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A conjecture on the norm of Lyapunov mapping

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Abstract: A conjecture that the norm of Lyapunov mapping L_A equals to its restriction to the symmetric set, S, i.e., $\|L_A\| = \|L_A|_S\|$ was proposed in [1]. In this paper, a method for numerical testing is provided first. Then, some recent progress on this conjecture is presented.

Keywords: Lyapunov mapping; Invariant subspace; Norm

1 Introduction

This paper considers the conjecture proposed in [1] that the norm of Lyapunov mapping $L_A : X \mapsto XA + A^{\mathrm{T}}X$ equals its restriction to the symmetric set, namely S, precisely, and we introduce some notations first [2].

· $M_{m\times n}$: Set of $m \times n$ real matrices.

· M_n: Set of $n \times n$ real matrices.

· Let $A = (a_{ij}) \in M_{m\times n}$, then its row stacking form is

$$V_{\mathrm{r}}(A) = (a_{11}, \cdots, a_{1n}, \cdots, a_{m1}, \cdots, a_{mn})^{\mathrm{T}};$$

and its column stacking form is

$$V_{\mathrm{c}}(A) = (a_{11}, \cdots, a_{m1}, \cdots, a_{1n}, \cdots, a_{mn})^{\mathrm{T}}.$$

· Let $x \in \mathbb{R}^{n^2}$, and then

$$V_{\mathrm{r}}^{-1}(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n(n-1)+1} & x_{n(n-1)+2} & \cdots & x_{n^2} \end{bmatrix};$$

$$V_{\mathrm{c}}^{-1}(x) = \begin{bmatrix} x_1 & x_{n+1} & \cdots & x_{n(n-1)+1} \\ x_2 & x_{n+2} & \cdots & x_{n(n-1)+2} \\ \vdots & \vdots & & \vdots \\ x_n & x_{2n} & \cdots & x_{n^2} \end{bmatrix}.$$

· Let $\rho : M_{m\times n} \to M_{p\times q}$ be a linear mapping. $X \in M_{m\times n}$ and $Y = \rho(X) \in M_{p\times q}$. $x = V_{\mathrm{r}}(X)$ and $y = V_{\mathrm{r}}(Y)$ ($x = V_{\mathrm{c}}(X)$ and $y = V_{\mathrm{c}}(Y)$). Then we can find a matrix M_{r}^{ρ} (respectively, M_{c}^{ρ}) such that

$$y = M_{\mathrm{c}}^{\rho}x, \ \ (\text{respectively, } \ y = M_{\mathrm{r}}^{\rho}x). \tag{1}$$

In the stability and stabilization of linear systems, quadratic Lyapunov function plays a fundamental role [3,4]. It is well known that for any matrix $A \in M_n$,

$$PA + A^{\mathrm{T}}P = -Q \tag{2}$$

is called the Lyapunov equation. A is stable, iff for any $Q > 0$ the Lyapunov equation has a unique solution $P > 0$. In general, we may describe (2) as a mapping:

Definition 1 A Lyapunov mapping $L_A : M_n \to M_n$ is defined as

$$L_AX = XA + A^{\mathrm{T}}X, \ \ X \in M_n, \tag{3}$$

where $A \in M_n$ is a given matrix.

Proposition 1 Let $\rho = L_A$. Then

$$M_{\mathrm{r}}^{\rho} = M_{\mathrm{c}}^{\rho} = A^{\mathrm{T}} \otimes I_n + I_n \otimes A^{\mathrm{T}}. \tag{4}$$

Identifying M_n with $\mathbb{R}^{n^2}$, we can use the Euclid norm on M_n. In this paper, we investigate the norm of L_A.

Next, we cite some known results about the norm of L_A from [1].

· Let S and K be symmetric and skew-symmetric subspaces of M_n respectively. That is,

$$S_n = \{A \in M_n | A^{\mathrm{T}} = A\}, \ \ K_n = \{A \in M_n | A^{\mathrm{T}} = -A\}.$$

Then both S_n and K_n are invariant subspaces of L_A. The restrictions of L_A on them are denoted by L_A^S and L_A^K respectively.

· S and K are two orthogonal subspaces of $M_n \simeq \mathbb{R}^{n\times n}$. Moreover,

$$\|L_A\| = \max\{\|L_A^S\|, \|L_A^K\|\}. \tag{5}$$

· When $n = 2$, $\|L_A\| = \|L_A^S\|$.

The conjecture proposed in [1] is

$$\|L_A\| = \|L_A^S\|, \ \ \forall A \in M_n. \tag{6}$$

This paper will give some formulas for numerical analysis of the norm of Lyapunov mapping. A routine is provided. Using it, you can test the conjecture numerically tens of thousand of times with one click. Then a recent theoretical result shows under a mild assumption that the conjecture is true.

2 Numerical analysis

First, we give a numerical method to test the norm. In this approach we need to find the matrix expressions of L_A^S, and L_A^K. Choose an orthonormal basis of M_n as

$$\begin{cases} S_{ii} = d_{ii}, & i = 1, \cdots, n; \\ S_{ij} = \dfrac{1}{\sqrt{2}}(d_{i,j} + d_{j,i}), & i = 1, \cdots, n-1, \\ & j = i+1, \cdots, n; \\ K_{ij} = \dfrac{1}{\sqrt{2}}(d_{i,j} - d_{j,i}), & i = 1, \cdots, n-1, \\ & j = i+1, \cdots, n, \end{cases} \tag{7}$$

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where $K := (k_{p,q}) = d_{i,j}$ means

$$k_{p,q} = \begin{cases} 1, & p = i \text{ and } q = j, \\ 0, & \text{otherwise}. \end{cases}$$

It is obvious that $\{S_{ii}, S_{ij}\}$ and $\{K_{i,j}\}$ are the orthonomal bases of S_n and K_n respectively.

For notational ease, we denote the ordered bases as

$$\begin{aligned} \mathcal{S} &= [S_{11} \cdots S_{1n}\ S_{22} \cdots S_{2n}\ \cdots S_{nn}] \\ &:= [\mathcal{S}_1\ \mathcal{S}_2\ \cdots \mathcal{S}_p], \end{aligned}$$

where $p = \dfrac{n(n+1)}{2}$, and

$$\begin{aligned} \mathcal{K} &= [K_{12} \cdots K_{1n}\ K_{23} \cdots K_{2n}\ \cdots\ K_{(n-1)n}] \\ &:= [\mathcal{K}_1\ \mathcal{K}_2\ \cdots \mathcal{K}_q], \end{aligned}$$

where $q = \dfrac{n(n-1)}{2}$.

Next, we use $\{E_{ij} = d_{i,j} \mid i, j = 1, \cdots, n\}$ as a basis of M_n. Then for any $A = (a_{ij}) \in M_n$, it can be expressed as

$$A = \sum_{i,j=1}^{n} a_{ij} E_{ij}.$$

For the basis elements E_{ij} we have

$$L_{E_{ij}} \mathcal{S}_k = \mathcal{S} \lambda_{ij}^k, \quad i, j = 1, \cdots, n; k = 1, \cdots, p, \tag{8}$$

where $\lambda_{ij}^k \in \mathbb{R}^p, i, j = 1, \cdots, n, k = 1, \cdots, p$ are structure parameters of the restriction of Lyapunov mapping on S^n. Then we can construct a set of structure matrices as

$$M_{ij}^S = \left[\lambda_{ij}^1\ \lambda_{ij}^2 \cdots \lambda_{ij}^p\right] \in M_p.$$

Note that M_{ij}^S are independent of a particular matrix A. Then for any $A \in M_n$ the matrix expression of L_A^S is

$$M_A^S = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} M_{ij}^S. \tag{9}$$

Similarly, we have

$$L_{E_{ij}} \mathcal{K}_k = \mathcal{K} \mu_{ij}^k, \ \ i, j = 1, \cdots, n; k = 1, \cdots, q, \tag{10}$$

where $\mu_{ij}^k \in \mathbb{R}^q, i, j = 1, \cdots, n, k = 1, \cdots, q$ are structure parameters of the restriction of Lyapunov mapping on K^n. Then we can also construct a set of structure matrices as

$$M_{ij}^K = \left[\mu_{ij}^1\ \mu_{ij}^2 \cdots \mu_{ij}^q\right] \in M_q.$$

M_{ij}^K are also independent of A. For any $A \in M_n$ the matrix expression of L_A^K is

$$M_A^K = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} M_{ij}^K. \tag{11}$$

Example 1 Let $n = 3$. We have

$$M_{11}^K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ M_{12}^K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};\ M_{13}^K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix};$$

$$M_{21}^K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};\ M_{22}^K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ M_{23}^K = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$M_{31}^K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ M_{32}^K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ M_{33}^K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we can calculate $M_{i,j}^S,\ i, j = 1, 2, 3$ (to save space, we skip them here).

Now consider a particular matrix as

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Then $L_A = A^{\mathrm{T}} \otimes I_3 + I_3 \otimes A^{\mathrm{T}}$ is

$$L_A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 2 & -2 \end{bmatrix}.$$

Using (11), we have

$$\begin{aligned} M_A^K &= M_{11}^K + M_{13}^K - M_{22}^K + 2M_{23}^K + M_{32}^K - M_{33}^K \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}. \end{aligned}$$

Using (9), we have

$$M_A^S = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sqrt{2} & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & \sqrt{2} & 0 \\ 0 & 1 & 0 & 2\sqrt{2} & -2 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 2\sqrt{2} & -2 \end{bmatrix}.$$

It is easy to calculate that

$$\begin{aligned} &\|L_A\| = \|L_A^S\| = 5.19677069822662; \\ &\|L_A^K\| = 2.56155281280883. \end{aligned}$$

A routine in MatLab is provided at "http://lsc.amss.ac.cn/~dcheng" for calculating the bases of S^n and K^n for any $n \geqslant 2$. It can also be used to test whether $\|L_A\| = \|L_A^S\|$ numerically for arbitrary A assigned by you. For a randomly chosen A, you may test it as many times as you like.

Note that the numerical test is not a proof. Say, you may guess $\|L_A\| > \|L_A^K\|$, which completes the proof of the conjecture. Numerical test by choosing A randomly can hardly find a counter-example. However, a trivial counter-example is $A = I_n$.

3 An updated result

In this section we provide a new result about the conjecture. We need the concept of swap matrix and its properties.

Definition 2 [2] A swap matrix, $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ and its rows by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

When $m = n$ we simply denote $W_{[n,n]}$ by $W_{[n]}$.

Example 2 Let $m = 2$ and $n = 3$, and then the swap matrix $W_{[2,3]}$ is constructed as

$$W_{[2,3]} = \begin{array}{c} \begin{array}{cccccc} (11) & (12) & (13) & (21) & (22) & (23) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{array}.$$

Proposition 2 [2]

$$W^{\mathrm{T}}_{[m,n]} = W^{-1}_{[m,n]} = W_{[n,m]}. \tag{13}$$

Proposition 3 [2] Let $A \in M_{m\times n}$. Then

$$W_{[m,n]}V_{\mathrm{r}}(A) = V_{\mathrm{c}}(A),\ \ W_{[n,m]}V_{\mathrm{c}}(A) = V_{\mathrm{r}}(A). \tag{14}$$

Proposition 4 [2] Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then

$$W_{[m,p]}(A \otimes B)W_{[q,n]} = B \otimes A. \tag{15}$$

Now consider the norm of L_A. It is well known that

$$\|L_A\| = \lambda_{\max}(L^{\mathrm{T}}_A L_A) := \lambda_m.$$

Hence, let ξ be an eigenvector corresponding to λ_m, then

$$\|L_A\| = \frac{\|L_A\xi\|}{\|\xi\|}.$$

Hence, we have the following result immediately.

Lemma 1 For a given A if there is a symmetric ξ (i.e., $V^{-1}_{\mathrm{r}}(\xi) = V^{-1}_{\mathrm{c}}(\xi)$ is symmetric), which is an eigenvector with respect to λ_m, then

$$\|L_A\| = \|L^S_A\|.$$

For statement ease, if $\xi \in \mathbb{R}^{n\times n}$, the m-transpose of ξ, denoted by ξ^t, means the transpose of its corresponding matrix form, i.e.,

$$\xi^t := V_{\mathrm{r}}[(V^{-1}_{\mathrm{r}}(\xi))^{\mathrm{T}}]\ \text{ equivalently } V_{\mathrm{c}}[(V^{-1}_{\mathrm{c}}(\xi))^{\mathrm{T}}].$$

ξ is said to be m-symmetric (m-skew symmetric) if $\xi^t = \xi$ ($\xi^t = -\xi$).

Now we are ready to present our main result. Using the above notations, we have

Theorem 1 For a given A if there is a ξ, which is an eigenvector with respect to λ_m and is not m-skew symmetric, then

$$\|L_A\| = \|L^S_A\|.$$

Proof It is easy to calculate that

$$\begin{aligned} L^{\mathrm{T}}_A L_A &= (I_n \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I_n)^{\mathrm{T}}(I_n \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I_n) \\ &= (I_n \otimes A + A \otimes I_n)(I_n \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I_n) \\ &= I_n \otimes AA^{\mathrm{T}} + A^{\mathrm{T}} \otimes A + A \otimes A^{\mathrm{T}} + AA^{\mathrm{T}} \otimes I_n. \end{aligned} \tag{16}$$

It follows from Proposition 4 that

$$W_{[n]}(L^{\mathrm{T}}_A L_A)W_{[n]} = L^{\mathrm{T}}_A L_A. \tag{17}$$

Now assume ξ is an eigenvector with respect to λ_m. Then we claim that ξ^t is also an eigenvector with respect to λ_m. To prove this claim, note that because of Proposition 3

$$\xi^t = W_{[n]}\xi.$$

Using (17) and Proposition 2, we have

$$\begin{aligned} L^{\mathrm{T}}_A L_A W_{[n]}\xi &= W_{[n]}L^{\mathrm{T}}_A L_A W_{[n]}W_{[n]}\xi \\ &= W_{[n]}L^{\mathrm{T}}_A L_A\xi = \lambda_{\max}W_{[n]}\xi. \end{aligned}$$

Now one sees that $W_{[n]}\xi$ is also an eigenvector with respect to λ_m. Now if ξ is not m-skew symmetric, then $\xi + \xi^t$ is an m-symmetric eigenvector with respect to λ_m. The conclusion follows.

4 Conclusions

This paper revealed some new developments in the conjecture of the norm of Lyapunov mapping:

$$\|L_A\| = \|\, L_A|_S\|\,.$$

It is proven that if the largest eigenvalue λ_m of $L^{\mathrm{T}}_A L_A$ has a non-skew symmetric eigenvector, the conjecture is true. Some formulas for numerical analysis were provided.

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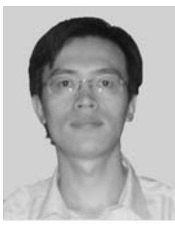

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