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DYNAMICS AND CONTROL OF SINGULAR BOOLEAN NETWORKS

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ABSTRACT

Consider a class of singular Boolean networks, which consist of two parts: difference part and algebraic part. Using the truth matrix of the algebraic part, the trajectories of the networks are obtained and certain properties are investigated. Then the results are extended to Boolean control networks and the controllability of singular Boolean control systems is investigated. Necessary and sufficient conditions are obtained.

Key Words: Singular Boolean (control) network, truth matrix, trajectory, controllability, mix-valued logical (control) network.

I. INTRODUCTION

Boolean networks were firstly introduced by Kauffman to describe the genetic regulatory networks [16, 17]. Since then, it has attracted much attention from biologists, physicians, and system scientists, because it has shown strong power on modeling and analyzing biological networks as well as social and circuit networks [1, 10].

Recently, a new research tend has risen, which is based on the algebraic state space representation (ASSR) of Boolean networks, using semi-tensor product (STP) [3]. Particularly, various control problems of Boolean networks have been investigated using ASSR. For instance, the controllability of Boolean control networks [2, 20]; the observability of Boolean networks [37, 21]; optimal control of Boolean control networks [22, 38, 12]; stability and stabilization [4, 24]; disturbance decoupling [5]; and some other applications such as observer design of finite automata [36], game theory [8, 33], and particularly industrial application in control of combustion engines [34, 35], just to name a few. The ASSR is also a basic tool for this paper.

When a Boolean network is used to describe a biological network or a social network, its dynamics is described as a difference equation. But, it happens that the nodes (or states) may be restricted by some algebraic constraints [29]. Hence, the difference-algebraic Boolean network (or singular Boolean network) becomes a theoretically interesting and practically meaningful topic. It has also been discussed widely [7, 11, 26, 27, 28].

It is well known that for a differential/difference-algebraic system in $\mathbb{R}^n$, a natural and convenient way to investigate it is to solve the algebraic equations to get the second part of variables as functions of the first part of variables. The existence of such a function is guaranteed by so called the Implicit Function Theorem. Then substituting them into the differential/difference equations transforms the system to a normal differential/difference system. If this solution-substitution process cannot be done, it is hard to get a closed-form solution and the numerical solutions are mostly adopted [13, 14]. Similarly, for difference-algebraic Boolean network, as the second part of variables can be solved the problem becomes easier. The necessary and sufficient conditions for the solvability of the second part variables of a Boolean or k-valued logical network were given by [7]. The corresponding conditions for a mix-valued logical network were presented in [30].

This paper is focused on a class of singular control Boolean networks, where the second part of variables

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may not be solvable from the algebraic equations. We start from the algebraic part, which is a static Boolean system as

$$\Sigma: \begin{cases} \varphi_1(\xi_1, \cdots, \xi_n) = c_1, \\ \vdots \\ \varphi_s(\xi_1, \cdots, \xi_n) = c_s, \end{cases} \tag{1}$$

where $\xi_i \in \mathcal{D} := \{0, 1\}$, $i = 1, \cdots, n$ are arguments; $c_i \in \mathcal{D}$, $i = 1, \cdots, s$ are constant parameters; $\varphi_i: \mathcal{D}^n \to \mathcal{D}$, $i = 1, \cdots, s$ are Boolean functions. Assume (X, Z) is a partition of $\Xi := \{\xi_1, \cdots, \xi_n\}$, precisely, $X = \{x_1, \cdots, x_p\} \subset \Xi$, $Z = \{z_1, \cdots, z_q\} \subset \Xi$, $X \cup Z = \Xi$ and $X \cap Z = \varnothing$. (Hence, $p + q = n$.)

The Ledley solution of system Σ (or (1)) was firstly introduced by Ledley [23], and discussed in detail in [19]:

Definition I.1 *Consider the static Boolean system Σ. Assume there exists a partition (X, Z) of the set of arguments $\Xi = \{\xi_1, \cdots, \xi_n\}$ such that z_i, $i = 1, \cdots, q$ are logical functions of x_i, $i = 1, \cdots, p$, denoted as*

$$\Gamma: \begin{cases} z_1 = f_1(x_1, \cdots, x_p), \\ \vdots \\ z_q = f_q(x_1, \cdots, x_p). \end{cases} \tag{2}$$

Moreover,

- *if $\Gamma \Rightarrow \Sigma$, then Γ is called an antecedence solution of Σ;*
- *if $\Sigma \Rightarrow \Gamma$, then Γ is called a consequence solution of Σ.*

The existence of the antecedence/consequence solutions has been discussed in detail in [30]. In this paper the antecedence solution will be used to transform a singular Boolean control network into a regular form. Then the latter is used to investigate some control problems of the original singular Boolean network.

The rest of this paper is organized as follows: Section 2 provides some necessary preliminaries, including (i) a brief review for semi-tensor product of matrices, which is one of the fundamental tools in our approach; (ii) truth matrix and the solution of static logical networks. In Section 3, we consider the trajectories of singular Boolean networks. Section 4 considers the controllability of singular Boolean networks, necessary and sufficient conditions are obtained. Section 5 is a brief conclusion.

II. PRELIMINARIES

2.1. Semi-tensor Product (STP) of Matrices

This section gives a brief review for STP, which is the fundamental tool in our analysis. The readers are referred to [3] or [6] for details.

First, we give some notations:

- $\mathbf{1}_n = [\underbrace{1, \cdots, 1}_{n}]^\top$.
- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of matrix M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the i-th column (row) of matrix M.
- δ_n^i: the i-th column of the identity matrix I_n.
- $\Delta_n := \left\{\delta_n^i | i = 1, \cdots, n\right\}$, $\Delta := \Delta_2$.
- A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(L) \subset \Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.
- If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}]$. For simplicity, it is briefly denoted as $L = \delta_n[i_1, i_2, \cdots, i_r]$.
- Let $B = (b_{i,j}) \in \mathcal{M}_{m\times n}$. If $b_{i,j} \in \mathcal{D}$, $\forall i, j$, then B is called a Boolean matrix. The set of $m \times n$ Boolean matrices is denoted by $\mathcal{B}_{m\times n}$. Particularly, $\mathcal{B}_m$ is the set of m-dimensional Boolean vectors.
- Let $B = (b_{i,j}) \in \mathcal{B}_{m\times n}$. The Hamming weight of B is defined as

$$w_H(B) := \sum_{i=1}^{m}\sum_{j=1}^{n} b_{i,j}.$$

Definition II.1 ([3]) *Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product (STP) of M and N, denoted by $M \ltimes N$, is defined as*

$$M \ltimes N := \left(M \otimes I_{\frac{t}{n}}\right)\left(N \otimes I_{\frac{t}{p}}\right) \in \mathcal{M}_{\frac{mt}{n}\times\frac{qt}{p}}, \tag{3}$$

where $\otimes$ is the Kronecker product.

When $n = p$, the STP coincides with the conventional matrix product. So the STP is a generalization of conventional matrix product. Fortunately, it keeps almost all the properties of the conventional matrix product unchanged. In addition, it has some new properties. The following properties are frequently used in the sequel.

Proposition II.2 ([3]) *Let $X \in \mathbb{R}^m$ be a column and M is an arbitrary matrix. Then*

$$X \ltimes M = (I_m \otimes M) X. \tag{4}$$

Definition II.3 ([18]) *Given $M \in \mathcal{M}_{m\times p}$, $N \in \mathcal{M}_{n\times p}$. The Khatri-Rao product of M and N is defined as*

$$\begin{aligned} M * N := [&\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N), \cdots, \\ &\mathrm{Col}_p(M) \ltimes \mathrm{Col}_p(N)] \in \mathcal{M}_{mn\times p}. \end{aligned} \tag{5}$$

Next, we consider the algebraic state space expression of logical dynamic systems.

Definition II.4 *(i) A function $f : \mathcal{D}^n \to \mathcal{D}$ is called a Boolean function. It can be expressed as*

$$y = f(x_1, x_2, \cdots, x_n), \quad y, x_1, \cdots, x_n \in \mathcal{D}. \tag{6}$$

(ii) A mapping $F : \mathcal{D}^n \to \mathcal{D}^m$ is called a Boolean mapping. A Boolean mapping F is composed of m Boolean functions, as

$$F : \begin{cases} y_1 = f_1(x_1, \cdots, x_n), \\ \vdots \\ y_m = f_m(x_1, \cdots, x_n). \end{cases} \tag{7}$$

Identifying

$$1 \sim \delta_2^1 = [1, 0]^\top, \quad 0 \sim \delta_2^2 = [0, 1]^\top,$$

and it is called the vector form of Boolean variable, then the Boolean function f becomes $f : \Delta^n \to \Delta$ and the Boolean mapping F becomes $F : \Delta^n \to \Delta^m$. In vector form we have the following algebraic form of expression.

Theorem II.5 ([6]) *Let $f : \mathcal{D}^n \to \mathcal{D}$ be a Boolean function. Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2\times 2^n}$, such that in vector form (6) can be expressed as*

$$f(x_1, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i, \tag{8}$$

where M_f is called the structure matrix of f.

Consider the Boolean mapping (7). According to Theorem II.5, there exist M_i, $i = 1, \cdots, m$ which are the structure matrices of the corresponding component functions. Then we have the following result.

Theorem II.6 ([6]) *Consider the Boolean mapping (7). In vector form, let $x = \ltimes_{i=1}^n x_i$, $y = \ltimes_{i=1}^m y_i$. Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2^m\times 2^n}$, such that (7) can be expressed as*

$$y = M_F x, \tag{9}$$

where

$$M_F = M_1 * M_2 * \cdots * M_m \tag{10}$$

is called the structure matrix of F.

The following propositions are useful in this paper.

Proposition II.7 *Let $X \in \Delta_m$ and $Y \in \Delta_n$. Then*

$$\left(I_n \otimes \mathbf{1}_n^\top\right) XY = X; \tag{11}$$

$$\left(\mathbf{1}_m^\top \otimes I_n\right) XY = Y. \tag{12}$$

Proof. We only prove (11), and it is similar for (12). Since for two vectors $X \ltimes Y = X \otimes Y$, we have

$$\left(I_n \otimes \mathbf{1}_n^\top\right) XY = (I_n X) \otimes (\mathbf{1}_n^\top Y) = X \otimes 1 = X,$$

where we use the property of Kronecker product: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, here A, B, C, D are of proper dimensions. □

Proposition II.8 *Define*

$$R_n^P := \mathrm{diag}\left(\delta_n^1, \delta_n^2, \cdots, \delta_n^n\right),$$

which is called the power-reducing matrix. Then for any $X \in \Delta_n$, we have

$$X^2 = R_n^P X. \tag{13}$$

2.2. Solution of Static Logical Systems

We start from investigating the solutions of a static Boolean system (1). First, we introduce the truth matrix. Motivated by [19], it was proposed and discussed in detail in [30].

Definition II.9 ([30]) *Consider the system (1) (or Σ). Let (X, Z) be a partition of the argument set $\Xi = \{\xi_1, \cdots, \xi_n\}$. $|X| = p$, $|Z| = q$, $p > 0$, $q > 0$, $p + q = n$. A matrix $T_\Sigma^{(X,Z)} = (t_{ij}) \in \mathcal{B}_{2^q\times 2^p}$ is called the truth matrix of system (1) (or of Σ) with respect to the partition (X, Z), if*

$$t_{i,j} = \begin{cases} 1, & x = \delta_{2^p}^j,\ z = \delta_{2^q}^i \text{ is a solution of (1)}, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

We give a simple example to depict this.

Example II.10 *Consider a Boolean system*

$$\begin{aligned} \Sigma : \varphi(\xi_1, \xi_2, \xi_3, \xi_4) = &(\xi_1 \wedge \xi_2 \wedge \xi_4) \\ &\vee (\xi_1 \wedge \neg\xi_2 \wedge \xi_3 \wedge \neg\xi_4) \vee (\neg\xi_1 \wedge \xi_2 \wedge \neg\xi_4) \\ &\vee (\neg\xi_1 \wedge \neg\xi_2 \wedge \xi_3 \wedge \xi_4) = 1. \end{aligned} \tag{15}$$

Assume the partition (X, Z) is given as $X = \{x_1 = \xi_1,\ x_2 = \xi_2\}$, $Z = \{z_1 = \xi_3,\ z_2 = \xi_4\}$. Denote $x =$

Table 1. Truth Matrix of (15)

$z\backslash x$	1, 1	1, 0	0, 1	0, 0
1, 1	1	0	0	1
1, 0	0	1	1	0
0, 1	1	0	0	0
0, 0	0	0	1	0

$x_1x_2 = \xi_1\xi_2$, $z = z_1z_2 = \xi_3\xi_4$. *Then it is easy to figure out the truth matrix shown in Table 1.*

Hence the truth matrix, denoted by $T_\Sigma^{(X,Z)} := T$, *is*

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{16}$$

Assume $A, B \in \mathcal{B}_{m\times n}$, $A = (a_{i,j})$ and $B = (b_{i,j})$. Then

$$A \le B \ (A < B) \Leftrightarrow a_{i,j} \le b_{i,j} \ (a_{i,j} < b_{i,j}), \ \forall i, j.$$

The main result in [30] is:

Theorem II.11 *Consider a static Boolean system (1) (or* Σ*) with partition* (X, Z) *as defined in Definition I.1.*

(i) Σ *with given partition has antecedence solution (2) (or* Γ*), if and only if there exists a logical matrix* $M \in \mathcal{L}_{2^q\times 2^p}$ *such that*

$$M \le T_\Sigma^{(X,Z)}. \tag{17}$$

(ii) If M *satisfies (17), then the corresponding antecedence solution (2) can be expressed in vector form as*

$$z = Mx. \tag{18}$$

Conversely, if (18) is an antecedence solution, then M *satisfies (17).*

(iii) Σ *with given partition has consequence solution (2) (or* Γ*), if and only if there exists a logical matrix* $M \in \mathcal{L}_{2^q\times 2^p}$ *such that*

$$T_\Sigma^{(X,Z)} \le M. \tag{19}$$

(iv) If M *satisfies (19), then the corresponding consequence solution (2) can be expressed in vector form as (18).*

(v) (1) is equivalent to (2), if and only if the truth matrix $T_\Sigma^{(X,Z)} := T$ *is a logical matrix. Moreover, the solution (2) is* $z = Tx$, *which is both antecedence and consequence solution of (1).*

Example II.12 *Recall Example II.10. It is clear that there are* 4 *logical matrices satisfying (17) as follows*

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence we have 4 *antecedence solutions of (15), which are*

$$z = M_i x, \quad i = 1, 2, 3, 4.$$

Using Proposition II.7, we have

Case 1:

$$\begin{aligned} z_1 &= \left(I_2 \otimes \mathbf{1}_2^\top\right) z = \left(I_2 \otimes \mathbf{1}_2^\top\right) M_1 x \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x. \end{aligned}$$

Similarly,

$$\begin{aligned} z_2 &= \left(\mathbf{1}_2^\top \otimes I_2\right) z = \left(\mathbf{1}_2^\top \otimes I_2\right) M_1 x \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} x. \end{aligned}$$

Back to logical form, we have

$$\begin{cases} z_1 = 1, \\ z_2 = x_1 \leftrightarrow x_2. \end{cases} \tag{20}$$

Case 2:

$$\begin{cases} z_1 = x_1 \vee [\neg(x_1 \vee x_2)], \\ z_2 = x_1 \leftrightarrow x_2. \end{cases} \tag{21}$$

Case 3:

$$\begin{cases} z_1 = \neg(x_1 \wedge x_2), \\ z_2 = x_1 \leftrightarrow x_2. \end{cases} \tag{22}$$

Case 4:

$$\begin{cases} z_1 = \neg x_2, \\ z_2 = x_1 \leftrightarrow x_2. \end{cases} \tag{23}$$

III. TRAJECTORIES OF SINGULAR LOGICAL NETWORKS

Consider a singular Boolean network:

$$\begin{cases} \xi_1(t+1) = f_1(\xi_1(t), \cdots, \xi_n(t)), \\ \vdots \\ \xi_p(t+1) = f_p(\xi_1(t), \cdots, \xi_n(t)); \\ \varphi_1(\xi_1(t), \cdots, \xi_n(t)) = c_1, \\ \vdots \\ \varphi_q(\xi_1(t), \cdots, \xi_n(t)) = c_q, \end{cases} \tag{24}$$

where $\xi_i(t) \in \mathcal{D}$, $i = 1, \cdots, n$ are state variables; $\varphi_j : \mathcal{D}^n \to \mathcal{D}$, $j = 1, \cdots, q$, are Boolean functions. We call the first part and the second part of (24) difference (D)-part and algebraic (A)-part respectively.

Assume the A-part of (24) has truth matrix with respect to the partition (X, Z) as $X = \{\xi_1, \cdots, \xi_p\}$, $Z = \{\xi_{p+1}, \cdots, \xi_n\}$ as $T^{(X,Z)}_{(24)\text{-A}} := T \in \mathcal{B}_{2^{n-p} \times 2^p}$. Using vector expression of ξ_i, and let $x_i(t) := \xi_i(t)$, $i = 1, \cdots, p$, $z_i(t) = \xi_{p+i}(t)$, $i = 1, \cdots, n-p$, $x(t) := \ltimes_{i=1}^p x_i$, $z(t) = \ltimes_{i=1}^{n-p} z_i(t)$, $\xi(t) = x(t)z(t)$, $c = \ltimes_{i=1}^q c_i$, then we can express (24) into its algebraic state space form as

$$\begin{cases} x(t+1) = L_{\mathrm{D}} x(t) z(t), \\ \Psi x(t) z(t) = c, \end{cases} \tag{25}$$

where L_{D} is the structure matrix of (24)-D.

Definition III.1 *Consider system (24) (equivalently, (25)). A sequence $\xi = \{\xi(t) = x(t)z(t) \mid t = 0, 1, \cdots\}$ is said to be a solution of (24) with initial value $\xi(0) = \xi_0$, where ξ_0 is pre-assigned, if the sequence $\{\xi(t)\}$ satisfies (24). Such a solution is also called a trajectory starting from ξ_0.*

Definition III.2 *Consider system (24). Let the successive state of $\xi(t)$ be denoted by $\xi(t+1) = x(t+1) \ltimes z(t+1)$, and assume $x(t+1) = \delta_{2^p}^\alpha$.*

(i) $x(t)$ is called a bifurcation point, if $w_H[\mathrm{Col}_\alpha(T)] > 1$;
(ii) $x(t)$ is called a stop point, if $w_H[\mathrm{Col}_\alpha(T)] = 0$;
(iii) $x(t)$ is called a regular point, if $w_H[\mathrm{Col}_\alpha(T)] = 1$.

Assume the truth matrix of (24)-A is T, then a partition can be obtained as

$$\Delta_{2^p} = \mathcal{S} \cup \mathcal{B} \cup \mathcal{R}.$$

It follows that if $x(t) \in \mathcal{S}$, it is a stop point; if $x(t) \in \mathcal{B}$, it is a bifurcation point; and if $x(t) \in \mathcal{R}$, it is a regular point.

We use a simple example to depict these three kinds of states.

Example III.3 *Consider the following network:*

$$\begin{cases} \xi_1(t+1) = \neg \xi_2(t), \\ \xi_2(t+1) = \xi_1(t) \wedge \xi_3(t), \\ [\xi_1(t) \wedge (\xi_2(t) \to \xi_3(t)] \vee \\ \quad [\xi_1(t) \leftrightarrow (\xi_2(t) \vee \xi_3(t))] = 1. \end{cases} \tag{26}$$

Consider the partition $(X, Z) = (\{\xi_1, \xi_2\}, \{\xi_3\})$. Then it is easy to figure out the truth matrix of (26)-A as:

Table 2. Truth Matrix of (26)-A

$z\backslash x$	1, 1	1, 0	0, 1	0, 0
1	1	1	0	0
0	1	1	0	1

From the definition and the truth matrix above, we have the predecessive points of $\delta_4^1 \sim (1,1)$ and $\delta_4^2 \sim (1,0)$ are bifurcation points, the predecessive point of $\delta_4^4 \sim (0,0)$ is a regular point, and the predecessive point of $\delta_4^3 \sim (0,1)$ is a stop point.

In the following, the trajectories of a singular logical network are investigated case by case.

3.1. Regular Points

Consider system (24) with its algebraic state space expression (25). Assume (24)-A has only regular points: That is, the truth matrix is a logical matrix. Then, its A-part is equivalent to

$$z = Tx, \tag{27}$$

where $T \in \mathcal{L}_{2^{n-p} \times 2^p}$. Substituting (27) into (25)-D yields

$$\begin{aligned} x(t+1) &= L_{\mathrm{D}} x(t) T x(t) = L_{\mathrm{D}} \left(I_{2^p} \otimes T \right) x^2(t) \\ &= L_{\mathrm{D}} \left(I_{2^p} \otimes T \right) R^P_{2^p} x(t) := M x(t). \end{aligned} \tag{28}$$

Then the trajectories are obtained as

$$\begin{cases} x(t) = M^t x(0), \\ z(t) = T M^t x(0), \quad t = 0, 1, \cdots. \end{cases} \tag{29}$$

We give an example to depict this.

Example III.4 *Consider the following network:*

$$\begin{cases} \xi_1(t+1) = \xi_2(t) \wedge \xi_3(t), \\ \xi_2(t+1) = \neg\xi_1(t) \vee \xi_3(t), \\ [\neg\xi_1(t) \vee (\xi_2(t) \leftrightarrow \xi_3(t))] \wedge [\xi_1(t) \vee \neg\xi_3(t)] = 0. \end{cases} \tag{30}$$

The partition is $(X, Z) = (\{\xi_1, \xi_2\}, \{\xi_3\})$. It is easy to figure out the truth matrix as

$$T = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the (30)-A is equivalent to $z = Tx$, which is expressed in logical form as

$$\xi_3(t) = \neg(\xi_1(t) \wedge \xi_2(t)).$$

Then (30)-D can be written as

$$\begin{cases} \xi_1(t+1) = \xi_2(t) \wedge \neg(\xi_1(t) \wedge \xi_2(t)), \\ \xi_2(t+1) = \neg\xi_1(t) \vee \neg(\xi_1(t) \wedge \xi_2(t)). \end{cases}$$

3.2. Regular and Stop Points

In this case we have to deal with stop points. We need the following new concepts.

Definition III.5 *Let $x = \ltimes_{i=1}^p x_i \in \Delta_{2^p}$, $\mathcal{E} \subset \Delta_{2^p}$, $z_j = f_j : \mathcal{E} \to \Delta_2$, $j = 1, \cdots, q$. z_j is called a restricted logical function of x. Setting $z := \ltimes_{j=1}^q z_j$, the algebraic expression of $z = F(x)$ is expressed as*

$$z = H^{\mathcal{E}} x, \tag{31}$$

where $H^{\mathcal{E}} \in \mathcal{M}_{2^q \times 2^p}$ is defined as follows: Assume $x = \delta_{2^p}^s$,

- *if $x \in \mathcal{E}$ and $z = \ltimes_{j=1}^q z_j(x) = \delta_{2^q}^\alpha$, then*
$$\mathrm{Col}_s(H^{\mathcal{E}}) := \delta_{2^q}^\alpha;$$
- *if $x \notin \mathcal{E}$, then*
$$\mathrm{Col}_s(H^{\mathcal{E}}) := \mathbf{0}_{2^q}.$$

If $L^{\mathcal{E}} \in \mathcal{M}_{m \times n}$ and $\mathrm{Col}(L^{\mathcal{E}}) \subset \Delta_m \cup \{\mathbf{0}_m\}$, then $L^{\mathcal{E}}$ is called a restricted logical matrix. The set of $m \times n$ dimensional restricted logical matrices is denoted by $\mathcal{L}^0_{m \times n}$.

Now, if a logical network is defined over $\mathcal{E} \subset \Delta_{2^p}$ only. Then it can be expressed as

$$x(t+1) = L^{\mathcal{E}} x(t), \tag{32}$$

where

$$\mathrm{Col}_j(L^{\mathcal{E}}) = \begin{cases} x(t+1), & x(t) = \delta_{2^p}^j \in \mathcal{E}, \\ \mathbf{0}_{2^p}, & x(t) = \delta_{2^p}^j \notin \mathcal{E}. \end{cases}$$

We give a new concept:

Definition III.6 *Consider (24). If $z = Tx$ implies (24)-A, where $T \in \mathcal{L}^0_{2^q \times 2^p} \backslash \mathcal{L}_{2^q \times 2^p}$, it is called a restricted antecedence solution (RAS) of (24)-A.*

The technique proposed for the case where the system has only regular points can be used to investigate the singular logical networks with stop points. Consider network (24) again, and it is clear that $z = Tx$ is an RAS of (24)-A, where $T \in \mathcal{L}^0_{2^{n-p} \times 2^p}$. Plugging it into (25)-D, a restricted dynamic network of $x(t)$ can be obtained. Note that $x(t) = \mathbf{0}_{2^p}$ implies that $x(t-1)$ corresponds to the successor of a stop point. That is, $z(t-1)$ does not exist. Therefore, $x(t-2)$ is a stop point.

We use an example to describe this.

Example III.7 *Consider the following system*

$$\begin{cases} x_1(t+1) = x_2(t), \\ x_2(t+1) = (x_1(t) \vee x_2(t)) \wedge z(t), \\ [x_1(t) \wedge (x_2(t) \leftrightarrow z(t))] \vee [\neg(x_1(t) \vee \\ \quad x_2(t) \vee z(t))] = 1. \end{cases} \tag{33}$$

It is easy to figure out that the truth matrix is:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

There are 3 regular points $\mathcal{R} := \{(1,1),\ (1,0),\ (0,0)\}$ and 1 stop point $\{(0,1)\}$. The algebraic form of (33)-D is

$$x(t+1) = \delta_4[1,2,3,4,1,2,4,4]x(t)z(t). \tag{34}$$

Plugging $z = Tx$ into (34) and using (28) yield

$$x(t+1) = L^{\mathcal{E}} x(t), \tag{35}$$

where $L^{\mathcal{E}} = \delta_4[1,4,0,4]$.

Note that in equation (32) it is reasonable to remove the stop points from the equation. Then we have the following S-reduced (stop point reduced) system as

$$x(t+1)|_{\mathcal{R}} = L^{\mathcal{R}} x(t)|_{\mathcal{R}}, \tag{36}$$

where $L^{\mathcal{R}}$ is obtained from $L^{\mathcal{E}}$ by deleting the rows and columns which correspond to stop points.

Example III.8 *Continue Example III.7. Since $\delta_4^3 \sim (0,1)$ is a stop point. We can delete the third row and third column of $L^{\mathcal{E}}$ in (35) to get the equation of regular points as*

$$\tilde{x}(t+1) = L^{\mathcal{R}}\tilde{x}(t), \tag{37}$$

where $L^{\mathcal{R}} = \delta_3[1,3,3]$. Note that in (37) $\tilde{x}(t) \in \Delta_3$ is the restriction $x(t)|_{\mathcal{R}}$. Precisely,

$$\begin{array}{lll} \tilde{x} = \delta_3^1 & \Leftrightarrow & (x_1,x_2,z) = (1,1,1) \\ \tilde{x} = \delta_3^2 & \Leftrightarrow & (x_1,x_2,z) = (1,0,0) \\ \tilde{x} = \delta_3^3 & \Leftrightarrow & (x_1,x_2,z) = (0,0,0). \end{array}$$

3.3. General Case

This subsection considers the general case, that is, in addition to regular and stop points, (24) has also bifurcation point(s).

Now assume the bifurcation happens at columns j_r, $r = 1,\cdots,\mu$. Precisely,

$$w_H(\mathrm{Col}_{j_r}(T)) = \ell_r > 1, \quad r = 1,\cdots,\mu.$$

Then we can split $\mathrm{Col}_{j_r}(T)$ into ℓ_r columns as

$$\mathrm{Col}_{j_r}(T) = \bigvee_{s=1}^{\ell_r} C_s^r,$$

where $C_s^r \in \Delta_{2^{n-p}}$. Then we can construct T as follows

$$T = \vee_{j_1=1}^{\ell_1} \vee_{j_2=1}^{\ell_2} \cdots \vee_{j_\mu=1}^{\ell_\mu} T_{i_1,i_2,\cdots,i_\mu}, \tag{38}$$

where each $T_{i_1,i_2,\cdots,i_\mu}$ has the same columns α of T if $\alpha \notin \{j_r \mid r = 1,\cdots,\mu\}$; and its j_r-th column is one of C_s^r, $r = 1,\cdots,\mu$. Note that each $T_{i_1,i_2,\cdots,i_\mu}$ is of the "regular and stop" type as discussed in previous section.

Using each $T_{i_1,i_2,\cdots,i_\mu}$, we can construct corresponding $L^{\mathcal{E}}_{i_1,i_2,\cdots,i_\mu}$ as what we did in previous section. Then we construct a matrix $L^T \in \mathcal{L}_{2^{n-p}\times \mu 2^{n-p}}$ as

$$\begin{aligned} L^T := [&L^{\mathcal{E}}_{1,1,\cdots,1}, L^{\mathcal{E}}_{1,1,\cdots,2}, \cdots, L^{\mathcal{E}}_{1,1,\cdots,\ell_\mu}, \\ &L^{\mathcal{E}}_{1,\cdots,2,1}, L^{\mathcal{E}}_{1,\cdots,2,2}, \cdots, L^{\mathcal{E}}_{1,\cdots,2,\ell_\mu}, \\ &\cdots, \\ &L^{\mathcal{E}}_{\ell_1,\cdots,\ell_{\mu-1},1}, \cdots, L^{\mathcal{E}}_{\ell_1,\cdots,\ell_{\mu-1},\ell_\mu}]. \end{aligned}$$

Using this L^T, we can build a control Boolean network as

$$\begin{array}{l} x(t+1) = L^T \ltimes_{i=1}^{\mu} u_i(t)x(t) \\ z(t) = Tx(t). \end{array} \tag{39}$$

From the construction one sees easily that

Proposition III.9 *Each controlled trajectory of (39) under the control sequence $\{u(t) \mid t = 0,1,\cdots\}$ corresponds to each RAS of (24).*

Proof. Assume a trajectory of (24) and a trajectory of (39) start from same initial state $(x(0), z(0))$. Consider a particular RAS of (24): when it goes through regular or stop point, the corresponding trajectory of (39) will go through the same point no matter what control is chosen. When it meets a bifurcation point, which is at column j_r of T, the RAS must choose one feasible direction to go. Assume the i-th 1 is chosen. Precisely, if the i-th 1 is at α-th position of the column, which means $z(t) = \delta_{2^{n-p}}^{\alpha}$. Then we can choose $u_r = \delta_{\ell_r}^i$, which forces $z(t) = \delta_{2^{n-p}}^{\alpha}$ too. That is, for each RAS, we can construct a proper control sequence $\{u(t) \mid t = 0,1,\cdots\}$ such that the controlled trajectory of (39) coincides with the RAS. Conversely, for each controlled trajectory, it is also easy to see that we can also find an RAS matching it. □

We also give an example to depict this.

Example III.10 *Consider the following system*

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = (x_1(t) \vee x_2(t)) \wedge z(t) \\ [x_1(t) \wedge (x_2(t) \wedge z(t))] \vee [x_1(t) \wedge \neg x_2(t)] \vee \\ \quad [\neg x_1(t) \wedge x_2(t) \wedge \neg z(t)] \vee [\neg x_1(t) \wedge \neg x_2(t)] = 1. \end{cases} \tag{40}$$

It is easy to figure out that the truth matrix is:

$$T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then the system can be expressed as a switched system, for $\tau(t) \in \{1,2,3,4\}$. As $\tau(t) = 1$ we have a system with the dynamic part as the original one and the algebraic part with

$$T_{1,1} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

that is,

$$z = x_1 \vee \neg x_2. \tag{41}$$

Plugging it into (40)-D, we have

$$\begin{cases} x_1(t+1) = x_2(t), \\ x_2(t+1) = x_1(t). \end{cases} \tag{42}$$

Similarly, for $\tau(t) = 2, 3, 4$, we have

$$T_{1,2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

then we have

$$z = x_1. \tag{43}$$

Plugging it into (40)-D, we have

$$\begin{cases} x_1(t+1) = x_2(t), \\ x_2(t+1) = (x_1(t) \vee x_2(t)) \wedge x_1(t); \end{cases} \tag{44}$$

$$T_{2,1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

then we have

$$z = x_1 \leftrightarrow x_2. \tag{45}$$

Plugging it into (40)-D, we have

$$\begin{cases} x_1(t+1) = x_2(t), \\ x_2(t+1) = (x_1(t) \vee x_2(t)) \wedge (x_1(t) \leftrightarrow x_2(t)); \end{cases} \tag{46}$$

$$T_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

then we have

$$z = x_1 \wedge x_2. \tag{47}$$

Plugging it into (40)-D, we have

$$\begin{cases} x_1(t+1) = x_2(t), \\ x_2(t+1) = (x_1(t) \vee x_2(t)) \wedge (x_1(t) \wedge x_2(t)); \end{cases} \tag{48}$$

According to Proposition III.9, each RAS of the system (40) corresponds to a trajectory of the following switched system:

$$\begin{cases} x(t+1) = f_\sigma(x(t)) \\ z(t+1) = g_\sigma(x(t+1)), \end{cases} \tag{49}$$

where $\sigma \in \{1, 2, 3, 4\}$, and f_1, f_2, f_3, f_4 are described as (42), (44), (46), and (48) respectively; and g_1, g_2, g_3, g_4 are described as (41), (43), (45), and (47) respectively.

IV. CONTROLLABILITY OF SINGULAR LOGICAL NETWORKS

A singular control logical network concerned in this paper is of the following form:

$$\begin{cases} \xi_1(t+1) = f_1(\xi_1(t), \cdots, \xi_n(t); u_1(t), \cdots, u_m(t)), \\ \vdots \\ \xi_p(t+1) = f_p(\xi_1(t), \cdots, \xi_n(t); u_1(t), \cdots, u_m(t)); \\ \varphi_1(\xi_1(t), \cdots, \xi_n(t)) = c_1, \\ \vdots \\ \varphi_q(\xi_1(t), \cdots, \xi_n)(t)) = c_q. \end{cases} \tag{50}$$

Since we are only concerning the controllability, the output of the system is ignored.

The algebraic state space representation of (50) is expressed as

$$\begin{cases} x(t+1) = Lu(t)x(t)z(t) \\ \Psi x(t)z(t) = c, \end{cases} \tag{51}$$

where $x(t) = \ltimes_{i=1}^{p} x_i(t) := \ltimes_{i=1}^{p} \xi_i(t)$, $z(t) = \ltimes_{i=1}^{n-p} z_i(t) := \ltimes_{j=p+1}^{n} \xi_j(t)$, $u(t) = \ltimes_{i=1}^{m} u_i(t)$.

A state ξ satisfying (50)-A is called a legal point. Denote the set of legal points by Ω.

Definition IV.1 *System (50) is*

1. *controllable from $p \in \Omega$ to $q \in \Omega$, if there is a finite control sequence $\{u(0), u(1), \cdots, u(T)\}$, such that the controlled RAS satisfies $\xi(0) = p$ and $\xi(T+1) = q$;*
2. *controllable at $p \in \Omega$, if for each $q \in \Omega$, there is a finite control sequence $\{u(0), u(1), \cdots, u(T)\}$, such that the controlled RAS satisfies $\xi(0) = p$ and $\xi(T+1) = q$;*
3. *controllable, if for any $p \in \Omega$ to $q \in \Omega$ there is a finite control sequence $\{u(0), u(1), \cdots, u(T)\}$, such that the controlled RAS satisfies $\xi(0) = p$ and $\xi(T+1) = q$;*

Assume (50)-A (equivalently, (51)-A) has only regular and stop points. Then we have a unique RAS

$$z(t) = Tx(t). \tag{52}$$

Plugging it into (51)-D yields

$$\begin{aligned} x(t+1) &= Lu(t)x(t)Tx(t) \\ &= Lu(t)(I_{2^p} \otimes T)x^2(t) \\ &= Lu(t)(I_{2^p} \otimes T)R_{2^p}^P x(t) \\ &= L(I_{2^{p+m}} \otimes T)(I_{2^m} \otimes R_{2^p}^P)u(t)x(t) \\ &:= \tilde{L}u(t)x(t) \end{aligned} \tag{53}$$

Note that in the algebraic deduction of (53) the following easily verified result has been used.

Proposition IV.2 *Assume the conventional matrix product of A and B is legal (that is, the column number of A equals the row number of B). Then for any two matrices C and D, we have*

$$(A \otimes C) \ltimes (B \otimes D) = (A \ltimes B) \otimes (C \ltimes D). \tag{54}$$

Split $\tilde{L}$ into 2^m equal blocks as

$$\tilde{L} = [\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_{2^m}],$$

where $\tilde{L}_i \in \mathcal{L}^0_{2^p \times 2^p}$, $\forall i$. Deleting their rows and columns, which are corresponding to $\mathcal{S}$ (set of stop points), we have $\tilde{L}_i^{\mathcal{R}}$, $\forall i$, which form $\tilde{L}^{\mathcal{R}}$. Then similar to (36), we have an S-reduced control system as

$$x(t+1)|_{\mathcal{R}} = \tilde{L}^{\mathcal{R}} u(t) x(t)|_{\mathcal{R}}. \tag{55}$$

In the following we use Boolean addition $\sum_{\mathcal{B}}$ (i.e., $1+1=1$) and Boolean power $A^{(s)}$, where $A \in \mathcal{B}_{n\times n}$ [19]. Define

$$M := \sum_{\mathcal{B}}{}_{i=1}^{2^m} \tilde{L}_i^{\mathcal{R}}, \tag{56}$$

and set

$$\mathcal{C} = \sum_{\mathcal{B}}{}_{j=1}^{r} M^{(j)}, \tag{57}$$

where $r = 2^p - |\mathcal{S}|$. Similar to regular case (refer to [3]), we can prove the following result:

Theorem IV.3 *Assume (50)-A has no bifurcation points, then*

(i) $(x_i, Tx_i) \in \Xi$ is controllable to $(x_j, Tx_j) \in \Xi$, if and only if $\mathcal{C}_{j,i} > 0$;
(ii) (50) is controllable at $(x_j, Tx_j) \in \Xi$, if and only if $\mathrm{Col}_j(\mathcal{C}) > 0$;
(iii) (50) is controllable, if and only if $\mathcal{C} > 0$.

Next, we consider the general case.

Since (50)-A is exactly the same as (24)-A, we have exactly the same decomposition of T as in (38). Next, we can use each $T_{i_1,i_2,\cdots,i_\mu}$ to replace the T of (50)-A, which turns (50) to be of previous type, i.e., its algebraic part has only regular and stop points. Using previous technique, we can construct $M_{r_1,r_2,\cdots,r_\mu}$ as in (56). Finally, an overall M can be defined as

$$M := \sum_{\mathcal{B}}{}_{r_1=1}^{\ell_1} \sum_{\mathcal{B}}{}_{r_2=1}^{\ell_2} \cdots \sum_{\mathcal{B}}{}_{r_\mu=1}^{\ell_\mu} M_{r_1,r_2,\cdots,r_\mu}. \tag{58}$$

Applying this M to (57), the controllability matrix $\mathcal{C}$ for general case is obtained. Arguing as in the proof of Proposition III.9, one sees easily that Theorem IV.3 remain true for general case.

Corollary IV.4 *Consider system (50). When M is the union of all possible $M_{r_1,r_2,\cdots,r_\mu}$ (as defined by (58)) and the controllability matrix $\mathcal{C}$ is constructed by this generalized M, the conclusions of Theorem IV.3 remain true for general case.*

Remark IV.5 *In fact, we can also convert the system into a standard control system where an additional control will be added. This control is used to "perform" the switch among different models. Eventually, the controllability of this extended system is exactly the same as what we did above.*

The following practical example is from [27] and initially proposed in [9, 15] for biochemical oscillator in the cell cycle.

Example IV.6 ([27]) *Consider a Boolean model for the biochemical oscillator in the cell cycle which includes cyclins (denoted by x_1), cyclin-dependent kinases (cdks) (x_2), and ligases (x_3). In addition, another cyclins (denoted by u), which may affect the cell cycle, is considered as the input. Thus we have the following singular Boolean control system:*

$$\begin{cases} x_1(t+1) = \neg x_3(t) \vee \neg u(t), \\ x_2(t+1) = x_1(t), \\ 1 = x_2(t) \leftrightarrow x_3(t). \end{cases} \tag{59}$$

The partition is $(X, Z) = (\{x_1, x_2\}, \{x_3\})$. The truth matrix can be figured out from the (59)-A part as

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

which is a logical matrix. Hence (59)-A part is equivalent to $x_3 = Tx_1x_2 = x_2$. Then (59) can be rewritten as

$$\begin{cases} x_1(t+1) = \neg x_2(t) \vee \neg u(t), \\ x_2(t+1) = x_1(t), \end{cases} \tag{60}$$

which is a classical Boolean control network. Its controllability matrix is obtained as

$$\mathcal{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} > 0.$$

Therefore the system (59) is controllable.

The following example gives a more general case.

Example IV.7 *Consider a control logical network as*

$$\begin{cases} x_1(t+1) = z(t), \\ x_2(t+1) = x_1(t) \wedge u(t), \\ [x_1(t) \wedge (x_2(t) \wedge z(t))] \vee [x_1(t) \wedge \neg x_2(t)] \vee \\ \quad [\neg x_1(t) \wedge x_2(t) \wedge \neg z(t)] \wedge [\neg x_1(t) \wedge \neg x_2(t)] = 1. \end{cases} \tag{61}$$

Since the algebraic part of (61) is exactly the same as (40)-A, we can use the decomposition of Example III.10 directly. Skipping the tedious computation, we can calculate the controllability matrix as

$$\mathcal{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus, (61) is not overall controllable. But it is controllable at (x_0, Tx_0), where $x_0 = \delta_4^2$, $T \in \{T_{1,1},\ T_{1,2},\ T_{2,1},\ T_{2,2}\}$ could be any one. Back to the logical value, we have that the system is controllable at $(1,0,1)$ as well as $(1,0,0)$.

V. CONCLUSION

The controllability of difference-algebraic logical control systems is considered in this paper. It is well known that for a differential-algebraic control system, the variables involved in algebraic part, in general, are not able to be solved out from algebraic equations. This paper reveals that for a difference-algebraic logical control system, the variables involved in algebraic part are always "solvable" from the algebraic equations in the sense that the solution could be empty or multiple depending on individual points. This property makes the investigation of its controllability always possible. First, the dynamics and controllability of systems, which have only regular and stop points are considered. Necessary and sufficient conditions are obtained. Then the results are extended to general case, where the bifurcation points exist.

It is worth noting that the technique proposed in this paper can be applied to mix-valued difference-algebraic logical control systems immediately, and all the results in this paper can be extended to this class of general logical systems with some obvious modification.

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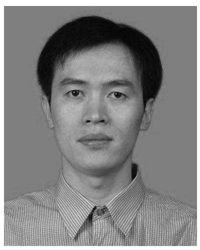

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