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STABILITY OF SWITCHED POLYNOMIAL SYSTEMS*

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Abstract This paper investigates the stability of (switched) polynomial systems. Using semi-tensor product of matrices, the paper develops two tools for testing the stability of a (switched) polynomial system. One is to convert a product of multi-variable polynomials into a canonical form, and the other is an easily verifiable sufficient condition to justify whether a multi-variable polynomial is positive definite. Using these two tools, the authors construct a polynomial function as a candidate Lyapunov function and via testing its derivative the authors provide some sufficient conditions for the global stability of polynomial systems.

Key words Global asymptotical stability, semi-tensor product, switched polynomial systems.

1 Introduction

Stability is a long standing and challenging topic for investigating nonlinear (control) systems. Lyapunov function is a fundamental tool for studying stability and stabilization of (control) systems. The stability of polynomial systems has been attracting special research interest[1−4]. It is because not only this kind of systems are practically important, but also constructing Lyapunov functions for them is relatively easier. For instance, Roser[3] constructed a homogeneous Lyapunov function for homogeneous system under the hypothesis that zero is locally asymptotically stable. M'Closkey and Murray[2] considered the problem of exponential stabilization of controllable, driftless systems using time-varying, homogeneous feedback. Grun[4] showed that for any asymptotically controllable homogeneous system in Euclidian space, there exists a homogeneous control Lyapunov function and a homogeneous, possibly discontinuous state feedback law stabilizing the corresponding sampled closed loop system. The Kronecker product is used for non-quadratic stability analysis and sufficient conditions for global asymptotic stability of polynomial systems are obtained in terms of LMI feasibility tests for the existence of homogeneous Lyapunov functions of even degree[1]. But in these investigations the homogeneity plays an important role. For instance, in [3–5] the system considered is homogeneous; in [1, 3] the Lyapunov function is homogeneous; in [2] the feedback is homogeneous; in [6] the derivative is homogeneous, etc. Obviously, homogeneity brings restriction and/or conservative to the application of the methods presented above.

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In this paper the polynomial systems considered are not assumed to be homogeneous. We first develop some results for homogeneous case and then extend them to non-homogeneous case.

The paper is organized as follows. Section 2 gives a brief review for semi-tensor product of matrixes. Converting polynomials and their derivatives into canonical forms is discussed in Section 3. Section 4 provides an easily verifiable sufficient condition for testing the positivity of homogeneous polynomials. Section 5 discusses the global stability of vector fields via two Lyapunov functions. In Section 6 some sufficient conditions are obtained for the stability of polynomial systems. Section 7 contains some concluding remarks.

2 Semi-Tensor Product

This section is a brief review on semi-tensor product of matrices, which plays a fundamental role in the following discussion. We restrict it to the definitions and some basic properties, which are useful in the sequel. In addition, only left semi-tensor product for multiple-dimension case is involved in the paper. We refer to [7–8] for right semi-tensor product, general dimension case, and many details. Through out this paper "semi-tensor product" means the left semi-tensor product.

Definition 2.1 1) Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then, we split X into p equal-size blocks as $X^1, X^2, \cdots, X^p$, which are $1 \times n$ rows. Define the STP, denoted by $\ltimes$, as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in \mathbb{R}^n, \\ Y^{\mathrm{T}} \ltimes X^{\mathrm{T}} = \sum_{i=1}^{p} y_i (X^i)^{\mathrm{T}} \in \mathbb{R}^n. \end{cases} \tag{1}$$

2) Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the STP of A and B, denoted by $C = A \ltimes B$, as the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, q,$$

where A^i is i-th row of A and B_j is the j-th column of B.

Remark 2.2 Note that when $n = p$ the STP coincides with the conventional matrix product. Therefore, the STP is only a generalization of the conventional product. For convenience, we may omit the product symbol $\ltimes$.

Some fundamental properties of the STP are collected in the following.

Proposition 2.3 *The STP satisfies* (*as long as the related products are well-defined*)

1) (*Distributive rule*)

$$\begin{aligned} &A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C, \\ &(\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{2}$$

2) (*Associative rule*)

$$\begin{aligned} &A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C, \\ &(B \ltimes C) \ltimes A = B \ltimes (C \ltimes A). \end{aligned} \tag{3}$$

Proposition 2.4 *Let $A \in M_{p\times q}$ and $B \in M_{m\times n}$. If $q = km$, then*

$$A \ltimes B = A(B \otimes I_k). \tag{4}$$

If $kq = m$, then

$$A \ltimes B = (A \otimes I_k)B. \tag{5}$$

Proposition 2.5 1) *Assume A and B are of proper dimensions such that $A \ltimes B$ is well-defined, then*

$$(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}; \tag{6}$$

2) *In addition, assume both A and B are invertible, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{7}$$

Proposition 2.6 *Assume $A \in M_{m\times n}$ is given.*
1) *Let $Z \in \mathbb{R}^t$ be a row vector, then*

$$A \ltimes Z = Z \ltimes (I_t \otimes A); \tag{8}$$

2) *Let $Z \in \mathbb{R}^t$ be a column vector, then*

$$Z \ltimes A = (I_t \otimes A) \ltimes Z. \tag{9}$$

For notational ease, hereafter we omit the symbol $\ltimes$.

3 Polynomials and Their Derivatives

In this section, we show how to convert a polynomial and its derivative along a trajectory of polynomial system into normal form. Note that when $\xi \in \mathbb{R}^n$ is a column or a row vector, $\underbrace{\xi \ltimes \xi \ltimes \cdots \ltimes \xi}_{k}$ is well-defined. We denote it briefly as

$$\xi^k := \underbrace{\xi \ltimes \xi \ltimes \cdots \ltimes \xi}_{k}.$$

Now let $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then x^k is well-defined. Using it, a k-th degree polynomial $P_k(x)$ can be expressed as

$$P_k(x) = Ex^k, \tag{10}$$

where E is a row of dimension n^k. Note that such E is not unique.

Let $P(x)$ be a polynomial with lowest degree k and highest degree $k+s$. Then, it can be expressed as

$$P(x) = E_k x^k + E_{k+1}x^{k+1} + \cdots + E_{k+s}x^{k+s}. \tag{11}$$

We call (11) the canonical form of a polynomial. Similarly, a polynomial system can be expressed as

$$\dot{x} = f(x) := F_k x^k + F_{k+1}x^{k+1} + \cdots + F_{k+s}x^{k+s}, \tag{12}$$

where F_i, $i = k, k+1, \cdots, k+s$, are $n \times n^i$ matrices.

Next we consider the derivative of a polynomial. For this purpose we need the swap matrix, which is also called the permutation matrix and is defined implicitly by Magnus and Neudecker[9]. Many properties can be found in [7–8]. The swap matrix $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ and its rows by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$. Then, its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

When $m = n$ we simply denote $W_{[n,n]}$ by $W_{[n]}$.

Let $A \in M_{m\times n}$, i.e., A is an $m \times n$ matrix. Denote by $V_r(A)$ the row stacking form of A, that is,

$$V_r(A) = (a_{11}\, a_{12} \cdots a_{1n} \ \cdots \ a_{m1}\, a_{m2} \cdots a_{mn})^{\mathrm{T}},$$

and by $V_c(A)$ the column stacking form of A, that is,

$$V_c(A) = (a_{11}\, a_{12} \cdots a_{m1} \ \cdots \ a_{1n}\, a_{2n} \cdots a_{mn})^{\mathrm{T}}.$$

The following "swap" property shows the meaning of the name.

Proposition 3.1 1) *Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns, then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X, \quad W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y. \tag{14}$$

2) *Let $A \in M_{m\times n}$, then*

$$W_{[m,n]} V_r(A) = V_c(A), \quad W_{[n,m]} V_c(A) = V_r(A). \tag{15}$$

Proposition 3.2 *Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$, then*

$$W_{[m,n]}^{\mathrm{T}} = W_{[m,n]}^{-1} = W_{[n,m]}. \tag{16}$$

Using swap matrix, we can prove that

Proposition 3.3 *If $X \in \mathbb{R}^n$, $Y^{\mathrm{T}} \in \mathbb{R}^m$, then*

$$XY = Y \ltimes W_{[n,m]} \ltimes X. \tag{17}$$

Now we consider how to calculate the differential form of a polynomial. We construct an $n^{k+1} \times n^{k+1}$ matrix Φ_k as

$$\Phi_k = \sum_{s=0}^{k} I_{n^s} \otimes W_{[n^{k-s},n]}. \tag{18}$$

Then, we have the following differential form of X^k, which is fundamental in later approach.

Proposition 3.4

$$D(X^{k+1}) = \Phi_k \ltimes X^k. \tag{19}$$

Now let

$$V(x) = \sum_{i=j}^{j+t} E_i x^i$$

be a candidate of Lyapunov function. We calculate its derivative with respect to system (13) as

$$\begin{aligned}\dot{V}|_{(13)} &= \Big(\sum_{i=j}^{j+t} E_i \Phi_{i-1} x^{i-1}\Big)\Big(\sum_{\alpha=k}^{k+s} F_\alpha x^\alpha\Big)\\ &= \big(E_j \Phi_{j-1} x^{j-1}\big) F_k x^k\\ &\quad + \big(E_{j+1} \Phi_j x^j F_k x^k + E_j \Phi_{j-1} x^{j-1} F_{k+1} x^{k+1}\big)\\ &\quad + \cdots + E_{j+t} \Phi_{j+t-1} x^{j+t-1} F_{k+s} x^{k+s}.\end{aligned} \tag{20}$$

Using Proposition 2.6, we can express the derivative into canonical form as

$$\begin{aligned}\dot{V}|_{(13)} &= E_j \Phi_{j-1}(I_{n^{j-1}} \otimes F_k) x^{j+k-1}\\ &\quad + (E_{j+1} \Phi_j (I_{n^j} \otimes F_k) + E_j \Phi_{j-1}(I_{n^{j-1}} \otimes F_{k+1}))\, x^{j+k}\\ &\quad + \cdots + E_{j+t} \Phi_{j+t-1}(I_{n^{j+t-1}} \otimes F_{k+s}) x^{j+t+k+s-1}\\ &:= D_{j+k-1} x^{j+k-1} + D_{j+k} x^{j+k} + \cdots + D_{j+t+k+s-1} x^{j+t+k+s-1}.\end{aligned} \tag{21}$$

4 Positivity of Homogeneous Polynomials

In this section we consider when a homogeneous polynomial is positive definite. In general, this is a very hard open problem. We give an easily verifiable sufficient condition. The argument is based on the following lemma.

Lemma 4.1[6] *Let $S \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$. Then, we have the following inequality:*

$$|x^S| \le \sum_{j=1}^{n} \frac{s_j}{|S|} |x_j|^{|S|}, \tag{22}$$

where $x^S = \prod_{i=1}^n (x_i)^{s_i}$, and $|S| = \sum_{i=1}^n s_i$.

Sometimes we need a modification. Assume $\lambda_i > 0$ and $\prod_{i=1}^n \lambda_i^{s_i} = 1$. Then, replacing x_i by $\lambda_i x_i$, we have a modification of (22) as

$$|x^S| \le \sum_{j=1}^{n} \frac{s_j}{|S|} \lambda_j^{|S|} |x_j|^{|S|}. \tag{23}$$

To use this lemma for a k-th homogeneous polynomial of $x \in \mathbb{R}^n$, we must know the powers of x_i in each component of x^k. Since x^k has n^k components, for each x_i, we use an n^k dimensional vector, denoted by V_k^i, to represent the powers of x_i in each component of x^k.

Example 4.2 Let $x \in \mathbb{R}^2$, then,

$$\begin{aligned}x^4 = (&x_1^4, x_1^3 x_2, x_1^2 x_2 x_1, x_1^2 x_2^2, x_1 x_2 x_1^2, x_1 x_2 x_1 x_2, x_1 x_2^2 x_1, x_1 x_2^3,\\ &x_2 x_1^3, x_2 x_1^2 x_2, x_2 x_1 x_2 x_1, x_2 x_1 x_2^2, x_2^2 x_1^2, x_2^2 x_1 x_2, x_2^3 x_1, x_2^4).\end{aligned}$$

Hence,

$$V_4^1 = [4,3,3,2,3,2,2,1,3,2,2,1,2,1,1,0]^{\mathrm{T}},$$

and

$$V_4^2 = [0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4]^{\mathrm{T}}.$$

Next, we consider the general form of V_k^i.

Lemma 4.3 *Let $x \in \mathbb{R}^n$. Then the powers of x_i in each components of x^k, expressed by V_k^i, are*

$$V_k^i = (\mathbf{1}_n)^{k-1}\delta_i^n + (\mathbf{1}_n)^{k-2}\delta_i^n\mathbf{1}_n + \cdots + \mathbf{1}_n\delta_i^n(\mathbf{1}_n)^{k-2} + \delta_i^n(\mathbf{1}_n)^{k-1}, \tag{24}$$

where $\mathbf{1}_n = (\underbrace{1, 1, \cdots, 1}_{n})^{\mathrm{T}}$, δ_i^n is the i-th column of I_n.

Proof First, we prove a recursive form as follows:

$$\begin{cases} V_1^i = \delta_i^n, \\ V_{s+1}^i = \mathbf{1}_n V_s^i + \delta_i^n \mathbf{1}_{n^{s-1}}, \quad s \geq 1. \end{cases} \tag{25}$$

Since $x^1 = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, x_i appears only on its i-th component with power 1, so $V_1^i = \delta_i^n$. Now assume V_s^i is known, it is a vector of dimension n^s. We may get V_{s+1}^i from V_s^i through the following two steps: First, repeating it n times to get an n^{s+1} vector. This is produced by multiplying x^s with $x_1, x_2, \cdots, x_n$, respectively. It is represented by $\mathbf{1}_n V_s^i$. Next, the i-th n^s dimensional block of x^{s+1} is obtained by multiplying x^s with x_i. Hence, in this block the power of x_i must be raised by 1. This is performed by $\delta_i^n \mathbf{1}_n$. Combining these two steps yields (25).

Using (25) repetitively, we can prove (24) easily. ∎

In x^k, the terms of highest degree of x_i, i.e., x_i^k, are particularly important. When $k = 2$, they are called the diagonal elements, because in quadratic form $x^{\mathrm{T}}Qx$, they correspond to diagonal elements of Q. It is well known that for quadratic form we have so-called diagonal dominating principle (DDP), that is, $x^{\mathrm{T}}Qx$ is positive definite if the diagonal elements are dominating, i.e.,

$$q_{ii} > \sum_{j \neq i} |q_{ij}|, \quad i = 1, 2, \cdots, n. \tag{26}$$

For $k > 2$, we still call x_i^k diagonal elements. The DDP has been extended to general case when $k > 2$ is even[6]. In the following we give a matrix expression of the cross row diagonal dominating principle (CRDDP) and DDP proposed by Cheng[6] for general case.

First, we want to figure out the positions of diagonal elements in x^k. It is easy to prove the following lemma.

Lemma 4.4 *Let $x \in \mathbb{R}^n$. The position of diagonal element x_i^k in x^k is on d_i-th, where*

$$d_i = (i-1)\frac{n^k - 1}{n - 1} + 1, \quad i = 1, 2, \cdots, n. \tag{27}$$

For instance, assume $x \in \mathbb{R}^4$ and $k = 2$. Using (27), we have $d_1 = 1$, $d_2 = 6$, $d_3 = 11$, and $d_4 = 16$. It is easy to verify this from Example 4.2.

For convenience, we define the position set of diagonal elements as $D_n^k = \{d_i | i = 1, 2, \cdots, n\}$, where d_i is the position of x_i^k in x^k.

Note that if an even degree homogeneous polynomial $P(x) = Fx^k$ is positive definite, then its diagonal elements x_i^k must have positive coefficients, that is,

$$F_{d_i} > 0, \quad i = 1, 2, \cdots, n.$$

Using Lemmas 4.1, 4.3, and 4.4, we have the following result.

Theorem 4.5 *Let k be even, and $P(x) = Fx^k$ be a k-th homogeneous polynomial with $x \in \mathbb{R}^n$.*

1) *Assume $F_{d_i} > 0$, $i = 1, 2, \cdots, n$. Define $\widetilde{F}$ by*

$$\widetilde{F}_i = \begin{cases} 0, & i \in D_n^k, \\ |F_i|, & \textit{otherwise}. \end{cases} \tag{28}$$

If

$$F_{d_i} > \frac{1}{k}\widetilde{F}V_k^i, \quad i = 1, 2, \cdots, n, \tag{29}$$

then $P(x)$ is positive definite.

2) *Assume $F_{d_i} < 0$, $i = 1, 2, \cdots, n$. Define $\widetilde{F}$ by*

$$\widetilde{F}_i = \begin{cases} 0, & i \in D_n^k, \\ |F_i|, & \textit{otherwise}. \end{cases} \tag{30}$$

If

$$-F_{d_i} > \frac{1}{k}\widetilde{F}V_k^i, \quad i = 1, 2, \cdots, n, \tag{31}$$

then $P(x)$ is negative definite.

Proof Using (22) to each term of $P(x)$, one sees that for each component x_i^k of x^k, its coefficient is $\frac{V_k^i}{k}$. Keeping diagonal elements x_i^k, $i = 1, 2, \cdots, n$, unchanged, and enlarging the absolute values of other terms by (22), it is easy to check that (29) assures the positivity. The argument for negativity is similar. ∎

We give some examples to describe this.

Example 4.6 1) Consider polynomial

$$P(x) = x_1^4 - x_1^2x_2^2 + 1.5x_1x_2^3 + 2x_2^4.$$

Express $P(x) = Fx^4$, then,

$$F = [1, 0, 0, -1, 0, 0, 0, 1.5, 0, 0, 0, 0, 0, 0, 0, 2].$$

Using (28), $\widetilde{F}$ is constructed as

$$\widetilde{F} = [0, 0, 0, 1, 0, 0, 0, 1.5, 0, 0, 0, 0, 0, 0, 0, 0].$$

It is easy to calculate that

$$\begin{aligned} V_4^1 &= \mathbf{1}_{22}^{31} + \mathbf{1}_{22}^{21}\mathbf{1}_2 + \mathbf{1}_{22}{}^1_2\mathbf{1}_2^2 + {}^1_2\mathbf{1}_2^3 \\ &= [4, 3, 3, 2, 3, 2, 2, 1, 3, 2, 2, 1, 2, 1, 1, 0]^{\mathrm{T}} \end{aligned}$$

and

$$\begin{aligned} V_4^2 &= \mathbf{1}_{22}^{32} + \mathbf{1}_{22}^{22}\mathbf{1}_2 + \mathbf{1}_{22}{}^2_2\mathbf{1}_2^2 + {}^2_2\mathbf{1}_2^3 \\ &= [0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4]^{\mathrm{T}}. \end{aligned}$$

Note that $F_{d_1} = 1$ and $F_{d_2} = 2$. Checking (29), we have

$$F_{d_1} - \frac{1}{4}\widetilde{F}V_4^1 = \frac{1}{8} > 0, \quad F_{d_2} - \frac{1}{4}\widetilde{F}V_4^2 = \frac{3}{8} > 0.$$

Thus, $P(x) > 0$, that is, $P(x)$ is positive definite.

2) Consider

$$Q(x) = x_1^4 + 6x_1^2x_2^2 + 1.5x_1x_3^3 + 2x_2^4.$$

Express $Q(x) = Hx^4$, then,

$$H = [1, 0, 0, 6, 0, 0, 0, 1.5, 0, 0, 0, 0, 0, 0, 0, 2].$$

Construct $\widetilde{H}$ as

$$\widetilde{H} = [0, 0, 0, 6, 0, 0, 0, 1.5, 0, 0, 0, 0, 0, 0, 0, 0].$$

Checking (29), we have

$$H_{d_1} - \frac{1}{4}\widetilde{H}V_4^1 = -\frac{19}{8} < 0, \quad H_{d_2} - \frac{1}{4}\widetilde{H}V_4^2 = -\frac{17}{8} < 0.$$

We can conclude nothing.

Comparing $P(x)$ with $Q(x)$, it is easy to see that $Q(x) \geq P(x)$. Therefore, for $Q(x)$ the inequality (29) is not sharp enough. The problem is that we don't need to enlarge positive semi-definite term $6x_1^2x_2^2$ in $Q(x)$. We can simply ignore it.

To find positive semi-definite terms, we construct the following matrix:

$$V_k = [V_k^1, V_k^2, \cdots, V_k^n].$$

Then, the (i, j)-th element of V_k is the power of x_j in the i-th component of x^k. For instance, in Example 4.6, we have

$$V_4^{\mathrm{T}} = \begin{bmatrix} 4 & 3 & 3 & 2 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 \end{bmatrix}.$$

If the elements in i-th row of V_k are all even, then the i-th term of $P(x) = Fx^k$ has all even powers. We call such terms the even power terms. Now, if the corresponding coefficient F_i of F is positive, i.e., $F_i > 0$, then in estimating the inequality such terms can be omitted. We, therefore, have the following corollary.

Corollary 4.7 *In Theorem* 4.5 *the positivity of* $P(x)$ *remains true when* (28) *is replaced by*

$$\widetilde{F}_i = \begin{cases} 0, & \text{even term with } F_i \geq 0, \\ |F_i|, & \text{otherwise}. \end{cases} \tag{32}$$

Similarly, the negativity of $P(x)$ *remains true when* (30) *is replaced by*

$$\widetilde{F}_i = \begin{cases} 0, & \text{even term with } F_i \leq 0, \\ |F_i|, & \text{otherwise}. \end{cases} \tag{33}$$

5 Global Stability via Two Lyapunov Functions

Consider a dynamic system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{34}$$

where $f(x)$ is a smooth vector field.

First, we define a kind of stability, called U stability.

Definition 5.1 Let U be a neighborhood of the origin. System (34) is said to be U-stable, if it is Lyapunov stable and for any $x_0 \in \mathbb{R}^n$,

$$\lim_{t\to\infty} d(x(x_0,t),U) = 0,$$

where $x(x_0,t)$ is the solution to (34) with initial point x_0 and d is the distance.

The following result is obvious.

Proposition 5.2 *Consider system* (34).

1) *Assume there is a positive definite radially unbounded function $V_1(x) > 0$. $U := \{x|V_1(x) < \alpha\}$, for some $\alpha > 0$, is a neighborhood of the origin. If*

$$\dot V_1|_{(34)} < 0, \quad x \in U^c,$$

then, system (34) *is U-stable.*

2) *Assume there is a positive definite function $V_2(x) > 0$. $W := \{x|V_2(x) \le \beta\}$, for some $\beta > 0$, is a neighborhood of the origin. If*

$$\dot V_2|_{(34)} < 0, \quad 0 \ne x \in W,$$

then, (34) *is asymptotically stable at the origin, and W is a region of attraction.*

3) *If there are $V_1(x)$ and $V_2(x)$, which are the same as in the items* 1 *and* 2, *respectively. Moreover, assume $U \subset W$, then, system* (34) *is globally asymptotically stable.*

6 Stability of Polynomial Systems

Consider a polynomial

$$P(x) := p_k x^k + p_{k+1}x^{k+1} + \cdots + p_{k+s}x^{k+s}.$$

We define its lowest degree terms and highest degree terms by

$$L_P(x) := p_k x^k, \quad H_P(x) := p_{k+s}x^{k+s}.$$

Similarly, for a polynomial vector field $f(x)$ the lowest and highest terms form two homogeneous vector fields, denoted by $L_f(x)$ and $H_f(x)$, respectively.

The following result is obvious.

Lemma 6.1 *Assume a polynomial $P(x)$ is positive definite, then the two polynomials $L_P(x)$ and $H_P(x)$ are positive semi-definite.*

Based on this lemma, we assume

Assumption 1 System (13) is an odd-ended system, i.e., both $\deg(L_f(x))$ and $\deg(H_f(x))$ are odd. Then, we can express (13) as

$$\begin{aligned} \dot x &= f(x) = F_{2i+1}x^{2i+1} + F_{2i+2}x^{2i+2} + \cdots + F_{2(i+j)+1}x^{2(i+j)+1} \\ &:= L_f(x) + f_{2i+2} + f_{2i+3} + \cdots + f_{2(i+j)} + H_f(x), \end{aligned} \tag{35}$$

where $f_k = F_k x^k$, $k = 2i+1, 2i+2, \cdots, 2(i+j)+1$.

Now assume we can find two positive definite homogeneous polynomials $V_1(x) > 0$ and $V_2(x) > 0$ with $\deg(V_1(x)) = 2p$ and $\deg(V_2(x)) = 2q$. Moreover,

$$\dot V_1|_{H_f(x)} < 0, \quad x \ne 0 \ \text{ and } \ \dot V_2|_{L_f(x)} < 0, \quad x \ne 0.$$

Next, we define a cub, C, as

$$C := \{x \in \mathbb{R}^n \mid |x_i| \le r_i,\ i = 1, 2, \cdots, n\},$$

where $r_i > 0$. We want to estimate $\dot{V}_1|_{f(x)}$ for $x \in C^c$ and $\dot{V}_2|_{f(x)}$ for $x \in C$. To use the result for the positivity (equivalently, negativity) of homogeneous polynomials, we want to convert them into homogeneous forms. We provide two algorithms for this purpose.

Algorithm 6.2

1) Calculate

$$\dot{V}_1|_{f_k(x)} = Z_{2p+k-1} x^{2p+k-1}, \tag{36}$$

where $k = 2i+1, 2i+2, \cdots, 2(i+j)$.

2) Remove negative semi-definite terms:

$$D_k(x) := \widetilde{Z}_{2p+k-1} x^{2p+k-1}, \tag{37}$$

where $k = 2i+1, 2i+2, \cdots, 2(i+j)$ and the components of $\widetilde{Z}_{2p+k-1}$ are defined as

$$\widetilde{Z}^i_{2p+k-1} = \begin{cases} 0, & \text{even power term with } Z^i_{2p+k-1} \le 0, \\ \left|Z^i_{2p+k-1}\right|, & \text{otherwise.} \end{cases}$$

3) Enlarge it to homogeneous case:

$$H_k(x) := \widetilde{Z}_{2p+k-1} |x|^{2p+k-1} \frac{\left(|x_1|^{2(i+j)-k+1} + \cdots + |x_n|^{2(i+j)-k+1}\right)}{\max\left\{r_s^{2(i+j)-k+1} | s = 1, 2, \cdots, n\right\}}, \tag{38}$$

where $x \in C^c,\ k = 2i+1, 2i+2, \cdots, 2(i+j)$.

Using Algorithm 6.2, we can define an estimation as

$$E_1(x) := \dot{V}_1|_{H_f(x)} + \sum_{k=2i+1}^{2(i+j)} H_k(x). \tag{39}$$

From the constructing of the algorithm, it is easy to see the following lemma.

Lemma 6.3

$$\dot{V}_1|_{f(x)} \le E_1(x), \quad x \in C^c. \tag{40}$$

Since E_1 is a homogeneous function, it is easy to use previous methods to check its negativity. Next, we check the negativity of $\dot{V}_2|_{f_k(x)}$.

Algorithm 6.4

1) Calculate

$$\dot{V}_2|_{f_k(x)} = Z_{2q+k-1} x^{2q+k-1}, \quad k = 2i+2, 2i+3, \cdots, 2(i+j)+1. \tag{41}$$

2) Remove negative semi-definite terms:

$$D_k(x) := \widetilde{Z}_{2q+k-1} x^{2q+k-1}, \tag{42}$$

where $k = 2i+2, 2i+3, \cdots, 2(i+j)+1$, and the components of $\widetilde{Z}_{2q+k-1}$ are defined as

$$\widetilde{Z}^i_{2q+k-1} = \begin{cases} 0, & \text{even power term with } Z^i_{2q+k-1} \le 0, \\ \left|Z^i_{2q+k-1}\right|, & \text{otherwise.} \end{cases}$$

3) Enlarge it to homogeneous case:

$$L_k(x) := \tfrac{1}{2q+k-1}\widetilde{Z}_{2q+k-1}\left[V^1_{2q+k-1}r_1^{k-1-2i}|x_1|^{2q+2i} + \cdots + V^n_{2q+k-1}r_n^{k-1-2i}|x_n|^{2q+2i}\right], \quad (43)$$

where $k = 2i+2, 2i+3, \cdots, 2(i+j)+1$.

Using Algorithm 6.4, we can define an estimation as

$$E_2(x) := \dot{V}_2|_{L_f(x)} + \sum_{k=2(i+1)}^{2(i+j)+1} L_k(x). \quad (44)$$

From the algorithm it is easy to see the following lemma.

Lemma 6.5

$$\dot{V}_2|_{f(x)} \le E_2(x), \quad x \in C. \quad (45)$$

Since E_2 is a homogeneous function, it is easy to use the tool developed in Section 4 to check its negativity.

Summarizing the above arguments, we have

Theorem 6.6 *Consider system* (35). *Assume that there exist homogeneous* $V_1(x) > 0$, $V_2(x) > 0$, *and an invariant cub* C, *such that*

$$\begin{aligned} &E_1(x) < 0, \quad x \in C^c, \\ &E_2(x) < 0, \quad 0 \ne x \in C. \end{aligned} \quad (46)$$

Then, the system is globally asymptotically stable.

Consider a switched polynomial system

$$\dot{x} = f_{\sigma(t)}(x), \quad (47)$$

where $\sigma(t) : [0, \infty) \to \Lambda = \{1, 2, \cdots, N\}$ is a switching signal, f_λ, $\lambda \in \Lambda$, are odd-ended polynomial vector fields.

Using Theorem 6.6, we have

Theorem 6.7 *Consider system* (47). *Assume that there exist homogeneous* $V_1(x) > 0$, $V_2(x) > 0$, *and an invariant cub* C, *such that for the* i*-th switching mode,*

$$\begin{aligned} &E_1^i(x) < 0, \quad x \in C^c, \\ &E_2^i(x) < 0, \quad 0 \ne x \in C. \end{aligned} \quad (48)$$

Then, the system is globally asymptotically stable under arbitrary switches.

7 An Illustrative Example

Example 7.1 Consider the following polynomial system:

$$\begin{cases} \dot{x}_1 = -\beta x_1 + x_1^2 + x_2^2 - \alpha x_1^3, \\ \dot{x}_2 = -\beta x_2 + 2x_1x_2 - \alpha x_2^3. \end{cases} \quad (49)$$

We can express the polynomial system (49) as

$$\dot{x} = A_1 x + A_2 x^2 + A_3 x^3, \tag{50}$$

where

$$A_1 = \begin{bmatrix} -\beta & 0 \\ 0 & -\beta \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1\,0\,0\,1 \\ 0\,1\,1\,0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -\alpha\; 0\,0\,0\,0\,0\,0 & 0 \\ 0 \;\; 0\,0\,0\,0\,0\,0 & -\alpha \end{bmatrix},$$
$$x = (x_1, x_2)^{\mathrm{T}}, \quad x^2 = (x_1^2, x_1x_2, x_2x_1, x_2^2)^{\mathrm{T}},$$
$$x^3 = (x_1^3, x_1^2x_2, x_1x_2x_1, x_1x_2^2, x_2x_1^2, x_2x_1x_2, x_2^2x_1, x_2^3)^{\mathrm{T}}.$$

Denote $f_1 := A_1x$, $f_2 := A_2x^2$, $f_3 := A_3x^3$, and choose candidate Lyapunov functions

$$V_1(x) = \frac{1}{4}(x_1^4 + x_2^4), \quad V_2(x) = \frac{1}{2}(x_1^2 + x_2^2). \tag{51}$$

It is obvious that $V_1(x)$ and $V_2(x)$ are two positive definite homogenous polynomials. Moreover,

$$\dot{V}_1|_{f_3(x)} = -\alpha(x_1^6 + x_2^6) < 0, \quad x \neq 0,\ \alpha > 0,$$
$$\dot{V}_2|_{f_1(x)} = -\beta(x_1^2 + x_2^2) < 0, \quad x \neq 0,\ \beta > 0.$$

Next, we define a cub C as

$$C := \{x \in R^2 | |x_i| \leq 1,\ i = 1, 2\}.$$

Now, using Algorithm 6.2, the estimation $E(x)$ can be obtained.

First, we have

$$\begin{aligned} \dot{V}_1|_{f_1(x)} &= -\beta(x_1^4 + x_2^4) = Z_1x^4, \\ \dot{V}_1|_{f_2(x)} &= x_1^3(x_1^2 + x_2^2) + 2x_1x_2^4 = Z_2x^5, \end{aligned} \tag{52}$$

where

$$\begin{aligned} Z_1 &= [-\beta, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\beta], \\ Z_2 &= [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]. \end{aligned} \tag{53}$$

Removing the negative semi-definite terms, we get

$$D_1(x) = \widetilde{Z}_1x^4, \quad D_2(x) = \widetilde{Z}_2x^5 = Z_2x^5, \tag{54}$$

where

$$\widetilde{Z}_1 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$\widetilde{Z}_2 = [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].$$

Thus, $H_1(x) = 0$. Then, we enlarge $D_2(x)$ to $H_2(x)$, where

$$\begin{aligned} H_2(x) &= \widetilde{Z}_2|x|^5(|x_1| + |x_2|) \\ &= x_1^6 + x_1^4x_2^2 + 2x_1^2x_2^4 + |x_1|^5|x_2| + |x_1|^3|x_2|^3 + 2|x_1||x_2|^5 \\ &= [1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, \\ &\qquad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]|x|^6. \end{aligned} \tag{55}$$

It is easy to see that

$$D_2(x) \le H_2(x).$$

Define

$$\begin{aligned} E(x) :&= \dot{V}_1|_{f_3(x)} + H_1(x) + H_2(x) = \dot{V}_1|_{f_1(x)} + H_2(x) \\ &= -\alpha(x_1^6 + x_2^6) + x_1^6 + x_1^4x_2^2 + 2x_1^2x_2^4 + |x_1|^5|x_2| + |x_1|^3|x_2|^3 + 2|x_1||x_2|^5 \\ &= E|x|^6, \end{aligned} \tag{56}$$

where

$$\begin{aligned} E = [&1-\alpha, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, \\ &0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\alpha]. \end{aligned} \tag{57}$$

Now we check its negativity. From (57),

$$E_{d_1} = 1 - \alpha, \quad E_{d_2} = -\alpha.$$

Choose $\alpha > 4$, from Lemma 4.3, then we have

$$\begin{aligned} V_6^1 &= [6, 5, 5, 4, 5, 4, 4, 3, 5, 4, 4, 3, 4, 3, 3, 2, 5, 4, 4, 3, 4, 3, 3, 2, 4, 3, 3, 2, 3, 2, 2, 1, \\ &\quad 5, 4, 4, 3, 4, 3, 3, 2, 4, 3, 3, 2, 3, 2, 2, 1, 4, 3, 3, 2, 3, 2, 2, 1, 3, 2, 2, 1, 2, 1, 1, 0]^{\mathrm{T}}, \\ V_6^2 &= [0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5, \\ &\quad 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5, 2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6]^{\mathrm{T}}, \\ \widetilde{E} &= [0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, \\ &\quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]. \end{aligned} \tag{58}$$

Thus, we can get the following inequalities:

$$E_{d_1} + \frac{1}{6}\widetilde{E}V_6^1 = 1 - \alpha + 3 < 0,$$

$$E_{d_2} + \frac{1}{6}\widetilde{E}V_6^2 = -\alpha + 4 < 0.$$

Using Theorem 4.5, $E(x)$ is negative definite.

Next, using Algorithm 6.4, we get the estimation of $F(x)$,

$$\begin{aligned} \dot{V}_2|_{f_3(x)} &= -\alpha(x_1^4 + x_2^4) = Z_3x^4, \\ \dot{V}_2|_{f_2(x)} &= x_1(x_1^2 + x_2^2) + 2x_1x_2^2 = Z_2x^3, \end{aligned} \tag{59}$$

where

$$\begin{aligned} Z_3 &= [-\alpha, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\alpha], \\ Z_2 &= [1, 0, 0, 3, 0, 0, 0, 0]. \end{aligned} \tag{60}$$

Removing the negative semi-definite terms, we get

$$D_3(x) = \widetilde{Z}_3x^4, \quad D_2(x) = \widetilde{Z}_2x^3 = Z_2x^3, \tag{61}$$

where

$$\widetilde{Z}_3 = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],$$
$$\widetilde{Z}_2 = [1,0,0,3,0,0,0,0].$$

Thus, $L_3(x) = 0$. Similarly, we enlarge $D_2(x)$ to $L_2(x)$,

$$\begin{aligned} L_2(x) &= \frac{1}{3}\widetilde{Z}_2(V_3^1, V_3^2)(x_1^2 + x_2^2) \\ &= \frac{1}{3}\widetilde{Z}_2(V_3^1 x_1^2, V_3^2 x_2^2) \\ &= 2x_1^2 + 2x_2^2, \end{aligned} \tag{62}$$

where $V_3^1 = [3,2,2,1,2,1,1,0]^{\mathrm{T}}$, $V_3^2 = [0,1,1,2,1,2,2,3]^{\mathrm{T}}$.

Then, we have

$$D_2(x) \le L_2(x), \quad x \in C.$$

Define an estimation as

$$\begin{aligned} F(x) :&= \dot{V}_2|_{f_1(x)} + L_2(x) + L_3(x) \\ &= \dot{V}_2|_{f_1(x)} + L_2(x) \\ &= -\beta(x_1^2 + x_2^2) + 2x_1^2 + 2x_2^2 \\ &= (2-\beta)x_1^2 + (2-\beta)x_2^2. \end{aligned} \tag{63}$$

When $\beta > 2$, $F(x)$ is negative.

Using Theorem 6.6, we conclude that system (49) is globally asymptotically stable when $\alpha > 4$, $\beta > 2$.

Particularly, choosing $\alpha = 5$, $\beta = 10$, we get a trajectory in Figure 1.

Remark 7.2 We may have an alternative way to enlarge $D_2(x)$ to $L_2(x)$ as

$$\begin{aligned} D_2(x) &= Z_2 x^3 \\ &= x_1^3 + 3x_1x_2^2 \\ &\le |x_1|^2 + 3|x_2|^2 := L_2(x), \quad x \in C. \end{aligned} \tag{64}$$

Based on Example 7.1, we can provide an illustrative example for switched polynomial system.

Example 7.3 Consider a switched polynomial system

$$\dot{x} = g_{\sigma(t)}(x), \tag{65}$$

where $\sigma(t) : [0,\infty) \to \Lambda = \{1,2\}$ is a switching signal, g_λ, $\lambda = 1,2$, are odd-ended polynomial vector fields. The subsystems are, respectively,

$$\dot{x} = g_1(x) = A_1^1 x + A_2^1 x^2 + A_3^1 x^3, \tag{66}$$

$$\dot{x} = g_2(x) = A_1^2 x + A_2^2 x^2 + A_3^2 x^3. \tag{67}$$

where

$$A_1^i = \begin{bmatrix} -\beta_i & 0 \\ 0 & -\beta_i \end{bmatrix}, \quad A_2^i = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_3^i = \begin{bmatrix} -\alpha_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_i \end{bmatrix}. \tag{68}$$

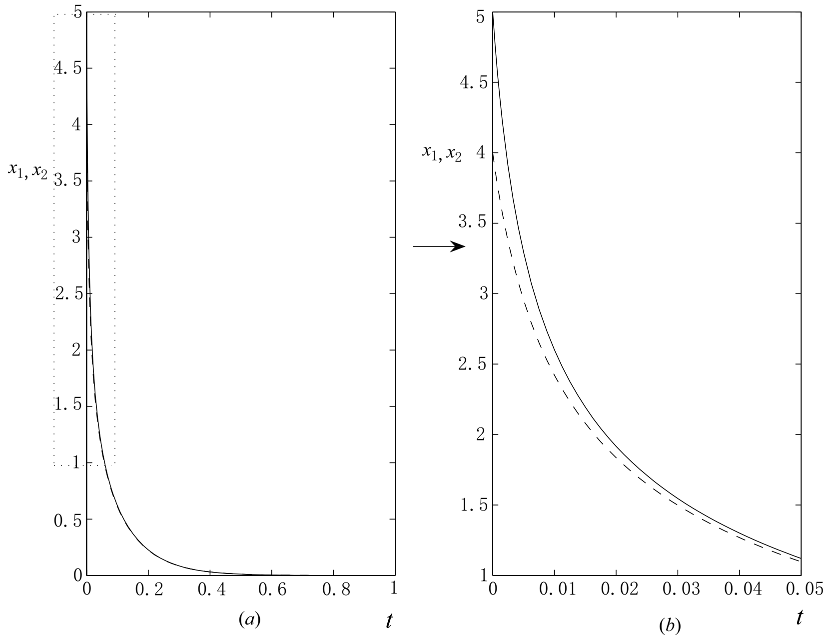

Figure 1 The trajectory of Example 7.1 when $\alpha = 5, \beta = 10$, and x(0)=[5,4]. (a) trajectory in interval [0, 1]; (b) trajectory in interval [0, 0.05].

and $\alpha_i > 4,\ \beta_i > 2,\ i = 1, 2.$

From Example 7.1 we know that, using two candidate Lyapunov functions (51), subsystems (66) and (67) are all globally asymptotically stable. In addition, the conditions in Theorem 6.7 are all satisfied. We conclude that the switched system (65) is globally asymptotically stable under arbitrary switches.

8 Conclusion

The stability problem of (switched) polynomial systems was investigated in this paper. The main results of this paper are the following. First, the semi-tensor product was used to convert multi-variable polynomials into canonical forms. As a generalization, the product of two polynomials can also be converted into the canonical form. Second, some sufficient conditions were obtained for verifying the positivity of homogenous polynomials by using semi-tensor product. Using them, a new method, called the two Lyapunov function approach, was proposed to justify the global stability of polynomial systems, which are not assumed to be homogeneous but only odd-ended. The method proposed is also applicable to the stability of odd-ended switched polynomial systems.

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