On Decomposed Subspaces of Finite Games

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***Abstract*—This note provides the detailed description of the decomposed subspaces of finite games. First, the basis of potential games and the basis of non-strategic games are revealed. Then the bases of pure potential and pure harmonic subspaces are also obtained. These bases provide an explicit formula for the decomposition, and are convenient for investigating the properties of the corresponding subspaces. As an application, we consider the dynamics of networked evolutionary games (NEGs). Three problems are considered: 1) the dynamic equivalence of evolutionary games; 2) the dynamics of near potential games; and 3) the decomposition of NEGs.**

***Index Terms*—Decomposition, harmonic game, non-strategic game, potential game, semi-tensor product of matrices.**

I. Preliminaries

Potential games have been firstly introduced by Rosenthal [16]. They not only play an important role in game-theoretic analysis, but also become a powerful tool in several control problems. We refer to [13] for the concept and general properties of potential games. Some of their applications to control problems are: i) consensus of multi-agent systems [10]; ii) optimization of distributed coverage of graphs [20]; iii) congestion control [19]; and iv) control of power networks [9]; etc.

It is well known that a noncooperative strategic form finite game can be described as a triple (N, S, c), where:

1) $N = \{1, 2, \dots, n\}$ is the set of players;
2) $S_i = \{1, 2, \dots, k_i\}$ is the set of strategies of player i, $i = 1, \dots, n$; $S = \prod_{i=1}^n S_i$ is called the set of strategy profiles;
3) $c = \{c_1, c_2, \dots, c_n\}$, where $c_i : S \to \mathbb{R}$ is the payoff function of player i.

The set of strategic form finite games with $|N| = n$, and $|S_i| = k_i$, $i = 1, \dots, n$, is denoted by $\mathcal{G}_{[n;k_1,\dots,k_n]}$. By introducing a game graph flow, and using the Helmholtz decomposition theorem from graph theory, an orthogonal decomposition for the vector space of finite games is obtained by Candogan *et al.* in their novel paper [2] as (1), where $\mathcal{P}$ is the pure potential subspace, $\mathcal{N}$ is the non-strategic subspace, and $\mathcal{H}$ is the pure harmonic subspace. All three subspaces are defined by corresponding combinatorial operators

$$\mathcal{G}_{[n;k_1,\dots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{\text{Harmonic}}}_{\text{Potential games}} \overbrace{\oplus\ \mathcal{H}}^{\text{games}}. \tag{1}$$

In this follow-up work, we give an alternative approach. First, using vector expression of payoffs, the $\mathcal{G}_{[n;k_1,\dots,k_n]}$ is naturally identified with $\mathbb{R}^{nk}$. Then the three subspaces $\mathcal{P}, \mathcal{N}$, and $\mathcal{H}$ are described as three subspaces of $\mathbb{R}^{nk}$ by their definitions directly. Moreover, the inner product and the orthogonality are inherited from $\mathbb{R}^{nk}$. Comparing with the original approach in [2], our framework is much more simplified and less involved. It is more suitable for engineering-based research as it provides a constructive mechanism to decompose games and is therefore more applicable in practice as can be seen in Section VI, where we apply the developed techniques to assess NEGs.

In this note, we provide precisely the bases of subspaces $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{H}$, which results in an explicit expression of the decomposed subspaces, while only implicit forms of Moore-Penrose inverses have been achieved via operator approach proposed in [2]. Furthermore, to show the usefulness of our algebraic framework, we consider evolutionary games. Three problems are investigated: First, the dynamic equivalence of evolutionary games is introduced. Then the near potential games are investigated. It is shown that when a near potential game and its nearest potential game are dynamically equivalent, players' strategies of the near potential game converge to a pure Nash equilibrium using asynchronous myopic best response adjustment, etc. Finally, the decomposition of an NEG is considered: We show that the decomposition of an NEG can be obtained by summarizing the decompositions of pairwise network games.

The decomposition has some engineering applications, because a near potential game may have the same convergence properties as potential games. Then the above applications of potential games may be extended to their neighboring games. As pointed out in [2], the Nash equilibria of harmonic games have special forms. This property may be used for control design of game-based control problems.

For statement ease, we introduce some notations first:

1) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
2) $\mathcal{D}_k := \{1, 2, \dots, k\}, k \ge 2$.
3) δ_n^i: the i-th column of the identity matrix I_n.
4) $\Delta_n := \{\delta_n^i | i = 1, \dots, n\}$.
5) $\mathbf{1}_\ell = (\underbrace{1, 1, \dots, 1}_{\ell})^T$; $\mathbf{0}_\ell = (\underbrace{0, 0, \dots, 0}_{\ell})^T$.
6) $\mathbf{0}_{p\times q}$: a $p \times q$ matrix with zero entries.
7) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L are in the form of δ_m^k. That is, $\mathrm{Col}(L) \subset \Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.
8) If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted as $L = \delta_n[i_1, i_2, \dots, i_r]$.
9) $\mathrm{Span}\{A_1, \dots, A_s\}$: The subspace spanned by the columns of A_i, $i = 1, \dots, s$.
10) $U \oplus V$: orthogonal sum of two vector spaces.

The semi-tensor product of matrices is defined as follows [4], [5]:

Definition 1.1: Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product (STP) of M and N is defined as

$$M \ltimes N := \left(M \otimes I_{\frac{t}{n}}\right)\left(N \otimes I_{\frac{t}{p}}\right) \in \mathcal{M}_{mt/n\times qt/p} \tag{2}$$

where $\otimes$ is the Kronecker product.

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The STP of matrices is a generalization of conventional matrix product, and all the computational properties of conventional matrix product remain available ([5, Ch. 2]). Throughout this note, the default matrix product is STP, so the product of two arbitrary matrices is well defined, and the symbol $\ltimes$ is mostly omitted.

To apply matrix expression to games, we identify

$$j \sim \delta_{k_i}^j, \quad j = 1, \dots, k_i$$

then the set of strategies $S_i \sim \Delta_{k_i}$, $i = 1, \dots, n$. It follows that the payoff functions can be expressed as

$$c_i(x_1, \dots, x_n) = V_i^c \ltimes_{j=1}^n x_j, \quad i = 1, \dots, n \tag{3}$$

where $V_i^c \in \mathbb{R}^k$ is a row vector called the structure vector of c_i, ($k = \prod_{i=1}^n k_i$). Define the structure vector of the given game G by

$$V_G^c = (V_1^c, V_2^c, \dots, V_n^c) \in \mathbb{R}^{nk}. \tag{4}$$

Then it is clear that the set of finite games has a natural vector space structure as

$$\mathcal{G}_{[n;k_1,\dots,k_n]} \sim \mathbb{R}^{nk}. \tag{5}$$

Note that for a given game $G \in \mathcal{G}_{[n;k_1,\dots,k_n]}$, its structure vector V_G^c completely determines G. So the vector space structure (5) is very natural and reasonable.

The rest of this note is built up as follows. In Section II we discuss the potential subspace $\mathcal{G}_P$, and its basis is provided. Section III considers the non-strategic subspace $\mathcal{N}$. It is defined slightly different from [2] and its basis is also revealed. The pure potential subspace $\mathcal{P}$ is investigated in Section IV. Section V considers the harmonic subspace. Then the orthogonal decomposition formulas are presented. As an application, Section VI considers the dynamic of networked evolutionary games (NEGs). Section VII is a brief conclusion.

II. Subspace of Potential Games

This section will provide the basis of the potential subspace $\mathcal{G}_P$. We first review some results in [6] with a mild and straightforward generalization. We need some notations:

- Let $|S_i| = k_i$, $i = 1, \dots, n$. Then

$$k^{[p,q]} := \begin{cases} \prod_{j=p}^q k_j, & q \geq p \\ 1, & q < p. \end{cases}$$

- $E_i := I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \in \mathcal{M}_{k \times k/k_i}, i = 1, \dots, n.$

Construct a linear equation, called the potential equation, as

$$\begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \tag{6}$$

where $\xi_i \in \mathbb{R}^{k/k_i}$. Then we have the following result:

Theorem 2.1 ([6]): A finite game $G \in \mathcal{G}_{[n;k_1,\dots,k_n]}$ is a potential game, if and only if the potential (6) has solution. Moreover, if ξ is a solution then the potential function can be expressed as

$$P(x_1, \dots, x_n) = V^P \ltimes_{j=1}^n x_j \tag{7}$$

where V^P, the structure vector of the potential function, is

$$V^P = V_1^c - \xi_1^T E_1^T.$$

From the above result, one sees easily that for a potential game the payoff of player i is the sum of a potential function and a term that does not depend on i's strategy.

Next, we define

$$E_P := \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{8}$$

Using (6), after some elementary calculation one sees easily that $\mathcal{G}_P = \text{Span}(E_P)$. Next, we delete the last column of E_n and denote the remaining part by E_n^0, and construct E_P^0 from E_P via replacing E_n by E_n^0. As discussed in [6], one sees easily that $\text{Span}(E_p) = \text{Span}(E_p^0)$, and E_p^0 has full column rank. Hence, we have the following result.

Theorem 2.2: The subspace of potential games is

$$\mathcal{G}_P = \text{Span}(E_P) \tag{9}$$

which has $\text{Col}(E_P^0)$ as its basis.

According to the construction of E_P^0 it is clear that $\dim(\mathcal{G}_P) = k + \sum_{j=1}^n (k/k_j) - 1$, $\dim(\mathcal{H}) = (n-1)k - \sum_{j=1}^n (k/k_j) + 1$, which are already shown in [2].

III. Non-Strategic Subspace

In this section, the non-strategic subspace is defined using exact strategic equivalence of games. Then its basis is obtained.

Definition 3.1: Let $G, \tilde{G} \in \mathcal{G}_{[n;k_1,\dots,k_n]}$. G and $\tilde{G}$ are said to be exact strategically equivalent (E-S equivalent for short), if for any $i \in N$, any $x_i, y_i \in S_i$, and any $x^{-i} \in S^{-i}$, (where $S^{-i} = \prod_{j \neq i} S_j$), we have

$$c_i(x_i, x^{-i}) - c_i(y_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) - \tilde{c}_i(y_i, x^{-i}). \tag{10}$$

Note that the exact strategic equivalence means that for any behavior of his opponents, the change in player i's payoff from any unilateral deviation is the same in G and $\tilde{G}$.

Lemma 3.2: Two games $G, \tilde{G} \in \mathcal{G}_{[n;k_1,\dots,k_n]}$ are E-S equivalent, if and only if for each $x^{-i} \in S^{-i}$ there exists $d_i(x^{-i})$ such that

$$\begin{aligned} &c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i}) = d_i(x^{-i}) \\ &\forall\, x_i \in S_i, \quad \forall\, x^{-i} \in S^{-i},\ i = 1, \dots, n. \end{aligned} \tag{11}$$

(Please refer to the Appendix for all proofs.)

Next, denote the structure vectors of c_i, $\tilde{c}_i$, and d_i by V_i^c, $\tilde{V}_i^c$, and V_i^d respectively, we express (11) in matrix form as

$$\begin{aligned} V_i^c \ltimes_{j=1}^n x_j - \tilde{V}_i^c \ltimes_{j=1}^n x_j &= V_i^d \ltimes_{j \neq i}^n x_j \\ &= V_i^d \left(I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i}^T \otimes I_{k^{[i+1,n]}} \right) \ltimes_{j=1}^n x_j. \end{aligned}$$

Finally, we have

$$B_N^i \left(V_i^d\right)^T = \left(V_i^c - \tilde{V}_i^c\right)^T \tag{12}$$

where

$$\begin{aligned} B_N^i &:= I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \\ &= E_i, \quad i = 1, \dots, n. \end{aligned} \tag{13}$$

The above argument yields the following result:

Theorem 3.3: G and $\tilde{G}$ are E-S equivalent if and only if

$$\left(V_G^c - V_{\tilde{G}}^c\right)^T \in \text{Span}\left(B_N\right) \tag{14}$$

where

$$B_N = \text{diag}(E_1, E_2, \ldots, E_n). \tag{15}$$

Definition 3.4: The subspace $\mathcal{N} := \text{Span}(B_N)$ is called the non-strategic subspace.

From Theorem 3.3, one sees easily that G and $\tilde{G}$ are E-S equivalent if and only if there exists an $\eta \in \mathcal{N}$, such that

$$\left(V_{\tilde{G}}^c\right)^T = \left(V_G^c\right)^T + \eta. \tag{16}$$

Since E_i has k/k_i linearly independent columns, we have $\dim(\mathcal{N}) = \sum_{i=1}^n (k/k_i)$, $\dim(\mathcal{P}) = k - 1$. Define

$$\tilde{E}_P := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{17}$$

Comparing (17) with (8), we can verify that

$$\mathcal{G}_P = \text{Span}(\tilde{E}_P) = \text{Span}(E_P). \tag{18}$$

Observing (17) again, it is obvious that $\mathcal{N} \subset \mathcal{G}_P$.

Constructing $\tilde{E}_P^0$ from $\tilde{E}_P$ via replacing E_n by E_n^0, it is clear that the columns of $\tilde{E}_P^0$ form a basis of $\mathcal{G}_P$.

For a $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$, if its payoff vector $(V_G^c)^T \in \mathcal{N}$ then G is called a non-strategic game. According to the construction of $B_\mathcal{N}$, it is clear that if G is a non-strategic game, then for each given $x^{-i} \in S^{-i}$, $c_i(x_i, x^{-i}) = \text{const}, \forall x_i \in S_i$.

IV. Pure Potential Subspace $\mathcal{P}$

This section provides the basis of pure potential subspace $\mathcal{P}$. Define

$$B_P = \begin{bmatrix} I_k - \frac{1}{k_1} E_1 E_1^T \\ I_k - \frac{1}{k_2} E_2 E_2^T \\ \vdots \\ I_k - \frac{1}{k_n} E_n E_n^T \end{bmatrix} \in \mathcal{M}_{nk \times k}. \tag{19}$$

Using (18), it is easy to verify that $\mathcal{G}_P = \text{Span}(B_P, B_N)$, and $B_P^T B_N = 0$. Hence we have $\mathcal{G}_P = \mathcal{P} \oplus \mathcal{N}$, where

$$\mathcal{P} = \text{Span}(B_P). \tag{20}$$

Since $\dim(\mathcal{P}) = k - 1$, to find the basis of $\mathcal{P}$ one column of B_p needs to be removed. Note that

$$\begin{aligned} \left(I_k - \frac{1}{k_i} E_i E_i^T\right)\mathbf{1}_k &= \left(I_{k^{[1,i-1]}} \mathbf{1}_{k^{[1,i-1]}}\right)\left[\left(I_{k_i} - \frac{1}{k_i}\mathbf{1}_{k_i \times k_i}\right)\mathbf{1}_{k_i}\right] \\ &\quad \times \left(I_{k^{[i+1,n]}} \mathbf{1}_{k^{[i+1,n]}}\right) \\ &= \mathbf{0}_k, \quad i = 1, \ldots, n. \end{aligned}$$

It follows that $B_p \mathbf{1}_k = \mathbf{0}_{nk}$.

Deleting any one column of B_P, say, the last column, and denoting the remaining matrix by B_P^0, then we know that

$$\mathcal{P} = \text{Span}\left(B_P\right) = \text{Span}\left(B_P^0\right)$$

where $\text{Col}(B_P^0)$ is a basis of $\mathcal{P}$.

V. Pure Harmonic Subspace $\mathcal{H}$

In this section, we will construct the basis of pure harmonic subspace $\mathcal{H}$.

Construct a set of vectors as

$$J_s := \begin{bmatrix} \left(\delta_{k_1}^1 - \delta_{k_1}^{i_1}\right)\delta_{k_2}^1 \delta_{k_3}^1 \cdots \delta_{k_s}^1 \left(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}\right)\delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \delta_{k_1}^{i_1}\left(\delta_{k_2}^1 - \delta_{k_2}^{i_2}\right)\delta_{k_3}^1 \cdots \delta_{k_s}^1 \left(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}\right)\delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \delta_{k_1}^{i_1}\delta_{k_2}^{i_2}\left(\delta_{k_3}^1 - \delta_{k_3}^{i_3}\right) \cdots \delta_{k_s}^1 \left(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}\right)\delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \vdots \\ \delta_{k_1}^{i_1}\delta_{k_2}^{i_2}\delta_{k_3}^{i_3} \cdots \left(\delta_{k_s}^1 - \delta_{k_s}^{i_s}\right)\left(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}\right) \cdots \delta_{k_n}^{i_n} \\ -\left(\delta_{k_1}^1 \cdots \delta_{k_s}^1 - \delta_{k_1}^{i_1} \cdots \delta_{k_s}^{i_s}\right)\left(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}\right)\delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \mathbf{0}_{(n-1-s)k} \end{bmatrix}_{\substack{(i_1,\ldots,i_s) \neq \mathbf{1}_s^T; i_{s+1} \neq 1 \\ s = 1,2,\ldots,n-1.}}.$$

Define

$$B_H := [J_1, J_2, \ldots, J_{n-1}]. \tag{21}$$

Then we can show $\text{Col}(B_H)$ is the basis of $\mathcal{H}$:

Theorem 5.1: B_H has full column rank and

$$\mathcal{H} = \text{Span}(B_H). \tag{22}$$

As an application of the bases, we consider the orthogonal decomposition of a finite game $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$.

Construct a matrix

$$B := \left[B_P^0, B_N, B_H\right] \tag{23}$$

and set $d_1 = \dim(\mathcal{P})$, $d_2 = \dim(\mathcal{N})$, and $d_3 = \dim(\mathcal{H})$, then the following result is obvious:

Proposition 5.2: Let

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := B^{-1}\left(V_G^c\right)^T \tag{24}$$

where $x_i \in \mathbb{R}^{d_i}$, $i = 1, 2, 3$. Then

$$\pi_\mathcal{P}(G) = B\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}; \pi_\mathcal{N}(G) = B\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}; \pi_\mathcal{H}(G) = B\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$$

$$\pi_{\mathcal{G}_P}(G) = B\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}; \pi_{\mathcal{G}_H}(G) = B\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}.$$

Remark 5.3: In fact, the inner product defined in [2] is

$$\langle X, Y \rangle := X^T Q Y \tag{25}$$

where $Q = \text{diag}(\underbrace{k_1, \ldots, k_1}_{k}, \underbrace{k_2, \ldots, k_2}_{k}, \ldots, \underbrace{k_n, \ldots, k_n}_{k})$. So the bases obtained in this note is not applicable to the subspaces in [2]. Getting the bases for the subspaces in [2] directly is not easy. But modifying our bases can easily yield their bases.

VI. Application to NEGs

A. Evolutionary Games

Assume a noncooperative finite game in strategic form is repeated infinitely. Then each player can update his strategy by using the game

TABLE I
PAYOFF MATRIX OF EXAMPLE 6.1

c\s	11	12	13	21	22	23	31	32	33
c_1	90	−12	48	−12	1	24	48	24	1
c_2	90	−12	48	−12	−1	24	48	24	−1

TABLE II
RESPONDING STRATEGIES OF EXAMPLE 6.1

s(t+1)\s(t)	11	12	13	21	22	23	31	32	33
f_1	1	3	1	1	3	1	1	3	1
f_2	1	1	1	3	3	3	1	1	1

historical knowledge. Assume the strategy updating rule (SUR) can be expressed as

$$x_i(t+1)=f_i\left(x_j(s), c_j(s) | j \in N, s=0,1,\ldots,t\right), \quad i=1,\ldots,n. \tag{26}$$

There are some commonly used SURs, including: i) unconditional imitation [14]; (ii) Fermi rule [17], [18]; (iii) myopic best response adjustment (MBRA) [21]. These three SURs are mostly useful for evolutionary games, because they are of Markov-type, i.e., only the information of the previous moment is used to update the strategies. Using Markov-type SUR, (26) becomes [7]

$$x_i(t+1) = f_i\left(x_1(t)\right), \ldots, x_n(t)), \quad i=1,\ldots,n. \tag{27}$$

Equation (27) [or (26)] is called the profile dynamics of an evolutionary game.

For our purpose, we introduce MBRA in detail. Assume

$$\begin{aligned} S_i^* := &\left\{ z^* \middle| c_i\left(x_i = z^*, x^{-i} = x^{-i}(t)\right) \right. \\ &\left. = \max_{z \in S_i} c_i\left(x_i = z, x^{-i} = x^{-i}(t)\right) \right\} \\ := &\left\{z_1^*, \ldots, z_r^*\right\}. \end{aligned} \tag{28}$$

We may use the following two options:

(i) Deterministic MBRA (D-MBRA): Choose one corresponding to an ordered priority. For instance (as a default)

$$x_i(t+1) = z_1^* \tag{29}$$

(note that $S_i = \mathcal{D}_{k_i}$). This method leads to a deterministic multi-valued logical dynamics.

(ii) Stochastic MBRA (S-MBRA): Choose any $z_j^* \in S_i^*$ with equal probability. That is

$$x_i(t+1) = z_\mu^*(t), \quad \text{with} \quad p_\mu^i = \frac{1}{r}, \quad \mu=1,\ldots,r. \tag{30}$$

This method leads to a probabilistic multi-valued logical dynamics.

Then the SUR can determine the profile dynamics of evolutionary games.

We use an example to illustrate it. The following example is from [3]:

Example 6.1: A game $G \in \mathcal{G}_{[2;3,3]}$, where $S_1 = S_2 = \{1,2,3\}$, and 1: work; 2: shirk at office; 3: shirk at home. The payoffs are described in the payoff matrix (Table I).

Using MBRA, we can get the best responding strategies, which are shown in Table II. (Since we have $|S_1^*| = |S_2^*| = 1$, D-MBRA and S-MBRA lead to the same result.)

That is

$$x_i(t+1) = f_i\left(x_1(t), x_2(t)\right) = M_i x(t), \quad i=1,2 \tag{31}$$

where $x(t) = \ltimes_{i=1}^2 x_i(t)$, $M_i, i=1,2$ are the structure matrices of f_i, which are

$$\begin{aligned} M_1 &= \delta_3[1,3,1,1,3,1,1,3,1] \\ M_2 &= \delta_3[1,1,1,3,3,3,1,1,1]. \end{aligned} \tag{32}$$

B. Convergence of Near Potential Games

It is well known that a potential game has many nice dynamical properties. For instance, we are particularly interested in the following convergence property: If at each time a single player is chosen randomly to update his strategy, the MBRA, called the asynchronous MBRA, will lead to a pure Nash equilibrium. The near potential games were firstly investigated in detail in [3]. Their basic idea is: "Intuitively, dynamics in potential games and dynamics in games that are "close" (in terms of the payoffs of the players) to potential games should be related." Their main result is: a near potential game will converge to an ϵ equilibrium, where ϵ is estimated by the distance between the game and its closest potential game.

Definition 6.2: Two evolutionary games are said to be dynamically equivalent if they have the same strategy profile dynamics.

The following proposition is straightforward verifiable.

Proposition 6.3: If two games are E-S equivalent then they are dynamically equivalent.

Proposition 6.4: If a game G and its closest potential game $\pi_P(G)$ are dynamically equivalent, then the asynchronous MBRA will lead G to a pure Nash equilibrium.

Remark 6.5:

(i) Proposition 6.4 is obvious. The key issue is how to verify the dynamical equivalence. Reference [7] provides a procedure to construct strategy profile dynamics.

(ii) There are many SURs which lead a potential game to a pure Nash equilibrium. Then it is obvious that they will also lead a game G to a pure Nash equilibrium provided G is dynamically equivalent to its closest potential game $\pi_P(G)$. For instance, it is easy to prove that for a cascading potential game

$$\begin{cases} x_1(t+1) = f_1\left(x_1(t), x_2(t), \ldots, x_n(t)\right) \\ x_2(t+1) = f_2\left(x_1(t+1), x_2(t), \ldots, x_n(t)\right) \\ \vdots \\ x_n(t+1) = f_1\left(x_1(t+1), \ldots, x_{n-1}(t+1), x_n(t)\right) \end{cases}$$

MBRA will lead it to a pure Nash equilibrium.

(iii) Most of the learning algorithms for potential games guarantee convergence to a pure Nash equilibrium. The fictitious play [12], the log-linear learning [1] and its relaxed version [11], etc. are such examples. They also guarantee the convergence of a near potential game G to a pure Nash equilibrium, provided G and $\pi_P(G)$ are dynamically equivalent.

Example 6.6: Recall Example 6.1. It is easy to check that this game is not a potential game, and

$$\begin{aligned} V_{\pi_P(G)} = [&89.7778, -11.8889, 48.1111, -11.8889, \\ &0.4444, 24.4444, 48.1111, 24.4444, 0.4444, \\ &90.2222, -12.1111, 47.8889, -12.1111, -0.4444, \\ &23.5556, 47.8889, 23.5556, -0.4444]. \end{aligned}$$

A straightforward computation shows that for this $\pi_P(G)$ the profile dynamics is the same as (31), (32). According to Proposition 6.4, G will also converge to a pure Nash equilibrium using asynchronous MBRA (or any other SURs mentioned in Remark 6.5).

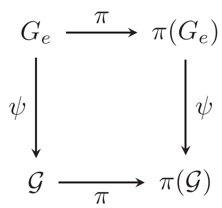

Fig. 1. Commutative mappings.

Note that [3] shows that G will converge to an ϵ-equilibrium and we prove that it will converge to a pure Nash equilibrium.

In general, it is expected that when a game is close enough to its certain projection, it has the same dynamic behaviors as the projected game as long as it is dynamically equivalent to its projection.

C. Decomposition of Networked Evolutionary Games

Definition 6.7 ([7]): A networked evolutionary game (NEG), denoted by $\mathcal{G} = ((N, E), G, \Pi)$, consists of three factors:

(i) a network graph: (N, E);

(ii) a fundamental network game (FNG): G with two players. Players i and j play this game provided $(i, j) \in E$. The payoff of j from the game is denoted by c_{ji}.

(iii) a local information based strategy updating rule (SUR), denoted by Π:

$$x_i(t+1) = f_i\left(x_j(t), c_j(t) | j \in U(i)\right), \quad i = 1, \ldots, n. \quad (33)$$

Let $U_2(i) := \{k | k \in U(j) \text{ and } j \in U(i)\}$. Then (33) can be expressed as

$$x_i(t+1) = f_i\left(x_j(t) | j \in U_2(i)\right), \quad i = 1, \ldots, n \quad (34)$$

where $c_j(t) = \sum_{i \in U(j)} c_{ji}(t)$.

Consider an NEG $\mathcal{G} = ((N, E), G, \Pi)$. Let $G_e,\ e = (i, j) \in E$ be the fundamental game performed over edge e. Define a natural inclusion mapping: $\psi : G_e \hookrightarrow \mathcal{G}$ as

$$c_\ell(x_1, \ldots, x_n) := \begin{cases} c_\ell(x_i, x_j), & \ell \in \{i, j\} \\ 0, & \text{Otherwise.} \end{cases}$$

The following lemma can be proved by a tedious but straightforward computation.

Lemma 6.8: Fig. 1 is commutative. That is

$$\psi \circ \pi(G_e) = \pi \circ \psi(G_e), \quad e \in E \quad (35)$$

where $\pi \in \{\pi_P, \pi_N, \pi_H, \pi_{\mathcal{G}_P}, \pi_{\mathcal{G}_H}\}$.

In the light of Lemma 6.8, the following decomposition of NEGs is obvious.

Proposition 6.9: Consider a networked evolutionary game $\mathcal{G} = ((N, E), G, \Pi)$. It has the orthogonal decomposition as

$$\mathcal{G} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{\text{Potential games}} \overbrace{\oplus \ \mathcal{H}}^{\text{Harmonic games}} \quad (36)$$

where $\mathcal{P} = \sum_{e \in E} \mathcal{P}^e$, $\mathcal{N} = \sum_{e \in E} \mathcal{N}^e$, $\mathcal{H} = \sum_{e \in E} \mathcal{H}^e$, $\mathcal{G}_P = \sum_{e \in E} \mathcal{G}_P^e$, and $\mathcal{G}_H = \sum_{e \in E} \mathcal{G}_H^e$, and $\mathcal{P}^e$, $\mathcal{N}^e$, $\mathcal{H}^e$, $\mathcal{G}_P^e$, and $\mathcal{G}_H^e$ are the corresponding subspaces of the game G over edge $e \in E$.

VII. Conclusion

In this note, we investigated the decomposition and the decomposed subspaces of non-cooperative strategic form finite games, $\mathcal{G}_{[n;k_1,\ldots,k_n]}$. First, $\mathcal{G}_{[n;k_1,\ldots,k_n]}$ was given a vector space structure as $\mathbb{R}^{nk}$ in a very natural way. Then the subspace of potential games, $\mathcal{G}_P$, and the non-strategic subspace, $\mathcal{N}$, defined in a straightforward way, are investigated and their subspace bases were obtained. Using them, the bases of the pure potential subspace $\mathcal{P}$ and the pure harmonic subspace $\mathcal{H}$ were obtained. Then the explicit form of orthogonal decomposition was presented. The difference between the decompositions in this note and in [2] was explained. Finally, the decomposition results were applied to investigate evolutionary games. Two results were obtained: i) the convergence of a near-potential game to a pure Nash equilibrium was revealed and ii) the decomposition of networked evolutionary games were obtained via the decomposition of their fundamental network games.

Based on functional operator and graph theory, [2] provided an elegant vector space structure with decomposed subspaces for finite games. But using their framework it is not easy to calculate decomposed subspaces directly. Based on linear algebra, our follow-up work provides bases for all subspaces respectively. They make the numerical computation much easier. Moreover, the explicit form of bases may help to reveal the subspaces' structures and properties.

Appendix

Proof of Lemma 3.2: (Necessity) Assume (11) fails. Then there exist an i and an $x^{-i} \in S^{-i}$, such that $c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i})$ depends on x_i. That is, there exists $a_i, b_i \in S_i$ such that

$$c_i(a_i, x^{-i}) - \tilde{c}_i(a_i, x^{-i}) \neq c_i(b_i, x^{-i}) - \tilde{c}_i(b_i, x^{-i}).$$

Then

$$c_i(a_i, x^{-i}) - c_i(b_i, x^{-i}) \neq \tilde{c}_i(a_i, x^{-i}) - \tilde{c}_i(b_i, x^{-i})$$

which violates (10).

(Sufficiency) From (11), we have

$$c_i(x_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) + d_i(x^{-i}), \quad \forall\, x_i \in S_i.$$

Plugging it into left-hand side of (10) yields the equality. ■

To prove Theorem 5.1, we first give the following lemma.

Lemma A.1: Given positive integers $p, q, r > 1$. Then the vectors $(\delta_p^1 - \delta_p^{i_p})(\delta_q^1 - \delta_q^{i_q})\delta_s^{i_s}$, $1 < i_p \leq p$, $1 < i_q \leq q$, $1 \leq i_s \leq s$, are linearly independent.

Proof of Lemma A.1: Let

$$\sum_{1 < i_p \leq p, 1 < i_q \leq q, 1 \leq i_s \leq s} a_{i_p, i_q, i_s} \left(\delta_p^1 - \delta_p^{i_p}\right)\left(\delta_q^1 - \delta_q^{i_q}\right)\delta_s^{i_s} = 0$$

where for all $1 < i_p \leq p, 1 < i_q \leq q, 1 \leq i_s \leq s$, a_{i_p, i_q, i_s} are real numbers.

Note that for all $1 \leq j \leq pq$, all distinct $1 \leq j_1, j_2 \leq s$, if the l-th entry of $\delta_{pq}^j \delta_s^{j_1}$ does not equal 0, then the l-th entry of $\delta_{pq}^j \delta_s^{j_2}$ must equal 0. Hence, for all $1 \leq i_s \leq s$

$$\sum_{1 < i_p \leq p, 1 < i_q \leq q} a_{i_p, i_q, i_s} \left(\delta_p^1 - \delta_p^{i_p}\right)\left(\delta_q^1 - \delta_q^{i_q}\right)$$

$$= \sum_{1 < i_p \leq p, 1 < i_q \leq q} a_{i_p, i_q, i_s} \left(\delta_p^1 \delta_q^1 - \delta_p^1 \delta_q^{i_q} - \delta_p^{i_p} \delta_q^1 + \delta_p^{i_p} \delta_q^{i_q}\right) = 0.$$

It is obvious that $\delta_p^1 \delta_q^1, \delta_p^1 \delta_q^{i_q}, \delta_p^{i_p} \delta_q^1, \delta_p^{i_p} \delta_q^{i_q}$, $1 < i_p \leq p, 1 < i_q \leq q$, are exactly all different columns of I_{pq}, hence linearly independent. Then for all $1 < i_p \leq p$, all $1 < i_q \leq q$, all $1 \leq i_s \leq s$, $a_{i_p, i_q, i_s} = 0$. ■

Using Lemma A.1, we prove Theorem 5.1.

Proof of Theorem 5.1: In J_1, only when $i_1 = 1$ or $i_2 = 1$ we get zero vectors. From Lemma A1, J_1 are linearly independent vectors.

The cardinality of J_1 is

$$|J_1| = \frac{k}{k_1 k_2}(k_1 - 1)(k_2 - 1).$$

In J_2, only when $(i_1, i_2) = (1, 1)$ or $i_3 = 1$ we have the third block is zero. And when the third block is zero, the first two blocks must be zero. From Lemma A1, the third block of J_2 are linearly independent vectors. Then J_2 are linearly independent vectors, and J_1, J_2 are also linearly independent vectors.

The cardinality of J_2

$$|J_2| = \frac{k}{k_1 k_2 k_3}(k_1 k_2 - 1)(k_3 - 1).$$

In general, for all $s = 1, \ldots, n-1$,

$$|J_s| = \frac{k}{k_1 k_2 \cdots k_{s+1}}(k_1 k_2 \cdots k_s - 1)(k_{s+1} - 1)$$

from Lemma A1, the s-th block of J_{s-1} are linearly independent vectors, then J_{s-1} are linearly independent vectors, and $J_1, J_2, \ldots, J_{s-1}$ are also linearly independent vectors.

Hence $J_1, J_2, \ldots, J_{n-1}$ are $\sum_{s=1}^{n-1} |J_s|$ linearly independent vectors.

Since

$$\sum_{i=1}^{n-1} |J_s| = (n-1)k - \sum_{i=1}^{n} \frac{k}{k_i} + 1$$

which is the dimension of $\mathcal{H}$, we conclude that $\mathcal{H} = \text{Span}(B_H)$. ■

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