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Synchronisation of a class of networked passive systems with switching topology

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This article considers the output synchronisation of a class of networked agents. Each agent is a passive system. Local information, used by each agent to adjust its movement, forms a network with switching adjacent topology. First, we consider the asymptotic stability of switched non-linear time-varying systems with delayed measurement feedback by using multiple Lyapunov function. A Barbalat-like lemma is obtained. Then the result is applied to the output synchronisation of a class of networked passive systems with switching topology. A verifiable sufficient condition is presented.

Keywords: output synchronisation; switched system; multiple Lyapunov function; Barbalat-like lemma

1. Introduction

Recently, the problem of synchronisation of networked dynamic agents has attracted increasing attentions from control community (Olfati-Staber 2004; Ren and Beard 2005; Hong, Hu, and Gao 2006). This is partly due to the broad synchronisation phenomena in biological, chemical, physical and social systems (Lü and Chen 2005; Chopra and Spong 2005; Yueh and Cheng 2006; Sun, Tian, Fu, and Qian 2007). Local and global adaptive synchronisation criteria for uncertain complex dynamical networks are proposed in Zhou, Lu, and Lü (2006). Li, Zhang, and Zhang (2006) investigate the global synchronisation of a class of complex networks with time-varying delays. It was shown that the synchronisation of delayed complex networks can be determined by their topologies. Consensus problems are discussed for networks of dynamic agents with fixed and switching topologies in Olfati-Staber (2004). Olfati-Staber (2007) provides a theoretical framework for analysis of consensus algorithms for multi-agent networked systems with an emphasis on the role of directed information flow, robustness to changes in network topology due to link failures, time-delay and performance guarantees.

Passivity is an important concept of system theory and has been widely used as a fundamental tool in the development of linear and non-linear feedback designs by Byrnes, Isidori, and Willems (1991). Recently, Chopra and Spong (2005) give a result for output synchronisation of dynamic agents, which are assumed to be non-linear passive systems and be affine in the control, with fixed regular graph structure. It is a motivation for this work. This article is a follow-up of Chopra and Spong (2005). We consider the case when the agents are moving and extend the main result in Chopra and Spong (2005) to the dynamic agents with switching topology.

Consider a non-linear affine system:

$$\begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \\ y_i = h_i(x_i), \quad i = 1, \dots, N, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$ represent agents, $f_i(.) \in \mathbb{R}^n$, $g_i(.) \in \mathbb{R}^{n\times m}$, $u_i \in \mathbb{R}^m$, $h_i(.) \in \mathbb{R}^m$ and smooth mappings, and $f_i(0) = 0$, $h_i(0) = 0$. We assume that a unique solution of (1) exists for all time. In the following, we cite some basic definitions from Chopra and Spong (2005) for the problem description.

Definition 1.1: System (1) is said to be output synchronised if

$$\|y_i - y_j\| \to 0 \text{ as } t \to \infty, \quad \forall\, i, j = 1, \dots, N. \tag{2}$$

Definition 1.2: System (1) is said to be passive with (u_i, y_i) as the input–output pair if there exists a C^1 storage function $S_i(x_i) \geq 0$, $S_i(0) = 0$, such that for all $t \geq 0$,

$$S_i(x_i(t)) - S_i(x_i(0)) \leq \int_0^t u_i^T(s) y_i(s) \mathrm{d}s. \tag{3}$$

In this article we consider the output synchronisation of system (1) under the assumption that the system is passive with respect to each sub-system (u_i, y_i).

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The synchronisation is based on the decentralised control constructed on the local information received by each agent, which forms a time-varying adjacent topology.

It is well known that such a time-varying topology leads to a switched system Jadbabaie, Lin, and Morse (2003). Meanwhile, when the switches and delay exist, the key technique used in Jadbabaie et al. (2003) fails to be applicable. To overcome this difficulty, we developed a new tool called the Barbalat-like lemma. It is applicable for switched non-linear time-varying systems with certain time delay.

The article is organised as follows. Section 2 develops a generalised Barbalat lemma for switched non-linear time-varying systems with delayed measurement feedback by using multiple Lyapunov function. The result obtained is applied to the problem of synchronisation of the system (1) in § 3. Synchronisation is achieved under certain additional conditions. An illustrative example is included. Section 4 is the conclusion.

2. Barbalat-like lemma

Consider a switched non-linear time-varying system with delayed measurement feedback:

$$\dot{x} = f_{\sigma(t)}(x(t), h(x(t-\tau)), t), \quad x \in \mathbb{R}^n, \tag{4}$$

where the switching law $\sigma(t): [0, \infty) \to \Lambda$ is a right-continuous piecewise constant mapping, $\Lambda = \{1, \ldots, N\}$ for some integer $N \geq 2$, time delay $\tau > 0$ is a constant parameter, and $f_i, i = 1, \ldots, N$, and h are smooth mappings.

It is obvious that the system (4) is the closed-loop of a switched non-linear time-varying system

$$\begin{cases} \dot{x} = f_{\sigma(t)}(x(t), u, t) \\ y = h(x(t-\tau)), \end{cases}$$

with delayed measurement feedback

$$u = h(x(t-\tau)).$$

To make the problem well posed, we assume that $u(t) = 0, t < \tau$. That is,

Assumption 1

$$h(x(t-\tau)) = 0, \quad t < \tau. \tag{5}$$

Under assumption A1, we denote the solution of system (4) with initial condition $x(0) = x_0$ and switching σ by $x(t) = \varphi_\sigma(x_0, t)$.

For switched system (4), a switching sequence is to specify when and to which mode the system should switch. A switching sequence is a countable ordered pair of active modes and ending dwell as

$$\{(i_0, t_0), (i_1, t_1), \ldots, (i_s, t_s), \cdots\}, \tag{6}$$

where $0 = t_0 < t_1 < \cdots < t_s < \cdots < \infty$, $\lim_{s \to \infty} t_s = \infty$ and $i_j \in \Lambda$. The undergoing switching sequence can be uniquely determined by switching function $\sigma(t)$, satisfying

$$\sigma(t) = i_k, \quad \text{for } t \in [t_{k-1}, t_k),\ i_k \in \Lambda,\ k = 1, 2, \ldots.$$

In this article, we assume that admissible switches have a positive dwell-time $T > 0$. That is,

Assumption 2

$$\min\{(t_{j+1} - t_j) \mid j = 0, 1, \ldots\} \geq T > 0. \tag{7}$$

The purpose of this section is to develop a Barbalat-like lemma for system (4). First, we state the original lemma.

Lemma 2.1: (*Barbalat*) (Slotine and Li 2006) *Let* $V(t)$ *be a differentiable function, and* $\lim_{t \to \infty} V(t) < +\infty$. *If* $\dot{V}(t)$ *is uniformly continuous, then*

$$\lim_{t \to \infty} \dot{V}(t) = 0.$$

Definition 2.2: A scalar time-varying function $V(x, t)$ is positive definite if $V(0, t) = 0$ and there exists a time-invariant positive definite function $V_0(x)$ such that

$$V(x, t) \geq V_0(x), \quad \forall t \geq 0, \quad x \in \mathbb{R}^n. \tag{8}$$

Similarly, we can define negative definiteness, positive (or negative) semi-definiteness as follows. A function $V(x, t)$ is negative definite if $-V(x, t)$ is positive definite; $V(x, t)$ is positive semi-definite if $V(x, t) \geq 0$, $\forall x \in \mathbb{R}^n$ and $\forall t \geq 0$; $V(x, t)$ is negative semi-definite if $-V(x, t)$ is positive semi-definite.

Recall the non-linear time-varying switched system (4) with delayed measurement feedback, we would like to provide a Barbalat-like lemma to it.

Theorem 2.3: *For the system* (4), *suppose the switching law satisfies* (6), (7) *and there exist* $V_i(x, t)$, $i \in \Lambda$, *which are differentiable, lower bounded and radially unbounded, such that*

(i) $\dot{V}_i(\varphi_{\sigma(t)}(t), t)$ *is negative semi-definite*, $\forall i \in \Lambda$,
(ii) $\dot{V}_i(\varphi_{\sigma(t)}(t), t)$ *is uniformly continuous in time*, $\forall i \in \Lambda$, $\forall\ x \in \mathbb{R}^n$,

(iii) *For every pair of switching times* $t_i < t_j$ *such that* $\sigma(t_i) = \sigma(t_j) = p$, *for each* $x_0 \in \mathbb{R}^n$ *and every solution* $\varphi_{\sigma(t)}(t) := \varphi_{\sigma(t)}(x_0, t)$,

$$V_p(\varphi_{\sigma(t)}(t_j), t_j) \leq V_p(\varphi_{\sigma(t)}(t_i), t_i), \tag{9}$$

then we have

$$\dot{V}_{\sigma(t)}(\varphi_{\sigma(t)}(t), t) \to 0, \quad t \to \infty.$$

Proof: Assume that $\dot{V}_i(\varphi_{\sigma(t)}(t), t)$ does not approach zero as $t \to \infty$. Then there exists $\varepsilon_0 > 0$, and an infinite sequence $\{t_{j_k}^k\}_{k=1}^{\infty}$ ($t_{j_k}^k \to \infty$, as $k \to \infty$, and $\sigma(t_{j_k}^k) = j_k$), such that

$$\left| \dot{V}_{j_k}\left(\varphi_{\sigma\left(t_{j_k}^k\right)}(t_{j_k}^k), t_{j_k}^k \right) \right| \geq \varepsilon_0. \tag{10}$$

Since $|\Lambda| < \infty$, there exists at least one special j_k, denoted by j^*, which appears an infinite number of times. We, therefore, can find a sub-sequence $\{t_{j^*}^k\}_{k=1}^{\infty}$ ($t_{j^*}^k \to \infty$, as $k \to \infty$), $\sigma(t_{j^*}^k) = j^*$ and $|\dot{V}_{j^*}(\varphi_{\sigma(t_{j^*}^k)}(t_{j^*}^k), t_{j^*}^k)| \geq \varepsilon_0$.

Because $L_{f_i} V_i(x, t)$ is uniformly continuous in time, $\forall i \in \Lambda$, there exists a $0 < \eta < T$ such that for any t' and t'' satisfying $|t' - t''| < \eta$, we have

$$\left| \dot{V}_i(\varphi_{\sigma(t')}(t'), t') - \dot{V}_i(\varphi_{\sigma(t'')}(t''), t'') \right| \leq \frac{\varepsilon_0}{2}, \quad \forall\, i \in \Lambda. \tag{11}$$

Suppose

$$\sigma(t) = j^*, \text{ for } t \in [t_{j_{k-1}}, t_{j_k}), \text{ and } t_{j^*}^k \in [t_{j_{k-1}}, t_{j_k}). \tag{12}$$

Because of (10) and (11), if $|t - t_{j^*}^k| < \eta$ and $t \in [t_{j_{k-1}}, t_{j_k})$, then

$$|\dot{V}_{j^*}(\varphi_{\sigma(t_{j^*})}(t_{j^*}), t_{j^*})| > \frac{\varepsilon_0}{2}. \tag{13}$$

Note that $\eta < T < |t_{j_k} - t_{j_{k-1}}|$, so either

$$\left[t_{j^*}^k,\, t_{j^*}^k + \frac{\eta}{2}\right] \subset \left[t_{j_{k-1}},\, t_{j_k}\right),$$

or

$$\left[t_{j^*}^k - \frac{\eta}{2},\, t_{j^*}^k\right] \subset \left[t_{j_{k-1}},\, t_{j_k}\right).$$

Without loss of generality, we assume that $[t_{j^*}^k, t_{j^*}^k + \frac{\eta}{2}] \subset [t_{j_{k-1}}, t_{j_k})$. Then

$$\left| \int_{t_{j^*}^k}^{t_{j^*}^k + \frac{\eta}{2}} \dot{V}_{j^*}(\varphi_{\sigma(\xi)}(\xi), \xi) \mathrm{d}\xi \right| = \int_{t_{j^*}^k}^{t_{j^*}^k + \frac{\eta}{2}} \left| \dot{V}_{j^*}(\varphi_{\sigma(\xi)}(\xi), \xi) \right| \mathrm{d}\xi \geq \frac{\varepsilon_0}{2} \times \frac{\eta}{2}. \tag{14}$$

Using the fact that $V_{j^*}(x, t)$ is lower bounded, the condition (iii), combined with the condition (i), ensures that the function V_{j^*} is non-increasing on the union of the intervals where subsystem j^* is active, V_{j^*} then approaches to a finite limiting value $V_{j^*}(\infty)$, then we have

$$\sum_{k=1}^{\infty} \left| \int_{t_{j^*}^k}^{t_{j^*}^k + \frac{\eta}{2}} \dot{V}_{j^*}(\varphi_{\sigma(\xi)}(\xi), \xi) \mathrm{d}\xi \right| = \sum_{k=1}^{\infty} \int_{t_{j^*}^k}^{t_{j^*}^k + \frac{\eta}{2}} \left| \dot{V}_{j^*}(\varphi_{\sigma(\xi)}(\xi), \xi) \right| \mathrm{d}\xi \leq V_{j^*}(0) - V_{j^*}(\infty). \tag{15}$$

Taking (14) into consideration, (15) is a contradiction. So

$$\dot{V}_{\sigma(t)}(\varphi_{\sigma(t)}(t), t) \to 0. \qquad \square$$

Now we are ready to study the synchronisation problem.

3. Synchronisation of agents with balanced graph structure

To begin with, we describe the information flow among agents by using concepts from graph theory.

Consider a system of N agents $\mathcal{V} = \{1, \ldots, N\}$ described by (1). Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a directed graph of order N with the set of agents $\mathcal{V} = \{1, \ldots, N\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and an adjacency matrix $A = [a_{ij}]$ with 0 and 1 adjacency elements a_{ij}. An edge of G is denoted by $e_{ij} = (i, j)$, it means that the agent j is transmitting its output to the agent i. The adjacent elements associated with the edges of the graph are of the value 1, i.e. $e_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} = 1$, otherwise $a_{ij} = 0$. Moreover, we assume $a_{ii} = 0$. The set of neighbours of agent i is denoted by $N_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$.

The in-degree and out-degree of agent i are, respectively, defined as follows:

$$\deg_{\text{in}}(i) = \sum_{j=1}^{n} a_{ji}, \quad \deg_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij}. \tag{16}$$

Here, $\deg_{\text{out}}(i) = |N_i|$.

Definition 3.1 (Balanced Graphs (Olfati-Staber 2004))**:** An agent i of a direct graph $G = (\mathcal{V}, \mathcal{E}, A)$ is said to be balanced if its in-degree and out-degree are equal. A graph $G = (\mathcal{V}, \mathcal{E}, A)$ is said to be balanced if all of its agents are balanced, i.e.

$$\sum_{j=1}^{n} a_{ji} = \sum_{j=1}^{n} a_{ij}, \quad \forall\, i. \tag{17}$$

Note that any undirected graph is balanced. An example of directed balanced graph is shown in Figure 1.

Definition 3.2 (Godsil and Royle 2001)**:** For a directed graph $G=(\mathcal{V},\mathcal{E},A)$, a weak path is a sequence $k_0,\ldots,k_r$ of distinct agents such that for $i=1,\ldots,r$ either (k_{i-1},k_i) or (k_i,k_{i-1}) in $\mathcal{E}$. A directed graph is weakly connected if and only if two agents can be joined by a weak path.

In the following, we develop control strategies for synchronisation of the passive agents which are networked with switched weak connected topology. First, we introduce a key result.

Theorem 3.3 (Lozano, Brogliato, Egeland, and Maschke 2000)**:** *Consider the non-linear system* (1). *The following statements are equivalent.*

(1) *There exists a* C^1 *storage function* $S_i(x_i)\geq 0$, $S_i(0)=0$ *and a function* $Q_i(x_i)\geq 0$ *such that for all* $t\geq 0$:

$$S_i(x_i(t))-S_i(x_i(0))=\int_0^t u_i^T y_i(s)\mathrm{d}s-\int_0^t Q_i(x_i(s))\mathrm{d}s. \tag{18}$$

(2) *There exists a* C^1 *scalar function* $S_i(x_i)\geq 0$, $S_i(0)=0$, *such that*

$$L_{f_i}S_i(x_i)=-Q_i(x_i); \tag{19}$$

$$L_{g_i}S_i(x_i)=h_i^T(x_i), \tag{20}$$

where $L_{f_i}S_i(x_i)=\frac{\partial S_i^T}{\partial x_i}f_i(x_i)$ *and* $L_{g_i}S_i(x_i)=\frac{\partial S_i^T}{\partial x_i}g_i(x_i)$.

(3) *The system is passive.*

The system is strictly passive if $Q_i(x_i)>0$, *passive if* $Q_i(x_i)\geq 0$ *and lossless if* $Q_i(x_i)=0$.

The agent dynamics of system (1) is assumed to be passive with positive definite storage functions $S_1(x_1),S_2(x_2),\ldots,S_N(x_N)$, respectively. As there are time delays in the network, the agents receive a delayed information of the outputs of other agents. Let the agents be coupled together by a control constructed by using local information

$$u_i=\begin{cases}\sum_{j\in N_i(t)}K(y_j(t-\tau)-y_i(t)), & t\geq\tau\\ 0, \quad t<\tau, \quad i=1,\ldots,N, & \end{cases} \tag{21}$$

where K is a positive constant, $N_i(t)$ is the set of $m_i(t)$ agents which are transmitting their outputs to the ith agent at time t (thus the cardinality of each set $N_i(t)$, $i=1,\ldots,N$ is $m_i(t)$), and τ is the constant time-delay in the network.

Since we are interested in long-term behaviour of the system, we do not need to worry about the period $t<\tau$. We will ignore this time period in later discussion.

Example 3.4: A system of networked four agents at time t is illustrated as in Figure 1.

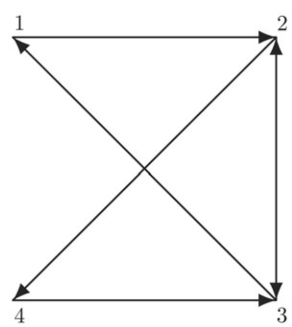

Figure 1. Networked four agents in Example 3.4.

Assume they have dynamics of (1), we construct the control as in (21). It is easy to calculate that $m_1(t)=1$, $m_2(t)=2$, $m_3(t)=2$, $m_4(t)=1$. Hence the controls are

$$\begin{aligned}u_1(t)&=K(y_2(t-\tau)-y_1),\\ u_2(t)&=K(y_4(t-\tau)-y_2)+K(y_3(t-\tau)-y_2),\\ u_3(t)&=K(y_2(t-\tau)-y_3)+K(y_1(t-\tau)-y_3),\\ u_4(t)&=K(y_3(t-\tau)-y_4), \quad t\geq\tau.\end{aligned} \tag{22}$$

Now consider system (1). Note that the relationship between neighbours (in other words, the interconnection topology) can change over time. We assume all possible topologies of balanced weakly connected graphs are $G^1,G^2,\ldots,G^M$, where $G^p=(\mathcal{V}^p,\mathcal{E}^p,A^p)$, $p=1,\ldots,M$, the adjacency matrix $A^p=[a_{ij}^p]$ with 0 and 1 adjacency elements a_{ij}^p, we denote $m_i^p=\sum_{j=1}^n a_{ij}^p$, $i=1,\ldots,n$.

Assume that (1) is passive. According to Theorem 3.3 there are $S_i(x_i)\geq 0$, $Q_i(x_i)\geq 0$ such that (18) holds. Using them we define a switching signal $\sigma(t):[0,\infty)\to W=\{1,\ldots,M\},\sigma(t)$ is a piecewise constant function satisfying condition (7), $\sigma(t)=p$ means that the topology is G^p at time t, then $m_i(t)=m_i^p$ and $N_i(t)=N_i^p$. Define

$$\begin{aligned}V_p(x_1(t),\ldots,x_N(t),t)&=\sum_{i=1}^N m_i^p K\int_{t-\tau}^t y_i^T y_i\,\mathrm{d}\xi\\ &+2\sum_{i=1}^N S_i(x_i)+2\sum_{i=1}^N\int_0^t Q_i(x_i(\xi))\mathrm{d}\xi,\\ &p\in W=\{1,\ldots,M\}.\end{aligned} \tag{23}$$

Note that here V_p is trajectory-depending. Then $2\sum_{i=1}^{N} S_i(x_i)$ is a time-invariant positive semi-definite function. Since

$$V_p(x_1(t),\dots,x_N(t),t) \geq 2\sum_{i=1}^{N} S_i(x_i), \tag{24}$$

$V_p(x_1(t),\dots,x_N(t),t)$ is a positive semi-definite candidate of Lyapunov function.

The following lemma comes from continuity:

Lemma 3.5: *Assume $f(x): \mathbb{R}^n \to \mathbb{R}^s$ is a continuous mapping, $\Omega \in \mathbb{R}^n$ is a bounded set, then $f(\Omega)$ is bounded.*

The following is our main result, which provides some sufficient condition to assure the output synchronisation of passive dynamic systems with time delay.

Theorem 3.6: *Assume the switching modes of the system* (1) *are passive with radially unbounded storage functions $S_i(x_i) > 0, i = 1,\dots,N$, all possible topologies of balanced weakly connected graph are $G^1, G^2, \dots, G^M$, the solution of system i under switching law σ and from initial state x_i^0 is denoted as $x_i(t) = \varphi_\sigma^i(x_i^0, t)$, $i = 1,\dots,N$, and the controls are chosen as in* (21), *the Lyapunov functions are defined as in* (23), *then if for every pair of switching times $t_i < t_j$ with $\sigma(t_i) = \sigma(t_j) = p$,*

$$V_p\big(\varphi_\sigma^1,\dots,\varphi_\sigma^N,t_j\big) \leq V_p\big(\varphi_\sigma^1,\dots,\varphi_\sigma^N,t_i\big), \tag{25}$$

then the non-linear systems described by (1) *can be output synchronised by controls* (21).

Proof: The derivative of (23) along trajectories of the system is given as

$$\begin{aligned}\dot V_p(x_1(t),\dots,x_N(t),t) = &\sum_{i=1}^{N} m_i^p K\big(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau)\big)\\ &+2\sum_{i=1}^{N}\big(L_{f_i}S_i + L_{g_i}S_i u_i\big)\\ &+2\sum_{i=1}^{N} Q_i(x_i(t)),\quad p=1,\dots,M.\end{aligned} \tag{26}$$

Using Theorem 3.3, the derivative can be reduced to

$$\begin{aligned}&\dot V_p(x_1(t),\dots,x_N(t),t)\\ &= \sum_{i=1}^{N} m_i^p K\big(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau)\big)\\ &\quad + 2\sum_{i=1}^{N}(y_i^T u_i)\sum_{i=1}^{N} m_i^p K\big(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau)\big)\\ &\quad + 2\sum_{i=1}^{N}\sum_{j\in N_i^p} y_i^T K\big(y_j(t-\tau) - y_i\big).\end{aligned} \tag{27}$$

The term $\sum_{i=1}^{N} m_i^p K(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau))$ can be written as $K\sum_{i=1}^{N}\sum_{j\in N_i^p}(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau))$. Plugging this into (27) yields

$$\begin{aligned}&\dot V_p(x_1(t),\dots,x_N(t),t)\\ &\quad = K\sum_{i=1}^{N}\sum_{j\in N_i^p}\big(y_i^T y_i - y_i^T(t-\tau)y_i(t-\tau)\big)\\ &\qquad + 2\sum_{i=1}^{N}\sum_{j\in N_i^p} y_i^T K(y_j(t-\tau) - y_i)\\ &\quad = K\sum_{i=1}^{N}\sum_{j\in N_i^p} y_i^T y_i - K\sum_{i=1}^{N}\sum_{j\in N_i^p}\big(y_i^T(t-\tau)y_i(t-\tau)\big)\\ &\qquad + 2K\sum_{i=1}^{N}\sum_{j\in N_i^p} y_i^T y_j(t-\tau) - 2K\sum_{i=1}^{N}\sum_{j\in N_i^p} y_i^T y_i.\end{aligned} \tag{28}$$

Since the graph is a balanced one, we have

$$\begin{aligned}&K\sum_{i=1}^{N}\sum_{j\in N_i^p}\big(y_i^T(t-\tau)y_i(t-\tau)\big)\\ &\qquad = K\sum_{i=1}^{N}\sum_{j\in N_i^p}\Big(y_j^T(t-\tau)y_j(t-\tau)\Big).\end{aligned} \tag{29}$$

Hence

$$\begin{aligned}\dot V_p(x_1(t),\dots,x_N(t),t) = -K&\sum_{i=1}^{N}\sum_{j\in N_i^p}(y_j(t-\tau) - y_i)^T\\ &\times(y_j(t-\tau) - y_i).\end{aligned} \tag{30}$$

It can be rewritten as

$$\begin{aligned}&\dot V_p(x_1(t),\dots,x_N(t),t)\\ &\quad = -K\sum_{i=1}^{N}\sum_{j\in N_i^p}[y_j(x_j(t-\tau)) - y_i(x_i(t))]^T\\ &\qquad \times [y_j(x_j(t-\tau)) - y_i(x_i(t))],\end{aligned}$$

hence $\dot V_p(x_1(t),\dots,x_N(t),t) \leq 0$.

Then V_p, $p = \{1,\dots,M\}$ are bounded, because $S_i(x_i)$, $i = 1,\dots,N$ are radially unbounded. It follows that x_i, $i = 1,\dots,N$ are bounded. Then using Lemma 3.5, $y_i = h_i(x_i), \dot y_i, f_i, g_i$, $i = 1,\dots,N$ are all bounded. Suppose

$$\|\dot x_i\| < \Upsilon_1,\quad i = 1,\dots,N \tag{31}$$

and

$$\|\dot y_i\| < \Upsilon_2,\quad i = 1,\dots,N, \tag{32}$$

then for any $t_1 > 0$, $t_2 > 0$,

$$|y_i(t_1) - y_i(t_2)| < \Upsilon_1\Upsilon_2|t_1 - t_2|, \quad i = 1, \ldots, N. \tag{33}$$

Hence, y_i, $i = 1, \ldots, N$ are uniformly continuous in time.

Using Theorem 2.3, $\dot{V}_{\sigma(t)}(x_1(t), \ldots, x_N(t), t) \to 0$ as $t \to \infty$.

Consider the set

$$E_p = \left\{x_i \in \mathbb{R}^n, i = 1, \ldots, N \middle| \dot{V}_p \equiv 0\right\}. \tag{34}$$

E_p is characterised by all trajectories such that

$$\Big\{\big(y_j(t-\tau) - y_i(t)\big)^T\big(y_j(t-\tau) - y_i(t)\big) \equiv 0,\ \forall j \in N_i^p,\ \forall i \in \Lambda\Big\}. \tag{35}$$

Since $\dot{V}_{\sigma(x_1(t),\ldots,x_N(t),t)}(t) \to 0$ as $t \to \infty$, it implies that the output of every ith agent asymptotically converges to that of its neighbours. Balanced weakly connectivity of the network then implies output synchronisation of system (1). □

Remark: If the topology of adjacent graph is fixed, Chopra and Spong (2005) defined a Lyapunov function

$$V(x_1(t), \ldots, x_N(t), t) = K\sum_{i=1}^{N}\sum_{j\in N_i(t)}\int_{t-\tau}^{t} y_j^T(s)y_j(s)\mathrm{d}s + 2(S_1 + \cdots + S_N),$$

and

$$\dot{V}(x_1(t), \ldots, x_N(t), t) = -K\sum_{i=1}^{N}\sum_{j\in N_i(t)}(y_j(t-\tau) - y_i)^T(y_j(t-\tau) - y_i) - 2\sum_{i=1}^{n}Q_i(x_i(t)). \tag{36}$$

Then using Barbalat's lemma, the same result of Theorem 3.6 can be obtained. Now for varying topology of adjacent graph, it is obvious that V is not a continuous function, so it cannot be chosen as a Lyapunov function. Moreover, $\dot{V}$, which exists a.e., cannot satisfy the requirement of Barbalat's lemma (Lemma 2.1) that $\dot{V}$ is uniformly continuous. That is why we need Theorem 2.3.

Example 3.7: Consider a multi-agent system with the dynamics of the agents being given as

$$\dot{x}_i = u_i, \quad y_i = x_i, \quad i = 1, 2, 3, 4. \tag{37}$$

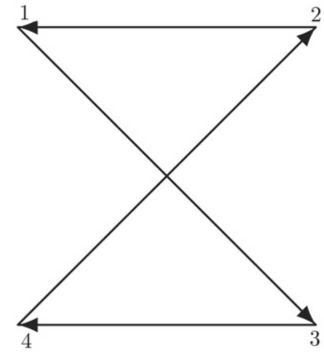

Figure 2. Networked four agents in Example 3.7.

The agents are passive with positive and radially unbounded storage functions

$$S_i(x_i) = \tfrac{1}{2}x_i^T x_i, \quad i = 1, 2, 3, 4. \tag{38}$$

Suppose the topologies are switched between G^1 (illustrated as in Figure 1) and G^2 (illustrated as in Figure 2). Using control (21), the closed-loop dynamics are described as follows: under topology G^1, the modes' dynamics are given as

$$\begin{aligned}
\dot{x}_1 &= K(x_2(t-\tau) - x_1),\\
\dot{x}_2 &= K(x_4(t-\tau) - x_2) + K(x_3(t-\tau) - x_2),\\
\dot{x}_3 &= K(x_2(t-\tau) - x_3) + K(x_1(t-\tau) - x_3),\\
\dot{x}_4 &= K(x_3(t-\tau) - x_4).
\end{aligned} \tag{39}$$

The Lyapunov function is given as

$$\begin{aligned}
V_1(x_1(t), \ldots, x_4(t), t) &= K\int_{t-\tau}^{t} x_1^T x_1\,\mathrm{d}s + 2K\int_{t-\tau}^{t} x_2^T x_2\,\mathrm{d}s\\
&\quad + 2K\int_{t-\tau}^{t} x_3^T x_3\,\mathrm{d}s + K\int_{t-\tau}^{t} x_4^T x_4\,\mathrm{d}s + \sum_{i=1}^{4} x_i^T x_i,
\end{aligned} \tag{40}$$

$$\begin{aligned}
\dot{V}_1(x_1(t), \ldots, x_4(t), t) = &- K[(x_2(t-\tau) - x_1(t)]^T[x_2(t-\tau) - x_1(t)]\\
&- K[(x_4(t-\tau) - x_2(t)]^T[x_4(t-\tau) - x_2(t)]\\
&- K[(x_3(t-\tau) - x_2(t)]^T[x_3(t-\tau) - x_2(t)]\\
&- K[(x_2(t-\tau) - x_3(t)]^T[x_2(t-\tau) - x_3(t)]\\
&- K[(x_1(t-\tau) - x_3(t)]^T[x_1(t-\tau) - x_3(t)]\\
&- K[(x_3(t-\tau) - x_4(t)]^T[x_3(t-\tau) - x_4(t)].
\end{aligned} \tag{41}$$

Under topology G^2, the modes' dynamics are given as

$$\begin{aligned}
\dot{x}_1 &= K(x_3(t-\tau) - x_1),\\
\dot{x}_2 &= K(x_1(t-\tau) - x_2),\\
\dot{x}_3 &= K(x_4(t-\tau) - x_3),\\
\dot{x}_4 &= K(x_2(t-\tau) - x_4).
\end{aligned} \tag{42}$$

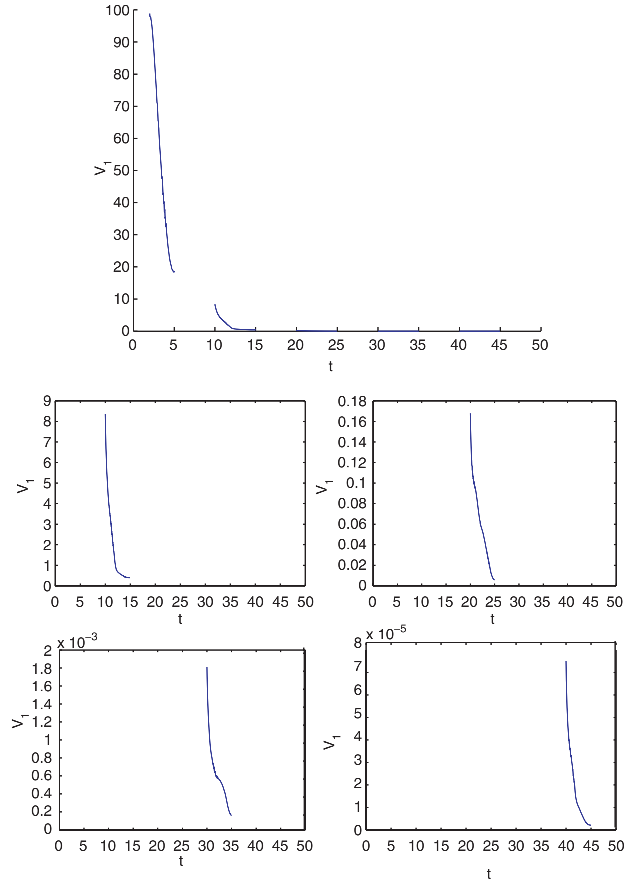

Figure 3. Lyapunov function V_1.

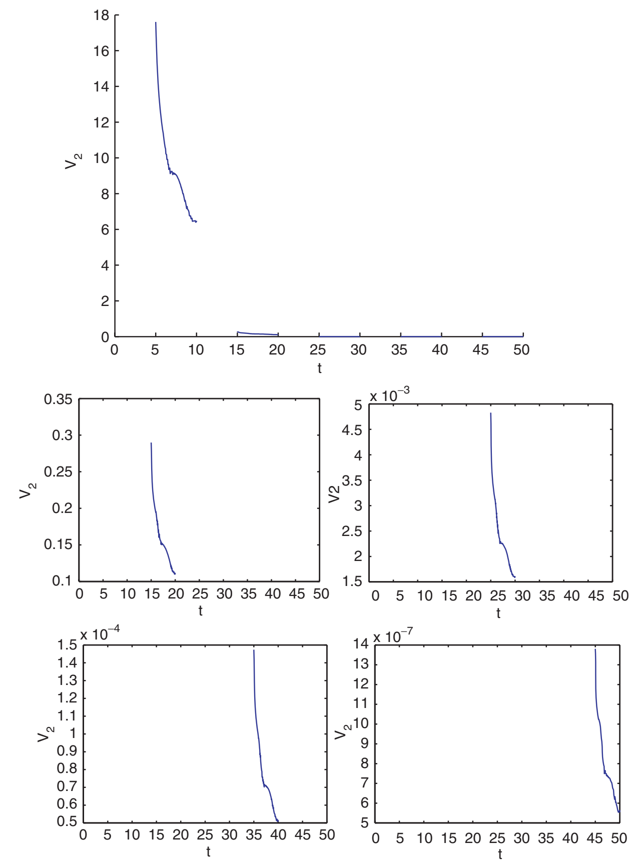

Figure 4. Lyapunov function V_2.

The Lyapunov function is given as

$$V_2(x_1(t), \dots, x_4(t), t)$$
$$= K\int_{t-\tau}^{t} x_1^T x_1 \, \mathrm{d}s + K\int_{t-\tau}^{t} x_2^T x_2 \, \mathrm{d}s + K\int_{t-\tau}^{t} x_3^T x_3 \, \mathrm{d}s$$
$$+ K\int_{t-\tau}^{t} x_4^T x_4 \, \mathrm{d}s + \sum_{i=1}^{4} x_i^T x_i, \tag{43}$$

$$\dot{V}_2(x_1(t), \dots, x_N(t), t) =$$
$$- K[(x_3(t-\tau) - x_1(t)]^T [x_3(t-\tau) - x_1(t)]$$
$$- K[(x_1(t-\tau) - x_2(t)]^T [x_1(t-\tau) - x_2(t)]$$
$$- K[(x_4(t-\tau) - x_3(t)]^T [x_4(t-\tau) - x_3(t)]$$
$$- K[(x_2(t-\tau) - x_4(t)]^T [x_2(t-\tau) - x_4(t)]. \tag{44}$$

We choose the switching law as

$$\sigma(t) = \begin{cases} G^1, & t \in \big[(2k-2)T, (2k-1)T\big), \\ G^2, & t \in \big[(2k-1)T, (2k)T\big), \\ & k = 1, 2, \dots, \end{cases} \tag{45}$$

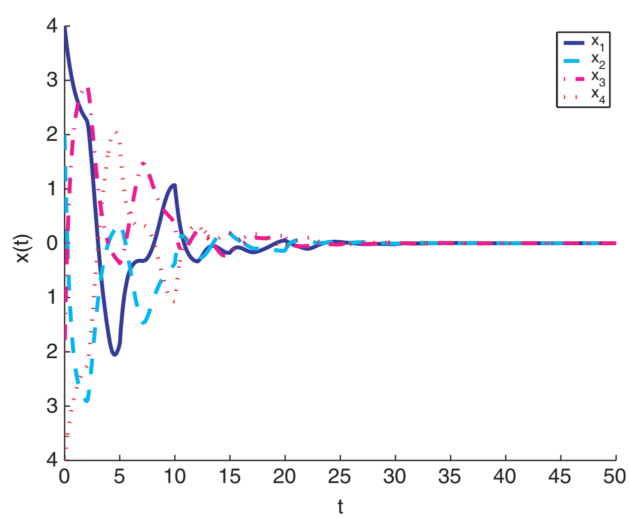

Figure 5. The synchronisation of the system.

where T is a sampling time. Choosing $T = 5$, the time-delay $\tau = 2s$, and the initial conditions are

$$x_1^0 = 4, \; x_2^0 = 2, \; x_3^0 = -2, \; x_4^0 = -4.$$

The corresponding V_1 is depicted in Figure 3 and V_2 is depicted in Figure 4, which shows the conditions of Theorem 3.6 are satisfied, so the system is synchronised.

Finally, Figure 5 gives the simulation for the system, which shows the synchronisation of the system.

4. Conclusion

Following the pioneer work in Chopra and Spong (2005), in this article we considered the problem of output synchronisation of a class of networked agents. The dynamics of each agent is a passive system and each agent can use its neighbourhood information to adjust its movement. The local information then forms a varying adjacent topology. We first investigated the stability of time-varying non-linear system by using multiple Lyapunov function. Some stability results were obtained. Particularly, an extended Lyapunov-like lemma was proved. Using this tool, certain sufficient conditions for the output synchronisation of the networked agents were obtained.

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