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A note on observability of Boolean control networks[☆]

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ABSTRACT

The observability of Boolean control networks is investigated. The pairs of states are classified into three classes: (i) diagonal, (ii) h-distinguishable, and (iii) h-indistinguishable. For h-indistinguishable pairs, we construct a matrix $\mathcal{W}$ called the transferable matrix, which indicates the control-transferability among h-indistinguishable pairs. Modifying $\mathcal{W}$ yields a Boolean matrix $\mathcal{U}^0$, which is used as the initial matrix for an iterative algorithm. After finite iterations a stable $\mathcal{U}^*$ is reached, which is called the observability matrix. It is proved that a Boolean control network is observable, if and only if, the last column of $\mathcal{U}^*$, $\mathrm{Col}_{r+1}(\mathcal{U}^*) = \mathbf{1}_r$. Some numerical examples are presented.

1. Introduction

The Boolean networks (BNs) were first proposed by S. Kauffman to describe genetic regulatory networks [1]. Since then the BN has attracted a considerable attention from systems biology, physics as well as systems science. In 2001, [2] pointed out that the genetic regulatory networks have input(s) and output(s), and they can be described as Boolean control networks (BCNs). Then the investigation of BCNs increases [3–6]. But during this period, most of the research were concentrated on control only. As pointed out by [5] that "One of the major goals of systems biology is to develop a control theory for complex biological systems". But because the genetic networks are logical and there were shortage of proper tools to deal with logical dynamic systems, the results on Boolean control networks (BCNs) were limited.

Using semi-tensor product (STP) of matrices, an algebraic state space approach to BNs and BCNs was proposed [7,8]. It stimulates the research on BNs and BCNs. We refer the reader to [9] for several dynamic and/or control problems of BNs, and to [10] for STP.

The controllability and observability are two fundamental problems in the control of BNs as well as in the control theory. The controllability of various types of BNs has been solved neatly. For instance, [7,11] solved the controllability of standard form of BCNs; the controllability of state restricted BCNs was solved in [12]; the controllability of probabilistic BCNs was solved in [13,14]. The controllability of time-varying BCNs [15], higher order BCNs [16,17], switched BCNs [18,19], time-delay BCNs [20,21], and periodic BCNs [22], etc., has also been investigated.

Similarly, the observability of BCNs has also been widely investigated. Though there is no dual relationship between controllability and observability such as for linear systems, as a convention, sometimes the observability of BCNs is still discussed simultaneously with controllability [7,21,23–26].

Unlike the controllability, the observability of BCNs has various definitions, and for the most general (sharp) definition, the necessary and sufficient condition was still not known until [27], see also [28].

First of all, [27] discussed four different definitions of observability in the recent literature. We first cite these four definitions in a uniform way, which might be different from the original ones in statement, but have been proved in [27] that the following four definitions are equivalent to their original ones.

Definition 1.1. A BCN is observable, if

(D1) [7] for any initial state x_0 there exists an input sequence $\{u_0, u_1, \ldots\}$ such that for any $\bar{x}_0 \neq x_0$ the corresponding output sequences $(y_0, y_1, \ldots) \neq (\bar{y}_0, \bar{y}_1, \ldots)$;

(D2) [11] for any two distinct states x_0, $\bar{x}_0$ there is an input sequence $\{u_0, u_1, \ldots, u_p\}, p \in \mathbb{Z}_+$, such that the corresponding output sequences $(y_0, y_1, \ldots, y_p) \neq (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_p)$;

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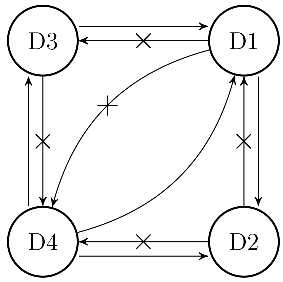

Fig. 1. The relationships of D1–D4.

(D3) [23] there exists an input sequence $\{u_0, u_1, \ldots, u_p\}$, $p \in \mathbb{Z}_+$, such that for any two distinct x_0, $\bar{x}_0$, the corresponding output sequences $(y_0, y_1, \ldots, y_p) \neq (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_p)$;

(D4) [24] for any two distinct states x_0, $\bar{x}_0$ and for any input sequence $\{u_0, u_1, \ldots\}$, the corresponding output sequences $(y_0, y_1, \ldots) \neq (\bar{y}_0, \bar{y}_1, \ldots)$.

The relationship among these four definitions is described in Fig. 1 [27].

In Fig. 1 "$\rightarrow$" means implication and "$\nrightarrow$" means not implication. Note that the "implication" means that if a BCN satisfies the preceding definition it also satisfies the following one. From Fig. 1 it is clear that D2 is the most sensitive (sharp) one. So as proposed by [27], we may take D2 as the standard one and concentrate on this definition. Hereafter, the observability of BCNs we concerned will be the one specified by D2.

In [11] only a sufficient condition was provided, while other papers deal with various other kinds of observability. Hence, the necessary and sufficient condition for observability of Boolean networks was still unknown until [27].

By resorting to formal language and finite automata, [27] (refer also to [28]) presents a necessary and sufficient condition for the observability of BCNs. Their result is like this: For each pair of distinct states $(x_0, \bar{x}_0)$, an algorithm is provided to construct a deterministic finite automata (DFA), denoted by $A_{(x_0,\bar{x}_0)}$. Then a system is not observable, if and only if, there is a pair of distinct states $(x_0, \bar{x}_0)$, such that the corresponding DFA, $A_{(x_0,\bar{x}_0)}$ can recognize its corresponding alphabet.

The result provided by [27] is the first theoretically verifiable necessary and sufficient condition for the observability of BCNs. But its computational complexity is a severe problem. As described in the paper, it is necessary to draw a DFA for each pair of h-indistinguishable pair of states, and then verify its recognizable languages. It can be practically done only for very small toy systems. Moreover, the knowledge about formal language and finite automata is required to understand their technique.

The purpose of this paper is to give an alternative set of necessary and sufficient conditions for the observability of BCNs. The necessary and sufficient conditions are easily verifiable and do not involve any additional auxiliary machines such as finite automata or so. Using the transition matrix of h-indistinguishable matrix $\mathcal{W}$ we construct a Boolean matrix, $\mathcal{U}^0$. That is a matrix with entries in $\{0, 1\}$. Then an algorithm is proposed to perform an iteration on $\{\mathcal{U}^i | i = 0, 1, \ldots\}$. After finite iterations a fixed matrix $\mathcal{U}^*$, called the observability matrix, will be reached. It is proved that the BCN is observable, if and only if, the last column of $\mathcal{U}^*$, which is the set of distinguishable indices of each rows respectively, is $\mathrm{Col}_{r+1}(\mathcal{U}^*) = \mathbf{1}_r$.

Though the approach seems completely different from [27], the initial idea was motivated by [27].

The paper is organized as follows: Section 2 presents some preliminaries. It consists of two subsections: one is a brief introduction to the semi-tensor product of matrices, and the other is for the algebraic state space representation of logical dynamic systems. Section 3 studies the observability of BCN. The h-indistinguishable matrix $\mathcal{W}$ is constructed. Using it, the algorithm is introduced. Then the main result is obtained as a necessary and sufficient condition. In Section 4 some illustrative examples are presented to demonstrate the algorithm and the main result. Some related topics are discussed in Section 5 as the concluding remarks.

2. Preliminaries

2.1. Semi-tensor product of matrices

This subsection gives a brief review for STP. The readers can refer to [10] for details.

First, we give some notations:

- $\mathbb{Z}_+$: the set of non-negative numbers.
- $\mathbf{1}_n = [\underbrace{1, \ldots, 1}_{n}]^T$.
- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the ith column (row) of M.
- $\mathcal{D} := \{0, 1\}$.
- δ_n^i: the ith column of the identity matrix I_n.
- $\Delta_n := \{\delta_n^i | i = 1, \ldots, n\}$, $\Delta := \Delta_2$.
- A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L, denoted by $\mathrm{Col}(L)$, are of the form δ_m^k, $1 \leq k \leq m$. That is,

  $\mathrm{Col}(L) \subset \Delta_m$.

  Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.
- If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_r}]$. For the sake of brevity, it is briefly denoted as $L = \delta_n[i_1, i_2, \ldots, i_r]$.

Definition 2.1. Let $M \in \mathcal{M}_{m\times n}$ and $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product (STP) of M and N, denoted by $M \ltimes N$, is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \quad (1)$$

where $\otimes$ is the Kronecker product.

When $n = p$, the STP coincides with the conventional matrix product. So the STP is a generalization of conventional matrix product. Fortunately, it keeps all the properties of the conventional matrix product unchanged. We, therefore, omit the symbol "$\ltimes$" mostly. In addition, it has some new properties. The following property is frequently used in the sequel.

Proposition 2.2. *Let $X \in \mathbb{R}^m$ be a column and M be any matrix. Then*

$$X \ltimes M = (I_m \otimes M)X. \quad (2)$$

Definition 2.3 ([29]). $M \in \mathcal{M}_{m\times p}$, $N \in \mathcal{M}_{n\times p}$. The Khatri–Rao product of M and N is defined as

$$M * N := [\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N), \ldots, \mathrm{Col}_p(M) \ltimes \mathrm{Col}_p(N)] \in \mathcal{M}_{mn\times p}. \quad (3)$$

2.2. Algebraic state space representation of Boolean networks

Definition 2.4. 1. A function $f : \mathcal{D}^n \to \mathcal{D}$ is called a Boolean function. It can be expressed as

$$y = f(x_1, x_2, \ldots, x_n), \quad y, x_1, \ldots, x_n \in \mathcal{D}. \quad (4)$$

2. A mapping $F : \mathcal{D}^n \to \mathcal{D}^m$ is called a Boolean mapping. A Boolean mapping F is composed of m Boolean functions, as

$$F: \begin{cases} y_1 = f_1(x_1, \ldots, x_n), \\ \vdots \\ y_m = f_m(x_1, \ldots, x_n). \end{cases} \tag{5}$$

Identifying

$$1 \sim \delta_2^1, \qquad 0 \sim \delta_2^2,$$

then a Boolean variable (also called a logical variable) x can be expressed as $\binom{x}{1-x} \in \Delta$, called the vector form of a Boolean variable x. Using vector forms, a Boolean function f becomes $f : \Delta^n \to \Delta$ and a Boolean mapping F becomes $F : \Delta^n \to \Delta^m$. Furthermore, we then have the following algebraic expression.

Theorem 2.5 ([10]). *Let $f : \mathcal{D}^n \to \mathcal{D}$ be a Boolean function. Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2\times 2^n}$, such that in vector form (4) can be expressed as*

$$f(x_1, \ldots, x_n) = M_f \ltimes_{i=1}^n x_i. \tag{6}$$

M_f is called the structure matrix of f.

Consider the Boolean mapping (5). According to Theorem 2.5, there exist $M_i, i = 1, \ldots, m$ which are the structure matrices of the corresponding component functions. Then we have the following result.

Theorem 2.6 ([10]). *Consider the Boolean mapping (5). In vector form, let $x = \ltimes_{i=1}^n x_i$, $y = \ltimes_{i=1}^m y_i$. Then there exists a unique logical matrix $M_F \in \mathcal{L}_{2^m\times 2^n}$, such that (5) can be expressed as*

$$y = M_F x, \tag{7}$$

where

$$M_F = M_1 * M_2 * \cdots * M_m \tag{8}$$

is called the structure matrix of F.

3. Observability of Boolean control networks

A Boolean control network can be described as follows:

$$\begin{cases} x_1(t+1) = f_1(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)), \\ \vdots \\ x_n(t+1) = f_n(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)), \\ \quad y_j(t) = h_j(x_1(t), \ldots, x_n(t)), \quad j = 1, \ldots, s, \end{cases} \tag{9}$$

where $x_i(t) \in \mathcal{D}$, $i = 1, \ldots, n$ are state variables; $u_i(t) \in \mathcal{D}$, $i = 1, \ldots, m$ are controls; and $y_i(t) \in \mathcal{D}$, $i = 1, \ldots, s$ are outputs.

Using Theorems 2.5 and 2.6, we have the following result.

Corollary 3.1. *In vector form the BCN (9) can be expressed as*

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{10}$$

where $x(t) = \ltimes_{i=1}^n x_i$, $u(t) = \ltimes_{i=1}^m u_i$, and $y(t) = \ltimes_{i=1}^s y_i$; $L \in \mathcal{L}_{2^n\times 2^{m+n}}$, $H \in \mathcal{L}_{2^s\times 2^n}$.

(10) is called the algebraic state space representation of (9). Since (9) and (10) are equivalent, hereafter we consider (10) only. Denote the state space as $\mathcal{X} := \Delta^n = \Delta_{2^n}$, we consider the pair of states $\{x, \bar{x}\} \in \mathcal{X} \times \mathcal{X}$ and construct a partition of $\mathcal{X} \times \mathcal{X}$ as

$$\mathcal{X} \times \mathcal{X} = D \cup \Xi \cup \Theta,$$

where

$$D := \left\{\{x, x\} | x \in \mathcal{X}\right\} \subset \mathcal{X} \times \mathcal{X};$$
$$\Xi := \left\{\{x, \bar{x}\} | x \neq \bar{x} \text{ and } Hx = H\bar{x}\right\} \subset \mathcal{X} \times \mathcal{X};$$
$$\Theta := \left\{\{x, \bar{x}\} | x \neq \bar{x} \text{ and } Hx \neq H\bar{x}\right\} \subset \mathcal{X} \times \mathcal{X}.$$

The pairs in D are called the diagonal pairs; the pairs in Ξ are called the h-indistinguishable pairs; and the pairs in Θ are called the h-distinguishable pairs. Note that we consider $\{x, y\} = \{y, x\}$, that is, the order of the pair is ignored.

Definition 3.2. A pair $\{x, \bar{x}\}$ is said transferable to $\{z, \bar{z}\}$, denoted by $\{x, \bar{x}\} \to \{z, \bar{z}\}$, if there exists a control u such that $z = Lux$ and $\bar{z} = Lu\bar{x}$ or $z = Lu\bar{x}$ and $\bar{z} = Lux$. That is, the first pair can be driven by some control to the second pair. Denote by

$$w_{\{x,\bar{x}\}\to\{z,\bar{z}\}}$$

the number of distinct controls, which can drive $\{x, \bar{x}\}$ to $\{z, \bar{z}\}$, is called the transferable index.

Next, we explain how to calculate the transferable index. Assume $x = \delta_{2^n}^p$, $\bar{x} = \delta_{2^n}^q$, $z = \delta_{2^n}^\alpha$, $\bar{z} = \delta_{2^n}^\beta$. We split L into 2^m equal size square matrices as

$$L = [L_1, L_2, \ldots, L_{2^m}].$$

Then it is easy to see the following result.

Proposition 3.3. *Using the above notation, $\{x, \bar{x}\}$ can be driven to $\{z, \bar{z}\}$ by control $\delta_{2^m}^j$, if and only if,*

$$\left\{\mathrm{Col}_p(L_j), \mathrm{Col}_q(L_j)\right\} = \left\{\delta_{2^n}^\alpha, \delta_{2^n}^\beta\right\}. \tag{11}$$

Using Proposition 3.3, calculating the transferable index $w_{\{x,\bar{x}\}\to\{z,\bar{z}\}}$ becomes a simple issue.

Given a BCN, we consider its observability. To begin with, assume

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_r\},$$

we construct a matrix $\mathcal{W} \in \mathcal{M}_{r\times(r+1)}$, called the transferable matrix, as

$$\mathcal{W} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,r} & w_{1,r+1} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,r} & w_{2,r+1} \\ \vdots & & & & \\ w_{r,1} & w_{r,2} & \cdots & w_{r,r} & w_{r,r+1} \end{bmatrix}, \tag{12}$$

where

$$w_{i,j} = w_{\xi_i\to\xi_j}, \quad j \neq r+1,$$

is the transferable index from ξ_i to ξ_j, and

$$w_{i,r+1} = w_{\xi_i\to D}$$

is the transferable index from ξ_i to any diagonal pair.

Second step, we define the row indistinguishable index d_i as

$$d_i = \sum_{j=1}^{r+1} w_{i,j}, \quad i = 1, \ldots, r. \tag{13}$$

Note that the row indistinguishable index d_i is the transferable index of $\xi_i \to \Xi \cup D$. That is,

$$d_i = w_{\xi_i\to\Xi\cup D}. \tag{14}$$

Using the row indistinguishable indices, we construct another matrix $\mathcal{U}^0 = (u_{i,j}) \in \mathcal{M}_{r\times(r+1)}$ as (For coding, just modify $\mathcal{W}$.)

$$u_{i,j}^0 = \begin{cases} 1, & w_{i,j} > 0, \\ 0, & w_{i,j} = 0, \quad i = 1, \ldots, r;\ j = 1, \ldots, r; \end{cases}$$

and

$$u^0_{i,r+1} = \begin{cases} 1, & d_i < 2^m, \\ 0, & d_i = 2^m, \end{cases} \quad i = 1, \ldots, r.$$

It is clear that $\mathcal{U}^0$ is a Boolean matrix.

Using $\mathcal{U}^0$, we can inductively construct $\mathcal{U}^k$ as follows:

Algorithm 3.4. Assume $\mathcal{U}^k = (u^k_{i,j})$ is known, we construct $\mathcal{U}^{k+1}$:

- For $i = 1, \ldots, r$, if $u^k_{i,r+1} = 1$,

$$\mathrm{Col}_{r+1}\left(\mathcal{U}^{k+1}\right) = \mathrm{Col}_{r+1}\left(\mathcal{U}^k\right) \vee \mathrm{Col}_i\left(\mathcal{U}^k\right). \quad (15)$$

- If

$$\mathcal{U}^{k^*+1} = \mathcal{U}^{k^*}, \quad (16)$$

set

$$\mathcal{U}^* := \mathcal{U}^{k^*}, \quad (17)$$

and stop.

Note that since at each efficient iteration, which means (16) does not hold, the norm $\| \mathrm{Col}_{r+1}(\mathcal{U}^k)\|_1$ (which is the sum of all elements) will increase at least one. So at most after $r-1$ iterations (16) becomes true. And then we have $\mathcal{U}^k = \mathcal{U}^{k^*}$, $\forall k > k^*$. We, therefore, can always get $\mathcal{U}^*$.

Theorem 3.5. *Consider the BCN (9) (equivalently, (10)).*

(1) *It is observable, if and only if,*

$$\mathrm{Col}_{r+1}\left(\mathcal{U}^*\right) = \mathbf{1}_r. \quad (18)$$

(2) $\xi_i = (x_0, \bar{x}_0)$ *is an indistinguishable pair, if and only if, the* $(i, r+1)$*th element of* $\mathcal{U}^*$ *satisfies*

$$u_{i,r+1} = 0. \quad (19)$$

Proof. We prove (1) first.

(Sufficiency) If $\{x_0, \bar{x}_0\} \in \Theta$, then this pair of points are obviously distinguishable. So we only need to worry about $\{x_0, \bar{x}_0\} \in \Xi$. For $\xi_i = \{x_0, \bar{x}_0\} \in \Xi$, if its row indistinguishable index $d_i < 2^m$, since

$$w_{\xi_i \to \Xi \cup D} + w_{\xi_i \to \Theta} = 2^m,$$

observing (14), it is clear that there is at least one control, which will drive ξ_i to Θ. Hence ξ_i is h-distinguishable. Now consider matrix $\mathcal{U}^0$. If $\xi_i, \xi_j \in \Xi$, $u^0_{i,r+1} = 0$, $u^0_{j,r+1} = 1$, and $u^0_{i,j} = 1$, then by construction, ξ_j is h-distinguishable and ξ_i is not. But since $u^0_{i,j} = 1$, which means there exists at least one control, which drives ξ_i to ξ_j. Since ξ_j is h-distinguishable, ξ_i is second step h-distinguishable. So in $\mathcal{U}^1$ we change $u^0_{i,r+1} = 0$ to $u^1_{i,r+1} = 1$. Continuing this process until $\mathcal{U}^{k^*} = \mathcal{U}^{k^*+1}$. If $\mathrm{Col}_{r+1}(\mathcal{U}^{k^*}) = \mathbf{1}_r$, it is obvious that the system is observable.

(Necessity) Let

$$\Gamma = \left\{\xi_i \in \Xi \,\middle|\, u^*_{i,r+1} = 0, \; i = 1, \ldots, r\right\}.$$

Then it is clear that Γ is a control-invariant set. That is, if $\gamma \in \Gamma$, then

$$Lu\gamma \in \Gamma, \quad \forall u \in \Delta_{2^m}.$$

Hence Γ is the set of indistinguishable pairs. It follows that $\Gamma = \emptyset$ is a necessary condition for observability. This fact implies $\mathrm{Col}_{r+1}(\mathcal{U}^*) = \mathbf{1}_r$.

From the proof of (1), it is easy to see that (2) is an immediate consequence. □

Remark 3.6. 1. From Theorem 3.5 it is clear that the ith component of $\mathrm{Col}_{r+1}\left(\mathcal{U}^*\right)$ indicates whether ξ_i is a distinguishable pair. So we call it the distinguishable index of ξ_i. Hence, we can call $\mathrm{Col}_{r+1}\left(\mathcal{U}^k\right)$ the vector of distinguishable indices, denote it by V^k_d.

2. In Algorithm 3.4 each time we updated only the vector of distinguishable indices. So (16) can be replaced by

$$\|V^{k^*+1}_d\|_1 = \|V^{k^*}_d\|_1. \quad (20)$$

4. Illustrative examples

The first example shows the detailed calculating process.

Example 4.1. Consider the following Boolean network

$$\begin{cases} x_1(t+1) = [u(t) \wedge \neg(x_1(t) \wedge x_2(t) \wedge x_3(t))] \vee \{\neg u(t) \\ \qquad \wedge [(x_1(t) \wedge (x_2(t) \vee \neg x_3(t))) \\ \qquad \vee (\neg x_1(t) \wedge x_2(t) \wedge \neg x_3(t))]\}, \\ x_2(t+1) = \{u(t) \wedge [(x_1(t) \wedge x_2(t) \wedge \neg x_3(t)) \\ \qquad \vee (\neg x_1(t) \wedge x_2(t) \wedge x_3(t)) \\ \qquad \vee \neg(x_1(t) \vee x_2(t) \vee x_3(t))]\} \\ \qquad \vee \{\neg u(t) \wedge [(x_1(t) \wedge x_2(t)) \\ \qquad \vee (\neg x_1(t) \wedge x_2(t) \wedge x_3(t))]\}, \\ x_3(t+1) = [u(t) \wedge \neg(x_2(t) \wedge x_3(t))] \\ \qquad \vee [\neg u(t) \wedge (\neg x_1(t) \wedge \neg x_2(t) \wedge x_3(t))], \\ y_1(t) = x_1(t) \vee \neg x_2(t) \vee x_3(t), \\ y_2(t) = \neg x_1(t) \vee x_2(t) \wedge \neg x_3(t). \end{cases} \quad (21)$$

Its algebraic form is

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \quad (22)$$

where

$$L = \delta_8[8 \;\; 1 \;\; 3 \;\; 3 \;\; 2 \;\; 3 \;\; 3 \;\; 1 \;\; 1 \;\; 4 \;\; 5 \;\; 3 \;\; 5 \;\; 3 \;\; 7 \;\; 7],$$
$$H = \delta_4[2 \;\; 1 \;\; 2 \;\; 2 \;\; 2 \;\; 3 \;\; 2 \;\; 1].$$

A straightforward computation shows that the h-indistinguishable pairs are

$$\Xi = \left\{\xi_1 = \{\delta^1_8, \delta^3_8\}, \xi_2 = \{\delta^1_8, \delta^4_8\}, \xi_3 = \{\delta^1_8, \delta^5_8\}, \xi_4 = \{\delta^1_8, \delta^7_8\}, \right.$$
$$\xi_5 = \{\delta^2_8, \delta^8_8\}, \xi_6 = \{\delta^3_8, \delta^4_8\}, \xi_7 = \{\delta^3_8, \delta^5_8\}, \xi_8 = \{\delta^3_8, \delta^7_8\},$$
$$\left. \xi_9 = \{\delta^4_8, \delta^5_8\}, \xi_{10} = \{\delta^4_8, \delta^7_8\}, \xi_{11} = \{\delta^5_8, \delta^7_8\}\right\}.$$

Using Proposition 3.3, the transferable matrix $\mathcal{W}$ is expressed in Table 1, where the last column is for the row indistinguishable index d_i.

Using $\mathcal{W}$, we can construct $\mathcal{U}_0$, which are shown in Table 2. The last column of Table 2 shows whether the corresponding row pair is observable, "1″ means "Yes" and "0″ means "No".

Next, since $u^0_{i,r+1} = 1$, for $i \in J_0 := \{1, 2, 4, 7, 9, 11\}$, we have

$$\begin{aligned} V^1_d &= \mathrm{Col}_{r+1}\left(\mathcal{U}^0\right) \bigcup \cup_{i \in J_0} \mathrm{Col}_i\left(\mathcal{U}^0\right) \\ &= [1 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1]^T. \end{aligned}$$

We have $u^1_{i,r+1} = 1$, for $i \in J_1 := \{1, 2, 4, 6, 7, 8, 9, 11\}$, and it follows that

$$\begin{aligned} V^2_d &= \mathrm{Col}_{r+1}\left(\mathcal{U}^1\right) \bigcup \cup_{i \in J_1} \mathrm{Col}_i\left(\mathcal{U}^1\right) \\ &= [1 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]^T. \end{aligned}$$

It turns out that $u^2_{i,r+1} = 1$, for $i \in J_2 := \{1, 2, 4, 6, 7, 8, 9, 10, 11\}$, and then

$$\begin{aligned} V^3_d &= \mathrm{Col}_{r+1}\left(\mathcal{U}^2\right) \bigcup \cup_{i \in J_2} \mathrm{Col}_i\left(\mathcal{U}^2\right) \\ &= [1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]^T. \end{aligned}$$

Table 1
Transferable matrix $\mathcal{W}$.

P	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	D	d_i
ξ_1	0	0	1	0	0	0	0	0	0	0	0	0	1
ξ_2	1	0	0	0	0	0	0	0	0	0	0	0	1
ξ_3	0	0	1	0	1	0	0	0	0	0	0	0	2
ξ_4	0	0	0	1	0	0	0	0	0	0	0	0	1
ξ_5	0	0	0	0	0	0	0	0	0	1	0	1	2
ξ_6	0	0	0	0	0	0	1	0	0	0	0	1	2
ξ_7	0	0	0	0	0	0	0	0	0	0	0	1	1
ξ_8	0	0	0	0	0	0	0	0	0	0	1	1	2
ξ_9	0	0	0	0	0	0	1	0	0	0	0	0	1
ξ_{10}	0	0	0	0	0	0	0	1	0	0	0	1	2
ξ_{11}	0	0	0	0	0	0	0	0	0	0	1	0	1

Table 2
Observability matrix $\mathcal{U}_0$.

P	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	V_d^0
ξ_1	0	0	1	0	0	0	0	0	0	0	0	1
ξ_2	1	0	0	0	0	0	0	0	0	0	0	1
ξ_3	0	0	1	0	1	0	0	0	0	0	0	0
ξ_4	0	0	0	1	0	0	0	0	0	0	0	1
ξ_5	0	0	0	0	0	0	0	0	0	1	0	0
ξ_6	0	0	0	0	0	0	1	0	0	0	0	0
ξ_7	0	0	0	0	0	0	0	0	0	0	0	1
ξ_8	0	0	0	0	0	0	0	0	0	0	1	0
ξ_9	0	0	0	0	0	0	1	0	0	0	0	1
ξ_{10}	0	0	0	0	0	0	0	1	0	0	0	0
ξ_{11}	0	0	0	0	0	0	0	0	0	0	1	1

Finally, we have $u^3_{i,r+1} = 1$, for $i \in J_3 := \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11\}$, and then

$$V_d^* = V_d^4 = \mathrm{Col}_{r+1}\left(\mathcal{U}^3\right) \bigcup \cup_{i\in J_3} \mathrm{Col}_i\left(\mathcal{U}^3\right) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T.$$

We conclude that the BN (21) is observable.

The second example is a Boolean model for biological system.

Example 4.2. Consider a Boolean model for the *lac* operon in the bacterium *Escherichiacoli* [30]:

$$\begin{cases} x_1(t+1) = \neg x_3(t) \wedge x_7(t) \neg x_8(t), \\ x_2(t+1) = x_1(t), \\ x_3(t+1) = \neg x_4(t) \wedge \neg x_9(t), \\ x_4(t+1) = x_2(t) \wedge x_5(t), \\ x_5(t+1) = \neg u_1(t) \wedge u_2(t) \wedge x_6(t), \\ x_7(t+1) = \neg u_1(t), \\ x_8(t+1) = x_3(t) \vee (\neg x_4(t) \wedge \neg x_9(t)), \\ x_9(t+1) = x_5(t) \vee x_{10}(t), \\ x_{10}(t+1) = \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_6(t))). \end{cases} \tag{23}$$
$$y_i(t) = x_i(t), \quad i = 1, 2, 6, 7;$$
$$y_3(t) = x_3(t) \wedge x_8(t), y_4(t) = x_4(t) \vee x_9(t),$$
$$y_5(t) = x_5(t) \wedge x_{10}(t)$$

where the variable meanings are as follows:

x_1: *lac* mRNA;
x_2: *lac*β-galactosidase;
x_3, x_8: the repressor protein LacI and medium LacI resp.;
x_4, x_9: allolactose and medium allolactose resp.;
x_5, x_{10}: lactose and medium resp.;
x_6: lac Permease;
x_7: the catabolite activator protein(CAP);
u_1, u_2, u_3: extracellular glucose, high extracellular lactose and medium extracellular lactose resp.;
y_i: outputs.

Then it is easy to figure out its algebraic form (22) with

$$L = \delta_{1024}[682, 682, 682, 682, \ldots, 1014, 1016, 882, 884] \in \mathcal{L}_{1024\times 8192};$$
$$H = \delta_{128}[1, 5, 1, 5 \cdots 120, 120, 128, 128] \in \mathcal{L}_{128\times 1024}.$$

Next, it is easy to figure out Ξ, which is

$$\Xi = \left\{\{\delta^1_{1024}, \delta^3_{1024}\}, \{\delta^1_{1024}, \delta^{65}_{1024}\}, \{\delta^2_{1024}, \delta^4_{1024}\}, \{\delta^2_{1024}, \delta^{33}_{1024}\}, \ldots \{\delta^{1021}_{1024}, \delta^{1022}_{1024}\}, \{\delta^{1023}_{1024}\delta^{1024}_{1024}\}\right\}$$

and $|\Xi| = 7488$. After the iteration algorithm the remained indistinguishable pairs, denoted by Ξ_r, is

$$\Xi_r = \left\{\{\delta^1_{1024}, \delta^3_{1024}\}, \{\delta^1_{1024}, \delta^{65}_{1024}\}, \{\delta^2_{1024}, \delta^4_{1024}\}, \{\delta^2_{1024}, \delta^{33}_{1024}\}, \ldots, \{\delta^{1020}_{1024}, \delta^{1024}_{1024}\}\right\},$$

where $|\Xi_r| = 2032$. So the system is not observable.

5. Concluding remarks

Motivated by Zhang & Zhang's work [27,28], an alternative approach to the observability of Boolean networks was proposed in this note. It significantly simplified the necessary and sufficient condition in [27,28], which is based on finite automata. Since our approach is straightforward, which does not involve finite automata or any other auxiliary machine, it is much simpler and easily understandable.

In our approach, the transferable matrix $\mathcal{W}$ of h-indistinguishable pairs is constructed. Then $\mathcal{W}$ is modified into a Boolean matrix $\mathcal{U}^0$, called the observability matrix. Using it an iterative algorithm is proposed to update an index set. After finite (less than the number of h-indistinguishable pairs, say r) very simple iterations, the index set is stable. Then the system is observable, if and only if, the last column of the observability matrix $\mathcal{U}^{k^*}$ is $\mathbf{1}_r$.

The following are some further discussions.

5.1. Multi-valued logical dynamic systems

Observe network (9) again. Now assume $x_i, u_j, y_\ell \in \mathcal{D}_k$, $k > 2$, then the system is called a k-valued logical control network. More general, we may have $x_i \in \mathcal{D}_{k_i}$, $i = 1, \ldots, n$, $u_i \in \mathcal{D}_{p_i}$, $i = 1, \ldots, m$, $y_i \in \mathcal{D}_{q_i}$, $i = 1, \ldots, s$, then the system is called a multi-valued logical control network or multi-valued logical control system. Multi-valued logical control system could come from biological (economical) systems, or game-based model [31]. Then they do not have logical form (9), but they can be described as (10).

The method developed in this note can be used for multi-valued logical control systems directly. We give an example to demonstrate it.

Example 5.1. 1. Consider a multi-valued logical dynamic system, assume that $x_1(t),\ x_3(t),\ u(t)\ \in\ \Delta_2$, $x_2(t)\ \in\ \Delta_3$. First, we consider the case when there is only one output node $y(t)\in\Delta_3$. Its algebraic form is

$$\begin{aligned} x(t+1) &= Lu(t)x(t),\\ y(t) &= Hx(t), \end{aligned} \tag{24}$$

where

$L\ =\ \delta_{12}[2,9,5,4,7,12,9,6,7,12,3,2,9,12,9,5,8,1,12,1,10,1,6,9]\in\mathcal{L}_{12\times24}$;

$H=\delta_3[2,1,2,3,3,2,2,1,3,2,3,1]\in\mathcal{L}_{3\times12}$.

It is easy to calculate that

$$\Xi=\left\{\{\delta_{12}^1,\delta_{12}^3\},\{\delta_{12}^1.\delta_{12}^6\},\ldots,\{\delta_{12}^8,\delta_{12}^{12}\},\{\delta_{12}^9,\delta_{12}^{11}\}\right\},$$

and $|\Xi|\ =\ 19$, the number of h-indistinguishable pairs is 2. After first iteration the number of h-indistinguishable pairs is 1 and then the algorithm stopped. The only remained h-indistinguishable pair is $\{\delta_{12}^6,\delta_{12}^{10}\}$. The system is unobservable.

2. Consider the system (24) again. Now assume we have two outputs, where $y_1\ =\ y$ is the old one hence $H_1\ =\ H$. We add a new output $y_2(t)\in\Delta$ and assume its structure matrix is

$H_2=\delta_2[2,1,1,1,1,1,2,1,1,2,1,1]$.

Then we have the overall output $y\ =\ y_1\ltimes y_2$, which has the structure matrix as

$$\begin{aligned} H &= H_1 * H_2\\ &= \delta_6[4,1,3,5,5,3,4,1,5,4,5,1]\in\mathcal{L}_{6\times12}. \end{aligned}$$

It is easy to obtain that

$$\Xi=\left\{\{\delta_{12}^1,\delta_{12}^7\},\{\delta_{12}^1.\delta_{12}^{10}\},\ldots,\{\delta_{12}^8,\delta_{12}^{12}\},\{\delta_{12}^9,\delta_{12}^{11}\}\right\},$$

where $|\Xi|\ =\ 13$. Moreover, we also have $\mathrm{Col}_{14}(\mathcal{U}^0)\ =\ \mathbf{1}_{13}$. Hence, there are no h-indistinguishable pairs. We conclude that the system is observable.

5.2. Observability subspace

For a BCN (similarly, for a multi-valued logical control system), a set of states, $\mathcal{O}\subset\Delta_{2^n}$ (correspondingly, $\mathcal{O}\subset\Delta_k$, $k=\prod_{i=1}^n k_i$), we may define the observable subspace as follows.

Definition 5.2. For a BCN (or multi-valued logical control system) $\mathcal{O}\ \subset\ \Delta_{2^n}$ (correspondingly, $\mathcal{O}\ \subset\ \Delta_k$) is called an observable subspace, if

(i) any two distinct points $x_0,\ \bar{x}_0\in\mathcal{O}$ are distinguishable;
(ii) if $\mathcal{U}\supset\mathcal{O}$ and $\mathcal{U}\neq\mathcal{O}$, then $\mathcal{U}$ is unobservable.

It is worth noting that the sets satisfying the two conditions in Definition 5.2, are far no unique. We give an example to depict this.

Example 5.3. Consider the following system

$$\begin{cases} x_1(t+1)=x_2(t)\wedge u_1(t),\\ x_2(t+1)=x_3(t)\vee x_4(t),\\ x_3(t+1)=x_4(t)\bar{\vee}x_5(t),\\ x_4(t+1)=\neg x_5(t),\\ x_5(t+1)=x_1(t)\leftrightarrow u_2(t),\\ \quad y_1(t)=x_1(t)\rightarrow x_3(t),\\ \quad y_2(t)=x_2(t)\vee x_5(t),\\ \quad y_3(t)=x_1(t)\wedge x_4(t). \end{cases} \tag{25}$$

We skip the detailed discussion and give the eventually h-indistinguishable pairs as

$$\left\{\{\delta_{32}^{17},\delta_{32}^{21}\},\{\delta_{32}^{18},\delta_{32}^{22}\},\{\delta_{32}^{25},\delta_{32}^{29}\},\{\delta_{32}^{26},\delta_{32}^{28}\},\{\delta_{32}^{26},\delta_{32}^{30}\}\{\delta_{32}^{28},\delta_{32}^{30}\}\right\}.$$

Then you may have

$$\begin{aligned} \mathcal{O}_1 &= \Delta_{32}\setminus\{\delta_{32}^{21},\delta_{32}^{22},\delta_{32}^{25},\delta_{32}^{26},\delta_{32}^{30}\};\\ \mathcal{O}_2 &= \Delta_{32}\setminus\{\delta_{32}^{17},\delta_{32}^{18},\delta_{32}^{28},\delta_{32}^{29},\delta_{32}^{30}\};\\ \mathcal{O}_3 &= \Delta_{32}\setminus\{\delta_{32}^{17},\delta_{32}^{22},\delta_{32}^{25},\delta_{32}^{26},\delta_{32}^{28}\};\ \cdots \end{aligned} \tag{26}$$

We may call such sets the sup-observable subspaces. Note that when you mention the observable subspace, it is not clearly defined.

For a linear or nonlinear system, the unobservable subspace (sub-manifold) is well defined. But for logical systems, "unobservable subspace" seems very confusing. For instance, if we consider the complement of $\mathcal{O}_1$ in (26), which is

$$\mathcal{O}_1^c=\{\delta_{32}^{21},\delta_{32}^{22},\delta_{32}^{25},\delta_{32}^{26},\delta_{32}^{30}\}. \tag{27}$$

It is almost observable except one pair $\{\delta_{32}^{26},\delta_{32}^{30}\}$. In other words, it is obvious that either $\{\delta_{32}^{21},\delta_{32}^{22},\delta_{32}^{25},\delta_{32}^{26}\}$ or $\{\delta_{32}^{21},\delta_{32}^{22},\delta_{32}^{25},\delta_{32}^{30}\}$ is observable. So for logical systems, "unobservable subspace" is not a well defined concept. Therefore, if $\mathcal{O}$ is a sup-observable subspace, its complement $\mathcal{O}^c$ may be called co-observable subspace.

According to the argument in Example 5.3, it is obvious that several concepts discussed in [32,33], which involve the observability subspace, may need to be reconsidered. A fundamental problem is: how to convert the set sense subspace discussed above forward/backward to the logical function sense subspace, discussed in [32,33] or [9].

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