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Dynamics of opinions with social biases[☆]

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ABSTRACT

This paper aims to provide a systemic analysis to social opinion dynamics subject to individual biases. As a generalization of the classical DeGroot social interactions, defined by linearly coupled dynamics of peer opinions that evolve over time, biases add to state-dependent edge weights and therefore lead to highly nonlinear network dynamics. Previous studies have dealt with convergence and stability analysis of such systems for a few specific initial node opinions and network structures, and here we focus on how individual biases affect social equilibria and their stabilities. Two categories of equilibria, namely the boundary and interior equilibria, are defined. For a few fundamental network structures, some important interior network equilibria are presented explicitly for a wide range of system parameters, which are shown to be locally unstable in general. Particularly, the interval centroid is proven to be unstable regardless of the bias level and the network topologies. Next, we prove that when the initial network opinions are polarized towards one side of the state space, node biases will drive the opinion evolution to the corresponding interval boundaries. Such polarization attraction effect continues to hold under even directed and switching network structures. Finally, a number of numerical examples are provided to validate our study and advance the understanding of the nonlinearity inherited within the biased opinion evolution.

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1. Introduction

Understanding how opinions of the members in our society evolve during their interactions that take place online or in daily lives is becoming increasingly important in many aspects ranging from political decisions to marketing strategies (Easley & Kleinberg, 2010; Friedkin, Proskurnikov, Tempo, & Parsegov, 2016; Jackson, 2010; Scott, 2017). In various cases, social opinions can be represented by real numbers, and by individuals averaging those numbers with neighbors the classical DeGroot's model was established (DeGroot, 1974). When the social network structure admits sufficient connectivity, it was shown that DeGroot type of social interactions often leads to convergence to a common opinion, namely agreement or consensus, across the entire society, e.g., Jadbabaie, Lin, and Morse (2003), Moreau (2005), Semonsen, Griffin, Squicciarini, and Rajtmajer (2018) and Tsitsiklis, Bertsekas, and Athans (1986). The significance of social agreement can be made clear through the notion of naive learning in the sense that a social agreement, even not at the perfect average, implies asymptotic learning of a hidden variable with sufficiently flat interconnections, when nodes' opinions are independently sampled in the first place, e.g. Golub and Jackson (2010).

In practical social networks, however, DeGroot social interactions are arguably rare since it is difficult to observe social agreement, e.g., Bindel, Kleinberg, and Oren (2015), DeMarzo, Vayanos, and Zwiebel (2003), Friedkin et al. (2016) and Lawrence, Sides, and Farrell (2010). As a result, a number of generalized models were proposed to capture different psychological effects behind social interactions. Peers might put weight on their initial opinions throughout the entire social interactions as memory effects (Friedkin & Johnsen, 1990); nodes might only interact

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with peers that hold opinions within a given range compared to their own opinions (Hegselmann & Krause, 2002; Li, Scaglione, Swami, & Zhao, 2013); a portion of nodes may be stubborn who never revise their initial beliefs (Acemoglu, Como, Fagnani, & Ozdaglar, 2013; Yildiz, Ozdaglar, Acemoglu, Saberi, & Scaglione, 2013); nodes might adjust the weights timely with the changes of their opinions (Bhawalkar, Gollapudi, & Munagala, 2013); nodes may tend to be repulsive towards enemies by carrying out negative interactions (Altafini, 2012, 2013; Shi, Johansson, & Johansson, 2013; Shi, Proutiere, Johansson, Baras, & Johansson, 2016). It turned out, beyond asymptotic stability, social dynamics can exhibit complex behaviors such as clustering and oscillation (Altafini, 2013; Blondel, Hendrickx, & Tsitsiklis, 2009; Shi et al., 2016), being consistent with studies from social and political science (Mas, Flache, & Helbing, 2010; McCarty, Poole, & Rosenthal, 2016). In fact, nonlinear bifurcations can arise from collective dynamics of interconnected agents as a way of gaining survival advantage (Leonard, 2014).

In the real world, individuals' examination of new information and rationale of decision making are often biased. Studies from social psychology show that people are more likely to accept confirming evidence as viewpoints given by someone similar to themselves (Lord, Ross, & Lepper, 1979). In Centola (2011), it was shown that people are more willing to accept health behaviors being put in a group of homophilous members. A convincing model for this biased opinion assimilation was proposed in Dandekar, Goel, and Lee (2013) as a natural interpretation of confirmation bias:

(i) Nodes hold real-valued opinions, and receive linear combinations of neighboring nodes in a social network as evidence;
(ii) Each individual node uses its current opinion to generate a nonlinear confidence weight towards the evidence it receives, depending on how biased this node is.

Then nodes update their states as a weighted sum of their current states and the evidence from neighbors.

However, due to the nonlinear mechanism of this biased opinion dynamics, analytical understandings of the behaviors of such social systems become extremely difficult. Note that, any serious attempt of modeling confirmation bias relies on making the confidence level towards external evidence depending on the node's current state and the evidence. Therefore, such nonlinearity is inherently aligning with such effort. In Dandekar et al. (2013), asymptotic behaviors of the biased opinion evolution were only established for *special initial values* and for some very *special network structure*, i.e, the two-island network model as well. Advancing such results will provide insight to this interesting social opinion dynamic model, but also shed light on perspectives of treating complex network dynamics with state-dependent edge weights.

In this paper, we attempt to provide a systemic analysis to a social opinion dynamical model with bias assimilation. Particularly, we focus on how individual biases affect social equilibria and their stabilities. First of all, we investigate the bias-induced equilibria of the collective nonlinear network dynamics. For fundamental network structures such as complete, star, and cycle graphs, the equilibria are presented explicitly for a wide range of system parameters. The given equilibria are also shown to be locally unstable in general. Particularly, the interval centroid is shown to be always unstable regardless the choice of bias level and network topologies. These results add to new understandings of the stability analysis in Dandekar et al. (2013), going beyond specific initial node opinions despite the high nonlinearity of the network dynamics. Next, we prove that when the initial network opinions are polarized towards one side of the state interval, such polarization will be persisted and amplified by node biases during the opinion evolution in the sense that all node states will converge to the corresponding interval boundaries. Such polarization attraction is shown to exist under even directed and switching network structures. Moreover, we provide several numerical examples to do some intuitive analyses.

The remainder of the paper is organized as follows. Section 2 defines the specific model and raises problems of interests we will analyze. Section 3 investigates the new equilibria that arise from the nonlinear network dynamics for both their positions and stabilities. Section 4 moves on to discuss the polarization attraction effect including the generalizations to directed and switching network structures. Furthermore, Section 5 provides some numerical examples to better illustrate our model and results. Finally some concluding remarks are given in Section 6.

Notation. For a vector $X = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, let $\|X\| = \sqrt{\sum_{i=1}^n x_i^2}$. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ represents the largest integer that is no larger than x, and $\lceil x \rceil$ represents the smallest integer that is no smaller than x.

2. Problem formulation

2.1. The model

To account for social opinion dynamics, consider a social network with n individuals (nodes) indexed in the set $\mathrm{V} = \{1, \dots, n\}$. The structure of the social network is represented by an undirected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, where each edge $\{i, j\} \in \mathrm{E}$ is an unordered pair of two different nodes in the set V. The graph G is assumed to be connected without loss of generality. Each $i \in \mathrm{V}$ holds an opinion $\mathbf{x}_i(t) \in \mathbb{R}$ at slotted time $t = 0, 1, 2, \dots$. Node i interacts with the neighbors in the set $\mathrm{N}_i := \{j \in \mathrm{V} : \{i, j\} \in \mathrm{E}\}$. The influence strength between two neighboring nodes i and j is represented by $w_{ij} > 0$ and then $d_i := \sum_{j \in \mathrm{N}_i} w_{ij}$ is the total weight of influence applied to node i. Note that with connectivity, N_i is non-empty for any i and thus $d_i > 0, i \in \mathrm{V}$. The node i's self-confidence is represented by $w_{ii} > 0$. Let

$$\mathbf{s}_i(t) := \sum_{j \in \mathrm{N}_i} w_{ij} \mathbf{x}_j(t)$$

be the weighted external evidence received by node i at time t. Let b_i be a positive number associated with node i as a bias parameter. The evolution of the $\mathbf{x}_i(t), \ i \in \mathrm{V}$ is described by Dandekar et al. (2013):

$$\mathbf{x}_i(t+1) = \frac{w_{ii}\mathbf{x}_i(t) + \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t)}{w_{ii} + \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t) + \left(1 - \mathbf{x}_i(t)\right)^{b_i}\left(d_i - \mathbf{s}_i(t)\right)}. \tag{1}$$

This model describes the opinion dynamics with a well-known phenomenon in social psychology named biased assimilation. For any node i, the weighted external evidence $\mathbf{s}_i(t)$ can be regarded as the weighted sum of all neighbors support for extreme opinion 1; correspondingly, $d_i - \mathbf{s}_i(t)$ represents the weighted sum of all neighbors support for extreme opinion 0. Node i then weights $\mathbf{s}_i(t)$ by a factor $\mathbf{x}_i^{b_i}(t)$ and weights $d_i - \mathbf{s}_i(t)$ by a factor $\left(1 - \mathbf{x}_i(t)\right)^{b_i}$. It can be easily shown that all $\mathbf{x}_i(t)$ will always be within the interval $[0, 1]$ if they all start from this interval at $t = 0$. Thus, the larger b_i is, the smaller $\mathbf{x}_i^{b_i}(t)$ is. As a result, node i will put a lower confidence towards the external evidence $\mathbf{s}_i(t)$. This is to say, b_i indicates the bias level of node i, and (1) is a description of biased opinion evolution.

2.2. Problems of interest

As an intriguing generalization to the DeGroot model, the complex network dynamics (1) is highly nonlinear. For nonlinear systems in general we are interested in the following aspects:

(i) Are there any equilibria? If yes, then how many?
(ii) Are the equilibria, if exist, locally stable or unstable?
(iii) What is the asymptotic behavior of the node states in terms of convergence to equilibria, limit cycles or chaotic attractors?

Unfortunately, a complete answer to these questions is rather challenging, despite of their significance in interpreting novel social behaviors. In fact, the original work in Dandekar et al. (2013) which introduces this model is only capable of establishing a few convergence conditions with quite special initial values or network structures. Moreover, we would like to note that advancing the tools for analyzing (1) may also give insight to the treatment of complex network dynamics with state-dependent edge coefficients, a key feature of the nonlinearity in (1).

3. The induced equilibria

In this section, we investigate the bias-induced equilibria of the system (1). Clearly, the total number of degrees of freedom is too high to facilitate a meaningful analysis given the bias levels b_i and the node weights w_{ij}. To ease the presentation, we impose the following assumption throughout the remaining part of this section.

Assumption 1. The following conditions hold for system (1):

(i) There is a $b > 0$ such that $b_i = b$ for all $i \in \mathrm{V}$;
(ii) $w_{ij} = 1$ for all $\{i, j\} \in \mathrm{E}$;
(iii) $w_{ii} = w \geq 0$ for all $i \in \mathrm{V}$.

Let $\mathbb{E}$ be the set of equilibria of (1) and $\mathbb{E}_{\text{bdy}} = \{\mathrm{X} \in [0, 1]^n \backslash (0, 1)^n : \mathrm{X} \in \mathbb{E}\}$ be the set of boundary equilibria. It is clear that every point in $\{0, 1\}^n$ is a boundary equilibrium. we introduce $\mathbb{E}_{\text{int}} = \{\mathrm{X} \in (0, 1)^n : \mathrm{X} \in \mathbb{E}\}$ as the set of interior equilibria, which is certainly of more interest. Furthermore, denote $\mathbf{X}(t) = \big(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)\big)^\top$. Recall the following definition (Khalil, 2002).

Definition 1. The equilibrium $\mathrm{X} = (\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_{n-1}, \mathrm{x}_n)^\top$ of system (1) is called locally stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|\mathbf{X}(t) - \mathrm{X}\| < \varepsilon$ for all $t \geq 0$ whenever $\|\mathbf{X}(0) - \mathrm{X}\| < \delta$. Otherwise, the equilibrium is called to be unstable.

3.1. Equilibria distribution

For any equilibrium X, there holds that for all $i \in \mathrm{V}$

$$\mathrm{x}_i = \frac{w_{ii}\mathrm{x}_i + \mathrm{x}_i^b\big(\sum_{j\in \mathrm{N}_i} w_{ij}\mathrm{x}_j\big)}{w_{ii} + \mathrm{x}_i^b\big(\sum_{j\in \mathrm{N}_i} w_{ij}\mathrm{x}_j\big) + \big(1-\mathrm{x}_i\big)^b\big(d_i - (\sum_{j\in \mathrm{N}_i} w_{ij}\mathrm{x}_j)\big)},$$

which is equivalent to

$$p_i(\mathrm{x}_1, \dots, \mathrm{x}_n) := \mathrm{x}_i^b\Big(\mathrm{x}_i - 1\Big)\Big(\sum_{j\in \mathrm{N}_i} w_{ij}\mathrm{x}_j\Big) + \mathrm{x}_i\Big(1-\mathrm{x}_i\Big)^b\Big(d_i - (\sum_{j\in \mathrm{N}_i} w_{ij}\mathrm{x}_j)\Big) = 0,\ \ i \in \mathrm{V}. \quad (2)$$

Here each $p_i(\mathrm{x}_1, \dots, \mathrm{x}_n)$ is a polynomial function.

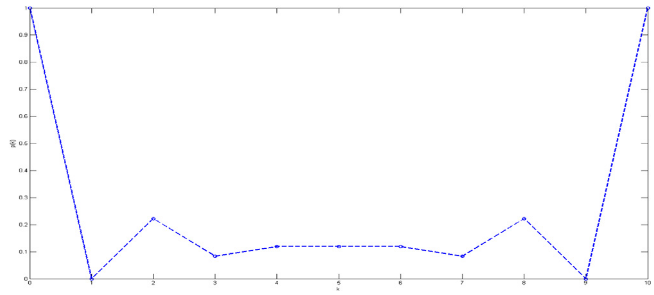

Fig. 1. The ratio of stable equilibria in the set $\mathbb{E}_{\text{ver}}^k$.

Remark 1. It is obvious that (2) is a system of polynomial equations, which is difficult to solve even numerically from a computational point of view. The Buchberger's algorithm in Cox, Little, and O'shea (2007) provides a form of exact solvers to find the recursive ideal generated by the n polynomials in (2) over the polynomial ring.

The 2^n equilibria in the set $\{0, 1\}^n$ are also quite interesting as they are vertex equilibria in the opinion space. Their stabilities would reflect stubborn and extreme social formations. To illustrate this, we provide the following example.

Example 1. Consider a cycle graph with 10 nodes. Let $w = 1$ and $b = 3$ in (1). The 2^{10} vertex equilibria are denoted by $\mathbb{E}_{\text{ver}} := \Big\{(\mathrm{x}_1\ \dots\ \mathrm{x}_{10})^\top : \mathrm{x}_i \in \{0, 1\}\Big\}$. The stability of each equilibrium in the set $\mathbb{E}_{\text{ver}}$ is tested by randomization, where around each equilibrium a total of 100 initial values are selected randomly. The algorithm is run for 10^4 steps and if the distance between the resulting outcome and the equilibrium is always within three times of the initial distance for the 100 initial values, the equilibrium is considered as stable. Denote by $\mathbb{E}_{\text{ver}}^k := \Big\{(\mathrm{x}_1\ \dots\ \mathrm{x}_{10})^\top : \mathrm{x}_i \in \{0, 1\},\ \sum_{i=1}^{10} \mathrm{x}_i = k\Big\}$ for $k = 0, 1, \dots, 10$. The subset of stable equilibria of $\mathbb{E}_{\text{ver}}^k$ is denoted by $\overline{\mathbb{E}}_{\text{ver}}^k$. Define $p(k) := \big|\overline{\mathbb{E}}_{\text{ver}}^k\big| / \big|\mathbb{E}_{\text{ver}}^k\big|$ as the ratio of stable equilibria in the set $\mathbb{E}_{\text{ver}}^k$. The plot of $p(k)$ is shown in Fig. 1. From the computation result, when $k = 1$ or 9, no vertex equilibria is stable.

The numerical result illustrates that for most k's, both stable and unstable equilibria exist in the set $\mathbb{E}_{\text{ver}}^k$. Moreover, $p(k)$ is symmetric with respect to $k = 5$, which seems natural in view of the construction of the set $\mathbb{E}_{\text{ver}}^k$ and the symmetry of a cycle graph.

3.2. Main results

Note that G is a complete graph if $\{i, j\} \in \mathrm{E}$ for all $i, j \in \mathrm{V}$ and $i \neq j$; a star graph if $\mathrm{E} = \big\{\{1, i\} :\ i = 2, \dots, n.\big\}$; and a cycle graph if $\mathrm{E} = \big\{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\big\}$. For notational simplicity, we use node $nm + k$ to represent node $k \in \mathrm{V}$ for all $m \in \mathbb{Z}$.

It can be easily seen that the opinion space centroid $(1/2, 1/2, \dots, 1/2)^\top$ is always an unstable interior equilibrium, as is described in the following proposition.

Proposition 1. *Let Assumption 1 hold. Then $\mathrm{X} = (1/2, 1/2, \dots, 1/2)^\top$ is always an unstable equilibrium of system (1).*

Proof. See the Appendix for the proof of this proposition and those for the remaining theorems in this section.

This proposition tells us that the centroid equilibrium is unstable under general conditions regardless of how the social network is structured. A social interpretation to Proposition 1 is that, it is almost impossible for everyone in a society to remain neutral for a long time.

When the underlying network structure is a complete graph, we have the following results.

Theorem 1. *Let* G *be a complete graph with* $n \geq 3$ *subject to Assumption 1. If* $b \leq 1$ *or* $b = 2$*, then the set of interior equilibria contains only the singleton* $(1/2, \dots, 1/2)^\top$*, i.e.,* $\mathbb{E}_{\text{int}} = \{(1/2, 1/2, \dots, 1/2)^\top\}$*. Moreover, the equilibrium* $(1/2, \dots, 1/2)^\top$ *is unstable.*

For star and cycle graphs, a variety of new interior equilibria arise from the nonlinear network dynamics, as presented in the following two results.

Theorem 2. *Let* G *be a star graph subject to Assumption 1. Then the following statements hold.*

(i) $\mathbb{E}_{\text{int}} = \left\{(1/2, x_2, x_3, \dots, x_n)^\top : \sum_{i=2}^n x_i = (n-1)/2, x_i \in (0,1) \text{ for all } i \in \mathrm{V}\backslash\{1\}\right\}$ *if* $b = 1$*;*
(ii) $\mathbb{E}_{\text{int}} = \left\{(1 - x_2, x_2, \dots, x_2)^\top : x_2 \in (0,1)\right\}$ *if* $b = 2$*;*
(iii) $\mathbb{E}_{\text{int}} = \left\{(1/2, 1/2, \dots, 1/2)^\top\right\}$ *if* $b \neq 1, 2$*.*

Moreover, any equilibrium $\mathrm{X} \in \mathbb{E}_{\text{int}}$ *is unstable.*

Theorem 3. *Let* G *be a cycle graph subject to Assumption 1. Then the following statements hold.*

(i) If $b = 1$ *and* $n \equiv 1, 2$ *or* $3 (\mathrm{mod}\ 4)$*,* $\mathbb{E}_{\text{int}} = \{(1/2, 1/2, \dots, 1/2)^\top\}$*;*
(ii) If $b = 1$ *and* $n \equiv 0 (\mathrm{mod}\ 4)$*,* $\mathbb{E}_{\text{int}} = \left\{(a_1, a_2, 1 - a_1, 1 - a_2, a_1, \dots, 1 - a_1, 1 - a_2)^\top : a_1, a_2 \in (0,1)\right\}$*;*
(iii) If $b = 2$ *and* $n \equiv 1 (\mathrm{mod}\ 2)$*,* $\mathbb{E}_{\text{int}} = \left\{(1/2, 1/2, \dots, 1/2)^\top\right\}$*;*
(iv) If $b = 2$ *and* $n \equiv 0 (\mathrm{mod}\ 2)$*,* $\mathbb{E}_{\text{int}} = \left\{(a, 1-a, a, 1-a, \dots, a, 1-a)^\top : a \in (0,1)\right\}$*.*

Moreover, any equilibrium $\mathrm{X} \in \mathbb{E}_{\text{int}}$ *is unstable for* $b = 1$ *or* $b = 2$*.*

It is extremely difficult to generalize these results to networks with less common structures. The reason comes from the fact that the ideal generated by the polynomial in (2) depends on the network structure in a highly nontrivial manner. While as we explained above, solving such systems of polynomial equations is equivalent to solving such equations on the generated ideals of the polynomials (Cox et al., 2007). Nonetheless, Proposition 1 and Theorems 1–3 show that intriguing equilibria can indeed arise for the system (1). We conjecture that a majority of the interior equilibria should be unstable.

These results imply that without external factors, the opinions of a group are likely to be extreme. Therefore, some interventions might be necessary if one wants to observe neutral public opinions.

4. Polarization attraction

In this section, we establish the polarization effect of the system (1) when individual opinions are collectively polarized towards one side of the opinion space.

4.1. Exponential polarization

We present the following result.

Theorem 4. *Let* $b_i > 0$ *for all* $i \in \mathrm{V}$*.*

(i) Suppose $\mathbf{x}_i(0) \in [0, 1/2)$ *for all* $i \in \mathrm{V}$*. Then* $\lim_{t\to\infty} \mathbf{x}_i(t) = 0$ *for all* $i \in \mathrm{V}$ *with*

$$\mathbf{x}_i(t) \leq \left(1 - \frac{\alpha}{2}\right)^t \max_{j\in\mathrm{V}} \mathbf{x}_j(0),$$

where $\alpha = \min_{k\in\mathrm{V}} \frac{d_k}{w_{kk}+d_k}\left[\left(1 - \max_{j\in\mathrm{V}} \mathbf{x}_j(0)\right)^{b_k} - \left(\max_{j\in\mathrm{V}} \mathbf{x}_j(0)\right)^{b_k}\right] \in (0, 1]$*.*

(ii) Suppose $\mathbf{x}_i(0) \in (1/2, 1]$ *for all* $i \in \mathrm{V}$*. Then* $\lim_{t\to\infty} \mathbf{x}_i(t) = 1$ *for all* $i \in \mathrm{V}$ *with*

$$\left|\mathbf{x}_i(t) - 1\right| \leq \left(1 - \frac{\beta}{2}\right)^t \left|\min_{j\in\mathrm{V}} \mathbf{x}_j(0) - 1\right|,$$

where $\beta = \min_{k\in\mathrm{V}} \frac{d_k}{w_{kk}+d_k}\left[\left(\min_{j\in\mathrm{V}} \mathbf{x}_j(0)\right)^{b_k} - \left(1 - \min_{j\in\mathrm{V}} \mathbf{x}_j(0)\right)^{b_k}\right] \in (0, 1]$*.*

Proof. We consider statement (i) at first and divide its proof into two steps.

Step 1. Let $\mathbf{y}(t) = \max_{i\in\mathrm{V}}\{\mathbf{x}_i(t)\}$. In this step, we prove that $\mathbf{y}(t)$ is decreasing. We define

$$\begin{aligned}
f_1^i(t) &= w_{ii} + \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t) + \left(1 - \mathbf{x}_i(t)\right)^{b_i}\left(d_i - \mathbf{s}_i(t)\right),\\
f_2^i(t) &= w_{ii}\left(\mathbf{x}_i(t) - \mathbf{y}(t)\right),\\
f_3^i(t) &= \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t)\left(1 - \mathbf{y}(t)\right) - \left(1 - \mathbf{x}_i(t)\right)^{b_i}\left(d_i - \mathbf{s}_i(t)\right)\mathbf{y}(t),\\
f_4^i(t) &= \mathbf{y}(t)\left(1 - \mathbf{y}(t)\right)\left[\mathbf{x}_i^{b_i}(t) - \left(1 - \mathbf{x}_i(t)\right)^{b_i}\right].
\end{aligned}$$

For the $f_j^i(t), j = 1, 2, 3, 4$, the following facts can be established.

(a) From $\mathbf{s}_i(t) = \sum_{j\in\mathrm{N}_i} w_{ij}\mathbf{x}_j(t) \leq \sum_{j\in\mathrm{N}_i} w_{ij}\mathbf{y}(t) = d_i\mathbf{y}(t)$ and $\mathbf{x}_i(t) \leq \mathbf{y}(t) < 1/2$, there holds that $f_1^i(t) > 0$.
(b) The definition of $\mathbf{y}(t)$ implies that $f_2^i(t) \leq 0$.
(c) If $0 \leq \mathbf{s}_i(t) \leq d_i\mathbf{y}(t)$, there holds that $f_3^i(t) \leq d_i f_4^i(t)$.
(d) If $b_i > 0$ and $\mathbf{y}(t) < 1/2$ hold, we obtain that $\mathbf{x}_i(t) \leq \mathbf{y}(t) < 1/2$ and $f_4^i(t) \leq 0$.

From system (1), for all $i \in \mathrm{V}$ and $\mathbf{y}(t) < 1/2$, we obtain that

$$\mathbf{x}_i(t+1) - \mathbf{y}(t) = \frac{f_2^i(t) + f_3^i(t)}{f_1^i(t)} \leq \frac{f_2^i(t) + d_i f_4^i(t)}{f_1^i(t)} \leq 0, \tag{3}$$

where the first inequality holds with (a), (c) and the second one holds with (b), (d). Therefore we have proved that if $\mathbf{y}(t) < 1/2$, there hold

$$\begin{aligned}
&\mathbf{x}_i(t+1) \leq \mathbf{y}(t), \forall i \in \mathrm{V},\\
&\mathbf{y}(t+1) = \max_{i\in\mathrm{V}}\left\{\mathbf{x}_i(t+1)\right\} \leq \mathbf{y}(t) < 1/2.
\end{aligned}$$

Hence, when $\mathbf{y}(0) < 1/2$, we conclude that $\left\{\mathbf{y}(t)\right\}$ is monotonically decreasing and therefore $\mathbf{y}(t) < 1/2$ for all $t \geq 0$.

Step 2. We will prove that $\left\{\mathbf{y}(t)\right\}$ converges to zero and establish a bound of the convergence rate. From (3), we know

$$\begin{aligned}
\mathbf{y}(t) - \mathbf{x}_i(t+1) &\geq -\frac{1}{w_{ii} + d_i}\left(f_2^i(t) + f_3^i(t)\right)\\
&\geq -\frac{d_i}{w_{ii} + d_i} f_4^i(t),
\end{aligned}$$

where the first inequality holds due to the facts that $\mathbf{x}_i^{b_i}(t) \leq 1$ and $\left(1 - \mathbf{x}_i(t)\right)^{b_i} \leq 1$, while the second inequality holds in view of the fact that $f_2^i(t) \leq 0$ for all $i \in \mathrm{V}$ and $t \geq 0$.

Because $\mathbf{x}_i(t) \leq \mathbf{y}(t) \leq \mathbf{y}(0) < 1/2$, there hold

$$\left(1 - \mathbf{x}_i(t)\right)^{b_i} - \mathbf{x}_i^{b_i}(t) \geq \left(1 - \mathbf{y}(0)\right)^{b_i} - \mathbf{y}^{b_i}(0) > 0,$$

and

$$-f_4^i(t) = -\mathbf{y}(t)\big(1-\mathbf{y}(t)\big)\Big[\mathbf{x}_i^{b_i}(t) - \big(1-\mathbf{x}_i(t)\big)^{b_i}\Big] \\ \geq \Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big]\mathbf{y}(t)\big(1-\mathbf{y}(t)\big).$$

This therefore gives us

$$\mathbf{x}_i(t+1) \leq \mathbf{y}(t) - \frac{d_i}{w_{ii}+d_i}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big] \\ \times \mathbf{y}(t)\big(1-\mathbf{y}(t)\big). \tag{4}$$

Introduce $\alpha = \min_{k\in\mathrm{V}} \frac{d_k}{w_{kk}+d_k}\Big[\big(1-\mathbf{y}(0)\big)^{b_k} - \mathbf{y}^{b_k}(0)\Big]$. Obviously $0 < \alpha \leq 1$. Because $\mathbf{y}(t+1) = \max_{i\in\mathrm{V}}\big\{\mathbf{x}_i(t+1)\big\}$ and (4) holds for all $i \in \mathrm{V}$, we obtain

$$\mathbf{y}(t+1) \leq \mathbf{y}(t) - \alpha\mathbf{y}(t)\big(1-\mathbf{y}(t)\big) \\ \leq (1-\alpha)\mathbf{y}(t) + \frac{\alpha}{2}\mathbf{y}(t) = \Big(1-\frac{\alpha}{2}\Big)\mathbf{y}(t).$$

Therefore, for all $i \in \mathrm{V}$,

$$\mathbf{x}_i(t) \leq \mathbf{y}(t) \leq \left(1-\frac{\alpha}{2}\right)^t \mathbf{y}(0) = \left(1-\frac{\alpha}{2}\right)^t \max_{j\in\mathrm{V}} \mathbf{x}_j(0).$$

This proves (i). The statement (ii) follows from a similar argument, whose details are omitted. Now we have completed the proof. □

Note that Theorem 4 demonstrates the fundamental difference between the DeGroot type of social interactions and the nonlinear opinion dynamics (1). Particularly, DeGroot model defines contraction mappings in the opinion space (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Cao, Morse, & Anderson, 2008; Tsitsiklis, 1984), where the metric $\max_{i\in\mathrm{V}} \mathbf{x}_i(t) - \min_{i\in\mathrm{V}} \mathbf{x}_i(t)$ is monotonically decreasing for any network structure. With a fixed interaction structure, convergence of DeGroot model can be explained by spectrum of the state transition matrix from standard linear systems theory (Xiao & Boyd, 2004), however, the contraction nature of the DeGroot dynamics is certainly beyond that which holds true even under random node interactions (Jackson, 2010; Shi, Anderson, & Johansson, 2015) or nonlinear edge weights (Bauso, Giarre, & Pesenti, 2006; Lin, Francis, & Maggiore, 2007; Moreau, 2005). The proof of Theorem 4 illustrates that $\max_{i\in\mathrm{V}} \mathbf{x}_i(t)$ is no longer contracting along (1). Instead, when $\max_{i\in\mathrm{V}} \mathbf{x}_i(t) < 1/2$, the entire network dynamics will be pushed to the boundary of the opinion space. From Theorem 4 and the definition of stability, we naturally obtain that boundary equilibrium $(0, 0, \ldots, 0)^\top$ and $(1, 1, \ldots, 1)^\top$ are stable.

Furthermore, Theorem 4 can be adjusted to include directed networks as well if we redefine the neighbor N_i. More analogous details can be found in the following subsection.

The results in Theorem 4 can well explain some realistic social phenomena. If there are a group of people who all have negative views (<0.5) to some objects, and each of them is influenced by others, then all of them will finally reach the extreme negative attitude (0) after sufficient communications. The similar phenomenon appears when they all have positive views (>0.5). These phenomena can usually be found in interest groups, boards of companies and political parties. As we mentioned before, people are likely to arrive extreme opinions with biased assimilation.

4.2. Directed and switching graph

We now generalize Theorem 4 to networks with directed and switching structures. To this end, let $\mathrm{G}(t) = \big(\mathrm{V}, \mathrm{E}(t)\big)$ be a time-varying directed graph where at time t, the edge set $\mathrm{E}(t)$ consists of some directed arcs as ordered pairs from the set V. Node i's self-confidence at time t is $w_{ii}(t)$, and the arc $(j, i) \in \mathrm{E}(t)$ holds a weight $w_{ij}(t)$. The neighbor set of node i at time t is in turn defined as $\mathrm{N}_i(t) := \big\{j : (j, i) \in \mathrm{E}(t)\big\}$. Let $\mathbf{s}_i(t) := \sum_{j\in\mathrm{N}_i(t)} w_{ij}(t)\mathbf{x}_j(t)$ and $d_i(t) := \sum_{j\in\mathrm{N}_i(t)} w_{ij}(t)$. In the rest of this section, we suppose that the graph $\mathrm{G}(t)$ can be either connected or disconnected, that is $d_i(t) \geq 0$ for all $t \in \mathbb{N}$. The network dynamics becomes

$$\mathbf{x}_i(t+1) \\ = \frac{w_{ii}(t)\mathbf{x}_i(t) + \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t)}{w_{ii}(t) + \mathbf{x}_i^{b_i}(t)\mathbf{s}_i(t) + \big(1-\mathbf{x}_i(t)\big)^{b_i}\big(d_i(t) - \mathbf{s}_i(t)\big)},\ i \in \mathrm{V}. \tag{5}$$

We impose the following assumption.

Assumption 2. The following conditions hold for the system (5):

(i) there exist $\mathrm{w}_{ii} \geq 0, i \in \mathrm{V}$ such that $w_{ii}(t) \leq \mathrm{w}_{ii}$ for all $t \geq 0$ and all $i \in \mathrm{V}$;
(ii) there exists $c > 0$ such that $d_i(t) \geq c$ whenever $d_i(t) > 0$ for all $i \in \mathrm{V}$;
(iii) there is $T \in \mathbb{N}^+$ such that $\sum_{s=t}^{t+T-1} d_i(s) > 0$ for any $t \geq 0$ and $i \in \mathrm{V}$.

It is worth emphasizing that the two conditions (*i*) and (*ii*) of Assumption 2 are just technical conditions which are consistent with standard DeGroot consensus algorithms (Blondel et al., 2005; Cao et al., 2008). On the other hand, the condition (*iii*) of Assumption 2 serves as a *connectivity* assumption. However, such connectivity is significantly weaker than the usual connectivity assumptions for DeGroot consensus algorithm in the sense that it only requires each node must be affected by some other node during the series of bounded time intervals. Actually, when Assumption 2 holds, the polarization effects continue to exist under this directed and time-varying node interactions, as shown in the following result.

Proposition 2. *Suppose Assumption 2 holds. Then the following statements hold true.*

(i) If $\mathbf{x}_i(0) \in [0, 1/2)$ for all $i \in \mathrm{V}$, then $\lim_{t\to\infty} \mathbf{x}_i(t) = 0$ for all $i \in \mathrm{V}$ with

$$\mathbf{x}_i(t) \leq \left(1-\frac{\alpha_*}{2}\right)^{\lfloor t/T\rfloor} \max_{j\in\mathrm{V}} \mathbf{x}_j(0),$$

where $\alpha_* = \min_{k\in\mathrm{V}} \frac{c}{\mathrm{w}_{kk}+c}\Big[\big(1 - \max_{j\in\mathrm{V}} \mathbf{x}_j(0)\big)^{b_k} - \big(\max_{j\in\mathrm{V}} \mathbf{x}_j(0)\big)^{b_k}\Big] \in (0, 1]$.

(ii) If $\mathbf{x}_i(0) \in (1/2, 1]$ for all $i \in \mathrm{V}$, then $\lim_{t\to\infty} \mathbf{x}_i(t) = 1$ for all $i \in \mathrm{V}$ with

$$\big|\mathbf{x}_i(t) - 1\big| \leq \left(1-\frac{\beta_*}{2}\right)^{\lfloor t/T\rfloor} \Big|\min_{j\in\mathrm{V}} \mathbf{x}_j(0) - 1\Big|,$$

where $\beta_* = \min_{k\in\mathrm{V}} \frac{c}{\mathrm{w}_{kk}+c}\Big[\big(\min_{j\in\mathrm{V}} \mathbf{x}_j(0)\big)^{b_k} - \big(1 - \min_{j\in\mathrm{V}} \mathbf{x}_j(0)\big)^{b_k}\Big] \in (0, 1]$.

Proof. (i). We continue to use the definition $\mathbf{y}(t) = \max_{i\in\mathrm{V}}\big\{\mathbf{x}_i(t)\big\}$. Furthermore, we define

$$\tilde{\mathbf{y}}(m) = \max_{mT\leq h\leq(m+1)T-1}\big\{\mathbf{y}(h)\big\} = \max_{\substack{mT\leq h\leq(m+1)T-1,\\ i\in\mathrm{V}}} \mathbf{x}_i(h),$$

where $m \in \mathbb{N}$. For all $i \in \mathrm{V}$, if $d_i(t) = s_i(t) = 0$, there holds $\mathbf{x}_i(t+1) = \mathbf{x}_i(t)$. When $d_i(t) > 0$ and $\mathbf{y}(t) < 1/2$, $\mathbf{x}_i(t+1) \leq \mathbf{y}(t)$ from (3). Therefore, from $\mathbf{y}(0) < 1/2$, we conclude that $\mathbf{y}(t)$ is monotonically decreasing and $\mathbf{y}(t) < 1/2$ for all $t \geq 0$. Then $\tilde{\mathbf{y}}(m) = \mathbf{y}(mT)$ holds and $\tilde{\mathbf{y}}(m)$ is monotonically decreasing.

We will prove that $\{\tilde{\mathbf{y}}(m)\}$ converge to zero and establish the convergence rate. Let $\tilde{t} \in [mT, \infty)$ where $m \in \mathbb{N}$ such that $d_i(\tilde{t}) \geq c > 0$. We see

$$\begin{aligned}&\mathbf{x}_i(\tilde{t}+1)\\ &\leq \mathbf{y}(\tilde{t}) - \frac{d_i(\tilde{t})}{\mathrm{w}_{ii}+d_i(\tilde{t})}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big]\mathbf{y}(\tilde{t})\big(1-\mathbf{y}(\tilde{t})\big).\end{aligned}$$

Furthermore, when $t \in [mT, \infty)$, in view of $\frac{d_i(t)}{\mathrm{w}_{ii}+d_i(t)}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big] \in (0, 1]$ and $\tilde{\mathbf{y}}(m) \geq \mathbf{y}(t)$, there holds

$$\begin{aligned}&\mathbf{y}(t) - \frac{d_i(t)}{\mathrm{w}_{ii}+d_i(t)}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big]\mathbf{y}(t)\big(1-\mathbf{y}(t)\big)\\ &\leq \tilde{\mathbf{y}}(m) - \frac{d_i(t)}{\mathrm{w}_{ii}+d_i(t)}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big]\\ &\times \tilde{\mathbf{y}}(m)\big(1-\tilde{\mathbf{y}}(m)\big).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathbf{x}_i(\tilde{t}+1) \leq& \tilde{\mathbf{y}}(m) - \frac{c}{\mathrm{w}_{ii}+c}\Big[\big(1-\mathbf{y}(0)\big)^{b_i} - \mathbf{y}^{b_i}(0)\Big]\\ &\times \tilde{\mathbf{y}}(m)\big(1-\tilde{\mathbf{y}}(m)\big).\end{aligned} \tag{6}$$

Because $\mathbf{x}_i(t+1) = \mathbf{x}_i(t)$ when $d_i(t) = 0$, we conclude that

$$\begin{aligned}\tilde{\mathbf{y}}(m+1) =& \max_{\substack{(m+1)T \leq h \leq (m+2)T-1,\\ i \in \mathrm{V}}} \mathbf{x}_i(h)\\ \leq& \max_{\substack{mT+1 \leq h \leq (m+2)T-1,\\ d_i(h)>0, i \in \mathrm{V}}} \mathbf{x}_i(h)\end{aligned} \tag{7}$$

for all $i \in \mathrm{V}$.

Introduce $\alpha_* = \min_{m \in \mathrm{V}} \frac{c}{\mathrm{w}_{mm}+c}\Big[\big(1-\mathbf{y}(0)\big)^{b_m} - \mathbf{y}^{b_m}(0)\Big]$. Obviously there holds $0 < \alpha_* \leq 1$. Due to (6) and (7), we thus have

$$\begin{aligned}\tilde{\mathbf{y}}(m+1) &\leq \tilde{\mathbf{y}}(m) - \alpha_* \tilde{\mathbf{y}}(m)\big(1-\tilde{\mathbf{y}}(m)\big)\\ &\leq (1-\alpha_*)\tilde{\mathbf{y}}(m) + \frac{\alpha_*}{2}\tilde{\mathbf{y}}(m) = \Big(1-\frac{\alpha_*}{2}\Big)\tilde{\mathbf{y}}(m).\end{aligned}$$

for all $m \in \mathbb{N}$. Therefore, for all $i \in \mathrm{V}$, $\tilde{\mathbf{y}}(m) \leq (1-\frac{\alpha_*}{2})^m \tilde{\mathbf{y}}(0)$. From the definition of $\tilde{\mathbf{y}}(m)$, we know

$$\mathbf{x}_i(t) \leq \Big(1-\frac{\alpha_*}{2}\Big)^{\lfloor t/T \rfloor} \max_{j \in \mathrm{V}} \mathbf{x}_j(0).$$

(ii). The statement follows from the same analysis as in the proof of (i). We thus have completed the proof. □

Proposition 2 weakens the conditions of Theorem 4 obviously. In the first place, the mutual influences of two persons can be different. Secondly, each person does not have to be affected by others at every time but during bounded time intervals. Therefore it can conform to more situations in reality and allow an extended scope of applicability of the obtained results.

5. Numerical examples

In this section, we will provide three examples to not only show the visual impressions of the results we proved before, but also go deeper into the nonlinearity of our opinion dynamics model with bias assimilation. First, we provide two sets of phase portraits of system (1) taking over complete graph and star graph with different bias parameter b.

Example 2. Suppose system (1) takes over a 3-node network, in which $w_{ij} = 1$ for all $\{i, j\} \in \mathrm{E}$ and $w_{ii} = 1, b_i = b \geq 0$ for all $i \in \mathrm{V}$. Now we can represent the evolutions of the nodes' opinions by phase portraits. In Fig. 2, the phase portraits are displayed for complete graph and star graph, respectively, under different values of b. It is evident that the node states tend to converge to an interior equilibrium when b is small, and diverge to the boundaries when b grows large. This result matches the social phenomenon where biased assimilation is more likely to bring out polarizations in Miller, Mchoskey, Bane, and Dowd (1993) and Taber and Lodge (2006) well.

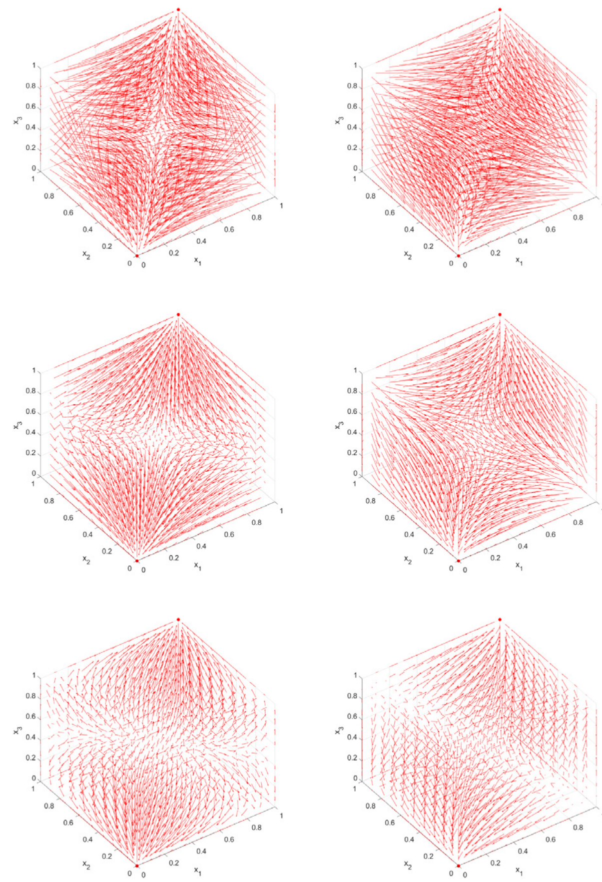

Fig. 2. Phase portraits when system (1) takes over complete graph (left) and star graph (right) with $b = \frac{1}{2}$, 1 and 2 in Example 2.

Next, we present an example illustrating equilibria stabilities.

Example 3. Consider a set of 12-node random geometric graphs with $r = r_0, 2r_0, \ldots, 10r_0$ where $r_0 = 0.02$. Then for each of r, we take 10 000 samples, that is, 12 nodes are uniformly and independently placed in $[0, 1]^2$ and any two nodes are connected by an edge if and only if their distance is smaller than r in each sample. For each realization, let q_r be the ratio of stable ones among equilibria $(1/2, 1/2, \ldots, 1/2)^\top$. The numerical mean of q_r based on the 10 000 samples for different r is shown in Fig. 3. From the plot, it is clear that higher connectivity weakens stability in our model.

Finally, we provide the following example to show the polarization phenomena in Theorem 4, where the entire network dynamics will be pushed to the boundary of the opinion space if $\max_{i \in \mathrm{V}} \mathbf{x}_i(t) < 1/2$.

Example 4. Let $\mathrm{V} = \{1, 2, \ldots, 50\}$, $\mathrm{E}_k = \{\{i, j\} | i - j \equiv -k, \ldots, k (mod\ 50), i \neq j\}$ and $\mathrm{G}_k = (\mathrm{V}, \mathrm{E}_k)$ for $k = 1, 2, 3, 4, 5$. Besides, $w_{ii} = 1, b_i = 0.2$ for all $i \in \mathrm{V}$ and $w_{ij} = 1$ for all $\{i, j\} \in \mathrm{E}_k$, $k = 1, 2, 3, 4, 5$. We use $\mathbf{e}(t) = \sum_{i=1}^n \mathbf{x}_i^2(t)$ to denote the deviation degree of $(0, \ldots, 0)^\top$ at time t. For all k, suppose

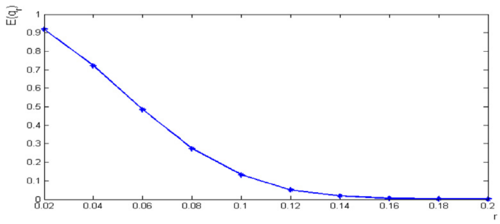

Fig. 3. Stability ratio of different r in Example 3.

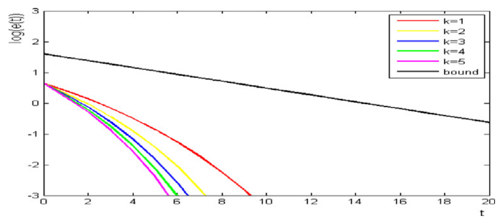

Fig. 4. $\log(\mathbf{e}(t))$ of $k = 1, 2, 3, 4, 5$ and the converge rate bound when $k = 5$ in Example 4.

$\mathbf{x}_i(0) = i/150, i \in \mathrm{V}$. Thus $e(0) = 1.908$ for $k = 1, \ldots, 5$. The evolutions of $\log(\mathbf{e}(t))$ when system (1) takes over different G_k are shown in Fig. 4. We also display the converge upper bound established in Theorem 4 when $k = 5$. We see from the result that when all the initial opinions are polarized ($\mathbf{x}_i(0) \in [0, 1/2)$ for all $i \in \mathrm{V}$), such polarization will be amplified persistently and opinions of all node converge to the corresponding boundaries. Moreover, the convergence rate to such polarization is at least faster than an exponential decay, as a validation to Theorem 4. This numerical result is consistent with Theorem 4.

6. Conclusions

We have provided a systemic analysis to social opinion dynamics subject to individual biases, which generated state-dependent edge weights and therefore highly nonlinear network dynamics. For a few fundamental network structures, some important interior network equilibria were presented for a wide range of system parameter in terms of their positions and stabilities, where the interval centroid was proven to be unstable regardless of the bias level and the network topologies. Furthermore, it was shown that when the initial network opinions are polarized towards one side of the state space, node biases would drive the opinion evolution to the corresponding interval boundaries under quite general network conditions. Future work includes studies of the distribution and stability of equilibria under more general network structures, especially those that are resilient subject to network structure switches as such structure change is common for real-world social networks.

Appendix A. Lemma

Before proving the statements in Section 3, we need the following lemma. We define the invariance potential function at first.

Definition 2. Let Assumption 1 hold. The invariance potential function of $\mathbf{x}_i(t) \in (0, 1)$ is defined as

$$\mathbf{s}^*\big(\mathbf{x}_i(t), b\big) = \frac{\big(1 - \mathbf{x}_i(t)\big)^{b-1}}{\mathbf{x}_i^{b-1}(t) + \big(1 - \mathbf{x}_i(t)\big)^{b-1}}.$$

This function helps us to provide an expression of the necessary and sufficient condition where one node's opinion remains unchanged. Now we present the following key technical lemma indicating the role of the invariance potential function concretely.

Lemma 1. *Suppose that $\mathbf{x}_i(t) \in (0, 1), i \in \mathrm{V}$, then under Assumption 1, the following statements hold:*

(i) $\mathbf{x}_i(t+1) = \mathbf{x}_i(t)$ if and only if $\mathbf{s}_i(t)/d_i = \mathbf{s}^*\big(\mathbf{x}_i(t), b\big)$;
(ii) $\mathbf{x}_i(t+1) > \mathbf{x}_i(t)$ if $\mathbf{s}_i(t)/d_i > \mathbf{s}^*\big(\mathbf{x}_i(t), b\big)$;
(iii) $\mathbf{x}_i(t+1) < \mathbf{x}_i(t)$ if $\mathbf{s}_i(t)/d_i < \mathbf{s}^*\big(\mathbf{x}_i(t), b\big)$.

Proof. (i). Since $\mathbf{x}_i(t) \in (0, 1)$, there hold $\big(1-\mathbf{x}_i(t)\big) \in (0, 1)$, $\mathbf{x}_i^b(t) > 0$ and $\big(1 - \mathbf{x}_i(t)\big)^b > 0$. As a result,

$$\begin{aligned}
& \mathbf{x}_i(t) = \mathbf{x}_i(t+1) \\
\iff\ & \mathbf{x}_i(t) = \frac{w\mathbf{x}_i(t) + \mathbf{x}_i^b(t)\mathbf{s}_i(t)}{w + \mathbf{x}_i^b(t)\mathbf{s}_i(t) + \big(1 - \mathbf{x}_i(t)\big)^b\big(d_i - \mathbf{s}_i(t)\big)} \\
\iff\ & \Big[\mathbf{x}_i^b(t)\big(1 - \mathbf{x}_i(t)\big) + \mathbf{x}_i(t)\big(1 - \mathbf{x}_i(t)\big)^b\Big]\mathbf{s}_i(t) \\
& = \mathbf{x}_i(t)\big(1 - \mathbf{x}_i(t)\big)^b d_i \\
\iff\ & \frac{\mathbf{s}_i(t)}{d_i} = \frac{\big(1 - \mathbf{x}_i(t)\big)^{b-1}}{\mathbf{x}_i^{b-1}(t) + \big(1 - \mathbf{x}_i(t)\big)^{b-1}} = \mathbf{s}^*\big(\mathbf{x}_i(t), b\big).
\end{aligned}$$

This proves (i).

(ii). We calculate the partial derivative of $\mathbf{x}_i(t+1)$ in system (1) and obtain that

$$\frac{\partial \mathbf{x}_i(t+1)}{\partial \mathbf{s}_i(t)} > 0, \tag{A.1}$$

when $\mathbf{x}_i(t) \in (0, 1)$. Due to (i) and (A.1), we obtain when $\mathbf{s}_i(t) > \mathbf{s}^*\big(\mathbf{x}_i(t), b\big)d_i$, $\mathbf{x}_i(t+1) > \mathbf{x}_i(t)$ holds.

(iii). the statement follows from the same analysis as in the proof of (ii). The desired lemma thus holds. □

Lemma 1 provides us with an explicit condition which can specify an equilibrium effectively. This is very important in the following proofs.

Appendix B. Proof of Proposition 1

When $\mathbf{x}_i(t) = 1/2$ for all $i = 1, 2, \ldots, n$, we know that $\mathbf{s}_i(t) = d_i/2$ for all $i \in \mathrm{V}$. Thus,

$$\mathbf{x}_i(t+1) = \frac{w/2 + (1/2)^b d_i/2}{w + (1/2)^b d_i/2 + (1/2)^b d_i/2} = 1/2$$

for all $i \in \mathrm{V}$. Therefore, we have proved that $\mathrm{X} = (1/2, 1/2, \ldots, 1/2)^\top$ is an equilibrium.

Next, we show that $\mathrm{X} = (1/2, 1/2, \ldots, 1/2)^\top$ is unstable. Let $\mathbf{X}(0) = (1/2 - \theta, 1/2 - \theta, \ldots, 1/2 - \theta)^\top$ where $\theta \in (0, 1/2)$. From Theorem 4, there holds $\lim_{t\to\infty} \mathbf{X}(t) = (0, 0, \ldots, 0)^\top$. It is clear from this point $(1/2, 1/2, \ldots, 1/2)^\top$ cannot be a stable equilibrium. We have proved the desired result. □

Appendix C. Proof of Theorem 1

From the definitions of complete graph and Lemma 1, when $(\mathrm{x}_1, \ldots, \mathrm{x}_n)^\top$ is an equilibrium point, there hold

$$\frac{\sum_{k=1,k\neq i}^n \mathrm{x}_k}{n-1} = \mathbf{s}^*(\mathrm{x}_i, b) = \frac{(1-\mathrm{x}_i)^{b-1}}{\mathrm{x}_i^{b-1} + (1-\mathrm{x}_i)^{b-1}} \tag{C.1}$$

and

$$\sum_{k=1}^n \mathrm{x}_k = \sum_{k=1,k\neq i}^n \mathrm{x}_k + \mathrm{x}_i = \frac{(n-1)(1-\mathrm{x}_i)^{b-1}}{\mathrm{x}_i^{b-1} + (1-\mathrm{x}_i)^{b-1}} + \mathrm{x}_i$$

for all $i \in \mathrm{V}$. Let

$$g_b(x) = \frac{(n-1)(1-x)^{b-1}}{x^{b-1}+(1-x)^{b-1}} + x$$

for $x \in (0,1)$. This immediately gives us that $\sum_{k=1}^{n} \mathrm{x}_k = g_b(\mathrm{x}_i)$ for all $i \in \mathrm{V}$, and

$$\frac{d}{dx}g_b(x) = 1 - \frac{(n-1)(b-1)(1-x)^{b-2}x^{b-2}}{\left[x^{b-1}+(1-x)^{b-1}\right]^2}.$$

When $x \in (0,1)$ and $b \le 1$, we conclude that $\frac{d}{dx}g_b(x) > 0$ and thus $g_b(x)$ is monotonic. Consequently, in view of $g_b(\mathrm{x}_i) = \sum_{k=1}^{n} \mathrm{x}_k = g_b(\mathrm{x}_j)$, there holds that $\mathrm{x}_i = \mathrm{x}_j$ for all $i, j \in \mathrm{V}$ when $\mathrm{X} \in \mathbb{E}_{\mathrm{int}}$. Hence, we can assume $\mathrm{x}_i = \tilde{x}$ for all $i \in \mathrm{V}$. According to (C.1) we know

$$\tilde{x} = \frac{(1-\tilde{x})^{b-1}}{\tilde{x}^{b-1}+(1-\tilde{x})^{b-1}},$$

which implies $\tilde{x} = 1/2$. We have now obtained that if $b \le 1$, the only interior equilibrium is $(1/2, 1/2, \ldots, 1/2)^\top$. When $x \in (0,1)$ and $b = 2$, we have $\frac{d}{dx}g_b(x) < 0$. The fact that $(1/2, 1/2, \ldots, 1/2)^\top$ is the unique equilibrium can be established using a similar analysis. Finally, the instability can be deduced from Proposition 1 directly. Now we have completed the proof. □

Appendix D. Proof of Theorem 2

According to the definition of star graph and Lemma 1, when $(\mathrm{x}_1, \ldots, \mathrm{x}_n)^\top$ is an equilibrium point, we obtain

$$\mathbf{s}^*(\mathrm{x}_1, b) = \frac{\sum_{i=2}^{n} \mathrm{x}_i}{n-1} \text{ and } \mathbf{s}^*(\mathrm{x}_i, b) = \mathrm{x}_1, i = 2, \ldots, n.$$

(i). When $b = 1$, for all $i \in \mathrm{V}\backslash\{1\}$, there holds

$$\mathrm{x}_i = \frac{w\mathrm{x}_i + \mathrm{x}_i\mathrm{x}_1}{w + \mathrm{x}_i\mathrm{x}_1 + (1-\mathrm{x}_i)(1-\mathrm{x}_1)}. \tag{D.1}$$

In view of $\mathrm{x}_i \neq 0$ or 1 for $i = 2, \ldots, n$, (D.1) immediately gives us $\mathrm{x}_1 = 1/2$. Besides, we have

$$\mathbf{s}^*(1/2, 1) = \frac{1}{2} = \frac{\sum_{i=2}^{n} \mathrm{x}_i}{n-1}. \tag{D.2}$$

From (D.2) it is easy to verify when $b = 1$, $\mathbb{E}_{\mathrm{int}} = \big\{(1/2, \mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_n)^\top : \sum_{i=2}^{n} \mathrm{x}_i = (n-1)/2, \mathrm{x}_i \in (0,1) \text{ for all } i \in \mathrm{V}\backslash\{1\}\big\}$.

Next, we prove the instability of any equilibrium $\mathrm{X} \in \mathbb{E}_{\mathrm{int}}$. For any equilibrium $\mathrm{X} = (1/2, \mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_n)^\top$ where $\sum_{i=2}^{n} \mathrm{x}_i = (n-1)/2$ and $\mathrm{x}_i \in (0,1)$ for all $i \in \mathrm{V} \setminus \{1\}$, let $\mathbf{x}_i(0) \in [0, \mathrm{x}_i)$ for all $i \in \mathrm{V}$. Then we will prove that $\mathbf{x}_i(t)$ is decreasing when for all $i \in \mathrm{V}$ there holds that $\mathbf{x}_i(t) \in [0, \mathrm{x}_i)$. We see

$$\begin{aligned}&\mathbf{x}_1(t+1) - \mathbf{x}_1(t)\\ =& \frac{\mathbf{x}_1(t)\big(1-\mathbf{x}_1(t)\big)\big(2\mathbf{s}_1(t) - d_1\big)}{w + \mathbf{x}_1(t)\mathbf{s}_1(t) + \big(1-\mathbf{x}_1(t)\big)\big(d_1 - \mathbf{s}_1(t)\big)} \le 0,\end{aligned} \tag{D.3}$$

where the inequality holds because $2\mathbf{s}_1(t) = 2\sum_{i=2}^{n} \mathbf{x}_i(t) < 2 \times \frac{n-1}{2} = d_1$. Similarly, for any $i \in \{2, 3, \ldots, n\}$, we obtain

$$\begin{aligned}&\mathbf{x}_i(t+1) - \mathbf{x}_i(t)\\ =& \frac{\mathbf{x}_i(t)\big(1-\mathbf{x}_i(t)\big)\big(2\mathbf{x}_1(t) - d_i\big)}{w + \mathbf{x}_i(t)\mathbf{x}_1(t) + \big(1-\mathbf{x}_i(t)\big)\big(d_i - \mathbf{x}_1(t)\big)} \le 0.\end{aligned} \tag{D.4}$$

Notice that $\mathbf{x}_i(0) \in [0, \mathrm{x}_i)$, $i \in \mathrm{V}$, we thus know $\mathbf{x}_i(t) \in [0, \mathrm{x}_i)$ is decreasing for all $i = 1, 2, \ldots, n$ and $t > 0$.

Let $\mathbf{z}(t) = \min_{i\in\mathrm{V}}\big\{\mathbf{x}_i(t)\big\}$. Then we have that $\mathbf{z}(t)$ is decreasing. From $\mathbf{z}(t) \ge 0$, there holds $\lim_{t\to\infty}\mathbf{z}(t) = \mathbf{z}^* \ge 0$. Next we will prove that $\mathbf{z}^* = 0$. We can conclude the following results easily

$$\mathbf{z}^* \le \mathbf{x}_i(t) \le \mathbf{x}_i(0) < \mathrm{x}_i \tag{D.5}$$

for all $i \in \mathrm{V}$. Due to (D.3) and (D.5), we obtain

$$\begin{aligned}&\big|\mathbf{x}_1(t+1) - \mathbf{x}_1(t)\big|\\ \ge& \frac{\mathbf{z}^*\big(1-\mathbf{x}_1(0)\big)\big((n-1) - \sum_2^n \mathbf{x}_i(0)\big)}{w + \frac{1}{2}\frac{n-1}{2} + (1-\mathbf{z}^*)^2(n-1)} = c_1\end{aligned}$$

where c_1 is a constant determined by w, $\mathbf{z}^*$ and $\mathbf{x}_i(0)$, $i \in \mathrm{V}$. When $\mathbf{z}^* > 0$, we have that $c_1 > 0$ and then $\lim_{t\to\infty}\mathbf{x}_1(t) = -\infty < 0$ monotonically. This immediately gives us a contradiction. Therefore, we know that $\mathbf{z}^* = 0$. It means that there exists $j \in \mathrm{V}$ such that $\lim_{t\to\infty}\mathbf{x}_j(t) = 0$. From the monotonic decrease of $\mathbf{x}_j(t)$, we know that when $\varepsilon = \frac{1}{2}\min_{i\in\mathrm{V}}\mathrm{x}_i > 0$, there exists $N \in \mathbb{N}$ such that $\mathbf{x}_j(t) < \varepsilon$ and $\big\|\mathbf{X}(t) - \mathrm{X}\big\| \ge \big|\mathbf{x}_j(t) - \mathrm{x}_j\big| > \frac{1}{2}\min_{i\in\mathrm{V}}\mathrm{x}_i = \varepsilon$ when $t > N$. For any small enough $\delta > 0$, let $\mathbf{x}_i(0) = \mathrm{x}_i - \frac{\delta}{n} \in [0, \mathrm{x}_i)$ and then $\big\|\mathbf{X}(0) - \mathrm{X}\big\| = \frac{\delta}{\sqrt{n}} < \delta$. But $\lim_{t\to\infty}\big\|\mathbf{X}(t) - \mathrm{X}\big\| > \varepsilon$. This immediately gives us that when $b = 1$ such equilibria are unstable from the definition of instability.

(ii). When $b \neq 1$, for all $i \in \mathrm{V}\backslash\{1\}$, there holds

$$\mathrm{x}_i = \frac{w\mathrm{x}_i + \mathrm{x}_i^b\mathrm{x}_1}{w + \mathrm{x}_i^b\mathrm{x}_1 + (1-\mathrm{x}_i)^b(1-\mathrm{x}_1)}.$$

Because $b \neq 1$ and $\mathrm{x}_i \in (0,1)$ for all $i \in \mathrm{V}$, we conclude that $w\mathrm{x}_i + \mathrm{x}_i^{b+1}\mathrm{x}_1 + \mathrm{x}_i(1-\mathrm{x}_i)^b(1-\mathrm{x}_1) = w\mathrm{x}_i + \mathrm{x}_i^b\mathrm{x}_1$, which implies

$$\mathrm{x}_i = \frac{1}{\big(\frac{\mathrm{x}_1}{1-\mathrm{x}_1}\big)^{1/(b-1)} + 1}$$

for all $i = 2, 3, \ldots, n$. Therefore, $\mathrm{x}_i = \mathrm{x}_2$ for all $i \in \mathrm{V}\backslash\{1\}$.

We now know that

$$\begin{aligned}\mathrm{x}_2 = \frac{(n-1)\mathrm{x}_2}{(n-1)} = \mathbf{s}^*(\mathrm{x}_1, b) &= \frac{(1-\mathrm{x}_1)^{b-1}}{(1-\mathrm{x}_1)^{b-1} + \mathrm{x}_1^{b-1}},\\ \mathrm{x}_1 = \mathbf{s}^*(\mathrm{x}_2, b) &= \frac{(1-\mathrm{x}_2)^{b-1}}{(1-\mathrm{x}_2)^{b-1} + \mathrm{x}_2^{b-1}}.\end{aligned}$$

Therefore, we obtain

$$\frac{\mathrm{x}_2}{1-\mathrm{x}_2} = \left(\frac{1-\mathrm{x}_1}{\mathrm{x}_1}\right)^{b-1}, \quad \frac{\mathrm{x}_1}{1-\mathrm{x}_1} = \left(\frac{1-\mathrm{x}_2}{\mathrm{x}_2}\right)^{b-1}. \tag{D.6}$$

If $b = 2$, it must be the case that $\mathrm{x}_1 + \mathrm{x}_2 = 1$ where $\mathrm{x}_1, \mathrm{x}_2 \in (0,1)$ from (D.6). Now, we can verify for $b = 2$ $\mathbb{E}_{\mathrm{int}} = \big\{(1-\mathrm{x}_2, \mathrm{x}_2, \ldots, \mathrm{x}_2)^\top : \mathrm{x}_2 \in (0,1)\big\}$.

Now, we prove the instability of any equilibrium $\mathrm{X} \in \mathbb{E}_{\mathrm{int}}$ when $b = 2$. For the equilibrium $\mathrm{X} = (1-a, a, \ldots, a)^\top$ where $a \in (0,1)$, let $\mathbf{x}_2(0) = \cdots = \mathbf{x}_n(0) \in [0, a)$ and $\mathbf{x}_1(0) \in [0, 1-a)$. We show that $\mathbf{x}_2(t)$ is decreasing when for all $i \in \mathrm{V}$ there holds that $\mathbf{x}_i(t) \in [0, \mathrm{x}_i)$. We have

$$\begin{aligned}&\mathbf{x}_2(t+1) - \mathbf{x}_2(t)\\ =& \frac{\mathbf{x}_2(t)\big(1-\mathbf{x}_2(t)\big)\Big[\mathbf{x}_2(t)\mathbf{x}_1(t) - \big(1-\mathbf{x}_2(t)\big)\big(1-\mathbf{x}_1(t)\big)\Big]}{w + \mathbf{x}_2^2(t)\mathbf{x}_1(t) + \big(1-\mathbf{x}_2(t)\big)^2\big(1-\mathbf{x}_1(t)\big)}\\ \le& 0,\end{aligned}$$

where the inequality holds because $0 < \mathbf{x}_2(t) < a < 1 - \mathbf{x}_1(t)$ and $0 < \mathbf{x}_1(t) < 1 - a < 1 - \mathbf{x}_2(t)$. Analogously, we have that for all $i \in \mathrm{V}$, $\mathbf{x}_i(t)$ is decreasing when $\mathbf{x}_2(0) = \cdots = \mathbf{x}_n(0) \in [0, a)$ and $\mathbf{x}_1(0) \in [0, 1-a)$. Then we can also prove that $\lim_{t\to\infty}\min_{i\in\mathrm{V}}\big\{\mathbf{x}_i(t)\big\} = 0$ and $\lim_{t\to\infty}\big\|\mathbf{X}(t) - \mathrm{X}\big\| > \frac{1}{2}\min\{a, 1-a\}$ for any $\mathbf{X}(0)$ satisfying that $\mathbf{x}_2(0) = \cdots = \mathbf{x}_n(0) \in [0, a)$ and $\mathbf{x}_1(0) \in [0, 1-a)$. We thus obtain that X is unstable similarly to the case in (i).

(iii). From (D.6), there holds

$$\frac{\mathrm{x}_1}{1-\mathrm{x}_1} = \left(\frac{\mathrm{x}_1}{1-\mathrm{x}_1}\right)^{(b-1)^2} \tag{D.7}$$

when $b \neq 1$. Recall that $b > 0$ and $b \neq 2$, we obtain $(b-1)^2 \neq 1$ and the solutions of Eq. (D.7) are given by $\frac{x_1}{1-x_1}$ be either 0 or 1. Due to $x_1 \in (0,1)$, this immediately gives us that $x_1 = 1/2$. We can therefore readily conclude that $x_2 = \cdots = x_n = 1/2$ from (D.6). Thus when $b \neq 1$ or 2, the only interior equilibrium is $(1/2, 1/2, \ldots, 1/2)^\top$. From Proposition 1, there holds that $(1/2, \ldots, 1/2)^\top$ is unstable.

We therefore have completed the proof. □

Appendix E. Proof of Theorem 3

In view of Lemma 1 and the definition of cycle graph, when $(x_1, \ldots, x_n)^\top$ is an equilibrium point, we have

$$\frac{x_{i-1} + x_{i+1}}{2} = \mathbf{s}^*(x_i, b) = \frac{(1-x_i)^{b-1}}{(1-x_i)^{b-1} + x_i^{b-1}}, \tag{E.1}$$

for all $i \in V$.

(i). When $b = 1$ and $n \equiv 1(mod\ 4)$, according to (E.1), we know

$$x_i + x_{i+2} = 1 \text{ and } x_{i+2} + x_{i+4} = 1, \tag{E.2}$$

for all $i \in V$. Therefore, we obtain that $x_i = x_{i+4k}$ for all $i \in V$ and $k \in \mathbb{Z}$. Due to $n \equiv 1(mod\ 4)$, let $n = 4m+1$ where $m \in \mathbb{N}$. Then we have that $x_i = x_{i+4m} = x_{i-1}$ for all $i \in V$. This gives us that $x_1 = x_2 = \cdots = x_n$. From (E.2), $x_i = 1/2$ holds for all $i \in V$. The only equilibrium is $(1/2, 1/2, \ldots, 1/2)^\top$ when $b = 1$ and $n \equiv 1(mod\ 4)$. If $n = 4m+2$ or $n = 4m+3$, the results can be obtained analogously. From Proposition 1, there holds that $(1/2, \ldots, 1/2)^\top$ is unstable.

(ii). When $b = 1$ and $n \equiv 0(mod\ 4)$, let

$$x_i = a_1, i \equiv 0(mod\ 4); x_i = a_2, i \equiv 1(mod\ 4);$$
$$x_i = a_3, i \equiv 2(mod\ 4); x_i = a_4, i \equiv 3(mod\ 4)$$

where $a_1, a_2, a_3, a_4 \in (0,1)$ for all $i \in V$. Noting (E.2), there hold $a_1 + a_3 = 1$ and $a_2 + a_4 = 1$. Now, we can verify when $b = 1$ and $n \equiv 0(mod\ 4)$ all interior equilibria are $X = (a_1, a_2, 1-a_1, 1-a_2, a_1, \ldots, 1-a_1, 1-a_2)^\top$ where $a_1, a_2 \in (0,1)$.

Next, we prove that when $b = 1$ and $n \equiv 0(mod\ 4)$, the equilibrium $X = (a_1, a_2, 1-a_1, 1-a_2, \ldots, 1-a_2)^\top$ where $a_1, a_2 \in (0,1)$ is unstable. Suppose $\mathbf{x}_{i+4k}(0) = \mathbf{x}_i(0)$ for all $k \in \mathbb{N}$ and $i = 1,2,3,4$. Besides, let $\mathbf{x}_1(0) < a_1, \mathbf{x}_2(0) < a_2, \mathbf{x}_3(0) < 1-a_1, \mathbf{x}_4(0) < 1-a_2$. We will prove that $\mathbf{x}_i(t)$ is decreasing for all $i = 1,2,3$ and 4. We first show that $\mathbf{x}_1(t)$ is decreasing when $\mathbf{x}_i(t) \in [0, x_i)$ for all $i \in V$. There holds

$$\begin{aligned}&\mathbf{x}_1(t+1) - \mathbf{x}_1(t)\\ &= \frac{2\mathbf{x}_1(t)\big(1-\mathbf{x}_1(t)\big)\big(\mathbf{s}_1(t)-1\big)}{w + \mathbf{x}_1(t)\mathbf{s}_1(t) + \big(1-\mathbf{x}_1(t)\big)\big(2-\mathbf{s}_1(t)\big)} \le 0,\end{aligned}$$

where the inequality holds because $\mathbf{s}_1(t) = \mathbf{x}_2(t) + \mathbf{x}_n(t) < a_2 + 1 - a_2 < 1$. Analogously, $\mathbf{x}_i(t), i = 2,3,4$ are decreasing when $\mathbf{x}_i(t) \in [0, x_i)$ for all $i \in V$. Therefore, we obtain that $\mathbf{x}_i(t)$ is decreasing when $\mathbf{x}_i(0) = \mathbf{x}_{i+4k}(0) \in [0, x_i), i = 1,2,3,4, k \in \mathbb{N}$. Then we can also prove that $\lim_{t\to\infty} \min_{i\in V}\{\mathbf{x}_i(t)\} = 0$ and $\lim_{t\to\infty} \|\mathbf{X}(t) - X\| > \frac{1}{2}\min\{a_1, a_2, 1-a_1, 1-a_2\}$ for any $\mathbf{X}(0)$ satisfying that $\mathbf{x}_i(0) = \mathbf{x}_{i+4k}(0) \in [0, x_i), i = 1,2,3,4, k \in \mathbb{N}$. We thus obtain that $X = (a_1, a_2, 1-a_1, 1-a_2, \ldots, 1-a_2)^\top$ where $a_1, a_2 \in (0,1)$ is unstable similarly to the proof of Theorem 2. Therefore, we prove that such equilibria are unstable.

(iii). We now discuss the interior equilibria when $b = 2$. Let $y_i = x_i + x_{i+1}$ for all $i \in V$. From (E.1) and $b = 2$, we obtain $\frac{x_{i-1}+x_{i+1}}{2} = 1 - x_i$, for all $i \in V$. Therefore, there holds $y_i + y_{i+1} = 2$, for all $i \in V$. Accordingly $y_i = y_{i+2k}$ holds for all $i \in V$ and $k \in \mathbb{Z}$. When $n = 2m+1$ where $m \in \mathbb{N}$, we obtain that $y_i = y_{i+2m} = y_{i-1}$ for all $i \in V$. Therefore, we have that $y_i = 1$ for all $i \in V$. That is, $x_i + x_{i+1} = 1$, for all $i \in V$. Because $n = 2m+1$ where $m \in \mathbb{N}$, we see that $x_i = x_{i+2m} = x_{i-1}$ for all $i \in V$. This immediately gives us $x_1 = x_2 = \cdots = x_n = 1/2$. From Proposition 1, there holds that $(1/2, \ldots, 1/2)^\top$ is unstable.

(iv). When $b = 2$ and $n = 2m$ where $m \in \mathbb{N}$, let $y_i = \tilde{a}, i \equiv 0(mod\ 2)$ and $y_i = 2 - \tilde{a}, i \equiv 1(mod\ 2)$ where $\tilde{a} \in (0,2)$ for all $i \in V$. In view of the definition of y_i, we know that $\sum_{i=1}^n x_i = \sum_{i=1}^m y_{2i-1} = \sum_{i=1}^m y_{2i}$. Therefore, we get that $m\tilde{a} = m(2-\tilde{a})$ and $\tilde{a} = 1$. This tells us $x_i + x_{i+1} = 1$ for all $i \in V$. Because $n = 2m$, we know that $x_i = x_{i+2k}$ for all $i \in V$ and $k \in \mathbb{N}$. We thus conclude that $\mathbb{E}_{\text{int}} = \{(a, 1-a, a, 1-a, \ldots, a, 1-a)^\top : a \in (0,1)\}$ when $b = 2$ and $n \equiv 0(mod\ 2)$. The instability of $(a, 1-a, a, 1-a, \ldots, a, 1-a)^\top$ can be proved similarly to the statement (ii) in Theorem 2. This completes the proof. □

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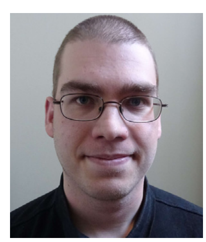

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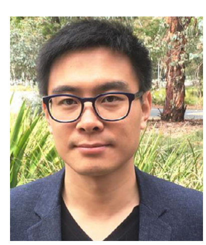

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