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The Strong Consistency of the Stochastic Gradient Algorithm of Adaptive Control

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Abstract—**By use of the technique of Chen [5] sufficient conditions are established for the strong consistency of the stochastic gradient (SG) algorithm for MIMO ARMAX stochastic systems without monitoring. This result is then used in conjunction with the method of disturbed adaptive controls introduced in [1], [2]. Hence it is shown that the SG algorithm generates strongly consistent parameter estimates while it is operating as a part of the SG algorithm of adaptive control of Goodwin, Ramadge, and Caines [6].**

I. Introduction

The stochastic gradient (SG) algorithm is probably the simplest method of parameter estimation for linear stochastic systems. It was used in [6] for the adaptive tracking control problem and later on in [2] for the adaptive tracking problem where disturbed controls were used for purposes of identification as explained below. In these papers the strong consistency of the SG algorithm was not established. In [4] the strong consistency of the estimates generated by the quasi-least-squares (QLS) method was proved for the case of a system subject to feedback control without monitoring.

In this note, by use of the technique given in [5], we first establish sufficient conditions for the strong consistency of the SG algorithm for MIMO stochastic systems without monitoring and then apply the method involving continually disturbed controls which was introduced in [1] and developed in [7] and [2]. Continually disturbing a system's control input provides a technique for ensuring that certain processes of regression vectors have the persistency of excitation property which is frequently required in recursive identification schemes (see, e.g., [9]).

The ordinary differential equation method and the associated hypotheses used in this paper should be compared to those (involving monitoring) used to obtain the related results of [8, Theorem 1]. Concerning proof techniques, we also remark that the stochastic Lyapunov—or super martingale—method used in the basic first lemma below is fundamental to the consistency proofs of [10] and [3], [4] and to all of the references of this note concerned with stochastic adaptive control.

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It will be shown that the SG algorithm used in [7] and [2] does in fact generate strongly consistent estimates. Hence, it is not necessary to introduce a second algorithm in order to generate consistent estimates, as in [2], where the approximate maximum likelihood (AML) algorithm of [10] was used in addition to the SG algorithm.

We consider the MIMO system

$$y_n + A_1 y_{n-1} + \cdots + A_p y_{n-p} = B_1 u_{n-1} + \cdots + B_q u_{n-q} + w_n + C_1 w_{n-1} + \cdots + C_r w_{n-r} \tag{1}$$

where y_n, u_n, and w_n are m-, l-, and m-dimensional, respectively, and $y_i = 0$, $u_j = 0$, $w_k = 0$ for all $i < 0, j < 0, k < 0$.

Let $\mathfrak{F}_n$ be a family of nondecreasing σ-algebras; assume that w_n and u_n are $\mathfrak{F}_n$-measurable and that

$$E(w_n/\mathfrak{F}_{n-1}) = 0, \quad E(\|w_n\|^2/\mathfrak{F}_{n-1}) \leqslant k_o r_{n-1}^{\epsilon}, \qquad 0 \leqslant \epsilon < 1 \tag{2}$$

where k_o is a positive constant and r_n is defined below in (10).

A_i, B_j, C_k, $i = 1, \cdots, p$, $j = 1, \cdots, q$, $k = 1, \cdots, r$ are the unknown matrix coefficients to be estimated.

Let us write

$$A(z) = I + A_1 z + \cdots + A_p z^p \tag{3}$$

$$B(z) = B_1 + B_2 z + \cdots + B_q z^{q-1} \tag{4}$$

$$C(z) = I + C_1 z + \cdots + C_r z^r \tag{5}$$

where z denotes the unit backward shift operator.

We shall adopt the following notation:

$$\theta^\tau = [-A_1, \cdots, -A_p, B_1, \cdots, B_q, C_1, C_r] \; mx(mp + lq + mr) \tag{6}$$

$$\begin{aligned} \phi_n^\tau = [&y_n^\tau, y_{n-1}^\tau, \cdots, y_{n-p+1}^\tau, u_n^\tau, \cdots, u_{n-q+1}^\tau \\ &y_n^\tau - \phi_{n-1}^\tau \theta_{n-1}, \cdots, y_{n-r+1}^\tau - \phi_{n-r}^\tau \theta_{n-r}] \end{aligned} \tag{7}$$

$$\phi_n^{o\tau} = [y_n^\tau, y_{n-1}^\tau \cdots y_{n-p+1}^\tau, u_n^\tau \cdots u_{n-1+1}^\tau, w_n^\tau \cdots w_{n-r+1}^\tau] \tag{8}$$

where θ_n is the estimate for θ given by the SG algorithm

$$\theta_{n+1} = \theta_n + \frac{\phi_n}{r_n}(y_{n+1}^\tau - \phi_n^\tau \theta_n), \tag{9}$$

$$r_n = 1 + \sum_{i=1}^{n} \|\phi_i\|^2, \qquad r_o = 1 \tag{10}$$

with ϕ_{-1} and θ_o deterministic and arbitrarily chosen.

The difference between the SG algorithm and the QLS algorithm lies in the fact that the residual term $y_n^\tau - \phi_{n-1}^\tau \theta_{n-1}$ in the SG algorithm is replaced by the term $y_n^\tau - \phi_{n-1}^\tau \theta_n$ in the QLS algorithm; in other words, the *a priori* prediction error is replaced by the *a posteriori* prediction error.

Set

$$\tilde{\theta}_n = \theta - \theta_n, \tag{11}$$

and

$$\xi_n = y_n - w_n - \theta_{n-1}^\tau \phi_{n-1}. \tag{12}$$

Then we have

$$\begin{aligned} C(z)(y_n - w_n - \theta_{n-1}^\tau \phi_{n-1}) &= \{(y_n - C(z) w_n) \\ &\quad + (C(z) - I)(y_n - \theta_{n-1}^\tau \phi_{n-1})\} - \theta_{n-1}^\tau \phi_{n-1} \\ &= \theta^\tau \phi_{n-1} - \theta_{n-1}^\tau \phi_{n-1} = \tilde{\theta}_{n-1}^\tau \phi_{n-1}, \end{aligned}$$

hence

$$C(z)\xi_n = \tilde{\theta}_{n-1}^\tau \phi_{n-1} \tag{13}$$

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and

$$\tilde{\theta}_{n-1}=\tilde{\theta}_n-\frac{\phi_n}{r_n}(\xi^\tau_{n+1}+w^\tau_{n+1}). \tag{14}$$

Using the now standard martingale convergence techniques (see, e.g., [5] or [6]) we may establish the following basic lemma.

Lemma 1: For the system and algorithm (1)-(10), if $C(z)-\frac{1}{2}I$ is strictly positive real, then

$$\sum_{i=0}^{\infty}\frac{\|\xi_{i+1}\|^2}{r_i}<\infty,\ \text{a.s.},\ \sum_{i=0}^{\infty}\frac{\|\tilde{\theta}^\tau_i\phi_i\|^2}{r_i}<\infty,\ \text{a.s.} \tag{15}$$

and

$$\operatorname{tr}\tilde{\theta}^\tau_n\tilde{\theta}_n \underset{n\to\infty}{\to} v<\infty \text{ on } [\omega: r_n \underset{n\to\infty}{\to}\infty]. \tag{16}$$

□

Set

$$\left.\begin{aligned} &F=\begin{bmatrix}-C_1 & I & 0 & \cdots & 0\\ \cdot & 0 & \ddots & & \cdot\\ \cdot & \cdot & \ddots & \ddots & \cdot\\ \cdot & \cdot & & \ddots & 0\\ \cdot & \cdot & & & I\\ -C_r & 0 & \cdots & \cdots & 0\end{bmatrix},\quad r>0\\ &F=\underbrace{[0]}_{m}\}m,\quad F^\circ=I,\quad r=0\end{aligned}\right\} \tag{17}$$

$$G=\begin{cases}\underbrace{[I\ \ 0\ \cdots\ 0]}_{mr} & r>0\\ I & r=0.\end{cases} \tag{18}$$

It is easy to see that there exists a random mr-dimensional vector η_o depending on the initial values of $\{\xi_i\}$ such that

$$\xi_n=G\eta_n,\ \eta_{n-1}=F\eta_n+G^\tau\tilde{\theta}^\tau_n\phi_n. \tag{19}$$

In other words, $\{\eta_n\}$ is the state process of a realization of the $\{\xi_n\}$ process. Clearly

$$\xi_{n+1}=G\sum_{i=0}^{n}F^{n-i}G^\tau\tilde{\theta}^\tau_i\phi_i+GF^{n+1}\eta_o. \tag{20}$$

From this and (14) we see that

$$\tilde{\theta}_{n+1}=\tilde{\theta}_o-\sum_{j=0}^{n}\frac{\phi_j}{r_j}\left[\sum_{i=0}^{j}\phi^\tau_i\tilde{\theta}_iGF^{\tau(j-i)}G^\tau+\eta^\tau_oF^{\tau(j+1)}G^\tau+w^\tau_{j+1}\right]. \tag{21}$$

Now we list some conditions that we shall refer to later on.

1) Either $r=0$, or $r>0$ with $C(z)-\frac{1}{2}I$ strictly positive real and with the zeros of det $C(z)$ lying outside the closed unit disk.

2) There exist random variables $\alpha>0$, $\beta>0$, and $T>0$ such that

$$\sum_{i=m(t)}^{m(t+\alpha)}\frac{\phi_i\phi^\tau_i}{r_i}\geq\beta I,\ \forall t\geq T,\ \forall\omega\in[\omega: r_n\to\infty], \tag{22}$$

where

$$m(t)=\max\,[n: t_n\leq t],\ t\geq 0,\ t_n=\sum_{i=0}^{n-1}\beta_i,\ t_0=0, \tag{23}$$

$$\beta_i=\frac{\|\phi_i\|^2}{r_i}. \tag{24}$$

(This condition first appeared in [3].)

3) There exists a random variable γ, $0<\gamma<\infty$ such that $\nu^{on}_{\max}/\nu^{on}_{\max}\leq\gamma$, for all $n\geq 0$ where $\nu^{on}_{\max}$ and $\nu^{on}_{\min}$ are maximum and minimum eigenvalues of $N_n\triangleq\Sigma^{n'}_{i=1}\phi^o_i\phi^{o\tau}_i+1/dI$, $d=mp+lq+mr$.

4) There exists a random variable γ, $0<\gamma<\infty$ such that $\nu^n_{\max}/\nu^n_{\min}\leq\gamma$, for all $n\geq 0$ where $\nu^n_{\max}$ and $\nu^n_{\min}$ are maximum and minimum eigenvalues of $\Sigma^n_{i=1}\phi_i\phi^\tau_i+1/dI$, $d=mp+lq+mr$.

5) B_1 is of full rank and the zeros of $B^+_1B(z)$ (B^+_1 denotes the pseudoinverse of B_1) are outside the closed unit disk.

6) $B^+_1A(z)$ and $B^+_1B(z)$ are left coprime and $B^+_1A_p$ and $B^+_1B_q$ are of full rank.

The next two items specify alternative adaptive control laws.

7) The $\mathfrak{F}_n$-measurable control u_n is selected such that

$$\theta^\tau_n\phi_n=y^*_{n+1} \tag{25}$$

where y^*_n is a bounded deterministic sequence.

8) The $\mathfrak{F}_n$-measurable control u_n is selected such that

$$\theta^\tau_n\phi_n=y^*_{n+1}+v_n \tag{26}$$

with v_n an $\mathfrak{F}_n$-measurable disturbance sequence. (This is the so-called continually disturbed control law which was introduced in [1].)

Lemma 2: For the SG algorithm, under the conditions of Lemma 1, Conditions 3) and 4) are equivalent on $[\omega: r_n\to\infty]$ and each of them implies Condition 2).

Proof: This fact is proved in [5] as Theorem 2, but ξ_i and ϕ_i in that paper should be understood to be defined by (7) and (12) above. □

Theorem 1: For the system and algorithm in (1)-(10), if Conditions 1) and 2) are satisfied, then for almost all $\omega\in[\omega: r_n\to\infty]$

$$\theta_n\underset{n\to\infty}{\to}\theta.$$

Proof: Since the zeros of det $C(z)$ lie outside the closed unit disk for $r>0$, there exist constants $\rho\in(0,1)$ and $k_3>0$ such that for both the $r=0$ and the $r>0$ cases $\|F^i\|\leq k_3\rho^i$.

Comparing (30) to (48) of [5] we find that the quantities $1/r_jI$ and $\tilde{\theta}^\tau_i\phi_i$, respectively, of the present note correspond to α_jR_j and $\tilde{\theta}^\tau_{i+1}\phi_i$ of that paper; in that paper $\|\alpha_jR_j\|$ is estimated by k_7/r_j and instead of (15) above we have

$$\sum_{i=0}^{\infty}\frac{\|\tilde{\theta}^\tau_{i+1}\phi_i\|^2}{r_i}<\infty.$$

With such a correspondence the analysis from [5] completely fits the present case for $\omega\in[\omega: r_n\to\infty]$. To be precise, we first introduce some notation by setting

$$G_{n+1,i}=-\frac{1}{\beta_i}\sum_{j=i}^{n}\frac{\phi_j}{r_j}\phi^\tau_i\tilde{\theta}_iGF^{\tau(j-i)}G^\tau,\ G_{n,n}=0$$

and we also denote the last two terms in (21) by J_{n+1} and H_{n+1}, respectively. This yields

$$\tilde{\theta}_{n+1}=\tilde{\theta}_o+\sum_{i=0}^{n}\beta_iG_{n-1,i}+J_{n+1}+H_{n-1}. \tag{27}$$

We now introduce two interpolating functions $X(t)$ and $\bar{X}(t)$ for any given matrix sequence $\{X_i\}$. These are given by the following:

1) the linear interpolation

$$X(t_n)=X_n$$

$$X(t)=\frac{t_{n+1}-t}{\beta_n}X_n+\frac{t-t_n}{\beta_n}X_{n+1},\ t\in[t_n,t_{n+1}]$$

2) the constant interpolation

$$\bar{X}(t)=X_n,\ t\in[t_n,\ t_{n+1}).$$

Let $G^o_{t,i}$ denote the linear interpolation of $\{G_{n,i}\}$ with the interpolating length $\{\beta_n\}$ for $t \geqslant t_i$ whenever i is fixed. When $t=t_k$, then $G^o_{t_k,i}=G_{k,i}$. Now for fixed t denote by $\bar{G}^o_{t,s}$ the constant interpolation for the sequence $\{G^o_{t,i}\}$ on $[0, t]$ with interpolating length $\{\beta_i\}$.

By the use of these two interpolations, the sum in (38) for $\tilde{\theta}_{n+1}$ turns into an integral of the function $\bar{G}^o_{t,s}$ when it appears in the formula for the interpolating function $\tilde{\theta}(t)$. To be specific, we have

$$\tilde{\theta}(t)=\tilde{\theta}_o+\int_o^t \bar{G}^o_{t,s}\ ds+J(t)+H(t) \tag{28}$$

where $J(t)$ and $H(t)$ are the linear interpolating function for the sequences $\{J_n\}$, $\{H_n\}$.

Notice that since $\omega\in[\omega:r_n\to\infty]$, $\tilde{\theta}(t)$ is defined for all $t\geqslant 0$ and we notice that

$$\tilde{\theta}(t_n)=\tilde{\theta}_n. \tag{29}$$

Define the family of matrix functions $\{\tilde{\theta}_n(t)\}$ by shifting the argument of $\tilde{\theta}(t)$ to the left as shown by

$$\tilde{\theta}_n(t)=\tilde{\theta}(t+n)\qquad t\geqslant 0. \tag{30}$$

We now invoke a major technical result of [5]; the proof of this lemma is valid in the present case subject only to the exchange of the symbols mentioned above.

Lemma 3: Under the conditions of Theorem 1, for any fixed $\omega\in[\omega:r_n\to\infty]$, $\{\tilde{\theta}_n(t)\}$ is uniformly bounded and equicontinuous. □

Hence, according to the Arzela–Ascoli theorem, there exists a subsequence $\{\tilde{\theta}_{n_k}(t)\}$ of $\{\tilde{\theta}_n(t)\}$ and a continuous matrix function $\theta(t)$ which is the uniform limit of $\{\tilde{\theta}_{n_k}(t)\}$ over any finite interval.

It follows from the proof of the lemma that

$$\|\theta(t+\Delta)-\theta(t)\|=\lim_{k\to\infty}\|\tilde{\theta}_{n_k}(t+\Delta)-\tilde{\theta}_{n_k}(t)\|\equiv\lim_{k\to\infty}\|\tilde{\theta}(t+\Delta+n_k)-\tilde{\theta}(t+n_k)\|=0$$

and hence $\tilde{\theta}(t)$ is a constant matrix $\tilde{\theta}^o$.

Now we show that $\tilde{\theta}^o=0$; for a fixed $t\in[0,\infty)$ we have, by the Schwartz inequality, the definition of α and β in Condition 2) above and the fact that $\beta_i\leqslant 1$,

$$\lim_{k\to\infty}\left\|\sum_{i=m(t+n_k)}^{m(t+n_k+\alpha)}\frac{\phi_i\phi_i^T}{r_i}\tilde{\theta}_i\right\|\leqslant\sqrt{2+\alpha}\lim_{k\to\infty}\left(\sum_{i=m(t-n_k)}^{m(t+n_k-\alpha)}\frac{\|\tilde{\theta}_i^T\phi_i\|^2}{r_i}\right)^{1/2}=0. \tag{31}$$

Notice that for all $i\in[0, 1,\cdots, m(t+n_k+\alpha)-m(t+n_k)]\triangleq S$

$$t<t_{m(t+n_k)+1}-n_k\leqslant t_{m(t+n_k)+1+i}-n_k=t_{m(t+n_k)}+\sum_{j=m(t+n_k)}^{m(t+n_k)+i}\beta_j-n_k$$

$$\leqslant t_{m(t+n_k)}+\sum_{j=m(t+n_k)}^{m(t+n_k+\alpha)}\beta_j-n_k\leqslant t+\alpha+2,$$

in other words, $t_{m(t+n_k)+i+1}-n_k\in[t,\ t+\alpha+2]$, for all $i\in S$. Hence,

$$\tilde{\theta}_{n_k}(t_{m(t+n_k)+i+1}-n_k)\to\tilde{\theta}^o \text{ as } k\to\infty \text{ and by (29), (30)}$$

$$\tilde{\theta}_{m(t+n_k)+i+1}\underset{k\to\infty}{\to}\tilde{\theta}^o \tag{32}$$

uniformly in $i\in S$. Consequently, we assert

$$\lim_{k\to\infty}\sum_{i=m(t+n_k)}^{m(t+n_k+\alpha)}\beta_i\|\tilde{\theta}_i-\tilde{\theta}^o\|=0.$$

From (31), (32) it is easy to conclude that

$$\tilde{\theta}^{oT}\left[\lim_{k\to\infty}\sum_{i=m(t+n_k)}^{m(t+n_k+\alpha)}\frac{\phi_i\phi_i^T}{r_i}\right]\tilde{\theta}^o=0. \tag{33}$$

[It is worth remarking that until this point all results have been obtained without invoking Condition 2)].

Now by using 2) it follows immediately from (33) that

$$\tilde{\theta}^o=0 \text{ on } [\omega:r_n\to\infty],$$

and $\tilde{\theta}(t+n_k)\to 0$ uniformly in $t\in[a, b]$, where $[a, b]$ is any finite interval. Since $\tilde{\theta}(t_n)=\tilde{\theta}_n$ it is easy to see that for any fixed $\omega\in[\omega:r_n\to\infty]$ there exists a subsequence $\tilde{\theta}_{m_k}\to 0$ as $k\to\infty$. From here by Lemma 1 we conclude that $\lim_{n\to\infty}\operatorname{tr}\tilde{\theta}_n^T\tilde{\theta}_n=\lim_{k\to\infty}\operatorname{tr}\tilde{\theta}_{m_k}^T\tilde{\theta}_{m_k}=0$, i.e.,

$$\theta_n\underset{n\to\infty}{\to}\theta\ \forall\omega\epsilon[\omega:r_n\to\infty]. \qquad\square$$

Let $\{y_n^*\}$ be a bounded deterministic reference sequence.

Theorem 2: Assume that for (1)–(10) Conditions 1, 5, and 7) are satisfied, and w satisfies

$$0<\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^n\|w_i\|^2<\infty \text{ a.s.} \tag{34}$$

Then $r_n\to\infty$ and

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^n\|u_i\|^2<\infty,\ \overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^n\|y_i\|^2<\infty \tag{35}$$

$$\overline{\lim_{n\to\infty}}\left\|\frac{1}{n}\sum_{i=1}^n(y_i-y_i^*)(y_i-y_i^*)^\tau-\frac{1}{n}\sum_{i=1}^n w_iw_i^\tau\right\|=0. \tag{36}$$

Proof: Suppose $\lim_{n\to\infty}r_n<\infty$, then $\phi_n\to 0$, $y_n\to 0$, $u_n\to 0$, and hence $w_i\to 0$. But, since $C(z)$ is asymptotically stable and (34) holds, this is seen to be an event of probability zero and so $r_n\to\infty$ a.s.

By 5) there exists constants (which may depend upon ω) such that

$$\frac{1}{n}\sum_{i=1}^n\|u_i\|^2\leqslant\frac{k_4}{n}\sum_{i=1}^n\|y_{i+1}\|^2+k_5.$$

Then using (12), (34), and 7), it can be shown (as in, e.g., [6]) that

$$\frac{1}{n}\sum_{i=1}^n\|y_{i+1}\|^2\leqslant\frac{k_8}{n}\sum_{i=1}^n\|\xi_{i+1}\|^2+k_q.$$

Consequently,

$$\overline{\lim_{n\to\infty}}\frac{r_n}{n}<\infty, \tag{37}$$

and hence via the Kronecker lemma we have

$$\frac{1}{n}\sum_{i=1}^n\|\xi_{i+1}\|^2\underset{n\to\infty}{\to}0 \text{ a.s.} \tag{38}$$

But by (12) and Condition 7)

$$(y_i-y_i^*)(y_i-y_i^*)^\tau-w_iw_i^\tau=\xi_i\xi_i^\tau+\xi_iw_i^\tau+w_i\xi_i^\tau$$

and from this and (34) we obtain (36), while (35) follows from (37). □

Now we introduce the continually disturbed controls of Condition 8).

Theorem 3: Let $\{w_i\}$, $\{v_i\}$ be two mutually independent i.i.d. sequences with $Ev_i = Ew_i = 0$, $Ew_iw_i^\tau = R_1 > 0$, $Ev_iv_i^\tau = R_2 > 0$, and let $\{y_i^*\}$ be a bounded deterministic reference sequence. For the system and algorithm (1)–(10), with $\mathfrak{F}_n \triangleq \sigma\{w_i, v_i, i \leqslant n\}$; if Conditions 1), 5), 6), and 8) are satisfied, then $r_n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 < \infty, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|y_i\|^2 < \infty$$

$$\cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (y_i - y^*_i)(y_i - y^*_i)^\tau = R_1 + R_2 \tag{39}$$

and

$$\theta_n \underset{n\to\infty}{\to} \theta \text{ a.s.}$$

Proof: (Sketch) From (12) and Condition 8) we have

$$y_{n+1} = \xi_{n+1} + y^*_{n+1} + (w_{n+1} + v_n). \tag{40}$$

By the strong law of large numbers we know that $1/n \sum_{i=1}^n w_iw_i^\tau \underset{n\to\infty}{\to} R_1$, and hence $r_n \underset{n\to\infty}{\to} \infty$ a.s. Recall that (37) and (38) still hold.

Consequently, (39) follows immediately.

To complete the proof it is sufficient to show that Condition 3) holds for the regression vector ϕ_n^o in (8), since then, by Lemma 2, Condition 2) is true and consequently Theorem 1 may be applied.

But this follows using the method of analysis contained in [7] and [2]. □

Concerning the sequence $\{y_n^*\}$, we observe that only the boundedness property is used and not the existence of the limits $\lim_{n\to\infty} 1/n \sum_{i=1}^n y^*_{i-k} y^{*\tau}_{i-l}$, for all k, l as assumed in the two papers cited above.

We also note that the i.i.d. hypotheses on w and v in this theorem were only adopted for simplicity and that Theorem 3 holds, with random $R > 0$ and $\gamma < \infty$, if w and v are taken to be mutually uncorrelated ergodic martingale difference sequences as in [7] and [2].

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