Evolutionarily Stable Strategy of Networked Evolutionary Games

Daizhan Cheng, *Fellow, IEEE*, Tingting Xu, and Hongsheng Qi

***Abstract*— The evolutionarily stable strategy (ESS) of networked evolutionary games (NEGs) is studied. Analyzing the ESS of infinite popular evolutionary games and comparing it with networked games, a new verifiable definition of ESS for NEGs is proposed. Then, the fundamental evolutionary equation (FEE) is investigated and used to construct the strategy profile dynamics (SPDs) of homogeneous NEGs. Two ways for verifying the ESS are proposed: 1) using the SPDs to verify it directly. The SPDs provides complete information about the NEGs, and then necessary and sufficient conditions are revealed. It can be used for NEGs with small size and 2) some sufficient conditions are proposed to verify the ESS of NEGs via their FEEs. This method is particularly suitable for large scale networks. Some illustrative examples are included to demonstrate the theoretical results.**

***Index Terms*— Evolutionarily stable strategy (ESS), fundamental evolutionary equation (FEE), networked evolutionary game (NEG), semitensor product (STP) of matrices.**

I. Introduction

THE LAST few decades have witnessed the increasing applications of concepts from game theory to the study of evolutions in biological systems [3], [28]. Particularly, it was shown that evolution could lead to cooperation [1], [2], [17]. Based on this observation and accompanying the development of network theory, networked evolutionary game (NEG) becomes a hot topic, because it is very likely that there are some topological structures, precisely the spacial relations, which decide the interactions among the players in evolutionary games [15], [20], [24].

Recently, the investigation of NEGs has attracted much attention from biologists, physicists, and system scientists, and so on, some new approaches and interesting results have been reported. For instance, some interesting develops are: 1) how the effective payoffs in the prisoner's dilemma game facilitate cooperation [25]; 2) the impact of link deletions on cooperation for public goods game [13] and for prisoner's dilemma game [26]; 3) social dilemmas on evolving random networks [27] and cooperation on scale-free networks [16]; and 4) evolution of public goods game on two interdependent networks [30], [31]. A comprehensive review on the evolutionary dynamics of NEGs can be found in [18].

Stability is one of the most important issues in this paper of the evolutionary games, because it shows where an evolution will go [23]. To study this property, the evolutionarily stable strategy (ESS) was first proposed in [21]. An ESS is a strategy such that, if all the members of a population adopt it, then no mutant strategy could invade the population under the influence of natural selection. ESS has then been studied and used widely. A detailed discussion is presented in [22]. Some new developments can be found in [12].

In the classical literature, the ESS was investigated under the assumption that all the members in the group play with each other or randomly. But for an NEG, a particular connected topology (or an adjacent graph) is given and each member can only play with its neighborhood members. So for NEGs the ESS should be different from the ESS of classical evolutionary games. So the ESS of NEGs is still an open problem.

Recently, a new mathematical tool, namely, semitensor product (STP) of matrices, has been used to investigate logical networks, including Boolean networks. The basic topological structures of Boolean networks have been studied [4], [6], [8]. Then, the control of Boolean networks have also been investigated [5], [14]. We also refer to [7] for a comprehensive introduction to STP approach for Boolean networks.

In this paper, the STP will be used to investigate the ESS of NEGs. We first review the fundamental evolutionary equation (FEE) of NEGs, proposed in [10]. Then, we consider how to use it to produce the strategy profile dynamics (SPDs) of NEGs. In fact, SPD is the overall network dynamics. After that a definition of ESS of NEGs is proposed, and demonstrates that the definition coincides with the classical definition. Finally, two ways for verifying the ESS of the NEGs are proposed: 1) using the SPDs to verify it directly. In principle, this method is universally applicable to any networks. However, because of the computational complexity, this method can be used for small-size NEGs and 2) some sufficient conditions are proposed to verify the ESS of NEGs via their FEEs. This method has much less computational load and can be used for large scale NEGs.

The rest of this paper is organized as follows. Section II provides some fundamental concepts and notations for describing NEGs. Particularly, the STP of matrices is briefly introduced. Section III reviews the FEE of an NEG, and then consider how to use FEE to produce the SPDs of an NEG.

Manuscript received April 21, 2013; accepted November 21, 2013. Date of publication December 12, 2013; date of current version June 10, 2014. This work was supported by the National Natural Science Foundation of China under Grant 61074114, Grant 61273013, Grant 61104065, and Grant 61333001.

D. Cheng is with the School of Control Science and Engineering, Shandong University, Jinan 250061, China, and also with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, China (e-mail: dcheng@iss.ac.cn).

T. Xu and H. Qi are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, China (e-mail: xutingting@amss.ac.cn; qihongsh@amss.ac.cn).

Digital Object Identifier 10.1109/TNNLS.2013.2293149

TABLE I
PAYOFF TABLE (HAWK–DOVE GAME)

$P_1 \backslash P_2$	H	D
H	$E(H,H)$	$E(H,D)$
D	$E(D,H)$	$E(D,D)$

The ESS of NEGs is defined and discussed in Section IV. In Section V, the ESS of both homogeneous and heterogeneous NEGs are investigated via their FEEs. Some illustrative examples are presented. Section VI consists of some concluding remarks.

II. Problem Formulation

A. General Definition of ESS

This subsection briefly reviews some basic concepts about ESS.

Definition 2.1 ([22]): Consider an evolutionary game. An ESS is a strategy such that, if all the members of a population adopt it, then no mutant strategy could invade the population under the influence of natural selection.

This definition has a very clear physical (or biological) meaning. As mentioned in [22]: the idea can be applied equally well to any kind of phenotypic variations. However, it does not provide a rigorous mathematical concept, which is easily verifiable. Let us observe an example, which was used in [22] as a detailed description of the ESS.

Example 2.2 ([22]): Consider the Hawk–Dove game. The game is symmetric and the payoff matrix is as in Table I, where $E(X, Y)$ is the payoff of individual adopting X against a Y opponent, $X, Y \in \{H, D\}$.

Let p be the frequency of H strategists in the population, $W(H)$ and $W(D)$ be the fitnesses of H and D strategists, respectively. In addition, before the contest, all individuals have fitness W_0.

Then if each individual engages in one contest, we have

$$\begin{cases} W(H) = W_0 + pE(H,H) + (1-p)E(H,D) \\ W(D) = W_0 + pE(D,H) + (1-p)E(D,D). \end{cases} \quad (1)$$

Suppose that individuals reproduce their kind asexually, in numbers proportional to their fitnesses. The frequency p' of hawks in the next generation is

$$p' = pW(H)/\overline{W} \quad (2)$$

where $\overline{W} = pW(H) + (1-p)W(D)$.

Now, assume H is an ESS and D is a mutant, then $(1-p)$ should be very small. Since H is stable, $W(H) > W(D)$. It follows that:

$$\begin{aligned} &\text{either } E(H,H) > E(D,H) \\ &\text{or } E(H,H) = E(D,H) \text{ and } E(H,D) > E(D,D). \end{aligned} \quad (3)$$

Equation (3) is referred to as the standard conditions for an ESS, however, it should be clear that they apply only to the particular model $\cdots$ with an infinite population, asexual inheritance, and pairwise contests [21].

Equation (3) may be used as a definition of ESS, which has mathematical rigorousness. However, its drawback lies on: 1) infinite population and 2) two alternative strategies. In addition, its necessity has not been proved.

When the NEGs are considered, a natural question is: is (3) applicable to NEGs? (3) seems to be a sufficient condition for ESS of evolutionary games. In addition, (2) is only one strategy updating rule (SUR). Comparing with most SURs of NEGs, (2) seems too artificial. In addition, it is based on a complete network graph, while general NEGs have different graph structures. We may conclude that (3) seems not applicable to general NEGs. That is, we need to seek a proper verifiable definition of ESS for NEGs.

B. STP of Matrices

Before reviewing the SPT of matrices, some concepts/notations are listed as follows.

1) $\mathcal{M}_{m\times n}$ is the set of $m \times n$ real matrices.
2) $\mathrm{Col}_i(M)$ is the ith column of matrix M; $\mathrm{Col}(M)$ is the set of columns of M.
3) $\mathcal{D}_k := \{1, 2, \ldots, k\}$.
4) $\delta_n^i := \mathrm{Col}_i(I_n)$, i.e., it is the ith column of the identity matrix.
5) $\Delta_n := \mathrm{Col}(I_n)$.
6) $M \in \mathcal{M}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(M) \subset \Delta_m$, the set of $m \times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$.
7) Assume $L \in \mathcal{L}_{m\times n}$, then $L = \left[\delta_m^{i_1}\ \delta_m^{i_2}\ \ldots\ \delta_m^{i_n}\right]$. Its shorthand form is $L = \delta_m\,[i_1\ i_2\ \ldots\ \delta_m]$.
8) $r = [r_1, \ldots, r_m]^T \in \mathbb{R}^m$ is called a probabilistic vector, if $r_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m r_i = 1$. The set of k dimensional probabilistic vectors is denoted by Υ_m.
9) If $M \in \mathcal{M}_{m\times n}$ and $\mathrm{Col}(M) \subset \Upsilon_m$, M is called a probabilistic matrix. The set of $m \times n$ probabilistic matrices is denoted by $\Upsilon_{m\times n}$.
10) $\mathbf{1}_n = (\underbrace{1, 1, \ldots, 1}_{n})^T$.

The STP of matrices is a basic tool in our approach. We give a brief review here and refer to [7], [9] for details.

Definition 2.3: Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote by $t := \mathrm{l\ cm}(n, p)$ the least common multiple of n and p. Then, we define the STP of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \quad (4)$$

Remark 2.4

1) When $n = p$, $A \ltimes B = AB$. Thus, the STP is a generalization of the conventional matrix product.
2) STP keeps all the major properties of the conventional matrix product unchanged.
3) Throughout this paper, the matrix product is assumed to be STP, and mostly, the notation $\ltimes$ is omitted.

The following is a special property for STP, which will be used in the sequel.

Proposition 2.5: Let $x \in \mathbb{R}^t$ be a column vector. Then for a matrix M

$$x \ltimes M = (I_t \otimes M) \ltimes x. \quad (5)$$

Definition 2.6: Define a matrix

$$W_{[n,m]} := \delta_{mn}\,[\,1,\ m+1,\ 2m+1,\ \ldots,\ (n-1)m+1,\ 2,\ m+2,\ 2m+2,\ \ldots,\ (n-1)m+2,\ldots,\ n,\ m+n,\ 2m+n,\ \ldots,\ mn] \in \mathcal{M}_{mn\times mn} \tag{6}$$

which is called a swap matrix.

The following is a fundamental property of swap matrix.

Proposition 2.7: Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{7}$$

Finally, we consider how to express a k-valued logical function into an algebraic form.

Theorem 2.8 [(9)]

1) Let $f\colon \mathcal{D}_k^n \to \mathcal{D}_k$ be a k-valued logical function, expressed as

$$y = f(x_1,\ldots,x_n). \tag{8}$$

Identify $i \sim \delta_k^i$, $i = 1,2,\ldots,k$. Then, there exists a unique logical matrix $M_f \in \mathcal{L}_{k\times k^n}$, called the structure matrix of f, such that under vector form (8) can be expressed as

$$y = M_f \ltimes_{i=1}^n x_i \tag{9}$$

where $y, x_i \in \Delta_k$, $i = 1,\ldots,n$. (9) is called the algebraic form of (8).

2) If in item 1) $\mathcal{D}_k$ is replaced by Υ_k, then the result remains true except that the structure matrix M_f in (9) is replaced by an $M_f \in \Upsilon_{k\times k^n}$.

C. Networked Evolutionary Game

We first briefly review the framework for NEGs, proposed in [10]. In this paper, only the symmetric case is considered. The basic concepts are summarized in the following definitions.

Definition 2.9

1) Given an undirected graph (N, E), where N is the set of nodes and $E \subset N \times N$ is the set of edges. In the graph x_1-x_2-…-x_k is called a path, if $(x_i, x_{i+1}) \in E$, $i = 1,\ldots,k-1$.
2) Let $x_0 \in N$. The d-neighborhood of x_0, denoted by $U_d(x_0)$, is defined as: $y \in U_d(x_0)$, if and only if there is a path from x_0 to y with length less than or equal to d, where $d \geq 1$. Briefly denote $U(x_0) := U_1(x_0)$.

Definition 2.10: A fundamental network game (FNG) is a special normal game [11]. It consists of three factors:

1) 2 players $N = \{i, j\}$;
2) both players have the same set of strategies: $S = \{1,\ldots,k\}$;
3) each player, say i, has its payoff function $c_{i,j}\colon S^2 \to \mathbb{R}$. Note that for a symmetric game the payoffs satisfy $c_{i,j}(s_i, s_j) = c_{j,i}(s_j, s_i)$, where $s_i, s_j \in S$.

Definition 2.11: A SUR, Π, is a rule, which decides the strategy of a player i at time $t+1$ by the strategies $\{x_j(t) | j \in U(i)\}$ and payoffs $\{c_j(t) | j \in U(i)\}$ of its neighborhood players at time t.

Now, we are ready to give the framework for NEGs.

Definition 2.12: A NEG, denoted by $((N, E), G, \Pi)$, consists of three ingredients:

1) an undirected graph (N, E);
2) an FNG, G, such that if $(i, j) \in E$, then i and j play FNG with strategies $x_i(t) \in S$ and $x_j(t) \in S$, respectively;
3) a local information based SUR.

In addition, assume $c_{i,j}$ is the payoff of the FNG between i and j for player i. Then, the overall payoff of player i is

$$c_i(t) = \frac{1}{|U(i)|-1} \sum_{j\in U(i)\backslash i} c_{ij}(t), \quad i \in N. \tag{10}$$

By definition, a SUR can be expressed as

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \,\big|\, j \in U(i)\}\right),\ t \geq 0,\ i \in N. \tag{11}$$

In addition, since $c_j(t)$ depends on the strategies of its neighborhood players, i.e., $\{x_k(t) \,|\, k \in U(j)\}$ only, it follows immediately that [10]:

$$x_i(t+1) = f_i\left(\{x_j(t) \,\big|\, j \in U_2(i)\}\right), \quad t \geq 0,\ i \in N. \tag{12}$$

In fact, the f_i in (11) and (12) are not the same. To avoid the notational mess, we use the same symbol for both. We call (12) the FEE of the NEG.

Remark 2.13

1) When the network graph is homogeneous, i.e., the degree of each node is unique, the FEEs for all nodes are the same.
2) Notice that (12) is a k-valued logical dynamic system. Using Theorem 2.8, we can express (12) into its algebraic form as

$$x_i(t+1) = M_i \ltimes_{j\in U_2(i)} x_j(t), \quad t \geq 0,\ i \in N. \tag{13}$$

Set $\ell = |U_2(i)|$, then in (13) the $M_i \in \mathcal{L}_{k\times k^\ell}$ when pure strategies are used; and $M_i \in \Upsilon_{k\times k^\ell}$ when mixed strategies are used.

We collect some SUR in the following example. Some of them are used in the sequel.

Example 2.14: The following are some commonly used SUR.

1) Π-*I:* Unconditional imitation [15] with fixed priority. The best strategy from strategies of neighborhood players $\{j \,|\, j \in U(i)\}$ at time t is selected as the strategy of player i at time $t+1$, denoted by $x_i(t+1)$. Precisely, if

$$j^* = \operatorname{argmax}_{j\in U(i)} c_j(x(t)) \tag{14}$$

then

$$x_i(t+1) = x_{j^*}(t). \tag{15}$$

When the players with the best payoff are not unique, say

$$\operatorname{argmax}_{j\in U(i)} c_j(x(t)) := \{j_1^*,\ldots,j_r^*\} \tag{16}$$

then we may choose the one corresponding to a priority as

$$j^* = \min\{\mu \mid \mu \in \mathrm{argmax}_{j\in U(i)}\, c_j(x(t))\}. \tag{17}$$

This method leads to a deterministic k-valued dynamics.

2) Π-*II:* Unconditional imitation with equal probability for best strategies. When the best payoff player is unique, it is the same as Π-I. When the players with best payoff are not unique, say, as in (16), then we randomly choose one with equal probability. That is

$$x_i(t+1) = x_{j^*_\mu}(t), \quad \text{with probability} \quad p^i_\mu = \frac{1}{r} \\ \mu = 1,\dots,r. \tag{18}$$

This method leads to a probabilistic k-valued dynamics.

3) Π-*III:* Simplified Fermi rule [24], [29]. That is, randomly choose a neighborhood $j \in U(i)$. Comparing $c_j(x(t))$ with $c_i(x(t))$ to determine $x_i(t+1)$ as

$$x_i(t+1) = \begin{cases} x_j(t), & c_j(x(t)) > c_i(x(t)) \\ x_i(t), & \text{otherwise.} \end{cases} \tag{19}$$

This method leads to probabilistic k-valued dynamics.

4) Π-IV (Π-V)*:* Myopic best response adjustment rule [32]

$$x_i(t+1) = \mathrm{argmax}_{x\in S}\left(c_i(x),\ c_j(x_j(t)),\ j \in U(i)\right). \tag{20}$$

If the right hand side of (20) is not unique, (17) or (18) can be used, respectively, to get two different SURs, denoted by Π-IV and Π-V, respectively.

Next, we give a topological structure to the set of profiles by defining a distance on this set.

Define a sequence of matrices $\pi^{[n,k]}_j \in \mathcal{L}_{k\times k^n}$, $j = 1,\dots,n$ as

$$\pi^{[n,k]}_i = \mathbf{1}^T_{k^{i-1}} \otimes I_k \otimes \mathbf{1}^T_{k^{n-i}}, \quad i = 1,\dots,n. \tag{21}$$

A straightforward computation shows that

Proposition 2.15: Let $x_i \in \Upsilon_k$, $i = 1,\dots,n$, such that $x = \ltimes^n_{i=1} x_i$. Then, x_i, $i = 1,\dots,n$, are uniquely determined by

$$x_i = \pi^{[n,k]}_i x, \quad i = 1,\dots,n. \tag{22}$$

Definition 2.16

1) Let $p, q \in \mathbb{R}^k$. Then the distance of p and q is defined as $d(p,q) = \sum_{i=1}^{k} |p_i - q_i|$.
2) Let $x, y \in \Upsilon_{k^n}$, $x_i = \pi^{[n,k]}_i x$, $y_i = \pi^{[n,k]}_i y$, $i = 1,\dots,n$. Then

$$\|x - y\| := \frac{1}{2}\sum_{j=1}^{n} d(x_j, y_j). \tag{23}$$

Remark 2.17: Consider a networked game with n players (nodes), and the common strategy set $S = \{1, 2,\dots,k\}$. Identifying $i \sim \delta^i_k$, $i = 1,\dots,k$, then a profile of pure strategies is $x = \ltimes^n_{i=1} x_i \in \Delta_{k^n}$. It is clear that for two profiles x, y: $\|x - y\| = r$, if and only if, in these two profiles there are r different strategies. That is, there are r players, who take different actions.

TABLE II
PAYOFF BI-MATRIX OF PRISONER'S DILEMMA

$P_1\backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

III. FEE AND SPD

A. FEE of NEGs

The FEE of an NEG, proposed in [10], has two equivalent forms. Equation (12) is its logical form, and (13) is its algebraic form.

According to the payoff bi-matrix, they can be calculated. We give a simple algorithm as follows.

Algorithm 3.1: Consider a node (player) i.

1) *Step 1:* For each $j \in U(i)$ consider $k \in U(j)$. According to $x_j(t)$ and $x_k(t)$, $c_{j,k}(t)$ can be calculated.
2) *Step 2:* Using formula (10), $c_j(t)$, $j \in U(i)$ can be calculated.
3) *Step 3:* Using the $c_j(t)$, $j \in U(i)$ and according to the SUR, $x_i(t+1)$ can be figured out.

We give an example to describe this algorithm.

Example 3.2: Assume there are countable players $\{i \mid i \in \mathbb{Z}\}$ on a line $\mathbb{R}$, where axis $\mathbb{R}$ is considered as its graph. Then for each player i, its neighborhood $U(i) = \{i-1, i, i+1\}$, and its second neighborhood is $U_2(i) = \{i-2, i-1, i, i+1, i+2\}$. Assume each i plays the prisoner's dilemma [19] with its neighborhood players. That is, $S_0 = \{1, 2\}$, where 1 means cooperate and 2 means defect. The payoff bi-matrix is as in Table II, where the profile is composed of the strategies of five players in the order of $(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$.

Set $P = -6$, $R = -5$, $S = -5$, and $T = -3$, then we can figure out the profile dynamics as in Table III. In the table, the first row (profile) means the strategies of $k \in U_2(i)$. For instance, 11 122 means player $i-2$, $i-1$, i take strategy 1 at t, and $i+1$, $i+2$ take strategy 2 at t; $c_{i-1}(t)$, $c_i(t)$, and $c_{i+1}(t)$ are the payoffs of players $i-1$, i, $i+1$ at time t, respectively; and $x_i(t+1)$ is the strategy of player i at time $t+1$, which is determined by the SUR.

Collecting the values of $x_i(t+1)$ in Table III, the FEE can be obtained as

$$x_i(t+1) = M \ltimes^2_{j=-2} x_{i+j}(t) \tag{24}$$

where the structure matrix is

$$M = \delta_2[1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2 \\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]. \tag{25}$$

B. Calculation of SPD via FEE

This subsection considers how to use FEE to produce the SPD for an NEG. We need the following lemma, which can be proved by a straightforward computation.

Lemma 3.3: Assume $X \in \Upsilon_p$ and $Y \in \Upsilon_q$. We define two dummy matrices: 1) $D^{[p,q]}_f$, named by front-maintaining operator (FMO) and 2) $D^{[p,q]}_r$, named by rear-maintaining operator (RMO), respectively, as follows:

$$D^{[p,q]}_f = I_p \otimes \mathbf{1}^T_q \\ D^{[p,q]}_r = \mathbf{1}^T_p \otimes I_q.$$

TABLE III
FROM PAYOFFS TO NEXT STRATEGY

Profile	11111	11112	11121	11122	11211	11212	11221	11222
$c_{i-1}(t)$	-5	-5	-5	-5	-5	-5	-5	-5
$c_i(t)$	-5	-5	-5	-5	-3	-3	-4.5	-4.5
$c_{i+1}(t)$	-5	-5	-3	-4.5	-5	-5	-4.5	-6
$x_i(t+1)$	1	1	2	2	2	2	2	2
Profile	12111	12112	12121	12122	12211	12212	12221	12222
$c_{i-1}(t)$	-3	-3	-3	-3	-4.5	-4.5	-4.5	-4.5
$c_i(t)$	-5	-5	-5	-5	-4.5	-4.5	-6	-6
$c_{i+1}(t)$	-5	-5	-3	-4.5	-5	-5	-4.5	-6
$x_i(t+1)$	2	2	2	2	2	2	2	2
Profile	21111	21112	21121	21122	21211	21212	21221	21222
$c_{i-1}(t)$	-5	-5	-5	-5	-5	-5	-5	-5
$c_i(t)$	-5	-5	-5	-5	-3	-3	-4.5	-4.5
$c_{i+1}(t)$	-5	-5	-3	-4.5	-5	-5	-4.5	-6
$x_i(t+1)$	1	1	2	2	2	2	2	2
Profile	22111	22112	22121	22122	22211	22212	22221	22222
$c_{i-1}(t)$	-4.5	-4.5	-4.5	-4.5	-6	-6	-6	-6
$c_i(t)$	-5	-5	-5	-5	-4.5	-4.5	-6	-6
$c_{i+1}(t)$	-5	-5	-3	-4.5	-5	-5	-4.5	-6
$x_i(t+1)$	2	2	2	2	2	2	2	2

Then we have

$$D_f^{[p,q]}XY = X. \tag{26}$$

$$D_r^{[p,q]}XY = Y. \tag{27}$$

The function of (26) and (27) is to add some dummy factors at the rear [front] of the original variables. Now, we give another algorithm to describe how to calculate the SPDs using FEE.

We need one more concept.

Definition 3.4 ([9]): Let $M \in \mathcal{M}_{p\times n}$, $N \in \mathcal{M}_{q\times n}$. Then, the Khatri–Rao product of M and N, denoted by $M * N \in \mathcal{M}_{pq\times n}$, is defined column by column as follows:

$$\mathrm{Col}_i(M * N) = \mathrm{Col}_i(M) \ltimes \mathrm{Col}_i(N), \quad i = 1, \dots, n. \tag{28}$$

Algorithm 3.5

1) *Step 1:* From the FEE (12) to calculate its algebraic form (13) as

$$x_i(t+1) = M_i \ltimes_{j\in U_2(i)} x_j(t), \quad i = 1, \dots, n \tag{29}$$

where $M_i \in \mathcal{L}_{k\times k^{|U_2(i)|}}$.

2) *Step 2:* Use Lemma 3.3 [equivalently, formulas (26) and (27)] to add some dummy factors such that the product in (29) can be a product of all factors, x_i, $i = 1, \dots, n$ as

$$x_i(t+1) = W_i \ltimes_{j=1}^n x_j, \quad i = 1, \dots, n. \tag{30}$$

3) *Step 3:* Denote by $x := \ltimes_{j=1}^n x_j$. The SPDs can be constructed as [7]

$$x(t+1) = Lx(t) \tag{31}$$

where $L \in \mathcal{L}_{k^n\times k^n}$ is determined by

$$L = W_1 * W_2 * \dots * W_n. \tag{32}$$

Equation (31) is called the algebraic form of the SPDs. It is the dynamics of the NEG.

An NEG is said to be homogeneous, if each node has the same degree [10]. Otherwise, it is heterogeneous. A homogeneous NEG has a universal FEE, and its SPDs is uniquely determined by its unique FEE. This is demonstrated in the following example.

Example 3.6: Recall Example 3.2. Assume the network graph $\mathbb{Z} \in \mathbb{R}$ is replaced by S_6. It is clearly homogeneous. In addition, the FEE of each player is (24) and (25). We construct its SPDs as follows.

It is easy to calculate that

$$\begin{aligned} x_1(t+1) &= Mx_5x_6x_1x_2x_3 \\ &= M\big(D_r^{[2,4]}x_4x_5x_6\big)x_1x_2x_3 \\ &= MD_r^{[2,4]}W_{[8,8]}x_1x_2x_3x_4x_5x_6 \\ &= W_1x(t) \end{aligned}$$

where $x(t) = \ltimes_{i=1}^6 x_i(t)$ and

$$\begin{aligned} W_1 &= MD_r^{[2,4]}W_{[8,8]} \\ &= \delta_2[1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]. \end{aligned}$$

Similarly, we have

$$x_i(t+1) = W_ix(t), \quad i = 2, 3, 4, 5, 6$$

and

$$\begin{aligned} W_2 &= MD_r^{[2,2]}W_{[4,16]} \\ &= \delta_2[\,1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2 \\ &\quad 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2 \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] \end{aligned}$$

$$\begin{aligned} W_3 &= MD_f^{[32,2]} \\ &= \delta_2[\,1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ &\quad 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ &\quad 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] \end{aligned}$$

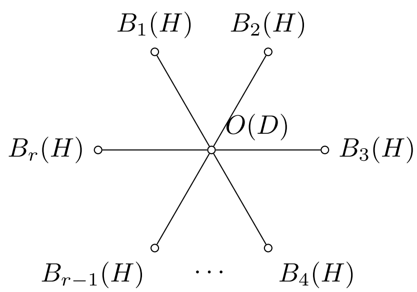


Fig. 1. Network with a mutant at O.

$$\begin{aligned}W_4 &= MD_r^{[2,32]}\\ &= \delta_2[\,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\ &\quad 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\ &\quad 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\ &\quad 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2]\\ W_5 &= MD_f^{[32,2]}W_{[4,16]}\\ &= \delta_2[\,1,2,2,2,2,2,2,2,1,2,2,2,2,2,2,2\\ &\quad 1,2,2,2,2,2,2,2,1,2,2,2,2,2,2,2\\ &\quad 1,2,2,2,2,2,2,2,1,2,2,2,2,2,2,2\\ &\quad 1,2,2,2,2,2,2,2,1,2,2,2,2,2,2,2]\\ W_6 &= MD_f^{[32,2]}W_{[8,8]}\\ &= \delta_2[\,1,2,2,2,1,2,2,2,1,2,2,2,1,2,2,2\\ &\quad 1,2,2,2,1,2,2,2,1,2,2,2,1,2,2,2\\ &\quad 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\ &\quad 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2].\end{aligned}$$

Finally, the SPDs is

$$x(t+1) = Lx(t)$$

where

$$\begin{aligned}L &= W_1 * W_2 * W_3 * W_4 * W_5 * W_6\\ &= \delta_{64}[1,52,24,56,15,64,32,64,13,64,32,64,15,64\\ &\quad 32,64,41,60,64,64,47,64,64,64,45,64,64,64\\ &\quad 47,64,64,64,50,52,56,56,64,64,64,64,62,64\\ &\quad 64,64,64,64,64,64,58,60,64,64,64,64,64,64\\ &\quad 62,64,64,64,64,64,64,64].\end{aligned}$$

IV. ESS of NEG

A. Comparing ESSs of NEGs With Classical Evolutionary Games

As mentioned in Section I, the ESS is a fundamental concept for evolutionary games. It is natural to extend it to the NEGs. Consider the ESS defined by (3). Assume on an NEG all nodes reach the strategy H, and then a mutant appears at node O, we refer to Fig. 1 for this.

First, we recall (3) to see whether it works for the NEGs. Since the probability for mutants is small, we assume there is no other mutant within $U_2(O)$. Ignoring W_0, we assume $U(O) = \{O, B_1, \dots, B_r\}$ and $|U(B_i)| - 1 = \ell_i$, $i = 1, \dots, r$

$$c_i(B_i) = \frac{1}{\ell_i}\left[(\ell_i - 1)E(H,H) + E(H,D)\right], \quad i = 1,\dots,r$$
$$c_0(O) = E(D,H).$$

Assume we use Π-I as the SUR, which is basically the simple best neighborhood choice. One sees easily that even if we assume $E(H,H) > E(D,H)$, we are not able to assure $c_i(B_i) > c_0(O)$, which means the mutant may not be eliminated at the next iteration and so forth.

Next, we consider the SUR described by (2). It can not be applicable to NEGs too, because for NEGs, a player can only get the information of its neighborhood players. Hence, we need a new precise definition of the ESS for NEGs. It is not difficult to verify that the following definition verifies the general Definition 2.1.

Definition 4.1

1) For a given NEG a strategy $\xi \in S$ is called an ESS, if there exists a $\mu \geq 1$, such that as long as the initial strategy profile y_0 satisfies
$$\|y_0 - x_0\| \leq \mu \tag{33}$$
we have
$$\lim_{t\to\infty} y(t, y_0) = x_0 \tag{34}$$
where $x_0 = \xi^n$. Moreover, ξ is called the ESS of level μ.
2) If for any $i \in N$, the strategies within $U_k(i)$ allow up to μ mutants, precisely
$$\left\|(y_0 - x_0)\big|_{U_k(i)}\right\| \leq \mu, \quad \forall i \in N \tag{35}$$
we have (34), then ξ is called the ESS of level $\mu/[k]$.

Remark 4.2

1) When the population n is finite, (34) can be replaced by: there exists a $T > 0$ such that
$$y(t, y_0) = x_0, \quad t \geq T. \tag{36}$$
2) Observing Remark 2.17, Definition 4.1 means at least one mutant could not affect the population under the evolutionary updating rule. We, therefore, usually choose μ as a positive integer.
3) It is clear that the μ can be used to measure the robustness of the stability. So the higher the level the more robust the ESS.

B. Verifying ESS via SPDs

Since the SPDs is the dynamics of overall NEG, it can be used to verify ESS directly. We give an example to describe this.

Example 4.3: Assume the NEG is almost the same as the one in Example 3.2, except that the network graph is S_7. Using FEE obtained in Example 3.2 and similar calculation in Example 3.6, we can get the SPDs as

$$x(t+1) = Lx(t) \tag{37}$$

where $x(t) = \ltimes_{i=1}^{7} x_i(t)$, and

$$
L = \delta_{128}
\begin{array}{l}
[1\ \ 68\ \ 8\ \ 72\ \ 15\ \ 80\ \ 16\ \ 80\ \ 29\ \ 96\ \ 32\ \ 96 \\
31\ \ 96\ \ 32\ \ 96\ \ 57\ \ 124\ \ 64\ \ 128\ \ 63\ \ 128\ \ 64\ \ 128 \\
61\ \ 128\ \ 64\ \ 128\ \ 63\ \ 128\ \ 64\ \ 128\ \ 113\ \ 116\ \ 120\ \ 120 \\
127\ \ 128\ \ 128\ \ 128\ \ 125\ \ 128\ \ 128\ \ 128\ \ 127\ \ 128\ \ 128\ \ 128 \\
121\ \ 124\ \ 128\ \ 128\ \ 127\ \ 128\ \ 128\ \ 128\ \ 125\ \ 128\ \ 128\ \ 128 \\
127\ \ 128\ \ 128\ \ 128\ \ 98\ \ 100\ \ 104\ \ 104\ \ 112\ \ 112\ \ 112\ \ 112 \\
126\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 122\ \ 124\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 126\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
114\ \ 116\ \ 120\ \ 120\ \ 128\ \ 128\ \ 128\ \ 128\ \ 126\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 122\ \ 124\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
126\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128].
\end{array}
$$

It is easy to calculate that

$$
L^k = \delta_{128}
\begin{array}{l}
[1\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128 \\
128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128\ \ 128]
\end{array}
$$

where $k \geq 3$.

It is clear that unless $x(0) = \delta_{128}^1$, which leads to $x(\infty) = x(3) = \delta_{128}^1 \sim (1, 1, 1, 1, 1, 1, 1)$, any other initial states converge to $\delta_{128}^{128} \sim (2, 2, 2, 2, 2, 2, 2)$.

We conclude that $\xi = \delta_2^2 \sim 2$ (i.e., strategy 2) is an ESS. In addition, it is so strong that we can choose $\mu = 6$, and as long as $|y_0 - x_0| \leq \mu$, (where $x_0 = \xi^7$), and setting $T = 3$, (36) holds. Hence, the ESS is of level 6.

Meanwhile, if we consider another strategy $\eta = \delta_2^1 \sim 1$. Then, it is clear that η is not an ESS, because for any mutant $|y_0 - \eta^7| \geq 1$, we have $y(t, y_0) \to \xi$. That is, the mutant strategy will invade and eventually dominate the population.

Remark 4.4: Observing Example 4.3, we set $H = \delta_2^2 \sim 2$ and $D = \delta_2^1 \sim 1$. Then one sees that

$$E(H, H) = -6 < E(D, H) = -5.$$

That is, (3) is not true. So in NEG case, (3) is also not necessary for the ESS.

We have calculated S_i, for $5 \leq i \leq 13$, and shown that for all for them δ_2^1 is not an ESS and δ_2^2 is stable. One may guess this is also true for all $i \geq 5$. But as i is large, this method is not applicable because of the computational complexity.

Remark 4.5:

1) Even if a network is heterogeneous, its SPDs is still computable provided that the network size is small. Hence, the ESS of a heterogeneous NEG can also be verified via its SPDs.
2) If the network size is not small, the SPDs for either homogeneous or heterogeneous NEGs can hardly be calculated, because of the computational complexity. Thus, some other techniques need to be developed.

V. Verification of ESS via FEE

In Section IV, we considered how to verify the ESS of a strategy using the SPDs of the NEG. For an NEG with large population, constructing the SPDs becomes difficult. For instance, in Example 4.3, when $n > 15$ or so, it is almost impossible to construct the structure matrix of the SPDs in personal computers. If 2-D case, say, $S_m \times S_n$, or even higher dimensional cases are investigated, verifying ESS via SPDs seems hopeless because of the computational complexity.

In this section, we consider how to verify the ESS of a strategy via the FEE of the NEG. To begin with, we consider the iteration of FEE.

A. Iteration of FEE in Homogeneous Case

For homogeneous case we have unique FEE. Consider the strategy of a fixed point i, to get an updated strategy $x_i(t_0+1)$ via FEE, we need to know $\{x_j(t_0) \mid j \in U_2(i)\}$, where t_0 is a given initial moment. We express it as

$$x_i(t_0 + 1) = f(x_j(t_0) \,\big|\, j \in U_2(i)). \tag{38}$$

Equation (38) is exactly the ESS for i. If we want to get two step updated strategy $x_i(t_0+2)$ via FEE, we surely need all the information about the strategies over $U_4(z_0)$. This procedure can be described as

$$\begin{aligned} x_i(t_0 + 2) &= f(x_j(t_0 + 1) \,\big|\, j \in U_2(i)) \\ &= f^2(x_k(t_0) \,\big|\, k \in U_4(i)). \end{aligned} \tag{39}$$

Keep going like this, we can find r step updated strategy $x_i(t_0+r)$ over a set of nested neighborhoods (refer to Fig. 2) as

$$x_i(t_0 + r) = f^r(x_k(t_0) \,\big|\, k \in U_{2r}(i)), \quad r = 1, 2, \ldots \tag{40}$$

It is worth noting that the iterative expression (40) is formally different from conventional mapping. However, it is only a specification of a general mapping

$$x_i(t_0 + 1) = f(x_j(t_0) \mid j \in N), \quad i \in N \tag{41}$$

and

$$x_i(t_0 + r) = f^r(x_k(t_0) \mid k \in N), \quad i \in N. \tag{42}$$

However, only part of $\{x_k(t_0) \mid k \in N\}$ are used. Thus, (38) and (40) are the same as (41) and (42), respectively. The only difference is the arguments which have been specified.

Next, we consider the heterogeneous case. For $x_i(t_0 + 1)$ we have

$$x_i(t_0 + 1) = f_i(x_j(t_0) \,\big|\, j \in U_2(i)). \tag{43}$$

It is almost the same as (38). As for 2 step case, we have

$$\begin{aligned} x_i(t_0 + 2) &= f_i(x_j(t_0 + 1) \mid j \in U_2(i)) \\ x_j(t_0 + 1) &= f_j(x_k(t_0) \mid k \in U_2(j)), \quad j \in U_2(i). \end{aligned} \tag{44}$$

We briefly rewrite (44) as

$$x_i(t_0 + 2) = f_i^2(x_k(t_0) \mid k \in U_4(i)). \tag{45}$$

It means two iterations by using the information $\{x_k(t_0) \mid k \in U_4(i)\}$ is enough. However, note that $\{f_j \mid j \in U_2(i)\}$ are also used. Similarly, we can denote

$$x_i(t_0 + r) = f_i^r(x_k(t_0) \mid k \in U_{2r}(i)). \tag{46}$$

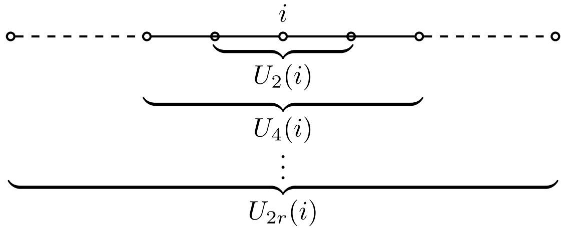


Fig. 2. Iteration over nested neighborhoods.

B. ESS of Homogeneous NEGs

Note that for a homogeneous NEG the FEE is unique. In addition, $|U_s(i)|$, $s = 1, 2, \dots$ are independent of i, we, therefore, can denote $d_s := |U_s(i)|$.

Using the iterative expression, we have the following result.

Theorem 5.1: Given a homogeneous NEG. Assume its unique FEE is

$$x_i(t+1) = f\left(\{x_j(t) \,\big|\, j \in U_2(i)\}\right), \quad t \geq 0,\ i \in N. \quad (47)$$

1) A strategy ξ is ESS of level $\mu/[2r]$ ($\mu \geq 1$), if there exists an integer $r \geq 1$, such that for any $i \in N$, and the strategies x_k, $k \in U_{2r}(i)$, satisfying $\sum_{k\in U_{2r}(i)} |x_k - \xi| \leq \mu$, we have

$$f^r(x_k \,\big|\, k \in U_{2r}(i)) = \xi, \quad \forall i \in N. \quad (48)$$

2) A strategy ξ is not ESS, if for any x_j, $j \in U_2(i)$, satisfying $\sum_{j\in U_2(i)} |x_j - \xi| \geq 1$, we have

$$f(x_j \mid j \in U_2(i)) \neq \xi. \quad (49)$$

Proof

1) First, notice that (48) is independent of i, because the NEG is homogeneous, as long as (48) holds for any one i, it is also true for all i. Next, according to (40), (48) means that even if in the $U_{2r}(i)$ there are up to μ mutants, after r generations (i.e., iterations), and all the mutants will disappear.
2) According to (49) one sees that if there is one or more mutant(s), then after one step we still have at least one mutant. Thus, the mutant(s) will never disappear.

■

It is obvious that as long as the mutants are distributed sparsely, then the requirement $\sum_{k\in U_{2r}(i)} |x_k - \xi| \leq \mu$ is satisfied.

Particularly, we can easily check the case when $\mu = 1$.

Proposition 5.2: Assume the mutants are so sparse that for any two mutants i, j

$$U_{2r}(i) \cup U_{2r}(j) = \emptyset \quad (50)$$

Then for any i, $\sum_{k\in U_{2r}(i)} |x_k - \xi| \leq 1$ is satisfied.

In fact, $\mu = 1$ is the most useful case.

Example 5.3: Recall Example 4.3. Its FEE was obtained in Example 3.2. Using a similar argument as in Example 4.3, it is ready to see that $\xi = \delta_2^2$ verifies (48) with $r = 1$, $\mu = 4$. Hence, for any S_n, $n > 5$, or even $n = \infty$, $\xi = \delta_2^2$ is an ESS of level 4. Similarly, $\eta = \delta_2^1$ verifies (49), hence $\eta = \delta_2^1$ is not ESS for any S_n.

C. ESS of General NEGs

This subsection considers the general case, where the network is, in general, heterogeneous. Houwever, the results obtained are also applicable to homogeneous case. We need the following assumption.

A1: There exist two numbers p and q satisfying $1 \leq p \leq q < \infty$, such that

$$p \leq \text{degree}(i) \leq q, \quad \forall i \in N. \quad (51)$$

As mentioned in Remark 4.5 for general case, when the network size is small, the ESS can be verified via its SPDs. So the NEGs concerned in this subsection are of large size. To deal with such networks, we propose a method called the decomposition approach. We describe it via the following example.

Example 5.4: Observing the network in Fig. 3(a), there is a node $O \in N$ with

$$\begin{aligned} U(O) &= \{O, A, C, F\} \\ U_2(O) &= \{O, A, B, C, D, E, F, G, H, I\} \\ U_r(O) &= \dots,\ r = 3, 4, \dots. \end{aligned}$$

According to $U(O)$, we split the network into three branches, namely O_1, O_2, and O_3, which are depicted in Fig. 3(b)–(d), respectively. Notice that the number of branches equals to $|U(O)| - 1$.

Assume $\xi \in S$ is a strategy. We hope that if ξ is an ESS in each branch, then it is also an ESS for overall network.

Under the assumption A1, one sees easily that for each node there are at most $q - p + 1$ different kinds of branches. Consider a node O, then O has $|U(O)| - 1$ branches and one of its branches, denoted by i, is depicted in Fig. 4. Setting $s = |U(A)| - 2$, we denote

$$x_O^i = x_O \ltimes x_A \ltimes x_{B_1} \ltimes \cdots \ltimes x_{B_s}.$$

The corresponding FEE for the ith branch of node O is denoted as

$$\begin{aligned} f_O^i&(x_O, x_A, x_{B_1}, \dots, x_{B_s}) \\ &= M_O^i \ltimes x_O \ltimes x_A \ltimes x_{B_1} \ltimes \cdots \ltimes x_{B_s}, \\ &\qquad\qquad i = 1, \dots, |U(O)| - 1 \end{aligned} \quad (52)$$

where $M_O^i \in \mathcal{L}_{k\times k^{s+2}}$, $i \in U(O)\backslash\{O\}$.

Let $O \in N$ and $i \in U(O)\backslash\{O\}$. Over each branch of O, say i, we set $U_r^i(O)$ the rth neighborhood of O with respect to branch O_i. $d_r^i := \left|U_r^i(O)\right|$. Then, we compute the iterative expression of FEE as

$$\begin{aligned} x_O^i(t_0+1) &= f_O^i(x_j(t_0)\big|j \in U_2^i(O)) \\ x_O^i(t_0+2) &= (f_O^i)^2(x_k(t_0)\big|k \in U_4^i(O)) \\ &\ \vdots \\ x_O^i(t_0+r) &= (f_O^i)^r(x_k(t_0)\big|k \in U_{2r}^i(O)) \\ &\qquad i \in U(O)\backslash\{O\}. \end{aligned} \quad (53)$$

Then, we have the following result.

Theorem 5.5: Consider a general NEG and assume A1. Let $O \in N$ be any node. If there exist two integers $\mu \geq 1$ and

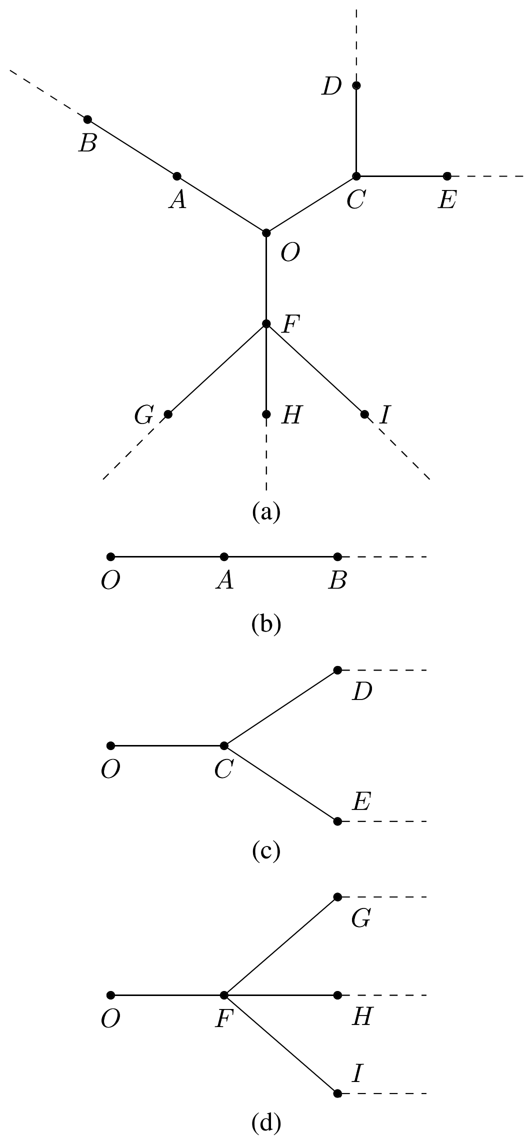


Fig. 3. (a) $U_2(O)$. (b) O_1. (c) O_2. (d) O_3.

$r \geq 1$, such that any $x_j(t_0)$, $j \in U_{2r}(O)$, which is the initial strategy profile of $\{j | j \in U_{2r}(O)\}$ with $\sum_{j \in U_{2r}(O)} |x_j(t_0) - \xi| \leq \mu$, satisfying

$$x_O^i(t_0 + \ell) = x_O^j(t_0 + \ell) \quad \forall i, j \in U(O) \backslash \{O\} \ell = 1, \dots, r - 1 \quad (54)$$

$$x_O^i(t_0 + r) = x_O^j(t_0 + r) = \xi \quad \forall i, j \in U(O) \backslash \{O\} \quad (55)$$

then ξ is an ESS of level $\mu/[2r]$.

Proof: According to (54) and (55) we can denote

$$x_O(t_0 + \ell) = x_O^i(t_0 + \ell), \quad \ell = 1, \dots, r. \quad (56)$$

For each branch O_i, according to the SUR, $x_O^i(t_0 + \ell) = x_O(t_0 + \ell)$ is the best selected strategy. Using the weighted average (10), it is clear that $x_O(t_0 + \ell)$ is also the best selected strategy for O at time $t_0 + \ell$ with respect to overall neighborhood $U_{2\ell}(O)$.

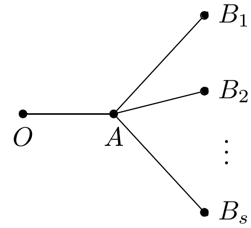


Fig. 4. Branch O_i.

TABLE IV
PROFILE DYNAMICS OF BRANCH (B)

Profile(OAB)	122	212	221
C_O	-6	-3	-6
C_A	-4.5	-6	-4.5
f_O^1	2	2	2

TABLE V
PROFILE DYNAMICS OF BRANCH (C)

Profile(OCDE)	1222	2122	2212	2221
C_O	-6	-3	-6	-6
C_C	-5	-6	-4.5	-4.5
f_O^2	2	2	2	2

TABLE VI
PROFILE DYNAMICS OF BRANCH (D)

Profile	12222	21222	22122	22212	22221
C_O	-6	-3	-6	-6	-6
C_F	-5.25	-6	-5.25	-5.25	-5.25
f_O^3	2	2	2	2	2

Finally, (54) and (55) assure that the strategy profile of each branch O_i evolutes as if it is independent of the other branches. Hence, the strategy of O at overall network converges to ξ after r generations, which is exactly the same as it converges to ξ at each branch. ∎

Remark 5.6: The conditions (54) and (55) seem too restrictive. Our conjecture is: if in every branch O converges to ξ then in the overall NEG O converges to ξ too.

Example 5.7: Consider an NEG, which is constructed as follows:

1) the FNG is the prisoner's dilemma as in Example 3.2 with parameters: $P = -6$, $R = -5$, $S = -6$, and $T = -3$;
2) the network is heterogeneous and could be huge, we do not care about its shape, but for any node $i \in N$, $2 \leq \text{degree}(i) \leq 4$. Then, there are only three kinds of branches (refer to Fig. 3);
3) the SUR is assumed to be unconditional imitation Π-I (or Π-II).

We try to use Theorem 5.5 to find ESS. For the sake of simplicity, we only use the information of $U_2(O)$. Hence, we assume $\mu = 1$. That is, we allow only one mutant at each kind of branches $U_2^i(O)$, $i = 1, 2, 3$. For each kind of branches we check f_O^i, and we need only to check the profile with one mutant case. This is done in Tables IV–VI, respectively.

It is clear that the conditions (54) and (55) of Theorem 5.5 with $r = 1$ are satisfied. Therefore, Theorem 5.5 assures that

$\xi = \delta_2^2$ is an ESS of level $\mu/[2r] = 1/[2]$. (The level could be higher if we check $U_4^i(O)$ for all i or so.)

When the higher dimension homogeneous cases are considered, Theorem 5.5 can provide a very simple and useful result.

Corollary 5.8: Assume both A and B are same type of homogeneous NEGs, i.e., their FEEs and SURs are the same, and ξ is an ESS for both NEGs of level μ (or $\mu/[k]$). Then, μ is also an ESS of same level for the product NEGs.

Example 5.9: Consider $S_m \times S_n$, where m and n are large, could be ∞. Assume the prisoner's dilemma with parameters: $P = -6$, $R = -5$, $S = -5$, and $T = -3$ is played. According to Example 5.3, $\xi = \delta_2^2$ is an ESS of level $\mu = 4$ over both S_n and S_m. According to Corollary 5.8, $\xi = \delta_2^2$ is also an ESS of level $\mu = 4$ for the product NEG.

VI. Conclusion

When an evolutionary game is considered, the ESS is of fundamental importance, because it indicates the target situation of the evolution. When the NEGs are considered, ESS is also a key issue to be investigated. Using STP of matrices, [10] proposed a rigorous mathematical model for NEGs. This paper is a follow up of [10], and it contains mainly of the following works.

1) Algorithms are proposed to get the SPDs of NEGs from their FEEs.
2) The concept of ESS has been proposed for NEGs. A comparison with the ESS of classical evolutionary games has been done, certain differences have been revealed.
3) Two ways for verifying ESSs were presented as:
   a) using the SPDs to verify the ESS. It provides a necessary and sufficient condition for a strategy to be ESS. But because of the computational complexity, it can be used for only small size NEGs;
   b) using the FEEs to verify the ESS. Only a sufficient condition has been obtained. It can be used for both homogeneous and heterogeneous cases. But it is efficient for large scale NEGs.

Though the method proposed in this paper covers both pure strategy and mixed strategy cases, this paper is mainly working with pure strategy case. More investigation for mixed strategy case needs to be considered. There is also room for improving the sufficient condition using FEEs.

Finally, the most challenging problem is to apply the results to real biological systems. We leave this for further study.

References

[1] R. Axelrod and W. D. Hamilton, "The evolution of cooperation," *Science*, vol. 211, pp. 1390–1396, Mar. 1981.

[2] R. Axelrod and D. Dion, "The further evolution of cooperation," *Science*, vol. 242, pp. 1385–1390, Dec. 1988.

[3] E. L. Charnov, *The Theory of Sex Allocation*. Princeton, NJ, USA: Princeton Univ. Press, 1982.

[4] D. Cheng, "Input-state approach to Boolean networks," *IEEE Trans. Neural Netw.*, vol. 20, no. 3, pp. 512–521, Mar. 2009.

[5] D. Cheng and H. Qi, "Controllability and observability of Boolean control networks," *Automatica*, vol. 45, pp. 1659–1667, Mar. 2009.

[6] D. Cheng and H. Qi, "State-space analysis of Boolean networks," *IEEE Trans. Neural Netw.*, vol. 21, no. 4, pp. 584–594, Apr. 2010.

[7] D. Cheng, H. Qi, and Z. Li, *Analysis and Control of Boolean Networks—A Semi-Tensor Product Approach*. London, U.K.: Springer-Verlag, 2011.

[8] D. Cheng, H. Qi, and Z. Li, "Model construction of Boolean network via observed data," *IEEE Trans. Neural Netw.*, vol. 22, no. 4, pp. 525–536, Apr. 2011.

[9] D. Cheng, H. Qi, and Y. Zhao, *An Introduction to Semi-Tensor Product of Matrices and its Applications*. Singapore: World Scientific, 2012.

[10] D. Cheng, F. He, H. Qi, and T. Xu, "Modeling, analysis and control of networked evolutionary games," *IEEE Trans. Autom. Control* [Online]. Available: http://lsc.amss.ac.cn/~dcheng/preprint/NTGAME02.pdf

[11] R. Gibbons, *A Primer in Game Theory*. Glasgow, Scotland: Bell & Bain, 1992.

[12] H. Gintis, *Game Theory Evolving*. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[13] L. Jiang, M. Perc, W. Wang, Y. Lai, and B. Wang, "Impact of link deletions on public cooperation in scale-free networks," *Europhys. Lett.*, vol. 93, no. 4, pp. 40001-1–40001-6, Feb. 2011.

[14] D. Laschov and M. Margaliot, "A maximum principle for single-input Boolean control networks," *IEEE Trans. Autom. Control*, vol. 56, no. 4, pp. 913–917, Apr. 2011.

[15] M. A. Nowak and R. M. May, "Evolutionary games and spatial chaos," *Nature*, vol. 359, pp. 826–829, Oct. 1992.

[16] M. Perc, "Evolution of cooperation on scale-free networks subject to error and attack," *New J. Phys.*, vol. 11, p. 033027, Mar. 2009.

[17] M. Perc and A. Szolnoki, "Coevolutionary games—A mini review," *BioSystems*, vol. 99, pp. 109–125, Oct. 2010.

[18] M. Perc, J. G. Gardeñes, A. Szolnoki, L. M. Floria, and Y. Moreno, "Evolutionary dynamics of group interactions on structured populations: A review," *J. R. Soc. Inter.*, vol. 10, p. 20120997, Jan. 2013.

[19] E. Rasmusen, *Games and Information: An Introduction to Game Theory*. 4th ed. Oxford, U.K.: Blackwell, 2007.

[20] F. C. Santos, M. D. Santos, and J. M. Pacheco, "Social diversity promotes the emergence of cooperation in public goods games," *Nature*, vol. 454, pp. 213–216, Jul. 2008.

[21] J. M. Smith and G. R. Price, "The logic of animal conflict," *Nature*, vol. 246, pp. 15–18, Nov. 1973.

[22] J. M. Smith, *Evolution and the Theorem of Games*. Cambridge, U.K.: Cambridge Univ. Press, 1982.

[23] S. Suzuki and E. Akiyama, "Evolutionary stability of first-order-information indirect reciprocity in sizable groups," *Theor. Popul. Biol.*, vol. 73, pp. 426–436, May 2008.

[24] G. Szabo and C. Toke, "Evolutionary prisoner's dilemma game on a square lattice," *Phys. Rev. E*, vol. 58, pp. 69–73, Jul. 1998.

[25] A. Szolnoki and M. Perc, "Towards effective payoffs in the prisoner's dilemma game on scale-free networks," *Phys. A, Statist. Mech. Appl.*, vol. 387, pp. 2075–2082, Nov. 2008.

[26] A. Szolnoki and M. Perc, "Emergence of multilevel selection in the prisoner's dilemma game on coevolving random networks," *New J. Phys.*, vol. 11, p. 093033, Sep. 2009.

[27] A. Szolnoki and M. Perc, "Resolving social dilemmas on evolving random networks," *Europhys. Lett.*, vol. 86, pp. 30007–30012, May 2009.

[28] P. D. Taylor and L. B. Jonker, "Evolutionary stable strategies and game dynamics," *Math. Biosci.*, vol. 40, pp. 145–156, Jul. 1978.

[29] A. Traulsen, M. A. Nowak, and J. M. Pacheco, "Stochastic dynamics of invasion and fixation," *Phys. Rev. E*, vol. 74, pp. 011909-1–011909-5, Jul. 2006.

[30] Z. Wang, A. Szolnoki, and M. Perc, "Evolution of public cooperation on interdependent networks: The impact of biased utility functions," *Europhys. Lett.*, vol. 97, pp. 48001–48006, Feb. 2012.

[31] Z. Wang, A. Szolnoki, and M. Perc, "Interdependent network reciprocity in evolutionary games," *Sci. Rep.*, vol. 3, p. 1183, Jan. 2013.

[32] H. P. Young, "The evolution of conventions," *Econometrica*, vol. 61, pp. 57–84, Jan. 1993.

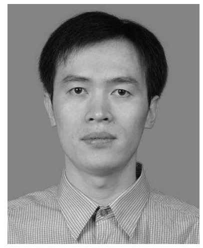
Daizhan Cheng (F'06) received the Degree from Tsinghua University, Beijing, China, the M.S. degree from the Graduate School, Chinese Academy of Sciences, Beijing, and the Ph.D. degree from Washington University, St. Louis, WA, USA, in 1970, 1981, and 1985, respectively.

He has been a Professor with the Institute of Systems Science, AMSS, Chinese Academy of Sciences, Beijing, since 1990. He is the author or co-author of 12 books, over 240 journal papers, and over 130 conference papers. His current research interests include nonlinear control systems, switched systems, Hamiltonian systems, and Boolean control networks.

Dr. Cheng was the Chairman of Technical Committee on Control Theory, Chinese Association of Automation from 2003 to 2010, and a member of the IEEE CSS Board of Governors in 2009. He was an IFAC Fellow in 2008 and IFAC Council Member from 2011 to 2014. He received the Second Grade National Natural Science Award of China in 2008, the Automatica 2008-2010 Best Theory/Methodology Paper Award, bestowed by IFAC, in 2011.

Tingting Xu received the B.S. degree in information and computing sciences from the China University of Mining and Technology, Beijing, China, in 2011.

She joined the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, in 2011. Her current research interests include game theory, genetic regulatory network, and multiagent system.



Hongsheng Qi received the B.S. degree in mathematics and applied mathematics from Anhui University, Anhui, China, and the Ph.D. degree in systems theory from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2003 and 2008, respectively.

He was a Post-Doctoral Fellow with the Key Laboratory of Systems Control, Chinese Academy of Sciences, from 2008 to 2010. Since 2010, he has been an Assistant Professor with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. His current research interests include nonlinear control, complex systems, and game theory.

Dr. Qi was a recipient of the "Automatica" 2008–2010 Best Theory/Methodology Paper Award in 2011.