this case, if locations are parts of agents' states, then the relative distance can be viewed as a function of the relative state. If the relative-distance-dependent noises have different intensities for different state components, then the model of the measurement noises comes down to the case with non-linear matrix functions of the relative state.

In this paper, we consider the distributed averaging of high-dimensional first-order agents with relative-state-dependent measurement noises. There are N agents in the network. The dynamics of agents are described by n-dimensional first-order integrators, that is, there are n state components and n control channels for each agent. The information interaction of agents is described by an undirected graph. Each agent can measure or receive its neighbours' state information with random noises. Different from our previous work for the case with vector noise intensity functions [19], here, the noise intensity is a non-linear matrix-valued function of relative states of agents. By the tools of stochastic differential equations and algebraic graph theory, we give sufficient conditions to ensure mean square and almost sure average consensus. The convergence rate and the steady-state error for average consensus are quantified. Especially, if the noise intensity function depends linearly on the relative distance with intensity coefficient constant σ, then a positive control gain k which satisfies $[nk\sigma^2(N-1)/N] < 1$ can ensure asymptotically unbiased mean square and almost sure average consensus.

The remainder of this paper is organised as follows. In Section 2, we formulate the models of agents, the measurement, the network and the problem to be investigated. In Section 3, we give sufficient conditions on the control and network parameters to ensure mean square and almost sure average consensus, and the case with special noise intensity functions and the case with two agents are investigated, respectively. In Section 4, we give some concluding remarks.

Throughout this paper, we use the following notations. $\mathbb{R}_+$ denotes the set of non-negative real numbers. $\mathbf{1}$ denotes a column vector with all ones. J_N denotes the matrix $\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. I_N denotes the N-dimensional identity matrix. O_N denotes the N-dimensional zero matrix. For a given matrix or vector A, its transpose is denoted by A^{T}, its trace is denoted by $\mathbf{tr}(A)$, its trace norm $\sqrt{\mathbf{tr}(A^{\mathrm{T}}A)}$ is denoted by $\|A\|_F$ and its Euclidean norm is denoted by $\|A\|$. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. For a given random variable or vector X, the mathematical expectation of X is denoted by $\mathbb{E}[X]$.

2 Problem formulation

2.1 Dynamic network model

In this paper, we consider the distributed averaging for a network of agents with the dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$. Here, each agent has n control channels, and each component of $x_i(t)$ is controlled by a control channel. Denote $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $u(t) = [u_1^{\mathrm{T}}(t), \ldots, u_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. The information flow structure among different agents is modelled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes with i representing the ith agent, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly. The dynamic systems together with the information flow graph is called a dynamic network, and is denoted by $(\mathcal{G}, x)$ [22]. The Laplacian matrix of $\mathcal{G}$ is denoted by $\mathcal{L}$. The ith agent can receive information from its neighbours with random perturbation in the form

$$y_{ji}(t) = x_j(t) + f_{ji}(x_j(t) - x_i(t))\xi_{ji}(t), \quad j \in N_i \tag{2}$$

where $N_i = \{j \in \mathcal{V} \mid a_{ij} = 1\}$ denotes the set of neighbours of agent i, $y_{ji}(t)$ denotes the measurement of the agent j's state $x_j(t)$ by agent i, and $\xi_{ji}(t) \in \mathbb{R}^n$ denotes the measurement noise. The noise intensity function $f_{ji}(\cdot)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}$ which satisfies $f_{ji}(0) = O_n$. The model (2) can also be regarded as a measurement model for the relative state, which can be written as

$$z_{ji}(t) = x_j(t) - x_i(t) + f_{ji}(x_j(t) - x_i(t))\xi_{ji}(t), \quad j \in N_i \tag{3}$$

where $z_{ji}(t)$ is the agent i's measurement of the relative state $x_j(t) - x_i(t)$ with measurement noises. For the measurement/communication model (2) and (3), we will use the following assumptions.

Assumption 2.1: The noise processes $\{\xi_{ji}(t), i, j = 1, 2, \ldots, N\}$ are independent n-dimensional Gaussian white noises, that is, $\int_0^t \xi_{ji}(s)\,\mathrm{d}s = w_{ji}(t)$, $t \geq 0$, where $\{w_{ji}(t), \ i, j = 1, 2, \ldots, N\}$ are independent n-dimensional Brownian motions.

Assumption 2.2: There exists a positive constant $\gamma_{\max}$ such that $\|f_{ji}(x)\|_F \leq \gamma_{\max}\|x\|$, $\forall x \in \mathbb{R}^n$, $i \neq j$, $i, j = 1, 2, \ldots, N$.

Remark 1: In distributed averaging with precise communication, it is always assumed that the states and control inputs of agents are scalars. This assumption will not loose any generality for the case with precise communication and with non-state-dependent measurement noises, since the state components of agents are decoupled. However, for the case with relative-state-dependent measurement noises, from model (2), one can see that the noise intensity of different state components will be generally coupled together. This leads to an essential difference between scalar and high-dimensional models for the case with relative-state-dependent measurement noises. The cross interferences of communication channels of different state components on consensus conditions will be investigated in Section 3.

Remark 2: Here, the measurement model is different from that of [19], where for a given link (j, i), the measurement noises of different state components are the same one-dimensional Brownian motion $\xi_{ji}(t)$. Here, for a given link (j, i), the measurement noises of different state components are independent Brownian motions, which form an n-dimensional Brownian motion $\xi_{ji}(t)$ and coupled together by the matrix function $f_{ji}(\cdot)$. Note that Assumptions 2.1 and 2.2 do not come down to a special case of Assumptions 2.1 and 2.2 of [19] and vice versa.

Remark 3: The distributed averaging of first-order integrator multi-agent systems can be viewed as a kind of information fusion algorithm. Here, we consider the case with

n-dimensional state components, which means that the information state to be exchanged is a vector, not a scalar. If we construct the averaging algorithm for each state component and the communication channels of different state components are independent of each other, then the closed-loop system degenerates to a special case considered in [19] (Theorem 4.2 of [19]). However, for the real communication environment, the communication channels of different state components may not be independent. This is a reason why we study high-dimensional distributed averaging algorithms. Particularly, for the measurement model (2), the communication channels of different state components are coupled together and the noises of different state components are not the same one, which cannot be covered by Li *et al.* [19].

2.2 Consensus protocol

We call the group of controls $u = \{u_i, i = 1, 2, \ldots, N\}$ a measurement-based distributed protocol, if $u_i(t) \in \sigma(x_i(s), y_{ji}(s), 0 \le s \le t, j \in N_i)$, $t \ge 0$, $i = 1, 2, \ldots, N$ [3].

Definition 1 [3]: A distributed protocol u is called a mean square (or almost sure) consensus protocol if it renders that the system (1) and (2) have the following properties: for any given $x(0) \in \mathbb{R}^{Nn}$, there is a random vector $x^* \in \mathbb{R}^n$, such that $\lim_{t\to\infty} E[\|x_i(t) - x^*\|^2] = 0$, $i = 1, 2, \ldots, N$ (or $\lim_{t\to\infty} x_i(t) = x^*$ a.s. $i = 1, 2, \ldots, N$). Particularly, if $\mathbb{E}(x^*) = (1/N)\sum_{j=1}^N x_j(0)$, $\mathbb{E}[\|x^*\|^2] < \infty$, then u is called an asymptotically unbiased mean square (or almost sure) average-consensus protocol, and $\mathbb{E}[\|x^* - \frac{1}{N}\sum_{j=1}^N x_j(0)\|^2]$ is called the mean square steady-state error.

Note that the asymptotically unbiased mean square (or almost sure) average-consensus implies that strong consensus can be achieved: the states of agents will converge to a common value in mean square (or almost surely). This is essentially different from 'practical consensus' or 'approximate consensus', which means that the final states of agents may not converge to the same value.

In this paper, we consider the following distributed protocol given by

$$u_i(t) = K\sum_{j=1}^N a_{ij}(y_{ji}(t) - x_i(t)), \quad t \ge 0, i = 1, 2, \ldots, N \quad (4)$$

where $K \in \mathbb{R}^{n\times n}$ is the control gain matrix to be designed.

In the following, we will find the sufficient conditions on the control gain matrix, the parameters of the network structure and noise processes to ensure the asymptotically unbiased mean square (or almost sure) average consensus.

3 Mean square and almost sure consensus

Denote $\delta(t) = [(I_N - J_N) \otimes I_n]x(t)$. Let $\delta(t) = [\delta_1^{\mathrm{T}}(t), \ldots, \delta_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, where $\delta_i(t) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$. Define the unitary matrix $T_{\mathcal{L}} = [(\mathbf{1}/\sqrt{N}), \phi_2, \ldots, \phi_N]$, where ϕ_i is the unit eigenvector of $\mathcal{L}$ associated with $\lambda_i(\mathcal{L})$, that is, $\phi_i^{\mathrm{T}}\mathcal{L} = \lambda_i(\mathcal{L})\phi_i^{\mathrm{T}}$, $\|\phi_i\| = 1$, $i = 2, \ldots, N$. Denote $\phi = [\phi_2, \ldots, \phi_N]$. Denote $\widetilde{\delta}(t) = (T_{\mathcal{L}}^{-1} \otimes I_n)\delta(t)$ and Let $\widetilde{\delta}(t) = [\widetilde{\delta}_1^{\mathrm{T}}(t), \ldots, \widetilde{\delta}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, then it can be verified that $\widetilde{\delta}_1(t) \equiv 0$. Denote $\overline{\delta}(t) = [\widetilde{\delta}_2^{\mathrm{T}}(t), \ldots, \widetilde{\delta}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, which is an $(N-1)n$ dimensional column vector. Denote $\Lambda_{\mathcal{L}}^0 = \mathrm{diag}(\lambda_2(\mathcal{L}), \lambda_3(\mathcal{L}), \ldots, \lambda_N(\mathcal{L}))$.

From (1) and (2), we have

$$\mathrm{d}x(t) = -(\mathcal{L} \otimes K)x(t) + F_K(t)\mathrm{d}w(t)$$

where $w(t) = [w_1^{\mathrm{T}}(t), w_2^{\mathrm{T}}(t), \ldots, w_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{N^2 n}$, $w_i = [w_{1i}^{\mathrm{T}}, w_{2i}^{\mathrm{T}}, \ldots, w_{Ni}^{\mathrm{T}}]^{\mathrm{T}}$, $F_K(t) = \mathrm{diag}(F_{1,K}(t), \ldots, F_{N,K}(t))$ and $F_{i,K}(t) = [a_{i1}Kf_{1i}(x_1(t) - x_i(t)), a_{i2}Kf_{2i}(x_2(t) - x_i(t)), \ldots, a_{iN}Kf_{Ni}(x_N(t) - x_i(t))] \in \mathbb{R}^{n\times Nn}$. According to the definition of $\delta(t)$, it follows that

$$\mathrm{d}\delta(t) = -(\mathcal{L} \otimes K)\delta(t)\mathrm{d}t + ((I_N - J_N) \otimes I_n))F_K(t)\mathrm{d}w(t),$$

where we use the same $F_K(t) = \mathrm{diag}(F_{1,K}(t), \ldots F_{N,K}(t))$, where $F_{i,K}(t) = [a_{i1}Kf_{1i}(\delta_1(t) - \delta_i(t)), a_{i2}Kf_{2i}(\delta_2(t) - \delta_i(t)), \ldots, a_{iN}Kf_{Ni}(\delta_N(t) - \delta_i(t))]$ since $f_{ji}(\delta_j(t) - \delta_i(t)) = f_{ji}(x_j(t) - x_i(t))$. From the definition of $\overline{\delta}(t)$, we obtain

$$\mathrm{d}\overline{\delta}(t) = -(\Lambda_{\mathcal{L}}^0 \otimes K)\overline{\delta}(t)\mathrm{d}t + (\phi^{\mathrm{T}}(I_N - J_N) \otimes I_n)F_K(t)\,\mathrm{d}w(t). \quad (5)$$

Theorem 3.1: Suppose that Assumptions 2.1 and 2.2 hold. Apply the protocol (4) to the system (1) and (2). If

$$\frac{K^{\mathrm{T}} + K}{2} - \frac{(N-1)\gamma_{\max}^2\|K\|^2}{N}I_n$$

is positive definite, then the protocol (4) is an asymptotically mean square and almost sure average consensus protocol. Precisely, the closed-loop system under (4) satisfies: for any given $x(0) \in \mathbb{R}^{Nn}$, there is a random vector $x^* \in \mathbb{R}^n$ with $\mathbb{E}(x^*) = (1/N)\sum_{j=1}^N x_j(0)$, such that

$$\lim_{t\to\infty} \mathbb{E}[\|x_i(t) - x^*\|^2] = 0, \quad i = 1, 2, \ldots, N \quad (6)$$

$$\lim_{t\to\infty} x_i(t) = x^*, \quad \text{a.s. } i = 1, 2, \ldots, N \quad (7)$$

and the steady-state error is given by

$$\mathbb{E}\left[\left\|x^* - \frac{1}{N}\sum_{j=1}^N x_j(0)\right\|^2\right] \le \frac{\|K\|^2\gamma_{\max}^2\lambda_N(\mathcal{L})\|\delta(0)\|^2}{N^2\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^f(K))} \quad (8)$$

Moreover, the convergence rates of $E\|\delta(t)\|^2$ and $\delta(t)$ are given by

$$\mathbb{E}[\|\delta(t)\|^2] \le \exp\{-2\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^f(K))t\}\|\delta(0)\|^2 \quad (9)$$

and

$$\limsup_{t\to\infty} \frac{\log\|\delta(t)\|}{t} \le -\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^f(K)) \quad \text{a.s.} \quad (10)$$

Proof: By Lemma A.1 of [19], applying the Itô formula to the system (5) yields

$$\begin{aligned}\mathrm{d}\|\overline{\delta}(t)\|^2 = {} & [-2\overline{\delta}^{\mathrm{T}}(t)(\Lambda_{\mathcal{L}}^0 \otimes K)\overline{\delta}(t) \\ & + \mathbf{tr}(F_K^{\mathrm{T}}(t)[(I_N - J_N) \otimes I_n]F_K(t))]\,\mathrm{d}t + \mathrm{d}M(t)\end{aligned} \quad (11)$$

where $\mathrm{d}M(t) = 2\overline{\delta}^{\mathrm{T}}(t)[\phi^{\mathrm{T}}(I_N - J_N) \otimes I_n]F_K(t)\mathrm{d}w(t)$. By the definitions of J_N and $F_K(t)$ and Assumption 2.2, noting that

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$a_{ij}^2 = a_{ij}$, we obtain

$$\begin{aligned}
&\mathbf{tr}(F_K^{\mathrm{T}}(t)[(I_N - J_N) \otimes I_n]F_K(t)) \\
&= \frac{N-1}{N} \sum_{i=1}^{N} \mathbf{tr}(F_{i,K}^{\mathrm{T}}(t)F_{i,K}(t)) \\
&= \frac{N-1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \\
&\quad \times \mathbf{tr}(f_{ji}^{\mathrm{T}}(\delta_j(t) - \delta_i(t))K^{\mathrm{T}}Kf_{ji}(\delta_j(t) - \delta_i(t))) \\
&\le \frac{(N-1)\gamma_{\max}^2 \|K\|^2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \|\delta_j(t) - \delta_i(t)\|^2 \qquad (12)
\end{aligned}$$

This together with (11) gives (see equation at the bottom of the page)

which together with

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \|\delta_j(t) - \delta_i(t)\|^2 = 2\delta^{\mathrm{T}}(t)(\mathcal{L} \otimes I_n)\delta(t)$$

$$= 2\bar{\delta}^{\mathrm{T}}(t)[(\phi^{\mathrm{T}}\mathcal{L}\phi) \otimes I_n]\bar{\delta}(t) \qquad (13)$$

leads to (see equation at the bottom of the page)

where

$$\overline{\Psi}_{\mathcal{L}}^{f}(K) = \Lambda_{\mathcal{L}}^{0} \otimes \left(\frac{K^{\mathrm{T}} + K}{2}\right) - \frac{(N-1)\gamma_{\max}^2 \|K\|^2}{N} (\Lambda_{\mathcal{L}}^{0} \otimes I_n)$$

This together with the comparison theorem [23] and the definition of $\bar{\delta}(t)$ leads to (9).

By Assumption 2.2, it is obvious that the system (5) satisfying the linear growth condition, namely, there exist positive constants α_1 and α_2 such that $\|-(\Lambda_{\mathcal{L}}^{0} \otimes K)\bar{\delta}(t)\| \le \alpha_1 \|\bar{\delta}(t)\|$ and $\|(\phi^{\mathrm{T}}(I_N - J_N) \otimes I_n)F_K(t)\| \le \alpha_2 \|\bar{\delta}(t)\|$. By Mao [24, Theorem 4.2, p. 128], the mean square exponential stability implies the almost sure exponential stability for the system (5) with the form (10).

By the properties of the matrix $\mathcal{L}$, from (5), we have

$$\frac{1}{N}(\mathbf{1}^{\mathrm{T}} \otimes I_n)x(t) = \frac{1}{N}(\mathbf{1}^{\mathrm{T}} \otimes I_n)x(0) + \frac{1}{N} M_K(t) \qquad (14)$$

where

$$M_K(t) = \int_0^t (\mathbf{1}^{\mathrm{T}} \otimes I_n)F_K(s)\,\mathrm{d}w(s)$$

For any $t \ge 0$, by Assumption 2.2 and the Martingale isometry, noting that $a_{ij} = 0$ or 1, we obtain

$$\begin{aligned}
\mathbb{E}\|M_K(t)\|^2 &= \sum_{i=1}^{N} \mathbb{E}\left[\left\| \int_0^t F_{i,K}(s)\,\mathrm{d}w_i(s) \right\|^2\right] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}\left[\left\| \int_0^t a_{ij}Kf_{ji}(x_j(s) - x_i(s))\,\mathrm{d}w_{ji}(s) \right\|^2\right] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^t a_{ij}^2 \mathbb{E}\|Kf_{ji}(x_j(s) - x_i(s))\|_F^2\,\mathrm{d}s \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \int_0^t \mathbb{E}\|Kf_{ji}(\delta_j(s) - \delta_i(s))\|_F^2\,\mathrm{d}s \\
&\le \|K\|^2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \int_0^t \mathbb{E}\|f_{ji}(\delta_j(s) - \delta_i(s))\|^2\,\mathrm{d}t \\
&\le \|K\|^2 \gamma_{\max}^2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \int_0^t \mathbb{E}\|\delta_j(s) - \delta_i(s)\|^2 \mathrm{d}s
\end{aligned}$$

Then by the definition of $\bar{\delta}(t)$, it follows from (9) and (13) that

$$\begin{aligned}
\mathbb{E}\|M_K(t)\|^2 &\le 2\|K\|^2 \gamma_{\max}^2 \lambda_N(\mathcal{L}_{\Gamma}) \int_0^t \mathbb{E}\|\delta(s)\|^2\,\mathrm{d}s \\
&= \frac{2\|K\|^2 \gamma_{\max}^2 \lambda_N(\mathcal{L}_{\Gamma}) \|\delta(0)\|^2}{\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^{f}(K))} \\
&\quad \times [1 - \exp(-2\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^{f}(K)))t] \qquad (15)
\end{aligned}$$

By

$$\frac{K^{\mathrm{T}} + K}{2} - \frac{(N-1)\gamma_{\max}^2 \|K\|^2}{N} I_n$$

we know that $\overline{\Psi}_{\mathcal{L}}^{f}(K)$ is positive definite, which together with (15) implies that $M(t)$ is a square integrable continuous martingale. By the definition of the Itô integral, as $t \to \infty$, $M(t)$ converges to a random variable with finite second-order moment both in mean square and almost surely. Denote the limit of the right side of (14) by

$$x^* = \frac{1}{N}(\mathbf{1}^{\mathrm{T}} \otimes I_n)x(0) + \frac{1}{N} \lim_{t \to \infty} M_K(t) \qquad (16)$$

Then from (9) and (10), we have (6) and (7). Since $M(t)$ is a square-integrable continuous martingale, (16) implies that $\mathbb{E}(x^*) = (1/N) \sum_{j=1}^{N} x_j(0)$. Letting $t \to \infty$, (15) gives (8). □

$$\mathrm{d}\|\bar{\delta}(t)\|^2 \le \left[-2\bar{\delta}^{\mathrm{T}}(t)(\Lambda_{\mathcal{L}}^{0} \otimes K)\bar{\delta}(t) + \frac{(N-1)\gamma_{\max}^2 \|K\|^2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \|\delta_j(t) - \delta_i(t)\|^2 \right] \mathrm{d}t + \mathrm{d}M(t)$$

$$\begin{aligned}
\mathrm{d}\|\bar{\delta}(t)\|^2 &\le -\bar{\delta}^{\mathrm{T}}(t) \left[\Lambda_{\mathcal{L}}^{0} \otimes (K^{\mathrm{T}} + K) - \frac{2(N-1)\gamma_{\max}^2 \|K\|^2}{N} [(\phi^{\mathrm{T}}\mathcal{L}\phi) \otimes I_n] \right] \bar{\delta}(t)\,\mathrm{d}t + \mathrm{d}M(t) \\
&= -2\bar{\delta}^{\mathrm{T}}(t)\overline{\Psi}_{\mathcal{L}}^{f}(K)\bar{\delta}(t)\,\mathrm{d}t + \mathrm{d}M(t) \\
&\le -2\lambda_{\min}(\overline{\Psi}_{\mathcal{L}}^{f}(K))\|\bar{\delta}(t)\|^2\,\mathrm{d}t + \mathrm{d}M(t)
\end{aligned}$$

Remark 4: Theorem 3.1 tells us that for achieving asymptotically mean square and almost sure average consensus, we may select the control gain K such that

$$\frac{K^{\mathrm{T}}+K}{2}-\frac{(N-1)\gamma_{\max}^2\|K\|^2}{N}I_n$$

is positive definite. It can be verified that if $K=kI_n$, $k\in\mathbb{R}$, then

$$\frac{K^{\mathrm{T}}+K}{2}-\frac{(N-1)\gamma_{\max}^2\|K\|^2}{N}I_n$$

is positive definite if and only if $0<k<[N/(N-1)\gamma_{\max}^2]$. An interesting topic is whether we can select K such that the performances (that is, convergence rate and mean square steady-state error) of the closed-loop system are optimised. This topic has been discussed preliminarily for the case with linear vector-valued noise intensity functions in [19]. However, for the case with non-linear matrix-valued noise intensity functions, this problem becomes much more difficult.

For the case where the noise intensity function only depends on the amplitude of the relative states, we have the following measurement model and assumption

$$y_{ji}(t)=x_j(t)+f_{ji}(\|x_j(t)-x_i(t)\|)\xi_{ji}(t),\quad j\in N_i \tag{17}$$

where the noise intensity function $f_{ji}(\cdot)$ is a non-linear mapping from $\mathbb{R}_+$ to $\mathbb{R}^{n\times n}$.

Assumption 3.1: There exists a positive constant $\gamma_{\max}$ such that $\|f_{ji}(x)\|_F\le\gamma_{\max}x$, $\forall x\in\mathbb{R}_+$, $i,j=1,2,\dots,N$.

By Theorem 3.1, we obtain the following result directly.

Corollary 3.1: Suppose that Assumptions 2.1 and 3.1 hold. Apply the protocol (4) to the system (1) and (17). If

$$\frac{K^{\mathrm{T}}+K}{2}-\frac{(N-1)\gamma_{\max}^2\|K\|^2}{N}I_n$$

is positive definite, then the protocol (4) is an asymptotically mean square and almost sure average consensus protocol.

3.1 Linear dependency on relative distance

In the following, we will consider the case where the noise intensity function has linear dependency on the relative distance.

Assumption 3.2: $f_{ji}(x)=\Sigma_{ji}x$, $\forall x\in\mathbb{R}_+$, $i,j=1,2,\dots,N$, where Σ_{ji}, $i,j=1,2,\dots,N$ are $n\times n$ dimensional matrices.

Define

$$\mathcal{A}_\Sigma=\left[\frac{a_{ji}\mathbf{tr}(\Sigma_{ij}^{\mathrm{T}}K^{\mathrm{T}}K\Sigma_{ij})+a_{ij}\mathbf{tr}(\Sigma_{ji}^{\mathrm{T}}K^{\mathrm{T}}K\Sigma_{ji})}{2}\right]$$

as a new weighted adjacency matrix and denote the associated Laplacian matrix by $\mathcal{L}_\Sigma$. We also denote

$$\widehat{\Psi}_{\mathcal{L}}^f(K)=\Lambda_{\mathcal{L}}^0\otimes\left(\frac{K^{\mathrm{T}}+K}{2}\right)-\frac{N-1}{N}[(\phi^{\mathrm{T}}\mathcal{L}_\Sigma\phi)\otimes I_n]$$

Then we have the following theorem.

Theorem 3.2: Apply the protocol (4) to the system (1) and (17). Suppose that Assumptions 2.1 and 2.2 hold. If $\widehat{\Psi}_{\mathcal{L}}^f(K)$ is positive definite, then the protocol (4) is an asymptotically mean square and almost sure average consensus protocol.

Proof: The proof is similar to Theorem 3.1 and is omitted here. □

Corollary 3.2: Apply the protocol (4) to the system (1) and (17). Suppose that Assumptions 2.1 and 3.2 hold with $\Sigma_{ij}=\sigma I_n$. Then the protocol (4) with $K=kI_n$, $k>0$, is an asymptotically unbiased mean square and almost sure average consensus protocol if $[nk\sigma^2(N-1)/N]<1$, and the convergence rate is given by

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\|\delta(t)\|^2}{t}\ge-\left(k-\frac{(N-1)n}{N}k^2\sigma^2\right)\lambda_N(\mathcal{L})$$
$$\lim_{t\to\infty}\frac{\log\mathbb{E}\|\delta(t)\|^2}{t}\le-\left(k-\frac{(N-1)n}{N}k^2\sigma^2\right)\lambda_2(\mathcal{L})$$

Proof: If $\Sigma_{ij}=\sigma I_n$ and $K=kI_n$, then $\mathbf{tr}(\Sigma_{ji}^{\mathrm{T}}K^{\mathrm{T}}K\Sigma_{ji})=nk^2\sigma^2$, which implies that

$$\widehat{\Psi}_{\mathcal{L}}^f(K)=\left(k-\frac{(N-1)n}{N}k^2\sigma^2\right)(\Lambda_{\mathcal{L}}^0\otimes I_n)$$

Then by Theorem 3.2, the conclusion of this corollary holds. □

3.2 Case with two agents

For the following case with two agents, we give some necessary and sufficient conditions on the control gain to ensure mean square or almost sure average consensus.

Theorem 3.3: Apply the protocol (4) to the system (1) and (17) with $N=2$. Suppose that Assumptions 2.1 and 3.2 hold. Then the protocol (4) with $K=kI_n$, $k\in\mathbb{R}$, is an asymptotically unbiased mean square average-consensus protocol if and only if $0<k<\frac{4}{(\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2)}$. Moreover, the convergence rate is given by

$$\mathbb{E}\|x_1(t)-x_2(t)\|^2=\mathbb{E}\|x_1(0)-x_2(0)\|^2e^{-k[4-k(\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2)]t} \tag{18}$$

The control gain to optimise the convergence rate is $k^*=\frac{2}{\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2}$.

Proof: Under the conditions of the theorem, the closed-loop equation (5) can be rewritten as

$$\mathrm{d}x_1(t)=k(x_2(t)-x_1(t))\mathrm{d}t+kf_{21}(\|x_2(t)-x_1(t)\|)\mathrm{d}w_{21}(t)$$
$$\mathrm{d}x_2(t)=k(x_1(t)-x_2(t))\mathrm{d}t+kf_{12}(\|x_1(t)-x_2(t)\|)\mathrm{d}w_{12}(t)$$

Let $\widetilde{x}(t)=x_1(t)-x_2(t)$, which is equivalent to the definition of $\overline{\delta}(t)$, then we have

$$\begin{aligned}\mathrm{d}\widetilde{x}(t)=&-2k\widetilde{x}(t)\mathrm{d}t+kf_{21}(\|\widetilde{x}(t)\|)\mathrm{d}w_{21}(t)\\&-kf_{12}(\|\widetilde{x}(t)\|)]\mathrm{d}w_{12}(t)\end{aligned} \tag{19}$$

Then by Assumption 3.2, and the Itô formula, we obtain

$$\begin{aligned}\mathbb{E}\|\widetilde{x}(t)\|^2=&\|\widetilde{x}(0)\|^2-k[4-k(\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2)]\\&\times\int_0^t\mathbb{E}\|\widetilde{x}(s)\|^2\mathrm{d}s\end{aligned}$$

which implies (18). Then by direct calculation, we know that

$$\underset{k:\ 0<k<\frac{4}{(\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2)}}{\operatorname{argmax}} k[4-k(\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2)] = \frac{2}{\|\Sigma_{12}\|_F^2+\|\Sigma_{21}\|_F^2}$$

□

Remark 5: From Theorem 3.3, we can see that the consensus condition depends on the trace norm of Σ_{ij}. If the communication channels for different state components are not independent, then the non-diagonal elements of Σ_{12} and Σ_{21} will increase the norm, which means that the interference from communication channels of other state components increases the difficulty to achieve consensus.

Remark 6: In [19], it was shown that for the case where the noise intensity function is a linear vector function of the relative state, if the control and communication channels of different state components are independent and homogeneous, then $0 < k < N/[(N-1)\sigma^2]$ is a necessary and sufficient condition to ensure mean square consensus, where σ is coefficient of the noise intensity function. Here, it can be seen that for the case with relative-distance-dependent matrix noise intensity functions, even the control channels of different state components are independent and homogeneous, the number of control channels n still has an explicit impact on the consensus condition. This is mainly due to that the dependency of the noise intensity function on the relative distance is indeed a non-linear dependency on the relative state.

By Assumption 3.1, the following lemma holds (see [24, Lemma 4.3.2, p. 120]).

Lemma 1: For all $\widetilde{x}(0) \neq 0$ in the system (19), $\mathbb{P}\{\widetilde{x}(t) \neq 0$ on all $t \geq 0\} = 1$. That is, almost all the sample paths of any solution starting from a non-zero state will never reach the origin.

Theorem 3.4: Consider a connected two-agent undirected network. Suppose that Assumptions 2.1 and 3.2 hold with $\Sigma_{12} = b_{12}I_n$, $\Sigma_{21} = b_{21}I_n$, $b_{12} > 0$, $b_{21} > 0$. Then the protocol (4) with $K = kI_n$, $k \in \mathbb{R}$ is an asymptotically unbiased almost sure average-consensus protocol if and only if $-(k/2)[4-(n-2)k(b_{12}^2+b_{21}^2)] < 0$, and the convergence rate is given by

$$\lim_{t\to\infty} \frac{\log \|x_1(t)-x_2(t)\|}{t} = -\frac{k}{2}\left[4-(n-2)ka_{12}(b_{12}^2+b_{21}^2)\right] \quad \text{a.s.} \tag{20}$$

Proof: By Lemma 3.1, $\widetilde{x}(t) \neq 0$ for all $t \geq 0$ almost surely. Thus, applying the Itô formula to $\log \|\widetilde{x}(t)\|$ yields

$$2\mathrm{d}\log\|\widetilde{x}(t)\| = -k[4-(n-2)k(b_{12}^2+b_{21}^2)]\mathrm{d}t + \frac{2k\|\widetilde{x}(t)\|\widetilde{x}^{\mathrm{T}}(t)}{\|\widetilde{x}(t)\|^2}[b_{21}\mathrm{d}w_{21}(t) - b_{12}\mathrm{d}w_{12}(t)]$$

which implies that

$$\log\|\widetilde{x}(t)\| = \frac{1}{2}\log\|x(0)\| - \frac{k}{2}\left[4-(n-2)k(b_{12}^2+b_{21}^2)\right]t + M_1(t) - M_2(t) \tag{21}$$

where

$$M_1(t) = \int_0^t \frac{kb_{21}\|\widetilde{x}(s)\|\widetilde{x}^{\mathrm{T}}(s)}{\|\widetilde{x}(s)\|^2}\mathrm{d}w_{21}(s) \text{ and } M_2(t) = \int_0^t \frac{kb_{12}\|\widetilde{x}(s)\|\widetilde{x}^{\mathrm{T}}(s)}{\|\widetilde{x}(s)\|^2}\mathrm{d}w_{12}(s)$$

are two local martingales with $M_1(0) = M_2(0) = 0$ and the quadratic variations

$$\langle M_1, M_2\rangle_t = k^2b_{21}^2t \quad \text{and} \quad \langle M_1, M_2\rangle_t = k^2b_{12}^2t$$

It is obvious that

$$\frac{\langle M_1, M_2\rangle_t}{t} = k^2b_{21}^2 < \infty \quad \text{and} \quad \frac{\langle M_1, M_2\rangle_t}{t} = k^2b_{12}^2 < \infty$$

Applying the law of large numbers gives

$$\lim_{t\to\infty}\frac{M_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{M_2(t)}{t} = 0$$

which together with (21) leads to

$$\lim_{t\to\infty}\log\|\widetilde{x}(t)\| = -\frac{k}{2}\left[4-(n-2)k(b_{12}^2+b_{21}^2)\right]$$

which leads to the conclusion of the theorem. □

Remark 7: In this paper, we consider the case with pure multiplicative measurement noises, that is, the noise intensity function $f_{ji}(\cdot)$ satisfies $f_{ji}(0) = O_n$, otherwise, we may rewrite the measurement model (2) as

$$y_{ji}(t) = x_j(t) + (f_{ji}(x_j(t)-x_i(t)) - f_{ji}(0))\xi_{ji}(t) + f_{ji}(0)\xi_{ji}(t), \quad j \in N_i$$

which is both with additive and multiplicative measurement noises. For this case, the time-varying consensus gain function $a(t)$ may be introduced into the control protocol, which is given by

$$u_i(t) = a(t)K\sum_{j=1}^N a_{ij}(y_{ji}(t) - x_i(t)), \quad t \geq 0,\ i = 1, 2, \ldots, N$$

The combination of techniques of [3] and this paper may be useful for the closed-loop analysis for this kind of consensus systems. However, this is out of the scope of this paper and may be an interesting topic for future research.

4 Conclusions

In this paper, the distributed averaging of high-dimensional first-order agents with relative-state-dependent measurement noises has been considered. The information exchange among agents is described by an undirected graph. Each agent can measure or receive its neighbours' state information with random noises. The noise intensity function is a non-linear matrix-valued function of relative states of agents. By the tools of stochastic differential equations and algebraic graph theory, we give sufficient conditions to ensure mean square and almost sure average consensus and the convergence rate and the steady-state error for average consensus are quantified. Especially, if the noise intensity function depends linearly on the relative distance with intensity constant σ, then a positive control gain k which satisfies $[nk\sigma^2(N-1)/N] < 1$ can ensure asymptotically unbiased mean square and almost sure average consensus.

For future research, the results can be extended to the case with discrete-time dynamics, high-order dynamics and switching topologies.

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6 References

1 Huang, M., Manton, J.H.: 'Coordination and consensus of networked agents with noisy measurement: stochastic algorithms and asymptotic behavior', *SIAM J. Control Optim.*, 2009, **48**, (1), pp. 134–161
2 Kar, S., Moura, J.M.F.: 'Distributed consensus algorithms in sensor networks with imperfect communication: link failures and channel noise', *IEEE Trans. Signal Process.*, 2009, **57**, (1), pp. 355–369
3 Li, T., Zhang, J.F.: 'Mean square average consensus under measurement noises and fixed topologies: necessary and sufficient conditions', *Automatica*, 2009, **45**, (8), pp. 1929–1936
4 Li, T., Zhang, J.F.: 'Consensus conditions of multiagent systems with time-varying topologies and stochastic communication noises', *IEEE Trans. Autom. Control*, 2010, **55**, (9), pp. 2043–2057
5 Huang, M., Manton, J.H.: 'Stochastic consensus seeking with noisy and directed inter-agent communication: fixed and randomly varying topologies', *IEEE Trans. Autom. Control*, 2010, **55**, (1), pp. 235–241
6 Wang, J., Elia, N.: 'Distributed averaging under constraints on information exchange: emergence of Levy flights', *IEEE Trans. Autom. Control*, 2012, **57**, (10), pp. 2435–2449
7 Kashyap, A., Basar, T., Srikant, R.: 'Quantized consensus', *Automatica*, 2007, **43**, (7), pp. 1192–1203
8 Carli, R., Fagnani, F., Speranzon, A., Zampieri, S.: 'Communication constraints in the average consensus problem', *Automatica*, 2008, **44**, (3), pp. 671–684
9 Lavaei, J., Murray, R.M.: 'Quantized consensus by means of gossip algorithm', *IEEE Trans. Autom. Control*, 2012, **57**, (1), pp. 19–32
10 Carli, R., Bullo, F.: 'Quantized coordination algorithms for rendezvous and deployment', *SIAM J. Control Optim.*, 2009, **48**, (3), pp. 1251–1274
11 Frasca, P., Carli, R., Fagnani, F., Zampieri, S.: 'Average consensus on networks with quantized communication', *Int. J. Nonlinear Robust Control*, 2009, **19**, (16), pp. 1787–1816
12 Li, T., Fu, M., Xie, L., Zhang, J.F.: 'Distributed consensus with limited communication data rate,' *IEEE Trans. Autom. Control*, 2011, **56**, (2), pp. 279–292
13 Li, T., Xie, L.: 'Distributed consensus over digital networks with limited bandwidth and time-varying topologies', *Automatica*, 2011, **47**, (9), pp. 2006–2015
14 Liu, J., Liu, X., Xie, W.C., Zhang, H.: 'Stochastic consensus seeking with communication delays', *Automatica*, 2011, **47**, (12), pp. 2689–2696
15 Ma, C.Q., Li, T., Zhang, J.F.: 'Consensus control for leader-following multi-agent systems with measurement noises,' *J. Syst. Sci. Complexity*, 2010, **23**, (1), pp. 35–49
16 Cheng, L., Hou, Z.G., Tan, M., Wang, X.: 'Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises', *IEEE Trans. Autom. Control*, 2011, **56**, (8), pp. 1958–1963
17 Cheng, L., Hou, Z.G., Tan, M.: 'A mean square consensus protocol for linear multi-agent systems with communication noises and fixed topologies', *IEEE Trans. Autom. Control*, 2014, **59**, (1), pp. 261–267
18 Elia, N.: 'Remote stabilization over fading channels', *Syst. Control Lett.*, 2005, **54**, (3), pp. 237–249
19 Li, T., Wu, F.K., Zhang, J.F.: 'Multi-agent consensus with relative-state-dependent measurement noises', *IEEE Trans. Autom. Control*, 2014, **59**, (9), pp. 2463–2468
20 Bolognani, S., Favero, S.D., Schenato, L., Varagnolo, D.: 'Consensus-based distributed sensor calibration and least-square parameter identification in WSNs', *Int. J. Robust Nonlinear Control*, 2010, **20**, (2), pp. 176–193
21 Schenato, L., Fiorentin, F.: 'Average TimeSynch: a consensus-based protocol for clock synchronization in wireless sensor networks', *Automatica*, 2011, **47**, (9), pp. 1878–1886
22 Olfati-Saber, R., Murray, R.M.: 'Consensus problem in networks of agents with switching topology and time-delays', *IEEE Trans. Autom. Control*, 2004, **49**, (9), pp. 1520–1533
23 Michel, A.N., Miller, R.K.: 'Qualitative analysis of large scale dynamical systems' (Academic Press, New York, USA, 1977)
24 Mao, X.: 'Stochatic differential equations and applications' (Horwood Publishing, Chichester, England, 1997)

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