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Brief paper

Strongly consistent coefficient estimate for errors-in-variables models ☆

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Abstract

For the single-input-single-output (SISO) errors-in-variables system it is assumed that the system input is an ARMA process and that the driven noise of the system input and the observation noise are jointly Gaussian. The two-dimensional observation made on system input and output is represented as a two-dimensional (2D) ARMA system of minimum phase driven by a sequence of 2D i.i.d. Gaussian random vectors (innovation representation). It is shown that the resulting ARMA system is identifiable, i.e., its coefficients are uniquely defined under reasonable conditions. Recursive algorithms are proposed for estimating coefficients of the ARMA representation including those contained in the original SISO system. The estimates are proved to be convergent to the true values with probability one and the convergence rate is derived as well. The theoretical results are justified by numerical simulation.

Keywords: Errors-in-variables; ARMA; Identifiability; Strong consistency; Convergence rate

1. Introduction

We consider the problem of identifying a linear single-input-single-output (SISO) system described by the difference equation

$$A(z)y_k^0 = B(z)u_k^0, \tag{1}$$

where $A(z)$ and $B(z)$ are unknown polynomials and z denotes the backward-shift operator $zy_k = y_{k-1}$.

System (1) will later be referred as the original SISO system.

The measurements u_k and y_k of the system input u_k^0 and output y_k^0 are corrupted by noises η_k and ξ_k, respectively:

$$y_k = y_k^0 + \xi_k, \quad u_k = u_k^0 + \eta_k. \tag{2}$$

Estimation of parameters of $A(z)$ and $B(z)$ from observed data $\{y_k\}$ and $\{u_k\}$ is called the "errors-in-variables" problem.

It is well-known (Anderson, 1985; Anderson & Deistler, 1984; Deistler, 1986; Scherrer & Deistler, 1998; Stoica & Nehorai, 1987) that there does not exist a unique solution in general, if only second-order statistics are exploited. However, if some additional assumptions are imposed, for example, if high order cumulant statistics can be used, then it is, in principle, possible to achieve consistent estimates (Nikias & Pan, 1988; Scherrer & Deistler, 1998; Tugnait, 1992).

By assuming the input is non-Gaussian and the noises are Gaussian in Tugnait (1992), the square root of the magnitude of the fourth cumulant of a generalized error signal is taken as a performance criterion for parameter estimation, and the global minimizer of the proposed criterion $\sqrt{J_N(\theta)}$ is proved to be strongly consistent as $N \to \infty$. For a fixed N, a numerical algorithm is also proposed in Tugnait (1992) to search the minimizer of $\sqrt{J_N(\theta)}$. But, it is not clear how to guarantee the algorithm to converge to the desired global minimizer. Besides, it would be of interest to recursively estimate unknown parameters with increasing data size N.

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There have been developed many interesting numerical identification algorithms by using various methods, e.g., Mahata and Söderström (2002), Stoica, Cedervall, and Eriksson (1995), Söderström, Mahata, and Soverini (2002) among others. By using the innovation representation of the observed data, the original problem is reduced to identifying the resulting 2D process. Both parametric and non-parametric identification methods are proposed in Söderström et al. (2002), but no consistency is guaranteed. A survey of different approaches is given in Söderström, Soverini, and Mahata (2001).

Sharing the idea of 2D approach proposed in Söderström et al. (2002), in this paper we show that the resulting representation is identifiable under reasonable conditions. Then, the recursive identification algorithms are proposed for estimating matrix coefficients appearing in the representation, and at same time the estimates for coefficients of the original SISO system are derived too. Conditions guaranteeing strong consistency (convergence with probability one) of the estimates are given, and the convergence rate is obtained as well. As a result, all coefficients not only in the original SISO system but also in the innovation representation of the 2D observation process are asymptotically achieved.

The rest of the paper is organized as follows. The basic assumptions on the original SISO system and the 2D ARMA representation of the observation process are given in Section 2. Identifiability of the ARMA representation is established in Section 3. Recursive algorithms are given and their strong consistency is proved in Section 4. To justify theoretical assertions some numerical simulation results are demonstrated in Section 5. After concluding remarks an appendix is given to present some results we refer to in order to ease reading.

2. Representation of observation process

The objective of the paper is to design a recursive algorithm based on the noise-corrupted observations $\{u_k\}$ and $\{y_k\}$ to consistently estimate coefficients of $A(z)$ and $B(z)$.

We first list conditions to be imposed on the system, input, and observation noises.

A1. Polynomials $A(z)=1+a_1z+\cdots+a_sz^s$ and $B(z)=b_1z+\cdots+b_sz^s\triangleq zB_1(z)$ are coprime and both $A(z)$ and $B_1(z)$ are stable, i.e., their all roots are outside the closed unit disk.

A2. The input $\{u_k^0\}$ is an ARMA process

$$P(z)u_k^0=Q(z)\varepsilon_k \tag{3}$$

with

$$P(z)=1+p_1z+\cdots+p_sz^s,$$
$$Q(z)=1+q_1z+\cdots+q_sz^s,$$

where both $P(z)$ and $Q(z)$ are stable, and $Q(z)$ has no common root with both $P(z)$ and $A(z)$.

A3. $\Delta_k\triangleq[\xi_k,\eta_k,\varepsilon_k]^{\mathrm{T}}$ is a sequence of i.i.d. Gaussian random vectors $\Delta_k\in\mathcal{N}(0,R)$, where

$$R=\begin{bmatrix} r_1 & 0 & 0\\ 0 & r_2 & 0\\ 0 & 0 & r_3\end{bmatrix}\quad \text{with } r_1>0,\ r_2>0 \text{ and } r_3>0.$$

Here, all polynomials and the covariance matrix R are unknown, but the upper bound s for orders of polynomials is given.

By (1) and (3) it is clear that

$$A(z)P(z)y_k^0=B(z)Q(z)\varepsilon_k$$

and y_k^0 is Gaussian stationary by A1 and A3.

As in Söderström et al. (2002) we denote the 2D observation vector by z_k:

$$z_k=\begin{bmatrix} y_k\\ u_k\end{bmatrix}=\begin{bmatrix} y_k^0\\ u_k^0\end{bmatrix}+\begin{bmatrix} \xi_k\\ \eta_k\end{bmatrix}. \tag{4}$$

By A1–A3, z_k is a Gaussian stationary process.

Let

$$G(z)\triangleq\begin{bmatrix} A(z) & -B(z)\\ 0 & P(z)\end{bmatrix}=I+G_1z+\cdots+G_sz^s, \tag{5}$$

where I is the 2×2 identity matrix.

Then by (1) and (3)

$$G(z)z_k=\zeta_k, \tag{6}$$

where

$$\zeta_k\triangleq\begin{bmatrix} A(z)\xi_k-B(z)\eta_k\\ Q(z)\varepsilon_k+P(z)\eta_k\end{bmatrix} \tag{7}$$

which is a Gaussian stationary process by A2 and A3.

Let

$$\begin{aligned} S(z)&\triangleq\begin{bmatrix} A(z) & -B(z) & 0\\ & P(z) & Q(z)\end{bmatrix}\\ &=S_0+S_1z+\cdots+S_sz^s, \end{aligned} \tag{8}$$

where

$$S_0=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}. \tag{9}$$

From (7) it follows that

$$\zeta_k=S(z)\Delta_k \tag{10}$$

and the spectral density of $\{\zeta_k\}$ is

$$f_\zeta(\lambda)=\frac{1}{2\pi}f(\mathrm{e}^{-\mathrm{i}\lambda}),$$

where by definition

$$f(z)\triangleq S(z)RS^{\mathrm{T}}(z^{-1}). \tag{11}$$

Lemma 1. *Assume A1–A3 hold. The 2D process $\{\zeta_k\}$ defined by (10) can uniquely be represented as a 2D MA system*

$$\zeta_k=C(z)w_k,$$

where $\{w_k\}$ *is a sequence of 2D i.i.d. Gaussian random vectors with* $Ew_k=0$ *and* $Ew_k w_k^{\mathrm{T}} \triangleq R_w$, *and* $C(z)$ *is a stable polynomial*

$$C(z) = I + C_1 z + \cdots + C_s z^s, \tag{12}$$

i.e., $\det C(z) \neq 0\ \forall |z| \leqslant 1$. *As a result,* z_k *given by* (4) *is represented as an ARMA system of minimum phase*

$$G(z) z_k = C(z) w_k. \tag{13}$$

Proof. Notice that $f(z)$ given by (11) is rational and analytic on $|z| = 1$, and it has full rank almost everywhere since $R > 0$ and $A(z) \not\equiv 0$, $P(z) \not\equiv 0$, and $Q(z) \not\equiv 0$. Then (see, e.g., Anderson & Moore, 1979; Söderström, 1981) $f(z)$ can uniquely be factorized as $f(z) = H(z) R_w H^{\mathrm{T}}(z^{-1})$ and ζ_k can be represented as

$$\zeta_k = H(z) w_k, \tag{14}$$

where $H(z)$ is a 2×2 matrix of rational functions with $H(0)=I$ and both $H(z)$ and $H^{-1}(z)$ are stable, and $Ew_k=0$, $Ew_k w_s^{\mathrm{T}} = R_w \delta_{k,s}$ with $\delta_{k,s}=1$ if $k=s$ and $\delta_{k,s}=0$ if $k \neq s$.

Since $H^{-1}(z)$ is stable, w_k can be expressed via $\{\zeta_k, \zeta_{k-1}, \ldots\}$. Let $\mathscr{F}_k$ be the σ-algebra generated by $\{\Delta_j, j \leqslant k\}$. By (10) it is clear that $\zeta_k \in \mathscr{F}_k$, and hence $w_k \in \mathscr{F}_k$. By stability of $H(z)$, ζ_k can be represented as a moving average of infinite order:

$$\zeta_k = \sum_{i=0}^{\infty} C_i w_{k-i}, \quad C_0 = I. \tag{15}$$

By A3 and (10), ζ_k is independent of Δ_{k-i}, and hence independent of $\mathscr{F}_{k-i}$, $\forall i \geqslant s+1$. Noticing that $w_{k-i} \in \mathscr{F}_{k-i}$, we have

$$E\zeta_k w_{k-i}^{\mathrm{T}} = E\zeta_k E w_{k-i}^{\mathrm{T}} = 0, \quad \forall i \geqslant s+1.$$

Therefore, in (15) the summation ceases at s, i.e.,

$$\zeta_k = w_k + C_1 w_{k-1} + \cdots + C_s w_{k-s}. \tag{16}$$

This means that the rational matrix $H(z)$ in (14) coincides with the polynomial $C(z) : H(z)=C(z)$. Consequently, $C(z)$ is stable, which implies that w_k is Gaussian $w_k \in \mathscr{N}(0, R_w)$ from (16) since ζ_k is Gaussian. From (6) and (16) we derive (13). □

Remark 1. For the innovation representation (16) the Gaussian assumption is not necessary. If we remove the Gaussian assumption, then $\mathscr{F}_k$ should be replaced by the Hilbert space $\mathscr{H}_k$ spanned by $\{\Delta_j, j \leqslant k\}$ in the mean square sense. In this case, w_k is no longer i.i.d. but a sequence of zero mean uncorrelated random vectors with $Ew_k w_k^{\mathrm{T}} \triangleq R_w$, and (15) corresponds to the Wold decomposition of the stationary process $\{\zeta_k\}$.

3. Identifiability

If we can consistently estimate $G(z)$, then from (5) we see that the consistent estimates for $A(z)$ and $B(z)$ are obtained at the same time. So, the crucial issue is the identifiability of system (13).

The necessary and sufficient conditions are given in Stoica and Nehorai (1987) for nonuniqueness of coefficients in the expressions (1) and (3). Thus, any violation of these conditions gives sufficient conditions for uniqueness of coefficients in (1) and (3).

By the sufficient conditions given in Stoica and Nehorai (1987) we see that coefficients in (1) and (3) are uniquely defined. However, this uniqueness does not exclude system (13) from having a common left factor. In other words, the identifiability of system (13) is not automatically guaranteed. We need a mild condition imposed on $[G_s, C_s]$, the matrix being the coefficients for the highest orders of $G(z)$ and $C(z)$.

A4. The 2×4-matrix $[G_s, C_s]$ is of full-row-rank, where G_s and C_s are given in (5) and (16), respectively.

For identifiability we also need the following technical condition:

A5. $P(z)$ is coprime with the polynomial $g(z)$:

$$g(z) \triangleq r_1 A(z) z^s A(z^{-1}) + r_2 B(z) z^s B(z^{-1}),$$

where r_1 and r_2 are the variances of observation noises ξ_k and η_k, respectively.

Theorem 1. *Assume* A1–A5 *hold. Then the matrix polynomials* $G(z)$ *and* $C(z)$ *in* (13) *are uniquely defined, i.e., the system* (13) *is identifiable.*

Proof. We first recall that a matrix polynomial is called unimodular if its determinant is a constant. It is well known that a unimodular matrix can be expressed as a finite product of elementary transformations. In the case of 2×2-matrices, the elementary transformation corresponds to one of the following four matrices:

$$\begin{bmatrix} 1 & f(z) \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} f(z) & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ f(z) & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & f(z) \end{bmatrix}, \tag{17}$$

where $f(z)$ is an arbitrary polynomial.

We now show that with possible exception of a unimodular matrix the matrix polynomials $G(z)$ and $C(z)$ have no common left factor.

Assume the converse: there is a 2×2-matrix polynomial $L(z)$ with $\det L(z) \triangleq h(z)$ being not a constant such that

$$L(z)[G'(z), C'(z)] = [G(z), C(z)].$$

Let z_0 be a root of $h(z) : h(z_0) = 0$. Then we have

$$\det G(z_0) = h(z_0) \det G'(z_0) \tag{18}$$

and

$$\det C(z_0) = h(z_0) \det C'(z_0). \tag{19}$$

Thus z_0 is a common root of $\det G(z)$ and $\det C(z)$.

If we can show that $\det G(z)$ and $\det C(z)$ have no common root, then the obtained contradiction shows that $G(z)$ and $C(z)$ have no common left factor with possible exception of a unimodular matrix. Assume the converse: let z^0 be a common root of $\det G(z)$ and $\det C(z)$. Since $\det G(z) = A(z)P(z)$, z^0 must be a root of either $A(z)$ or $P(z)$. By stability of $A(z)$ and $P(z)$ we have $|z^0| > 1$.

In the proof of Lemma 1, based on Anderson and Moore (1979) we have shown

$$f(z) = S(z)RS^{\mathrm{T}}(z^{-1}) = C(z)R_w C^{\mathrm{T}}(z^{-1}), \quad \forall z \in \mathbb{C}, \tag{20}$$

and hence

$$\det C(z) \det R_w \det C^{\mathrm{T}}(z^{-1}) = \det(S(z)RS^{\mathrm{T}}(z^{-1})).$$

We then have

$$\begin{aligned} &\det C(z^0) \det R_w \det C^{\mathrm{T}}((z^0)^{-1}) \\ &\quad = [r_1 A(z^0)A((z^0)^{-1}) + r_2 B(z^0)B((z^0)^{-1})] \\ &\qquad \times [r_2 P(z^0)P((z^0)^{-1}) + r_3 Q(z^0)Q((z^0)^{-1})] \\ &\qquad - r_2^2 P(z^0)P((z^0)^{-1})B(z^0)B((z^0)^{-1}) \\ &\quad = r_1 r_2 A(z^0)A((z^0)^{-1})P(z^0)P((z^0)^{-1}) \\ &\qquad + r_1 r_3 A(z^0)A((z^0)^{-1})Q(z^0)Q((z^0)^{-1}) \\ &\qquad + r_2 r_3 B(z^0)B((z^0)^{-1})Q(z^0)Q((z^0)^{-1}). \end{aligned} \tag{21}$$

If z^0 is a root of $A(z)$, then by A1 and A2, $B(z^0) \neq 0$, $Q(z^0) \neq 0$ and the right-hand side of (21) equals $r_2 r_3 B(z^0)B((z^0)^{-1})Q(z^0)Q((z^0)^{-1})$. Since $|z^0| > 1$ and both $B_1(z)$ and $Q(z)$ are stable, $(z^0)^{-1}$ cannot be a root of $B(z)$ and $Q(z)$. Noticing that r_2 and r_3 are positive by A3, we conclude that the right-hand side of (21) is nonzero.

If z^0 is a root of $P(z)$, then $Q(z^0) \neq 0$ and z^0 cannot be a root of $g(z)$ by A5:

$$g(z^0) = (z^0)^s [r_1 A(z^0)A((z^0)^{-1}) + r_2 B(z^0)B((z^0)^{-1})] \neq 0.$$

Noticing $|z^0| > 1$, we then have

$$r_1 A(z^0)A((z^0)^{-1}) + r_2 B(z^0)B((z^0)^{-1}) \neq 0$$

and the right-hand side of (21) equals

$$\begin{aligned} &[r_1 A(z^0)A((z^0)^{-1}) + r_2 B(z^0)B((z^0)^{-1})] \\ &\quad \times r_3 Q(z^0)Q((z^0)^{-1}) \neq 0. \end{aligned}$$

This contradicts with the converse assumption that z^0 is a root of $\det C(z)$. Therefore, $\det G(z)$ and $\det C(z)$ have no common root, and hence we have proved the assertion that $G(z)$ and $C(z)$ have no common left factor with possible exception of a unmodular matrix.

We now show that except constant matrices a unimodular matrix can neither be a common left factor of $G(z)$ and $C(z)$.

Let $N(z)$ be a unimodular matrix polynomial of order $m \geqslant 1$ such that

$$[G(z), C(z)] = N(z)[G'(z), C'(z)], \tag{22}$$

where $G'(z) = I + G'_1 z + \cdots + G'_{s-m} z^{s-m}$ and $C'(z) = I + C' z + \cdots + C'_{s-m} z^{s-m}$.

First, let $N(z)$ be one of the matrices expressed in (17), say,

$$N(z) = \begin{bmatrix} 1 & f(z) \\ 0 & 1 \end{bmatrix},$$

where $f(z) = f_1 z + \cdots + f_m z^m$.

Then, equalizing the coefficients of z^s in both sides of (22), we derive

$$[G_s, C_s] = \begin{bmatrix} 0 & f_m \\ 0 & 0 \end{bmatrix} [G'_{s-m}, C'_{s-m}]$$

which clearly is not of full-row rank. This contradicts with A4.

Since a unimodular matrix is a finite product of matrices given by (17), continuing the argument given above leads to the conclusion that a nonconstant unmodular matrix cannot be a common left factor of $G(z)$ and $C(z)$. Noticing that $G(z)$ and $C(z)$ are monic, we conclude that the only possible common left factor is the identity matrix. This means that $G(z)$ and $C(z)$ are uniquely defined. □

Remark 2. We have succeeded in guaranteeing uniqueness of coefficients in (13) due to the coprimeness and stability conditions figured in A1, A2 and A5 and the full-row rank condition of $[G_s, C_s]$. Besides, the nondegeneracy of Δ_k is also crucial for achieving uniqueness. However, the Gaussian assumption on Δ_k is not needed for identifiability of the system (13). As a matter of fact, the Gaussian assumption is used in Section 4 for $\{w_k\}$ to be a martingale difference sequence, (which is guaranteed by Lemma 1,) in order to apply existing results concerning the extended least squares (ELS) algorithm.

4. Consistent parameter estimates

In the last section we have shown that the minimum phase system (13) is identifiable in the sense that its coefficients are uniquely defined under A1–A5 (without need for Gaussianity). It is clear that if consistent estimates for coefficients can somehow be derived, then the system coefficients are uniquely defined. However, the converse, in general, is not true. Uniqueness does not indicate how to derive consistent estimates for coefficients.

Since the right-hand side of (13) is a correlated process, the existing estimation methods like ELS, IV method, subspace method, and prediction error method for estimating

$$\theta^{\mathrm{T}} = [-G_1, \ldots, -G_s, C_1, \ldots, C_s]$$

either are not directly applicable or require additional conditions to guarantee consistency.

We will use the ELS algorithm (Chen & Guo, 1991) incorporating with the overparameterization technique. But, we first demonstrate the convergence result when the commonly used in practice ELS algorithm without modification is applied with some additional assumption.

Let us introduce the strictly positive real (SPR) condition

A6. $C^{-1}(z)-\frac{1}{2}I$ is SPR, i.e.,

$$C^{-1}(e^{i\lambda})+C^{-T}(e^{-i\lambda})-I>0,\quad \forall\lambda\in[0,2\pi].$$

It is known that A6 implies stability of $C(z)$. It is also known that ELS may not converge to the true coefficients if A6 fails even though $C(z)$ is stable. This is demonstrated by Example 2 in Section 5.

The ELS for the matrix θ is recursively defined by the following algorithm:

$$\hat{w}_{k+1}=z_{k+1}-\theta_{k+1}^{T}\phi_k, \tag{23}$$

$$\theta_{k+1}=\theta_k+a_kP_k\phi_k(z_{k+1}^{T}-\phi_k^{T}\theta_k), \tag{24}$$

$$P_{k+1}=P_k-a_kP_k\phi_k\phi_k^{T}P_k,\quad a_k=(1+\phi_k^{T}P_k\phi_k)^{-1}, \tag{25}$$

$$P_0=\alpha I,$$

$$\phi_k^{T}=[z_k^{T},\ldots,z_{k-s+1}^{T},\hat{w}_k^{T},\ldots,\hat{w}_{k-s+1}^{T}], \tag{26}$$

where $\alpha>0$ and θ_0 is arbitrary.

It is worth noting that in the special case $C(z)=I$ the ELS is nothing else but the recursive expression of the LS estimate. When $C(z)\not\equiv I$, LS is not directly applicable, since w_k is not available. The idea of ELS is to replace w_{k+1} with its estimate given by (23) in the regressor (Chen & Guo, 1991), and this results in (26).

Theorem 2. *Assume A1–A6 hold. Let θ_n be given by* (23)–(26). *Then θ_n is strongly consistent with the following convergence rate*

$$\|\theta_{n+1}-\theta\|^2=O\left(\frac{\log n}{n}\right)\quad a.s. \tag{27}$$

Proof. By Lemma 1, $\{w_k\}$ is a sequence of i.i.d. Gaussian random vectors, which satisfy the moment condition (i) required in Theorem A in Appendix. Then, (27) follows from Theorem A, if we observe that the remaining conditions required in Theorem A are satisfied by setting $\varepsilon_n\equiv 0$. □

We now apply the overparameterized technique proposed in Guo and Huang (1989) to estimate θ without A6. The idea of the technique consists in the following observation: Although $C^{-1}(z)-\frac{1}{2}I$ is not SPR, there always exists a matrix $\Gamma(z)$ such that $[\Gamma(z)C(z)]^{-1}-\frac{1}{2}I$ is SPR provided $C(z)$ is stable. Then multiplying (13) from the left by $\Gamma(z)$ we obtain an overparameterized system equivalent to system (13). The overparameterized system is SPR and is used to estimate w_k. The obtained estimate is applied to serve as $\hat{w}_k$ appearing in (26) ignoring (23). The resulting estimate for θ is strongly consistent.

We now give a detailed description. By stability of $C(z)$, we have

$$C^{-1}(z)=\sum_{j=0}^{\infty}\Gamma_j z^j,\quad \forall|z|\leqslant 1,$$

where $\|\Gamma_j\|\leqslant M\lambda^j$, for some $M>0$ and $\lambda\in(0,1)$.

Let m be a sufficiently large integer such that

$$m>[\log[\|C(z)\|_\infty^{-1}M^{-1}(1-\lambda)]/|\log\lambda|]-1,$$

where $\|C(z)\|_\infty^2=\max_{|z|=1}\|C(z)C^{T}(z^{-1})\|$. Denote

$$\Gamma(z)\triangleq\sum_{i=0}^{m}\Gamma_j z^j,\quad \Gamma_0=I.$$

It is known (Chen & Guo, 1991; Guo & Huang, 1989) that

$$[\Gamma(z)C(z)]^{-1}-\frac{1}{2}I$$

is SPR and $\Gamma(z)$ is stable.

From (13) it follows that

$$\Gamma(z)G(z)z_k=\Gamma(z)C(z)w_k \tag{28}$$

for which the SPR condition required for convergence of the ELS algorithm is satisfied. Let

$$M(z)\triangleq\Gamma(z)G(z)=I+M_1z+\cdots+M_pz^p,\quad p=ms,$$

$$F(z)\triangleq\Gamma(z)C(z)=I+F_1z+\cdots+F_pz^p$$

and

$$\bar{\theta}^{T}=[-M_1,\ldots,-M_p,F_1,\ldots,F_p].$$

We first estimate w_k by the ELS algorithm for system (28) and denote the estimate still by $\hat{w}_k$:

$$\hat{w}_{k+1}=z_{k+1}-\bar{\theta}_{k+1}^{T}\bar{\phi}_k, \tag{29}$$

$$\bar{\theta}_{k+1}=\bar{\theta}_k+\bar{a}_k\bar{P}_k\bar{\phi}_k(z_{k+1}^{T}-\bar{\phi}_k^{T}\bar{\theta}_k), \tag{30}$$

$$\bar{P}_{k+1}=\bar{P}_k-\bar{a}_k\bar{P}_k\bar{\phi}_k\bar{\phi}_k^{T}\bar{P}_k,\quad \bar{a}_k=(1+\bar{\phi}_k^{T}\bar{P}_k\bar{\phi}_k)^{-1}, \tag{31}$$

$$\bar{P}_0=\alpha I,$$

$$\bar{\phi}_k^{T}=[z_k^{T},\ldots,z_{k-p+1}^{T},\hat{w}_k^{T},\ldots,\hat{w}_{k-p+1}^{T}], \tag{32}$$

where $\alpha>0$ and $\bar{\theta}_0$ is arbitrary.

Using $\hat{w}_k$, we now estimate the unknown coefficients

$$\theta^{T}=[-G_1,\ldots,-G_s,C_1,\ldots,C_s] \tag{33}$$

by the following algorithm:

$$\phi_k^{T}=[z_k^{T},\ldots,z_{k-s+1}^{T},\hat{w}_k^{T},\ldots,\hat{w}_{k-s+1}^{T}],$$
$$a_k=(1+\phi_k^{T}P_k\phi_k)^{-1} \tag{34}$$

$$\theta_{k+1} = \theta_k + a_k P_k \phi_k (z_{k+1}^{\mathrm{T}} - \phi_k^{\mathrm{T}} \theta_k),$$
$$P_{k+1} = P_k - a_k P_k \phi_k \phi_k^{\mathrm{T}} P_k, \tag{35}$$

where $\{\hat{w}_k\}$ is given by (29)–(32) rather than by (23).

Theorem 3. *Assume A1–A5 hold. Let $\hat{w}_k$ be generated by* (29)–(32). *Then θ_n given by* (34) *and* (35) *is strongly consistent with the following rate of convergence*:

$$\|\theta_{n+1} - \theta\|^2 = O\left(\frac{\log n}{n}\right) \quad \textit{a.s.} \tag{36}$$

Proof. By Theorem 4.8 of Chen and Guo (1991) θ_n generated by (34) and (35) is with the following rate of convergence

$$\|\theta_{n+1} - \theta\|^2 = O\left(\frac{\log \lambda_{\max}^0(n)}{\lambda_{\min}^0(n)}\right) \quad \text{a.s.,} \tag{37}$$

where $\lambda_{\max}^0(n)$ and $\lambda_{\min}^0(n)$, respectively denote the maximum and minimum eigenvalue of $P_0^{-1} + \sum_{k=1}^n \phi_k^0 \phi_k^{0\mathrm{T}}$ with

$$\phi_k^{0\mathrm{T}} = [z_k^{\mathrm{T}}, \ldots, z_{k-s+1}^{\mathrm{T}}, w_k^{\mathrm{T}}, \ldots, w_{k-s+1}^{\mathrm{T}}].$$

By stability of $G(z)$ and the fact that $\{w_k\}$ is i.i.d. Gaussian, form (13) it is clear that $\lambda_{\max}^0(n) = O(n)$. On the other hand, from Remark A in Appendix $\liminf_{n\to\infty} \frac{1}{n} \lambda_{\min}^0(n) > 0$. Putting these estimates into (37) leads to the desired result (36). $\square$

Remark 3. By Theorems 2 and 3 the strongly consistent estimates $A_n(z)$ and $B_n(z)$ are derived for $A(z)$ and $B(z)$:

$$\lim_{n\to\infty} A_n(z) = A(z) \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} B_n(z) = B(z) \text{ a.s.}$$

Besides, the strongly consistent estimate $C_n(z)$ for $C(z)$, characterizing the innovation representation (16) is also obtained:

$$\lim_{n\to\infty} C_n(z) = C(z) \quad \text{a.s.}$$

Remark 4. It is worth noting that when estimating $G(z)$ and $C(z)$ we do not know if $C(z)$ satisfies A6 although the order s of $C(z)$ is known. Therefore, in practice we may first try to estimate θ by (23)–(26). If it turns out that the estimate is undesirable, then we try to use (29)–(35) with sufficiently large m, say, a multiple of s. In Section 5, $s = 1$ and we take $m = 10$ for Example 2. Of course, we may try different m.

5. Numerical examples

We give two examples corresponding to the cases: (i) A6 holds, (ii) A6 does not hold, respectively.

Example 1. In this example all conditions A1–A6 are met. By Theorem 1 the coefficients of system (13) are uniquely defined, and by Theorem 2 the ELS algorithm gives consistent estimates.

Let

$$A(z) = 1 + a_1 z, \quad B(z) = b_1 z,$$
$$P(z) = 1 + p_1 z, \quad Q(z) = 1 + q_1 z$$

with $a_1 = -0.8,\ b_1 = 0.2,\ p_1 = -0.5,\ q_1 = 0.8$, and let the covariance matrix R of Δ_k be a 3×3-identity matrix. Then

$$S(z) = \begin{bmatrix} 1 - 0.8z & -0.2z & 0 \\ 0 & 1 - 0.5z & 1 + 0.8z \end{bmatrix} \tag{38}$$

and the spectral density of $\zeta_k = S(z)\Delta_k$ is $f_\zeta(\lambda) = \frac{1}{2\pi} f(\mathrm{e}^{-\mathrm{i}\lambda})$, where

$$f(\mathrm{e}^{-\mathrm{i}\lambda}) = \begin{bmatrix} 1.68 & 0.1 \\ 0.1 & 2.89 \end{bmatrix} + \begin{bmatrix} -0.8 & 0 \\ -0.2 & 0.3 \end{bmatrix} \mathrm{e}^{\mathrm{i}\lambda} + \begin{bmatrix} -0.8 & -0.2 \\ 0 & 0.3 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\lambda}.$$

A direct computation shows that

$$f(\mathrm{e}^{-\mathrm{i}\lambda}) = C(\mathrm{e}^{-\mathrm{i}\lambda}) R_w C^{\mathrm{T}}(\mathrm{e}^{\mathrm{i}\lambda}),$$

where $C(z) = I + C_1 z$ with

$$C_1 \simeq \begin{bmatrix} -0.7326 & -0.0412 \\ -0.0109 & 0.1054 \end{bmatrix}, \quad \text{and}$$
$$R_w \simeq \begin{bmatrix} 1.0857 & 0.1124 \\ 0.1124 & 2.8584 \end{bmatrix}. \tag{39}$$

Thus, we have the innovation representation

$$\zeta_k = C(z) w_k.$$

Notice that $C(z)$ has roots approximately equal to -9.4397 and 1.3640. Therefore, $C(z)$ is stable. Furthermore,

$$C^{-1}(z) = \frac{1}{1 - 0.6272z - 0.0776z^2} \times \begin{bmatrix} 1 + 0.1054z & 0.0412z \\ 0.0109z & 1 - 0.7326z \end{bmatrix}$$

and it is easy to numerically verify that

$$C^{-1}(\mathrm{e}^{\mathrm{i}\lambda}) + C^{-\mathrm{T}}(\mathrm{e}^{-\mathrm{i}\lambda}) - I > 0 \quad \forall \lambda \in [0, 2\pi].$$

Since

$$G_1 = \begin{bmatrix} a_1 & -b_1 \\ 0 & p_1 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 0 & -0.5 \end{bmatrix}. \tag{40}$$

G_1 itself is of full-row-rank. Therefore, $[G_1, C_1]$ is of full-row-rank whatever C_1 is. In our case C_1 given by (39) is also of full-row-rank.

Further,

$$g(z) = A(z) z A(z^{-1}) + B(z) z B(z^{-1})$$
$$= -0.8z^2 + 1.68z - 0.8$$

and $g(z)$ is coprime with $P(z)$ $(=1 - 0.5z)$.

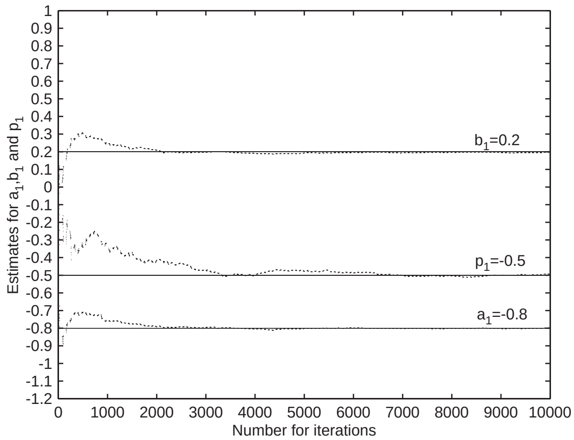

Fig. 1. Example 1 (ELS).

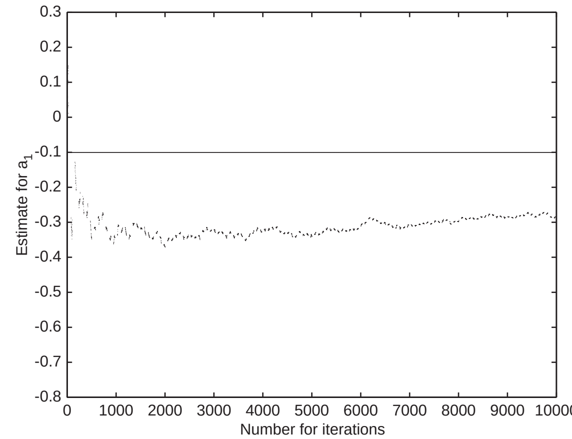

Fig. 2. Example 2 (ELS).

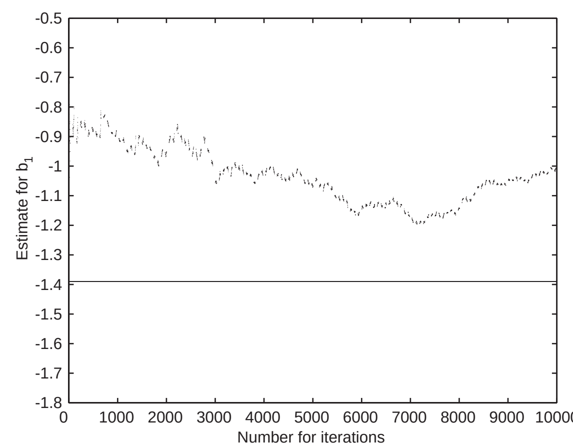

Fig. 3. Example 2 (ELS).

Thus, all conditions A1–A6 are satisfied and Theorem 2 is applicable. By (23)–(26) $\theta^{\mathrm{T}}=[-G_1, C_1]$ is estimated. The computational results given in Fig. 1 show that the estimates for a_1, b_1, and p_1, respectively, converge to the true values.

In what follows in all figures the true values are denoted by solid lines and their estimates by dashed lines.

Example 2. We take the same system as that considered in Example 1 but with different coefficients. Namely, let $a_1=-0.1$, $b_1=-1.39$, $p_1=0.89$, $q_1=0.6$. Let the covariance matrix R of Δ_k be a 3×3 diagonal matrix with diagonal elements $\{2, 0.5, 0.5\}$.

Since $\det G_1 \neq 0$, A4 holds.

Further, $g(z)=-0.2z^2+2.986z-0.2$ is with roots equal to 14.863 and 0.0673, and hence it is coprime with $P(z)=1-0.89z$. So, Conditions A1–A5 are satisfied, and by Theorem 1 system (13) is identifiable.

Notice that

$$S(z)=\begin{bmatrix} 1-0.1z & 1.39z & 0 \\ 0 & 1-0.89z & 1+0.6z \end{bmatrix} \tag{41}$$

and the spectral density of $\zeta_k=S(z)\Delta_k$ is $f_\zeta(\lambda)=\frac{1}{2\pi}f(\mathrm{e}^{-\mathrm{i}\lambda})$, where

$$f(\mathrm{e}^{-\mathrm{i}\lambda})=\begin{bmatrix} 2.9421 & -1.2371 \\ -1.2371 & 3.1521 \end{bmatrix}+\begin{bmatrix} -0.1 & 0 \\ 1.39 & -0.29 \end{bmatrix}\mathrm{e}^{\mathrm{i}\lambda} + \begin{bmatrix} -0.1 & 1.39 \\ 0 & -0.29 \end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda}.$$

A direct computation shows that

$$f(\mathrm{e}^{-\mathrm{i}\lambda})=C(\mathrm{e}^{-\mathrm{i}\lambda})R_w C^{\mathrm{T}}(\mathrm{e}^{\mathrm{i}\lambda}),$$

where $C(z)=I+C_1 z$ with

$$C_1 \simeq \begin{bmatrix} -0.6537 & -0.7531 \\ -0.1182 & -0.1486 \end{bmatrix},$$

$$R_w \simeq \begin{bmatrix} 1.8299 & -1.4555 \\ -1.4555 & 3.1090 \end{bmatrix}. \tag{42}$$

Since the roots of $\det C(z)=0.0081z^2-0.8023z+1$ are 97.5209 and 1.2626, $C(z)$ is stable. However, at $\lambda=0$ the determinant of $C^{-1}(\mathrm{e}^{\mathrm{i}\lambda})+C^{-\mathrm{T}}(\mathrm{e}^{-\mathrm{i}\lambda})-I$ equals -0.7193, and hence A6 is violated.

Figs. 2–4 show that in this case ELS give biased estimates.

According to Theorem 3 the algorithm (29)–(35) with $m=10$ gives strongly consistent estimates as shown in Fig. 5.

6. Concluding remarks

For the SISO errors-in-variables models the 2D observation process is represented as a 2D ARMA system driven by an i.i.d. random sequence. The identifiability of the represented system is proved. This is achieved under coprimeness and stability conditions. The consistency of parameter estimates together with the rate of convergence are derived with the additional Gaussian assumption. Numerical

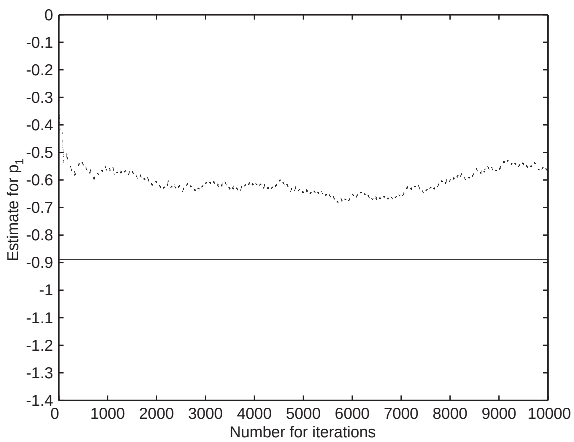

Fig. 4. Example 2 (ELS).

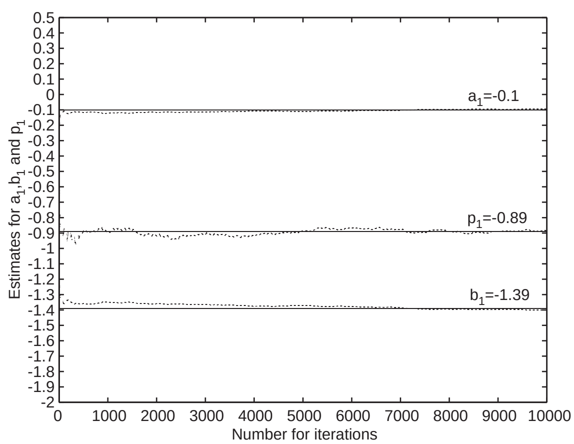

Fig. 5. Example 2 (ELS with overparameterization technique applied).

examples are given and justify the theoretical results given in the paper.

For further work it is desirable to extend results to the MIMO systems, and to consider more general noises. Furthermore, it is of interest to consider the input to be a general feedback control rather than an ARMA process.

Appendix A.

The following m-dimensional model is considered in Chen and Deniau (1994):

$$A(z)y_n = C(z)w_n + \varepsilon_n, \quad y_i = w_i = \varepsilon_i = 0, \ \ i < 0,$$

where $A(z) = I + A_1 z + \cdots + A_p z^p$, $C(z) = I + C_1 z + \cdots + C_r z^r$, $\{w_n, \mathscr{F}_n\}$ is a martingale difference sequence, and ε_n is the possibly existing model error.

The ELS algorithm is applied for estimating

$$\theta^{\mathrm{T}} = [-A_1, \ldots, -A_p, C_1, \ldots, C_r],$$

$$\begin{aligned}
&\hat{w}_{k+1} = y_{k+1} - \theta_{k+1}^{\mathrm{T}}\phi_k,\\
&\theta_{k+1} = \theta_k + a_k P_k \phi_k (y_{k+1}^{\mathrm{T}} - \phi_k^{\mathrm{T}}\theta_k),\\
&P_{k+1} = P_k - a_k P_k \phi_k \phi_k^{\mathrm{T}} P_k, \quad a_k = (1 + \phi_k^{\mathrm{T}} P_k \phi_k)^{-1},\\
&\phi_k^{\mathrm{T}} = [y_k^{\mathrm{T}}, \ldots, y_{k-p+1}^{\mathrm{T}}, \hat{w}_k^{\mathrm{T}}, \ldots, \hat{w}_{k-r+1}^{\mathrm{T}}].\\
&P_0 \neq 0.
\end{aligned}$$

The theorem proved in Chen and Deniau (1994) is presented here as

Theorem A. *Assume the following conditions hold*:

(i) *The martingale difference sequence* $\{w_k, \mathscr{F}_k\}$ *has the properties*:

$$\sup_n E[\|w_{k+1}\|^2 | \mathscr{F}_k] < \infty, \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} w_k w_k^{\mathrm{T}} = R > 0 \quad \textit{a.s.}$$

(ii) $det\, A(z) \neq 0, \ \forall |z| < 1$;

(iii) $A(z)$ *and* $C(z)$ *are left coprime and* $[A_p, C_r]$ *is of full rank*;

(iv) $C^{-1}(\mathrm{e}^{\mathrm{i}\lambda}) + C^{-\mathrm{T}}(\mathrm{e}^{-\mathrm{i}\lambda}) - I > 0, \ \forall \lambda \in [0, 2\pi]$;

(v) ε_n *is* $\mathscr{F}_{n-1}$ *-measurable and*

$$\delta_n \triangleq \frac{1}{n} \sum_{k=0}^{n} \|\varepsilon_{k+1}\|^2 \to 0 \quad \textit{a.s.}$$

Then the θ_n *given by ELS is strongly consistent with convergence rate*

$$\|\theta_n - \theta\|^2 = O(\delta_n) + O\left(\frac{\log(\log\log n)^c}{n}\right), \quad \forall c > 1 \ \textit{a.s.}$$

Remark A. In the proof of Theorem A the crucial step is to show that

$$\liminf_{n\to\infty} \frac{1}{n} \lambda_{\min}^0(n) > 0,$$

where $\lambda_{\min}^0(n)$ denotes the minimum eigenvalue of $P_0^{-1} + \sum_{k=1}^{n} \phi_k^0 \phi_k^{0\mathrm{T}}$ with

$$\phi_k^{0\mathrm{T}} = [y_k^{\mathrm{T}}, \ldots, y_{k-p+1}^{\mathrm{T}}, w_k^{\mathrm{T}}, \ldots, w_{k-r+1}^{\mathrm{T}}].$$

It is noticed that this estimate for the minimum eigenvalue takes place whatever δ_n is zero or not. Therefore, it holds true if y_i in ϕ_k^0 is replaced by z_i defined by (13) and both p and r are replaced by s.

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