SCIENCE CHINA
Information Sciences

• REVIEW •

Special Focus on Analysis and Control of Finite-Valued Network Systems

January 2018, Vol. 61 010201:1–010201:19
https://doi.org/10.1007/s11432-017-9265-2

From STP to game-based control

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Received 21 August 2017/Accepted 28 September 2017/Published online 12 December 2017

Abstract This paper provides a comprehensive survey on semi-tensor product (STP) of matrices and its applications to different disciplines. First of all, the STP and its basic properties are introduced. Meanwhile, its inside physical meaning is explained. Second, its application to conventional dynamic systems is presented. As an example, the region of attraction of stable equilibriums is discussed. Third, its application to logical systems is presented. Particularly, the algebraic state space representation of logical systems and the important role it plays in analysis and control of logical systems are emphasized. Fourth, its application to finite games is discussed. The most interesting problems include potential game, evolutionary game, and game theoretic control. Finally, the mathematical essence of STP is briefly introduced.

Keywords semi-tensor product of matrices, Boolean network, logical (control) system, finite game, game theoretic control

Citation Cheng D Z, Qi H S, Liu Z Q. From STP to game-based control. Sci China Inf Sci, 2018, 61(1): 010201, https://doi.org/10.1007/s11432-017-9265-2

1 An introduction: what is STP

This paper gives a comprehensive survey on semi-tensor product (STP) from its very beginning till nowadays. The STP of matrices is a generalization of conventional matrix product. It was born about twenty years ago. Then it has been developed promptly and has been applied to some related disciplines, including (1) nonlinear (control) systems, (2) logical (control) systems, (3) finite games.

This introduction consists of two parts. First, we discuss the multi-linear functions and use them to see the geometric inside of STP, and how the STP is used to express multi-linear functions into a matrix form. Second, some basic properties of STP are presented. It can be seen that as a generalization of conventional matrix product, it keeps all the properties of conventional matrix product available.

For statement ease, we first give some notations:

(1) $\mathbb{R}$: Field of real numbers.

(2) $\mathbb{R}^n$: n dimensional Euclidean space.

(3) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.

(4) $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$): the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$): the i-th column (row) of M.

(5) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L are of the form of δ_m^k. That is, $\mathrm{Col}(L) \subset \Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.

(6) A matrix $C > 0$ means all the entries are positive, that is, $c_{i,j} > 0$, $\forall i, j$.

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(7) Given a (C^ω) manifold M, the set of C^ω functions is $C^\omega(M)$; the set of vector fields is $V^\omega(M)$; the set of co-vector fields is $V^{*\omega}(M)$.

1.1 Geometric inside of STP

After unsuccessful submissions for several years, the first paper on STP of matrices was published in 2001 [1]. Since then, the criticism, questioning or doubts on STP had been lasted for years. The most frequently asked question is: is it reasonable to generalize the conventional matrix product to two arbitrary matrices? To answer this question we first give a motivation to reveal the insight structure of STP.

Consider a linear function $f \in L(\mathbb{R}^n, \mathbb{R})$, where $L(\mathbb{R}^n, \mathbb{R})$ is the set of linear functions from $\mathbb{R}^n$ to $\mathbb{R}$. Then there exists a row vector $a = (a_1, \dots, a_n)$ such that

$$f(x) = ax, \quad x \in \mathbb{R}^n. \tag{1}$$

Similarly, for a bilinear function $g \in L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$, there exists a matrix $A = (a_{i,j}) \in \mathcal{M}_{m\times n}$ such that

$$g(x, y) = x^{\mathrm{T}} Ay, \quad x \in \mathbb{R}^m, \ y \in \mathbb{R}^n. \tag{2}$$

To calculate (2), we may first consider $x^{\mathrm{T}}A$, which is

$$x^{\mathrm{T}} A = \sum_{i=1}^{m} x_i (a_{i,1}, a_{i,2}, \dots, a_{i,n}) \in L(\mathbb{R}^n, \mathbb{R}). \tag{3}$$

That is, a bilinear function is a linear combination of linear functions, with an argument x as its coefficients.

Since (3) is a linear mapping of x, intuitively, it may be expressed as a linear product form. Say, we write it into a "product form" as

$$x^{\mathrm{T}} A = [\mathrm{Row}_1(A), \dots, \mathrm{Row}_m(A)](?)x, \tag{4}$$

where (?) is an undefined product, which sum up the product of x_1 with the first block of the row stacking form of A, and x_2 with the second block, ..., and x_m with the last block.

Now if we consider a trilinear mapping $h(x, y, z) \in L(\mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. It is well known that the classical matrix theory cannot be used directly for this. In 1980's, the cubic matrix was proposed to describe it [2], which is depicted in Figure 1, where

$$d_{i,j,k} = h(\delta_p^i, \delta_m^j, \delta_n^k), \quad i = 1, \dots, p; \ j = 1, \dots, m; \ k = 1, \dots, n.$$

Unfortunately, the cubic expression needs many new formulas/rules in applications. Moreover, it cannot be extended to higher order multi-linear functions.

Now if we fix x, then $h(x, y, z) = h_x(y, z)$ becomes a bilinear function. Denote the cubic matrix as

$$C = [A_1, A_2, \dots, A_p]^{\mathrm{T}},$$

where A_i is the i-th layer of C. Then it is clear that

$$h_x = \sum_{i=1}^{p} A_i x_i = [A_1, \dots, A_p](?)x, \tag{5}$$

where the last "product form" is with the undefined product (?), which can be proceeded in the same way as for (4).

In fact, (?) is exactly the STP of matrices, which is formally defined as follows.

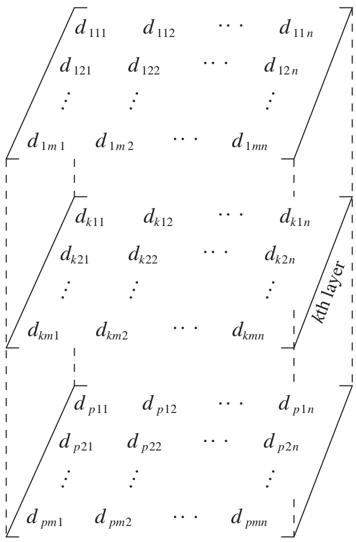

Figure 1 A cubic matrix.

Definition 1. Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$ and $t = \mathrm{lcm}(n,p)$ be the least common multiple of n and p. Then the STP of A and B is defined as follows:

$$A \ltimes B = \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right). \tag{6}$$

It is obvious that the operator (?) in (4) and (5) is exactly the "$\ltimes$". Hence, the above argument on bilinear and trilinear functions shows the geometric insight of STP. Moreover, continuing this order reducing process, any multi-linear function can be expressed into a matrix form.

Let $f \in L(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}, \mathbb{R})$ be a k-th order multi-linear function. Set the constants

$$d_{i_1,i_2,\ldots,i_k} := f(\delta_{n_1}^{i_1}, \ldots, \delta_{n_k}^{i_k}), \quad i_j = 1,\ldots,n_j;\ j = 1,\ldots,k. \tag{7}$$

Next, we arrange $\{d_{i_1,i_2,\ldots,i_k}\}$ into a row vector in an alphabetic order as

$$V_f = [d_{1,1,\ldots,1}, d_{1,1,\ldots,2}, \ldots, d_{n_1,n_2,\ldots,n_k}]. \tag{8}$$

Then we have a matrix form expression of f as follows.

Proposition 1. Let $f \in L(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}, \mathbb{R})$ be a k-th order multi-linear function. Then

$$f(x_1,\ldots,x_k) = V_f \ltimes_{i=1}^{k} x_i. \tag{9}$$

1.2 Some basic properties

According to the definition it is obvious that STP is a generalization of conventional matrix product. Fortunately, it keeps all major properties of conventional matrix product available. Because of this, the symbol $\ltimes$ is mostly omitted. Next, we list some main properties of STP, and refer to [3, 4] for details.

Proposition 2. (1) (Distributive law)

$$\begin{cases} F \ltimes (aG \pm bH) = aF \ltimes G \pm bF \ltimes H, \\ (aF \pm bG) \ltimes H = aF \ltimes H \pm bG \ltimes H, \quad a,b \in \mathbb{R}. \end{cases} \tag{10}$$

(2) (Associative law)

$$(F \ltimes G) \ltimes H = F \ltimes (G \ltimes H). \tag{11}$$

(3) (Transpose)

$$(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}. \tag{12}$$

(4) (Inverse) Assume both A and B are invertible. Then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{13}$$

There are some new properties for STP.

Proposition 3. (1) Let $X \in \mathbb{R}^t$ be a column, A an arbitrary matrix. Then

$$XA = (I_t \otimes A)X. \tag{14}$$

(2) Let $Y \in \mathbb{R}^t$ be a row, A an arbitrary matrix. Then

$$AY = Y(I_t \otimes A). \tag{15}$$

For improving the commutativity of STP, we define a swap matrix.

Definition 2. A swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ is defined as follows:

$$W_{[m,n]} = \delta_{mn}[1, m+1, \dots, (n-1)m+1; 2, m+2, \dots, (n-1)m+2; \dots; m, 2m, \dots, nm], \tag{16}$$

where $\delta_k[i_1, \dots, i_s]$ is a shorthand of $\left[\delta_k^{i_1}, \dots, \delta_k^{i_s}\right]$.

A swap matrix is an orthogonal one.

Proposition 4.

$$W_{[m,n]}^{\mathrm{T}} = W_{[m,n]}^{-1} = W_{[n,m]}. \tag{17}$$

Its main function is to exchange the factors.

Proposition 5. Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{18}$$

There are two major drawbacks of conventional matrix product comparing with scalar product: (1) conventional matrix product has dimension restriction for the factor matrices; (2) the matrix product is not commutative. Where the conventional matrix product is generalized to STP, the dimension restriction has been completely removed. Moreover, the STP has certain commutativity, called the pseudo-commutativity. In the light of swap matrix, the "commutativity" of STP has been improved.

The rest of this paper is organized as follows. Section 2 considers the application of STP to dynamic control systems. Using STP, the formulas for calculation Lie derivatives of smooth functions, vector fields, co-vector fields are obtained. Then they are applied to nonlinear systems. Particularly, as an example, the calculation of region of attraction of a stable equilibrium is presented. Section 3 discusses the application of STP to Boolean networks. The algebraic state space representation is discussed in detail. It converts a logical system to an algebraic system, and then the conventional tools for differential/difference equations can be used to the analysis and control design of logical systems. Section 4 generalizes the technique developed in Section 3 for Boolean networks to general logical systems. Section 5 is devoted to finite games. A finite game is essentially a mix-valued logical system, and hence the STP approach to logical systems is also applicable to finite games. Particularly, the potential game, the networked evolutionary game, and the game theoretic control are introduced. Section 6 briefly mentioned the mathematical essence of STP. It is revealed that STP is essentially an operator over equivalence class of matrices. Section 7 is a brief conclusion.

2 Application to dynamic control systems

2.1 Nonlinear computations

In this subsection some basic computation formulas/algorithms are provided via STP. They make STP powerful in analyzing the dynamics or designing control for nonlinear systems.

Differential formula. Let $f(x)$, $x=(x_1,\dots,x_n)^{\mathrm{T}}\in\mathbb{R}^n$, be an analytic function. Define its differential as

$$Df(x):=\left(\frac{\partial f(x)}{\partial x_1},\dots,\frac{\partial f(x)}{\partial x_n}\right). \tag{19}$$

Denote by $x^k=\underbrace{x\ltimes\cdots\ltimes x}_{k}$. Then we have the following fundamental formula.

Proposition 6.

$$D\left(x^{k+1}\right)=\Phi_k^n x^k, \tag{20}$$

where $\Phi_k^n=\sum_{i=0}^k I_{n^i}\otimes W_{[n^{k-i},n]}$.

Using Taylor expression and collecting terms, $f(x)$ can be expressed as

$$f(x)=F_0+F_1x+F_2x^2+\cdots. \tag{21}$$

Then using (20) to (21) term by term, the differential of $f(x)$ is computable.

Calculating Lie derivatives. Assume $h\in C^{\omega}(M)$, $X,Y\in V^{\omega}(M)$, $\alpha\in V^{*\omega}(M)$, where M is an n-dimensional analytic manifold. Express h, X, Y, α into their Taylor expansions as

$$\begin{aligned}
h &= h_0+h_1x+h_2x^2+\cdots;\\
X &= X_0+X_1x+X_2x^2+\cdots;\\
Y &= Y_0+Y_1x+Y_2x^2+\cdots;\\
\alpha^{\mathrm{T}} &= \alpha_0+\alpha_1x+\alpha_2x^2+\cdots.
\end{aligned}$$

Then the corresponding Lie derivatives can be calculated.

Proposition 7. (1)

$$L_Xh=\sum_{i=0}^{\infty}c_ix^i, \tag{22}$$

where $c_i=\sum_{k=0}^i h_{k+1}\Phi_k^n(I_{n^k}\otimes X_{i-k})$.

(2)

$$L_XY=\sum_{i=0}^{\infty}d_ix^i, \tag{23}$$

where $d_i=\sum_{k=0}^i\left[Y_{k+1}\Phi_k^n(I_{n^k}\otimes X_{i-k})-X_{k+1}\Phi_k^n(I_{n^k}\otimes Y_{i-k})\right]$.

(3)

$$(L_X\alpha)^{\mathrm{T}}=\sum_{i=0}^{\infty}e_ix^i, \tag{24}$$

where $e_i=\sum_{k=0}^i[\alpha_{k+1}\Phi_k^n(I_{n^k}\otimes X_{i-k})-V_c^{\mathrm{T}}(I_{n^k}\otimes X_{k+1}\Phi_k^n)^{\mathrm{T}}(I_{n^k}\otimes\alpha_{i-k})]$.

2.2 Analysis and control of nonlinear systems

Since the nonlinear control theory is based on the differential geometry, the technique developed in the previous subsection can be used to deal with nonlinear (control) systems [5]. Some applications including the numerical solution to Morgan's problem [1], some applications to physics [6] and algebra [7], to power systems [8–10], to fuzzy control systems [11, 12], and the application to control design [13].

Particularly, its application to estimate the region of attraction of a stable equilibrium is a theoretically meaningful and practically useful result. We briefly review this result [14].

Table 1 Truth table for unary operators

x	$\neg x$
1	0
0	1

Table 2 Truth table for binary operators

x	y	$x \vee y$	$x \wedge y$	$x \rightarrow y$	$x \leftrightarrow y$	$x \downarrow y$	$x \uparrow y$	$x \bar{\vee} y$
1	1	1	1	1	1	0	0	0
1	0	1	0	0	0	0	1	1
0	1	1	0	1	0	0	1	1
0	0	0	0	1	1	1	1	0

Consider the following nonlinear dynamic system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{25}$$

where $f(x)$ is an analytic field.

Let $x_{\rm s}$ be a stable equilibrium of (25). The region of attraction of $x_{\rm s}$ is denoted by $A(x_{\rm s})$, the boundary of $A(x_{\rm s})$ is denoted by $\partial A(x_{\rm s})$. An equilibrium $x_{\rm e}$ is said to be hyperbolic, if the Jacobi matrix of f at $x_{\rm e}$, $J_f(x_{\rm e})$, has no zero real part eigenvalues. A hyperbolic equilibrium is said to be of type-k, if $J_f(x_{\rm e})$ has exactly k positive real part eigenvalues.

It is well known that [15] if the state manifold is of dimension n, then the boundary of the stability region is of dimension $n-1$. Hence, the boundary is basically generated by the stable sub-manifolds of type-1 equilibriums.

Using STP, the following result is obtained to calculate the stable sub-manifolds of type-1 equilibriums.

Theorem 1 ([14]). Let $x_u = 0$ be an equilibrium of type-1 of system (25).

$$W^{\rm s}(e_u) = \{x \mid h(x) = 0\}. \tag{26}$$

Then $h(x)$ is uniquely determined by the following equations (27)–(29).

$$h(0) = 0, \tag{27}$$
$$h(x) = \eta^{\rm T} x + O(\|x\|^2), \tag{28}$$
$$L_f h(x) = \mu h(x), \tag{29}$$

where $L_f h(x)$ is the Lie derivative of $h(x)$ with respect to f, η is the eigenvector of $J_f^{\rm T}(0)$ with respect to its unique positive eigenvalue μ.

This result has then been extended for any type of equilibriums [16].

3 Application to logical systems

3.1 STP expression of logical functions

A logical variable x can only take two possible values, either true (1) or false (0). We can say $x \in \mathcal{D}$, where, $\mathcal{D} = \{0, 1\}$.

Some operators can be defined for logical variables. They can be expressed via truth table.

• Unary operators (Table 1).

• Binary operators. Disjunction ($\vee$), conjunction ($\wedge$), conditional ($\rightarrow$), bi-conditional ($\leftrightarrow$), nor or ($\downarrow$), not and ($\uparrow$), exclusive or ($\bar{\vee}$). Table 2 shows the truth table of some commonly used binary operators.

Table 3 Structure matrices of binary operators

σ	$\vee$	$\wedge$	$\rightarrow$	$\leftrightarrow$	$\downarrow$	$\uparrow$	$\bar{\vee}$
M_σ	$\delta_2[1,1,1,2]$	$\delta_2[1,2,2,2]$	$\delta_2[1,2,1,1]$	$\delta_2[1,2,2,1]$	$\delta_2[2,2,2,1]$	$\delta_2[2,1,1,1]$	$\delta_2[2,1,1,2]$

To use algebraic expression of logical functions, we identify $1 \sim \delta_2^1$, $0 \sim \delta_2^2$, where δ_n^k is the k-th column of the identity matrix I_n.

Under this vector expression it is easy to verify that

$$\neg x = M_n x, \tag{30}$$

where $M_n = \delta_2[2,1]$ is called the structure matrix of "negation". Note that $\delta_n[i_1,\dots,i_k]$ is a shorthand for $\left[\delta_n^{i_1},\dots,\delta_n^{i_k}\right]$, which is called a logical matrix.

Similarly, for binary operator σ, we can also find its corresponding structure matrix M_σ, such that

$$x\sigma y = M_\sigma xy. \tag{31}$$

Some commonly used binary operators and their structure matrices are listed in Table 3.

Using the structure matrices of fundamental operators and the STP, the structure matrix of general logical expression can easily be constructed.

Let $f : \mathcal{D}^n \to \mathcal{D}$ be a logical function. The following theorem gives its matrix expression.

Theorem 2. $y = f(x_1,\dots,x_n)$. Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f, such that when the logical variables x_i, $i = 1,\dots,n$, y are expressed into their vector forms we have

$$y = M_f \ltimes_{i=1}^n x_i. \tag{32}$$

Let $F : \mathcal{D}^n \to \mathcal{D}^m$ be a logical mapping consisting of m logical functions:

$$F: \begin{cases} y_1 = f_1(x_1,\dots,x_n),\\ y_2 = f_2(x_1,\dots,x_n),\\ \vdots\\ y_m = f_m(x_1,\dots,x_n). \end{cases} \tag{33}$$

According to Theorem 2, if the structure matrix of f_i is M_i, $i = 1,\dots,m$, then F can be expressed as:

$$F: \begin{cases} y_1 = M_1 \ltimes_{i=1}^n x_i,\\ y_2 = M_2 \ltimes_{i=1}^n x_i,\\ \vdots\\ y_m = M_m \ltimes_{i=1}^n x_i. \end{cases} \tag{34}$$

Denote by $y = \ltimes_{i=1}^m y_i$, $x = \ltimes_{i=1}^n x_i$. The following theorem gives an overall matrix expression of F.

Theorem 3. Assume a logical mapping F is given by (33) (equivalently, by (34)), then there exists a unique logical matrix $M_F \in \mathcal{L}_{2^m\times 2^n}$, called the structure matrix of F, such that in vector form we have

$$y = M_F x, \tag{35}$$

where $M_F = M_1 * M_2 * \cdots * M_m$. Note that $*$ is the Khatri-Rao product of matrices [4].

3.2 Algebraic representation of Boolean networks

A Boolean network is described as follows:

$$\begin{cases} x_1(t+1) = f_1(x_1(t),\dots,x_n(t)),\\ \quad\vdots\\ x_n(t+1) = f_n(x_1(t),\dots,x_n(t)), \end{cases} \tag{36}$$

where $x_i(t) \in \mathcal{D}$, $f_i : \mathcal{D}^n \to \mathcal{D}$ are logical functions.

Correspondingly, a Boolean control network is as follows:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)), \\ x_2(t+1) = f_2(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)), \\ \quad \vdots \\ x_n(t+1) = f_n(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)), \end{cases} \\ y_j(t) = h_j(x_1(t), \dots, x_n(t)), \quad j = 1, \dots, p, \tag{37}$$

where $x_i, u_j, y_k \in \mathcal{D}$, $i = 1, \dots, n$, $j = 1, \dots, m$, $k = 1, \dots, p$; $f_i : \mathcal{D}^{n+m} \to \mathcal{D}$, $i = 1, \dots, n$ and $h_k : \mathcal{D}^n \to \mathcal{D}$, $k = 1, \dots, p$ are logical functions.

Using vector expression of logical variables, the dynamics of both Boolean networks and Boolean control networks can be converted into an algebraic form.

Theorem 4. (1) There exists a unique logical matrix M, such that (36) can be expressed into its algebraic form as

$$x(t+1) = Mx(t), \tag{38}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$.

(2) There exist unique logical matrices L and H, such that (37) can be expressed into its algebraic form as

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{39}$$

where $u(t) = \ltimes_{j=1}^m u_j(t)$, $y(t) = \ltimes_{i=1}^p y_i(t)$.

3.3 Analysis and control of logical systems

The algebraic expression of Boolean networks and Boolean control networks converts the logical form of dynamics into an algebraic form. This converting makes the analysis and control design of Boolean networks much more convenient. In the following, we review some fundamental results using algebraic expression of Boolean networks.

3.3.1 *Topological structure of Boolean networks*

It seems that the dynamic equations of Boolean network (36) and Boolean control network (37) are similar to the discrete time equations of mechanical (control) systems. In fact, since the variables in Boolean (control) networks are logical, the techniques used for differential or difference equations are not applicable to solve the problems of analysis, synthesis and control design of Boolean networks. Once upon the time, the investigation for Boolean networks remained on “try and error” and “one by one”, and very limited systematic results were revealed. For instance, since there are finite nodes and each node can take only finite values, each trajectory will converge to a fixed point or a cycle, which is called an attractor.

Before the algebraic state space representation (ASSR), there were many papers discussing the attractors. But usually one paper considers only one or two special networks, without knowing general conclusion. Some of the papers are [17–19].

The topology of a Boolean network is determined by the following facts (1) number of attractors; (2) region of attraction of each attractor; (3) how long for each trajectory to enter its attractor. These questions are answered in [20]. For instance, the following theorem tells the number of attractors:

Theorem 5 ([20])**.** Consider a logical dynamic system (36). Its ASSR is (38). Denote by N_l the number of cycles of length l. Then N_l can be iteratively calculated by the following formulas (a fixed point is considered as a cycle of length 1).

$$\begin{cases} N_1 = \mathrm{Trace}(M), \\ N_l = \frac{\mathrm{Trace}(M^l) - \sum_{k \in \mathcal{P}(l)} kN_k}{l}, \quad 2 \leqslant l \leqslant 2^n, \end{cases} \tag{40}$$

where $\mathcal{P}(l)$ is the set of proper factors of l (including 1).

This general formula can be used to verify the results in [17–19]. Ref. [20] also pointed out some errors there.

3.3.2 *State space description of logical (control) systems*

The state space approach proposed by Kalman provides a framework for investigating dynamic control systems. It becomes one of three pillars of modern control theory. For logical systems, Ref. [21] proposed a dual approach, which defines the state space via the set of logical functions.

Definition 3. Consider the logical system (36) or logical control system (37). (1) The state space, denoted by $\mathcal{X}$, is defined as the set of logical functions of $X = \{x_1, \dots, x_n\}$, expressed as

$$\mathcal{X} := \mathcal{F}_\ell(x_1, \dots, x_n). \tag{41}$$

(2) Let $W = \{w_1, \dots, w_s\} \subset \mathcal{X}$. Then the subspace generated by W, denoted by $\mathcal{W}$, is defined as

$$\mathcal{W} := \mathcal{F}_\ell(w_1, \dots, w_s). \tag{42}$$

Definition 4. Consider the logical system (36) or logical control systems (37). Let $Z = \{z_1, \dots, z_n\} \subset \mathcal{X}$. $T : \mathcal{X} \to \mathcal{Z}$ is called a coordinate change, if T is one-to-one and onto.

Theorem 6. Let $Z = \{z_1, \dots, z_n\} \subset \mathcal{X}$. $\mathcal{Z}$ is the subspace generated by Z with its structure matrix Φ_Z, that is

$$z = \Phi_Z x,$$

where $\Phi_Z \in \mathcal{L}_{2^n \times 2^n}$. $T :\ X \to Z$ is a coordinate change, if and only if, Φ_Z is nonsingular.

Definition 5. Let $Z_0 = \{z_1, \dots, z_s\} \subset \mathcal{X}$. The subspace generated by Z_0, denoted by $\mathcal{Z}_0 = \mathcal{F}_\ell(Z_0)$, is called a regular subspace, if there exists $Z_1 = \{z_{s+1}, \dots, z_n\} \subset \mathcal{X}$, such that $T : X \to Z = \{Z_0, Z_1\}$ is a coordinate change.

Regular subspace is very important in structure analysis and control design of logical systems.

3.3.3 *Controllability of Boolean networks*

The first fundamental work on controllability and observability of logical control systems was presented in [3]. The following result is fundamental for controllability.

Consider system (37), its ASSR is (39). Split L into 2^m equal blocks as

$$L = [L_1, L_2, \dots, L_{2^m}],$$

where $L_i \in \mathcal{L}_{2^n \times 2^n}$, $i = 1, \dots, 2^m$. Define

$$M := \sum_{i=1}^{2^m} L_i,$$

$$\mathcal{M}_\mathcal{C} := \sum_{i=1}^{2^n}{}_{\mathcal{B}} M^i \in \mathcal{B}_{2^n \times 2^n},$$

where the operator $\sum_\mathcal{B}$ is a Boolean addition, that is, for Boolean variables a, b, $a + b = a \bar{\vee} b$. The matrix $\mathcal{M}_\mathcal{C}$ is called the controllability matrix. Then we have the following result.

Theorem 7 ([22])**.** Denote by $\mathcal{M}_{\mathcal{C}} = (c_{ij})$, then

(1) $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$, if and only if, $c_{i,j} > 0$;

(2) System (37) is controllable at $\delta_{2^n}^j$, if and only if, $\mathrm{Col}_j(\mathcal{M}_{\mathcal{C}}) > 0$;

(3) System (37) is controllable, if and only if, $\mathcal{M}_{\mathcal{C}} > 0$.

3.3.4 *Observability*

The observability of Boolean networks is more complicated than controllability. Recently, Ref. [23] mentioned 9 different definitions. Moreover, the following 4 definitions have been compared.

Definition 6 ([3])**.** System (37) is observable, if for any x_0, there exists a control sequence such that for any $\bar{x}_0 \neq x_0$, $H\bar{x}_0 = Hx_0$, the corresponding output sequences are different.

Definition 7 ([22])**.** System (37) is observable, if for any pair of initial values $x_0 \neq \bar{x}_0$, with $H\bar{x}_0 = Hx_0$, there exists infinite control sequence such that the corresponding output sequences are different.

Definition 8 ([24])**.** System (37) is observable, if there exists a finite control sequence such that for any pair of initial states $x_0 \neq \bar{x}_0$, with $H\bar{x}_0 = Hx_0$, the corresponding finite output sequences are different.

Definition 9 ([25])**.** System (37) is observable, if for a pair of initial values $x_0 \neq \bar{x}_0$, with $H\bar{x}_0 = Hx_0$, for any infinite control sequence the output sequences are different.

Ref. [23] has compared these 4 definitions of observability, and concluded that Definition 7 is the most reasonable one. They also given a necessary and sufficient condition for this observability by constructing corresponding finite automata.

The following result is motivated by [23] and is easily verifiable [26].

Divide the pair of states into 3 classes

$$\mathcal{X} \times \mathcal{X} = D \cup \Xi \cup \Theta,$$

where $D := \{\{x, x\} | x \in \mathcal{X}\} \subset \mathcal{X} \times \mathcal{X}$ called the diagonal pairs; $\Xi := \{\{x, \bar{x}\} | x \neq \bar{x} \text{ and } Hx = H\bar{x}\} \subset \mathcal{X} \times \mathcal{X}$ called the h-indistinguishable pairs; $\Theta := \{\{x, \bar{x}\} | x \neq \bar{x} \text{ and } Hx \neq H\bar{x}\} \subset \mathcal{X} \times \mathcal{X}$ called the h-distinguishable pairs.

Naming the h-indistinguishable pairs as $\Xi = \{\xi_1, \xi_2, \ldots, \xi_r\}$, we construct a matrix as

$$\mathcal{W} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,r} & w_{1,r+1} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,r} & w_{2,r+1} \\ & & \vdots & & \\ w_{r,1} & w_{r,2} & \cdots & w_{r,r} & w_{r,r+1} \end{bmatrix}, \tag{43}$$

where $w_{i,j}$ is the number of controls, which transfer ξ_i to ξ_j in one step ($j \leqslant r$), or transfer ξ_i to D ($j = r+1$). Define

$$d_i = \sum_{j=1}^{r+1} w_{i,j}, \quad i = 1, \ldots, r. \tag{44}$$

Construct a Boolean matrix of the size same as $\mathcal{W}$, denoted by $\mathcal{U}^0 = (u_{i,j}) \in \mathcal{B}_{r \times (r+1)}$ as

$$u_{i,j}^0 = \begin{cases} 1, & w_{i,j} > 0, \\ 0, & w_{i,j} = 0, \quad i = 1, \ldots, r, \ j = 1, \ldots, r, \end{cases}$$

and

$$u_{i,r+1}^0 = \begin{cases} 1, & d_i < 2^m, \\ 0, & d_i = 2^m, \quad i = 1, \ldots, r. \end{cases}$$

Then iteratively construct $\mathcal{U}$ as Algorithm 1.

Algorithm 1 Construct $\mathcal{U}$

Let $\mathcal{U}^k=(u^k_{i,j})$ be known.
for $i=1,\dots,r$ **do**
    if $u^k_{i,r+1}=1$ **then**
        $\mathrm{Col}_{r+1}(\mathcal{U}^{k+1})=\mathrm{Col}_{r+1}(\mathcal{U}^k)\vee \mathrm{Col}_i(\mathcal{U}^k)$.
    end if
    if $\mathcal{U}^{k^*+1}=\mathcal{U}^{k^*}$ **then**
        Set $\mathcal{U}:=\mathcal{U}^{k^*}$.
        Stop.
    end if
end for

Theorem 8. Consider system (37) (or (39)).

(1) System (37) is observable, if and only if,

$$\mathrm{Col}_{r+1}\,\mathcal{U}=\mathbf{1}_r. \tag{45}$$

(2) ξ_i is a pair of indistinguishable, if and only if, the $(i,r+1)$-th entry of $\mathcal{U}$, $u_{i,r+1}=0$.

3.3.5 *Stability and stabilization*

A logical system is globally stable if its each trajectory converges to a preassigned point. The following is a basic result.

Theorem 9 ([27])**.** System (36) is globally stable, if and only if it has exactly one attractor, which is a fixed point.

Consider the stabilization of system (37), we have the following result.

Theorem 10 ([27])**.** System (37) is stabilizable, if and only if, there exist a $1\leqslant k\leqslant 2^n$, and a $1\leqslant j\leqslant 2^m$, such that L^k_j is a constant matrix.

3.3.6 *Disturbance decoupling*

A logical control system with disturbances is described as

$$\begin{cases} x_1(t+1)=f_1(x_1(t),\dots,x_n(t);u_1(t),\dots,u_m(t);\xi_1(t),\dots,\xi_r(t)),\\ x_2(t+1)=f_2(x_1(t),\dots,x_n(t);u_1(t),\dots,u_m(t);\xi_1(t),\dots,\xi_r(t)),\\ \quad\vdots\\ x_n(t+1)=f_n(x_1(t),\dots,x_n(t);u_1(t),\dots,u_m(t);\xi_1(t),\dots,\xi_r(t)),\end{cases}\\ y_j(t)=h_j(x_1(t),\dots,x_n(t)),\quad j=1,\dots,p, \tag{46}$$

where $\xi_1,\dots,\xi_r$ are disturbances. The purpose of disturbance decoupling is the design controls such that the disturbances will not affect the outputs.

Ref. [28] proposes a procedure for solving disturbance decoupling problem via two steps.

(1) Find a y-friendly regular subspace: that is, find $\mathcal{Z}_2:=\mathcal{F}_\ell(z^2_1,z^2_2,\dots,z^2_q)$, such that $h_j\in\mathcal{Z}_2$, $\quad j=1,\dots,p$. Then find $\mathcal{Z}_1=\mathcal{F}_\ell(z^1_1,z^1_2,\dots,z^1_{n-q})$, such that $\mathcal{Z}=\mathcal{Z}_1\cup\mathcal{Z}_2=\mathcal{F}_\ell(z^1_1,z^1_2,\dots,z^1_{n-q},z^2_1,z^2_2,\dots,z^2_q)$ becomes a new coordinate frame. Then, under z coordinate frame, system (46) becomes

$$\begin{cases} z^1(t+1)=g^1(z^1(t),z^2(t),u(t),\xi(t)),\\ z^2(t+1)=g^2(z^1(t),z^2(t),u(t),\xi(t)),\end{cases}\\ y_j(t)=\eta_j(z^2(t)),\quad j=1,\dots,p. \tag{47}$$

(2) Design $u(t)=F(z^1(t),z^2(t))$, such that

$$g^2(z^1(t),z^2(t),F(z^1(t),z^2(t)),\xi(t))=\tilde g^2(z^2(t)). \tag{48}$$

Note that if (48) holds, the feedback control $u(t)=F(z^1(t),z^2(t))$ can solve the disturbance decoupling problem. About the solvability and the control design please refer to [28].

3.3.7 *Optimal control*

Optimization of Boolean control networks is another important problem. The first significant result is: the optimal trajectory can be found as a cycle [29]. Combining this result with the calculation of cycles, [30] obtained a feedback optimal control.

The above result can only be used for deterministic logical control systems. As for stochastic logical systems it is not applicable because there is no limit cycle. Combining dynamic programming with receding horizon control, [31] obtained a feedback optimal control for mix-valued logical dynamic control systems.

Another meaningful result is the Pontryagin maximum principle for logical control systems. As for classical control systems, it also provides necessary conditions for optimal control of logical control systems [32].

3.3.8 *Identification of logical systems*

The identification of Boolean network (36) means to identify the M matrix in ASSR (38) using a sequence of states. It is clear that each adjacent pair of states $x(t)$, $x(t+1)$ can identify a column of M: assume $x(t)=\delta_{2^n}^i$, $x(t+1)=\delta_{2^n}^j$, then one knows that the i-th column of M is $\mathrm{Col}_i(M)=\delta_{2^n}^j$ [33]. When the Boolean network is large, the above method requires a large number of data. In general, the in-degree of a Boolean network is much smaller than its size. Hence [33] proposes a method called the least in-degree identification, which can tremendously reduce the required data. This problem comes from the growth of cancer. The infecting process may be determined by observed data.

Consider the identification of Boolean control network (37), the purpose of the identification is to find the L and H of (39). To identify L from state-control data is similar to the identification of Boolean networks. But the identification of L and H from input-output data is more difficult. The first problem is the identifiability. If the system is controllable and observable, the identification is theoretically executable. Next, since the realization is not unique, the solution is not unique. For instance, under a coordinate change, the different L's and H's produce the same input-output relations. Ref. [34] provided an algorithm. Under the assumption of controllability and observability the algorithm converges to a realization within finite steps.

When the size of a network is large, the structure matrix will have a large dimension. Some techniques need to be developed to solve the computational complexity problem. For instance the aggregation seems to be an efficient method in certain circumstances [35, 36].

Before ending this section, we would like to mention that IET Control Theory and Applications is organizing a special issue on STP and its applications to logical systems. There are many updated materials in it. Particularly, we would like to recommend a survey paper in this special issue, which provides a comprehensive introduction for this new field [37].

4 From Boolean network to generalized logical system

4.1 What is a generalized logical system

Definition 10 ([26]). (1) A finite mapping $f:\prod_{i=1}^k\mathcal{D}_{n_i}\to\mathcal{D}_{n_0}$ is called a generalized logical mapping. Denote by

$$y=f(x_1,\ldots,x_k). \tag{49}$$

(2) A dynamic system

$$\begin{cases} x_1(t+1)=f_1(x_1(t),\ldots,x_n(t)),\\ \quad\vdots\\ x_n(t+1)=f_n(x_1(t),\ldots,x_n(t)),\end{cases} \tag{50}$$

where $\mathcal{D}_n := \{1, 2, \dots, n\}$, $x_i(t) \in \mathcal{D}_{n_i}$, and f_i, $i = 1, \dots, n$ are generalized logical mappings, is called a generalized logical network (or generalized logical dynamic system).

Similar to Boolean case, we identify $i \in \mathcal{D}_n$ with δ_n^i. Then

(1) There exists a unique logical matrix L_f, such that (49) can be expressed in its algebraic form as

$$y = L_f \ltimes_{i=1}^k x_i; \tag{51}$$

(2) There exists a unique logical matrix M, such that (50) can be expressed in its algebraic form as

$$x(t+1) = Mx(t), \tag{52}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$.

Note that similar to Boolean case, we can define the generalized logical control network and get its algebraic form, which is formally same as (39).

4.2 Applications of generalized logical (control) networks

Since the generalized logical (control) networks have the same algebraic expressions as for Boolean (control) networks, the technique developed for Boolean (control) networks can directly be used to solve the corresponding static or dynamic problems over finite sets. The following are some of the applications.

- Application to systems biology [35, 38–41].
- Application to graph theory and formation control [42–45].
- Application to circuit design and error detection [46–49].
- Application to fuzzy control [11, 12, 50–52].
- Application to finite automata and symbolic dynamics [53–58].
- Application to coding and numerical computation [30, 59–63].
- Application to network inquiry and tele-operation [64, 65].

Before ending this section, we should like to mention an important case, where the topology of a network is time-varying. This case can be treated by switched Boolean networks. We refer to [66–68] for switched Boolean networks.

5 STP approach to finite game theory

The symbol for the birth of modern game theory is the book of Neumann and Morgenstern [69]. Using STP to finite game theory is a very promising new direction. We, therefore, introduce it as a new section.

5.1 Vector space structure of finite games

Definition 11. A finite game is defined by a triple $G = (N, \mathcal{A}, U)$, where

(1) **(Players)** $N = \{1, 2, \dots, n\}$;

(2) **(Profile)** $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$, where $\mathcal{A}_i = \{a_1^i, \dots, a_{k_i}^i\} \sim \{1, \dots, k_i\}$, $i = 1, \dots, n$, is the set of actions of player i;

(3) **(Utility)** $U = (u_1, \dots, u_n)$, where $u_i : \mathcal{A} \to \mathbb{R}$ is the payoff function (utility) of player i.

Definition 12. Given a finite G. A profile $a = (x_1^*, \dots, x_n^*) \in \mathcal{A}$ is called a Nash equilibrium if

$$u_j(x_1^*, \dots, , x_j^*, \dots, x_n^*) \geqslant u_j(x_1^*, \dots, x_j, \dots, x_n^*), \quad j = 1, \dots, n. \tag{53}$$

Note that Nash equilibrium represents a scenario for which no player has a unilateral incentive to deviate.

Since $x_i \in \mathcal{D}_{n_i}$, it is natural to express it into a vector form in a natural way, that is $x_i = j \Leftrightarrow x_i = \delta_{n_i}^j$.

Denote the set of finite games $G = (N, \mathcal{A}, U)$, with $|N| = n$, $|\mathcal{A}_i| = k_i$, $i = 1, \dots, n$, by $\mathcal{G}_{[n;k_1,\dots,k_n]}$. When $k_i = \kappa$, $\forall i$, we denote $\mathcal{G}_{[n;\kappa]} = \mathcal{G}_{[n;\kappa,\dots,\kappa]}$.

Note that $u_i : \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_n} \to \mathbb{R}$ is a pseudo-logical mapping, then it is easy to see that there exists a unique row vector $V_i^c \in \mathbb{R}^k$ ($k = \prod_{i=1}^n k_i$), such that

$$u(x_1, \ldots, x_n) = V_i^c \ltimes_{i=1}^n x_i, \quad i = 1, \ldots, n. \tag{54}$$

Denote

$$V_G = [V_1^c, \ldots, V_n^c],$$

which is called the structure vector of G. Since $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$ is uniquely determined by u_i, $i = 1, \ldots, n$, then $\mathcal{G}_{[n;k_1,\ldots,k_n]}$ has a natural vector space structure as

$$G \in \mathcal{G}_{[n;k_1,\ldots,k_n]} \sim \mathbb{R}^{nk}. \tag{55}$$

The vector space structure of finite games was firstly proposed by [70]. The above vector space structure was proposed by [71], where the inner product is the conventional one for Euclidian space.

5.2 Decomposition of $\mathcal{G}_{[n;k_1,\ldots,k_n]}$

5.2.1 *Potential game*

Definition 13 ([72])**.** Consider a finite game $G = \{N, \mathcal{A}, U\}$. G is a potential game, if there exists a function $P : \mathcal{A} \to \mathbb{R}$, called potential function, such that for each $i \in N$ and $a_{-i} \in \mathcal{A}_{-i}$, $\forall \alpha, \beta \in \mathcal{A}_i$

$$u_i(\alpha, a_{-i}) - u_i(\beta, a_{-i}) = P(\alpha, a_{-i}) - P(\beta, a_{-i}), \quad i = 1, \ldots, n. \tag{56}$$

Denote

$$k^{[p,q]} := \begin{cases} \prod_{j=p}^q k_j, & q \geqslant p, \\ 1, & q < p, \end{cases} \tag{57}$$

and

$$E_i := I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \in \mathcal{M}_{k \times k/k_i}, \quad i = 1, \ldots, n. \tag{58}$$

Then the potential equation is defined as follows:

$$\begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ & & \vdots & & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} (V_2^c - V_1^c)^{\mathrm{T}} \\ (V_3^c - V_1^c)^{\mathrm{T}} \\ \vdots \\ (V_n^c - V_1^c)^{\mathrm{T}} \end{bmatrix}, \tag{59}$$

where $\xi_i \in \mathbb{R}^{k/k_i}$.

Briefly, we have

$$E\xi = b.$$

Theorem 11 ([73])**.** $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$ is potential, if and only if, potential equation (59) has solution. Moreover, assume ξ is a solution, then the potential function is

$$P(x_1, \ldots, x_n) = V^P \ltimes_{j=1}^n x_j, \tag{60}$$

where, V^P is calculated as

$$V^P = V_1^c - \xi_1^{\mathrm{T}} E_1^{\mathrm{T}}. \tag{61}$$

5.2.2 *Orthogonal decomposition*

Definition 14 ([74])**.** (1) A game $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$ is a non-strategic game, if for any $a_{-i} \in \mathcal{A}_{-i}$, and any $\alpha, \beta \in \mathcal{A}_i$, we have

$$u_i(\alpha, a_{-i}) = u_i(\beta, a_{-i}), \quad i = 1, \ldots, n. \tag{62}$$

(2) A game $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$ is a pure harmonic game, if

$$\begin{aligned} &\sum_{i=1}^{n} u_i(a) = 0, \quad a \in \mathcal{A}, \\ &\sum_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}) = 0, \quad i = 1, \ldots, n. \end{aligned} \tag{63}$$

Denoting the subspace of non-strategic games by $\mathcal{H}$ and the subspace of pure potential games by $\mathcal{P}$, then we have the following orthogonal decomposition:

$$\mathcal{G}_{[n;k_1,\ldots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{\text{Potential games}} \overbrace{\oplus \quad \mathcal{H}}^{\text{Harmonic games}}. \tag{64}$$

Note that $\mathcal{P} \oplus \mathcal{N}$ is the subspace of potential games and $\mathcal{H} \oplus \mathcal{N}$ is the subspace of harmonic games.

Note that such decomposition is not unique. Recently, based on symmetric games and skew symmetric games another orthogonal decomposition is proposed as in (65), where $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{E}$ are subspaces of symmetric games, skew symmetric games and asymmetric games respectively [75].

$$\mathcal{G}_{[n;\kappa]} = \mathcal{S}_{[n;\kappa]} \oplus \mathcal{K}_{[n;\kappa]} \oplus \mathcal{E}_{[n;\kappa]}. \tag{65}$$

5.3 Networked evolutionary game

A Markov-type evolutionary game with $G \in \mathcal{G}_{[n;k_1,\ldots,k_n]}$ is described as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad\vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \ldots, x_n(t)). \end{cases} \tag{66}$$

Since $x_i(t) \in \mathcal{A}_i = \mathcal{D}_{n_i}$, using STP, it can be expressed into the ASSR as

$$x(t+1) = Lx(t), \tag{67}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$. Then the STP approach can be used to the analysis and control design of evolutionary games. The first paper using STP approach to control design of evolutionary games is [76].

The investigation of networked evolutionary games (NEG) was then developed, which are defined as follows.

Definition 15 ([77])**.** A networked evolutionary game, denoted by $((N,E),G,\Pi)$, consists of (1) a network (graph) (N,E); (2) an FNG, G, such that if $(i,j) \in E$, then i and j play FNG with strategies $x_i(t)$ and $x_j(t)$ respectively; (3) a local information based strategy updating rule, Π.

A necessary and sufficient condition for an NEG to converge to a stationary profile was proposed, and a concept of dynamic equivalence of evolutionary games is proposed, which is very useful for near potential games.

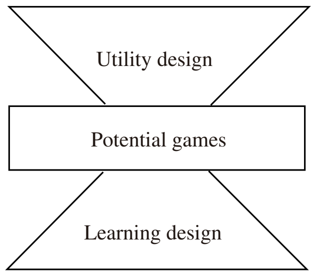

Figure 2 "Hourglass" architecture of game theoretic control.

5.4 Game theoretic control

Game theoretic control is a cross discipline between game theory and control theory. A framework for game theoretic control is presented in [78], and it is described as an hourglass in Figure 2.

The design process can be described as follows.

- Consider a multi-agent system with agents $N = \{1, \ldots, n\}$ and an objective function $\varphi(x_1, \ldots, x_n)$.
- Design the utility functions $u_i(x_1, \ldots, x_n)$ such that the system becomes a potential game with $\varphi(x_1, \ldots, x_n)$ as its potential function.
- Design a learning algorithm to assure the evolutionary game converges to a Nash equilibrium, which could be the maximum point.

Based on the potential equation and STP approach, some elementary studies have been done for this [79, 80]. As for the game theoretic control of networked games, Bayesian game may provide a nice framework, because in a networked game a player can hardly get complete information [81].

6 Mathematics behind STP

It is well known that the set of matrices with conventional product has rich and varied algebraic and geometric structures. It is natural to ask whether the STP can bring some new mathematical structures to us? This is an interesting and challenging problem. In the following we sketch some results presented in [82].

(1) Let $\mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m\times n}$. Then $(\mathcal{M}, \ltimes)$ is a monoid (semi-group with identity).

(2) $A, B \in \mathcal{M}$ are called equivalent, and denoted by $A \sim B$, if there exist I_p and I_q such that

$$A \otimes I_p = B \otimes I_q. \tag{68}$$

The equivalence class of A is denoted by $\langle A \rangle$.

(3) Over equivalence class

$$\langle A \rangle \ltimes \langle B \rangle := \langle A \ltimes B \rangle . \tag{69}$$

Eq. (69) is well defined.

(4) If $A = B \otimes I_s$, then we say that $B \prec A$. $\langle A \rangle$ with the order $\prec$ is a lattice.

(5) $(\{\langle A \rangle \,|A \text{ is invertible}\}, \ltimes)$ forms a Lie group.

(6) $\{\langle A \rangle \,|A \text{ is square}\}$ with $[\cdot,\cdot]$ forms a Lie algebra, where $[\langle A \rangle , \langle B \rangle] := \langle A \rangle \ltimes \langle B \rangle - \langle B \rangle \ltimes \langle A \rangle$. We could have many new properties.

It is interesting and challenging to obtain new insights about mathematical structure of STP.

7 Conclusion

In the last decade, the STP has been developed rapidly. It has been applied to several related disciplines as described in this paper. It has also been applied to various engineering problems. Its vitality comes from its wide applications. At the beginning of this paper we reveal the geometric inside of STP. The various applications of STP may come from its physical rationality. This paper surveyed the applications of STP to dynamic systems, to logical systems, and to finite games. There are many problems in each field which need to be dug in deeply. We are also confident that more related problems might be investigated via STP.

So for, people are mainly interested in the applications of STP, and less people are aware of the mathematics behind STP. Ref. [82] started such an investigation. In fact, the STP-based mathematics could be a huge uncultivated land.

Acknowledgements This work was supported partly by National Natural Science Foundation of China (NSFC) (Grant Nos. 61333001, 61773371, 61733018).

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