and

$$G \triangleq \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & -a_2^{(2)} & \cdots & & -a_\beta^{(2)} \\ 0 & -a_3^{(2)} & & & 0 \\ \vdots & & & & \vdots \\ 0 & -a_\beta^{(2)} & 0 & \cdots & 0 \end{bmatrix}$$

and recursively defining $\beta$-dimensional

$$x_k \triangleq [x_{k,1} \ \cdots \ x_{k,\beta}]^T$$

by

$$x_{k+1} = Dx_k + H^T C(z) w_{k+1} \tag{33}$$

where $H = \underbrace{[1 \ 0 \ \cdots \ 0]}_{\beta}$, we have

$$x_k = G[\xi_k \ \xi_{k-1} \ \cdots \ \xi_{k-\beta+1}]^T$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 < \infty \ \text{ a.s.}$$

From (33), we have

$$D^{-(k+1)} x_{k+1} = x_0 + \sum_{i=1}^{k+1} D^{-i} H^T C(z) w_i \tag{34}$$

where $x_0$ is a deterministic vector defined by initial values $y_0, y_{-1}, \ldots, y_{-p}$. The right-hand side of (34) converges a.s. to a nonzero random vector.

On the other side, however, $\{\|x_k\|/\sqrt{k}\}$ is a bounded sequence. This means that

$$\left\| D^{-(k+1)} x_{k+1} \right\| \le c\lambda^{k+1} \frac{\|x_{k+1}\|}{\sqrt{k+1}} \sqrt{k+1} \underset{k\to\infty}{\longrightarrow} 0 \ \text{ a.s.}$$

where $\lambda \in (0, 1)$ and $c$ is a constant.

The obtained contradiction shows that no root of $A(z)$ can be explosive. □

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# Adaptive Regulator for Discrete-Time Nonlinear Nonparametric Systems

Han-Fu Chen and Qian Wang


***Abstract*—A direct adaptive regulator for nonlinear nonparametric systems with measurement corrupted by noise is proposed. Under reasonable conditions the state of the closed-loop system is adaptively regulated so that it converges to zero as time tends to infinity. An illustrative example, being an affine nonlinear system, with all imposed conditions satisfied is given. The method of proof is based on stochastic approximation techniques.**


***Index Terms*—Adaptive regulator, nonlinear nonparametric systems, stochastic approximation.**

## I. INTRODUCTION

For most of practical systems the linear model is merely an approximation to the true system dynamics. This probably is the reason why much research attention has been paid to the nonlinear systems for recent years. Various typical nonlinear models are considered in literature, for example, the nonlinear ARX model is considered in [12], bilinear model in [14] and the Hammerstein model in [17]. The common feature for all these models is that the system is parameterized and the parameters linearly enter the models. Therefore, when the parameters are unknown in these models, they may recursively be estimated by conventional methods, for example, the least-squares (LS) method, and the parameter estimates may be used to form adaptive controls [7], [6], [15], [13], [8], [9]. Although parameterization of system uncertainties simplifies forming adaptive control laws, it is not an easy task to analyze the resulting nonlinear adaptive control systems (see [12]).

To design and to analyze adaptive control for nonparametric nonlinear systems in a random environment is the topic of the present note. To the authors' knowledge this is the first attempt to make a rigorous analysis for this difficult problem. As a first step, we have to restrict ourselves to consider the relatively simple case, adaptive regulation, rather than the general adaptive control problem. The purpose of regulation is to control a system in order its state or output to reach a desired value. Since the system is unknown, one may intend to realize regulation adaptively. The resulting adaptive control system is then called adaptive regulator. Even for this rather simple task, we have to impose rather restrictive but reasonable conditions on the nonlinear dynamics of the system. The system state is observed with additive noise. By noticing the inherent connection between adaptive regulation and the problem of searching zero of an unknown nonlinear function, we will apply the stochastic approximation method to propose an adaptive regulator and prove the regulation error asymptotically tending to zero.

To solve the stated problem under general conditions is beyond the target of this note. This note aims at stimulating research on nonlinear stochastic adaptive control, pointing out the possibility of shifting from the parametrization framework to more natural nonparametric approach. It is worth noting that stochastic approximation only serves as a tool to solve the stated problem rather than a research topic in this note.

Manuscript received August 8, 1999; revised May 31, 2000 and March 25, 2001. Recommended by Associate Editor L. Dai. This work was supported by the National Natural Science Foundation of China and the National Key Project of China.

The authors are with the Systems and Control Laboratory, Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, P.R. China (e-mail: hfchen@iss03.iss.ac.cn).

Publisher Item Identifier S 0018-9286(01)10363-6.

This note is presented as follows. The adaptive regulator is defined and the conditions to be used are listed in Section II, and the main results are given in Section III where an example of nonlinear systems satisfying all conditions imposed in Theorem 1 is presented too. Some concluding remarks are given in the last section.

## II. Adaptive Regulator

Consider the following nonlinear nonparametric system:

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^n$ is the control input, $f(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown nonlinear function with $(0, u^0)$ being the unknown equilibrium pair for System (1).

The system state $x_k$ can be observed with noise

$$y_{k+1} = x_{k+1} + \varepsilon_{k+1} \tag{2}$$

where $\varepsilon_{k+1} \in \mathbb{R}^n$ is the measurement noise and may depend on $u_k$.

The purpose of adaptive regulation is to define adaptive control based on measurements in order the system state to reach the desired one. Without loss of generality, we may assume the specified state the system is regulated to is zero.

The adaptive control is given according to the following recursive algorithm:

$$u_{k+1} = (u_k - a_k y_{k+1}) I_{[\|u_k - a_k y_{k+1}\| < 2b]} \tag{3}$$

where $b$ is specified in A1) given below and the step size $\{a_i\}$ is nonincreasing with

$$a_i > 0, \quad a_i \to 0, \quad \sum_{i=1}^{\infty} a_i = \infty. \tag{4}$$

The system composed of (1)–(4) is the adaptive regulator to be considered in this note.

We need the following conditions.

A1) The upper bound $b$ for $u^0$ is known, i.e., $\|u^0\| < b$, and $u^0$ is a robust stabilizing control in the sense that for any $d_k \xrightarrow[k\to\infty]{} 0$ the state $x_k$ tends to zero for the following system:

$$x_{k+1} = f(x_k, u^0) + d_k.$$

A2) System (1) is BIBS stable, i.e., for any bounded input, the system state is also bounded.

A3) $f(x, \cdot)$ is continuous for bounded $x$, i.e., for any $a > 0$

$$\sup_{\|x\| \le a} \|f(x, u + \Delta u) - f(x, u)\| \xrightarrow[\|\Delta u\| \to 0]{} 0.$$

A4) System (1) is strictly input passive[21, 22], i.e., there are $\beta$ and $\epsilon > 0$ such that for any input $\{u_i\}$

$$\sum_{i=1}^{n} u_i^\tau x_{i+1} \ge \epsilon \sum_{i=1}^{n} \|u_i\|^2 + \beta, \qquad \forall n. \tag{5}$$

A5) The noise sequence $\{\varepsilon_i\}$ satisfies

$$\lim_{T\to 0} \limsup_{k\to\infty} \frac{1}{T} \left\| \sum_{i=n_k}^{m(n_k, t)} a_i \varepsilon_{i+1} \right\| = 0, \qquad \forall t \in [0, T] \tag{6}$$

along any convergent subsequence $\{u_{n_k}\}$, where $\{a_i\}$ is the step size used in (3) and $m(n, T)$ is the integer-valued function defined by

$$m(k, T) \triangleq \max\left\{ m: \sum_{i=k}^{m} a_i \le T \right\}. \tag{7}$$

*Remark 1:* If the noise $\{\epsilon_i\}$ is independent of $\{u_i\}$, then A5) is equivalent to

$$\lim_{T\longrightarrow 0} \limsup_{n\longrightarrow\infty} \frac{1}{T} \left\| \sum_{i=n}^{m(n, t)} a_i \epsilon_{i+1} \right\| = 0, \qquad \forall t \in [0, T]. \tag{8}$$

In the case where $\epsilon_{i+1}$ depends on the past control $\{u_j, j \ge i\}$, A5) is easier to be verified in comparison with (8). Further, if $\sum_{i=1}^{\infty} a_i \epsilon_{i+1} < \infty$, then (6) is clearly satisfied. This is the case if $(\epsilon_i, \mathcal{F}_i)$ is a martingale difference sequence with $\sup_i (E\|\epsilon_{i+1}\|^2 | \mathcal{F}_i) < \infty$ and if $\sum_{i=1}^{\infty} a_i^2 < \infty$.

## III. Main Results

In this section, we intend to show that the adaptive control given by (3) reaches the goal of regulation, i.e., it regulates the system state tending to the desired state, zero. As a matter of fact, we have the following result.

*Theorem 1:* Suppose A1)–A5) hold. Then the adaptive regulator (1)–(4) has the desired properties:

$$u_k \to u^0, \quad x_k \to 0, \quad k \to \infty \tag{9}$$

at sample paths where A5) is satisfied.

*Proof:* Before proceeding to the proof, we first note that the algorithm (3) defining $u_k$ is nothing else but a projected stochastic approximation algorithm [10], [11], [1], [4].

Assume A1)–A5) hold. We complete the proof by four steps.

*Step 1:* Let $u_{n_i}$ be a convergent subsequence of $\{u_k\}$ defined by (3) such that $u_{n_i} \xrightarrow[i\to\infty]{} \overline{u}$, and $\|\overline{u}\| < 2b$. We show that

$$u_{m+1} = u_m - a_m y_{m+1} \tag{10}$$

$$\|u_{m+1} - u_{n_i}\| \le ct, \qquad \forall m: n_i \le m \le m(n_i, t), \quad \forall t \in [0, T] \tag{11}$$

for sufficiently large $i$ and small enough $T$, where $c$ is a constant to be specified later on.

Since system (1) is BIBS, from $\|u_k\| < 2b$ it follows that there is $a > 0$ such that $\|x_k\| \le a, \forall k$.

By A5) for large $i$ and small $T > 0$,

$$\left\| \sum_{j=n_i}^{m} a_j \varepsilon_{j+1} \right\| \le at, \qquad \forall m: n_i \le m \le m(n_i, t), \ \forall t \in [0, T].$$

This implies that

$$\left\| \sum_{j=n_i}^{m} a_j y_{j+1} \right\| = \left\| \sum_{j=n_i}^{m} a_j (x_{j+1} + \varepsilon_{j+1}) \right\| \le a \sum_{j=n_i}^{m} a_j + \sum_{j=n_i}^{m} a_j \varepsilon_{j+1} \le 2at, \quad \forall m: n_i \le m \le m(n_i, t).$$

Let $i$ be large enough such that

$$\|u_{n_i} - \overline{u}\| < \tfrac{1}{2}(2b - \|\overline{u}\|)$$

and let $T$ be small enough such that

$$aT < \tfrac{1}{4}(2b - \|\overline{u}\|).$$

Then we have

$$\|u_{n_i} - a_{n_i} y_{n_i+1}\| \le \|u_{n_i} - \overline{u}\| + \|\overline{u}\| + \|a_{n_i} y_{n_i+1}\| < 2b$$

and hence there is no truncation in (3) for $k = n_i$, i.e., (10) holds for $m = n_i$. Therefore

$$\|u_{n_i+1} - u_{n_i}\| = \|a_{n_i} y_{n_i+1}\| \le 2at \triangleq ct.$$

Thus, we have shown that (10) and (11) hold for $m = n_i$.

Assume (10) and (11) are true for all $m$: $m \le k$, $n_i \le k < m(n_i, t)$. We now show that they are true for $m = k+1$ too.

Since

$$\begin{aligned}\|u_{k+1} - a_{k+1}y_{k+2}\| &= \left\| u_{n_i} - \sum_{j=n_i}^{k+1} a_j y_{j+1} \right\| \\ &\le \|u_{n_i} - \overline{u}\| + \|\overline{u}\| + \left\| \sum_{j=n_i}^{k+1} a_j y_{j+1} \right\| < 2b\end{aligned}$$

from the algorithm (3) it follows that $u_{k+2} = u_{k+1} - a_{k+1}y_{k+2}$, or (10) holds for $m = k+1$.

Hence

$$\|u_{k+2} - u_{n_i}\| = \left\| \sum_{j=n_i}^{k+1} a_j y_{j+1} \right\| \le 2at \triangleq ct$$

and (11) is true for $m = k+1$ indeed.

By mathematical induction, the assertions (10) and (11) have been proved.

*Step 2:* We now show that for any convergent subsequence $\{u_{n_k}\}$, $u_{n_k} \longrightarrow \overline{u} \ne u^0$ there is a $\delta > 0$ such that

$$\liminf_{k \longrightarrow \infty} \frac{1}{T} \sum_{i=n_k}^{m(n_k, T)} a_i (u_i - u^0)^\tau x_{i+1} \ge \delta \tag{12}$$

for all small enough $T > 0$. By A4), (5) is satisfied for any $u_i$, so it holds with $u_i$ replaced by $u_i - u^0$

$$\begin{aligned}\sum_{j=n_k}^{i} (u_j - u^0)^\tau x_{i+1} &\ge \epsilon \sum_{j=n_k}^{i} \|u_j - u^0\|^2 + \beta \\ &= \epsilon \sum_{i=n_k}^{i} \|\overline{u} - u^0 + (u_i - \overline{u})\|^2 + \beta \\ &= \epsilon \sum_{i=n_k}^{i} \|\overline{u} - u^0\|^2 + 2\epsilon \sum_{i=n_k}^{i} (\overline{u} - u^0)^\tau (u_i - \overline{u}) \\ &\quad + \epsilon \sum_{i=n_k}^{i} \|u_i - \overline{u}\|^2 + \beta.\end{aligned} \tag{13}$$

Let us restrict $i$ in (13) to $\{n_k, n_k+1, \ldots, m(n_k, T)\}$.

Then for small $T$ and large $k$, from (11) and (13) it follows that:

$$\begin{aligned}&\frac{1}{i - n_k + 1} \sum_{j=n_k}^{i} (u_j - u^0)^\tau x_{i+1} \\ &\quad \ge \epsilon \|\overline{u} - u^0\|^2 - 4\epsilon \|\overline{u} - u^0\| cT - \epsilon 4T^2 + \frac{\beta}{i - n_k + 1}\end{aligned}$$

for $i \in [n_k, \ldots, m(n_k, T)]$.

This implies that there exist a $\delta > 0$ and a sufficiently large $i_0$, which may depend on $\overline{u}$ but is independent of $k$, such that

$$\begin{aligned}&\frac{1}{i - n_k + 1} \sum_{j=n_k}^{i} (u_j - u^0)^\tau x_{i+1} > \delta \\ &\qquad \forall i \in [n_k + i_0, n_k + i_0 + 1, \ldots, m(n_k, T)\}\end{aligned} \tag{14}$$

for all sufficiently large $k$ and small enough $T > 0$.

Set

$$S_{n_k, i} = \sum_{j=n_k}^{i} (u_j - u^0)^\tau x_{j+1}, \qquad S_{n_k, n_k - 1} = 0.$$

Using a partial summation, by (14) we have

$$\begin{aligned}&\sum_{i=n_k}^{m(n_k, T)} a_i (u_i - u^0)^\tau x_{i+1} \\ &\quad = \sum_{i=n_k}^{m(n_k, T)} a_i (S_{n_k, i} - S_{n_k, i-1}) \\ &\quad = a_{m(n_k, T)} S_{n_k, m(n_k, T)} + \sum_{i=n_k}^{m(n_k, T)-1} (a_i - a_{i+1}) S_{n_k, i} \\ &\quad > a_{m(n_k, T)} \delta (m(n_k, T) - n_k + 1) \qquad (15) \\ &\qquad + \sum_{i=n_k}^{n_k + i_0 - 1} (a_i - a_{i+1}) S_{n_k, i} \\ &\qquad + \sum_{i=n_k + i_0}^{m(n_k, T)-1} (a_i - a_{i+1}) \delta (i - n_k + 1). \qquad (16)\end{aligned}$$

Since $\|u_i\| < 2b$, $\|x_i\| < a$, it is seen that

$$\begin{aligned}&\left\| \sum_{i=n_k}^{n_k + i_0 - 1} (a_i - a_{i+1}) S_{n_k, i} \right\| \\ &\quad \le \sum_{i=n_k}^{n_k + i_0 - 1} (a_i - a_{i+1}) i_0 (2b + \|u^0\|) a \\ &\quad = i_0 (2b + \|u^0\|) a (a_{n_k} - a_{n_k + i_0}) \underset{k \longrightarrow \infty}{\longrightarrow} 0.\end{aligned}$$

Then, (15) implies that

$$\begin{aligned}&\sum_{i=n_k}^{m(n_k, T)} a_i (u_i - u^0)^\tau x_{i+1} \\ &\quad > a_{m(n_k, T)} (\delta (m(n_k, T) - n_k + 1) - \delta (m(n_k, T) - n_k)) \\ &\qquad + a_{n_k + i_0} \delta (i_0 + 1) + \delta \sum_{i=n_k + i_0 + 1}^{m(n_k, T)-1} a_i + o(1) \\ &\quad = \delta a_{m(n_k, T)} + a_{n_k + i_0} \delta (i_0 + 1) + \delta \sum_{i=n_k + i_0 + 1}^{m(n_k, T)-1} a_i + o(1) \\ &\quad \underset{k \longrightarrow \infty}{\longrightarrow} \delta T.\end{aligned}$$

This proves (12).

*Step 3:* Define $V(u) = \|u - u^0\|^2$. We show that $V(u_k)$ cannot cross a nonempty interval infinitely many times.

Notice that

$$V(0) = \|u^0\|^2 < b^2 < \inf_{\|u\|=2b} \|u - u^0\|^2 = \inf_{\|u\|=2b} V(u).$$

Assume the contrary, i.e., there are two subsequences $\{u_{m_i}\}$ and $\{u_{l_i}\}$ and a nonempty interval $[\delta_1, \delta_2]$ such that $\delta_2 > \delta_1 > 0$, $V(u_{m_i}) \le \delta_1$, $\delta_1 < V(u_k) < \delta_2$ for $k$: $m_i < k < l_i$, and $V(u_{l_i}) \ge \delta_2$.

Without loss of generality we may assume $u_{m_i} \underset{i \to \infty}{\longrightarrow} \overline{u}$. From Step 1, for sufficiently large $i$ and small enough $T$, we have

$$\|u_{m_i + 1} - u_{m_i}\| \le ct$$

which tends to zero as $i \to \infty$ and $T \to 0$, and hence $u_{m_i+1} \underset{i \to \infty}{\longrightarrow} \overline{u}$. From the continuity of $V(\cdot)$, it follows that:

$$V(u_{m_i}) \to V(\overline{u}) \le \delta_1, \quad \text{and} \quad V(u_{m_i+1}) \to V(\overline{u}) \ge \delta_1$$

which imply $V(\overline{u}) = \delta_1$. Since $V(u^0) = 0$, we conclude that $\overline{u} \neq u^0$. By the Taylor's expansion and by (10) we have

$$\begin{aligned} V\left(u_{m(m_i,T)+1}\right) - V\left(u_{m_i}\right) &= -V_u^\tau(\tilde{u}) \sum_{j=m_i}^{m(m_i,T)} a_j(f(x_j, u_j) + \varepsilon_{j+1}) \\ &= -V_u^\tau(\tilde{u}) \sum_{j=m_i}^{m(m_i,T)} a_j\varepsilon_{j+1} - \sum_{j=m_i}^{m(m_i,T)} a_j V_u^\tau(u_j) x_{j+1} \\ &\quad + \sum_{j=m_i}^{m(m_i,T)} a_j (V_u(u_j) - V_u(\tilde{u}))^\tau x_{j+1}, \end{aligned} \tag{17}$$

where $\|\tilde{u} - u_{m_i}\| \leq cT$. By noting (11), (5), the continuity of $V_u(\cdot)$ and the boundedness of $x_k$, we see that the last term in (16) is the order of $o(T)$ as $T \to 0$, and by A5) the first term on the right-hand side of (17) is also the order of $o(T)$. From (12), it follows that there exists an $\varepsilon > 0$ such that

$$-\sum_{j=m_i}^{m(m_i,T)} a_j V_u^\tau(u_j) x_j \leq -\varepsilon T \tag{18}$$

for sufficiently small $T$.

Hence for $i$ large enough and $T$ small enough we have

$$V\left(u_{m(m_i,T)+1}\right) - V\left(u_{m_i}\right) \leq -\frac{\varepsilon}{2} T \tag{19}$$

and hence

$$\limsup_{i\to\infty} V\left(u_{m(m_i,T)+1}\right) \leq \delta_1 - \frac{\varepsilon}{2} T. \tag{20}$$

It follows from (11) that:

$$\max_{m_i \leq m \leq m(m_i,T)+1} \|V(u_m) - V(u_{m_i})\| \xrightarrow[T\to 0]{} 0$$

which implies that $V(u_{m(m_i,T)+1}) \in [\delta_1, \delta_2]$ for small $T$. However, this contradicts (20). The contradiction proves that $V(u_k)$ cannot cross a nonempty interval infinitely often (i.o.). As a consequence, the algorithm (3) will cease to truncate after a finite number of times, because $V(0) = \|u^0\|^2 < b^2 < \inf_{\|u\|=2b} \|u - u^0\|^2 = \inf_{\|u\|=2b} V(u)$.

*Step 4:* Denote $v_1 \triangleq \liminf_{n\to\infty} V(u_n) \leq \limsup_{n\to\infty} V(u_n) \triangleq v_2$.

If $v_1 < v_2$, then $V(u_n)$ will cross some interval $[\delta_1, \delta_2]$ with $\delta_1 > 0$ infinitely many times. From Step 3, this is impossible. So, $v_1 = v_2$, or $V(u_n)$ converges.

If $u_k$ does not converge to $u^0$, then there is a convergent subsequence $\{u_{n_i}\}$ such that $u_{n_k} \xrightarrow[k\to\infty]{} \overline{u} \neq u^0$. Replacing $m_i$ in Step 3 by $n_k$, we again have (17)–(19). Since $V(u_n)$ converges, taking limit in both sides of (19) we arrive at $0 \leq -(\varepsilon/2)T$, which is impossible. Hence, $u_n \to u^0$.

Write (1) as $x_{k+1} = f(x_k, u^0) + f(x_k, u_k) - f(x_k, u^0)$. By A3) and the boundedness of $\{x_k\}$ we have $d_k \triangleq f(x_k, u_k) - f(x_k, u^0) \xrightarrow[k\to\infty]{} 0$, and by A1), we conclude $x_k \to 0$. □

*Remark 2:* It is easy to see that A5) is also necessary if A1–A4 and (9) hold. This is because for large $k$ the observation noise can be expressed as

$$\epsilon_{k+1} = \frac{u_{k+1} - u_k}{a_k} + f(x_k, u_k)$$

and hence

$$\sum_{i=n_k}^{m(n_k,T)} a_i \epsilon_{i+1} = \sum_{i=n_k}^{m(n_k,T)} (u_{i+1} - u_i) + \sum_{i=n_k}^{m(n_k,T)} a_i f(x_i, u_i)$$

which tends to zero by (9).

Theorem 1 remains valid if Condition A4) is replaced by the weaker condition either (12) or (14), because in the proof we only use (12) which in turn is implied by (14). We formulate this as Theorem 2.

*Theorem 2:* Suppose that System (1) satisfies A1), A2), A3), and A5), and for any convergent subsequence $\{u_{n_k}\}$, $u_{n_k} \to \overline{u} \neq u^0$ one of the following conditions is fulfilled:

i) there is a $\delta > 0$ such that (12) holds for all small enough $T > 0$;
ii) there exist a $\delta > 0$ and a sufficiently large $i_0$, which may depend on $\overline{u}$ but is independent of $k$, such that (14) holds for all $i \in [n_k + i_0, \ldots, m(n_k, T)]$ if $k$ is sufficiently large and $T > 0$ is small enough.

Then (9) holds at sample paths where A5) is satisfied.

*Remark 3:* The quadratic $V(\cdot)$ can be replaced by a continuously differentiable function $V(\cdot): R^n \longrightarrow R$ such that $V(u^0) = 0$, and $V(0) < \inf_{\|u\|=2b} V(u)$. Then, Theorem 2 remains valid if $(u_j - u^0)^\tau$ in (12) and (14) is replaced by $V_u^\tau(u_j)$, where $V_u$ denotes the gradient of $V$.

*Example:* Let the nonlinear system be affine

$$x_{k+1} = g(x_k)(u_k - u^0)$$

where the scalar nonlinear function $g(\cdot)$ is bounded from above and from below by positive constants: $0 < \alpha \leq g(x) \leq \beta < \infty, \forall x \in \mathbb{R}^n$.

Note that $(u_j - u^0)^\tau x_{j+1} = g(x_j)\|u_j - u^0\|^2 \geq \alpha\|u_j - u^0\|^2$, and hence (14) holds, if $u_{n_k} \longrightarrow \overline{u} \neq u^0$. Assume $b$ is known: $\|u^0\| < b$. Then A1), A2), and A3) are satisfied. Therefore, if $\{\epsilon_i\}$ satisfies A5), then $\{u_k\}$ given by (3) leads to $u_k \longrightarrow u^0$ and $x_k \longrightarrow 0$.

## IV. Conclusion

This note concerns the adaptive control for general nonlinear nonparametric systems. Based on stochastic approximation methods we presented a solution to the adaptive regulation problem under reasonable conditions on the nonlinear dynamics as well as on the measurement noise. In the further study, it may be of importance to consider the stochastic dynamic system where the noise may appear not only in observations but also in the state equation. To weaken the conditions required in theorems may also be of interest. Our results may serve as an initial step toward solving the general adaptive control problem.

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# Comments on “Parameterization of Stabilizing Compensators by Using Reduced-Order Observers”

Zhiwei Gao and Daniel W. C. Ho

***Abstract*—In this note, a counterexample of the above mentioned paper[1] is given. It is shown that the parameterization of all proper compensators internally stabilizing the plant is incomplete. Modified parameterization of all proper stabilizing compensators is also presented.**

***Index Terms*—Coprime factorization, parameterization, proper stabilizing compensators.**

## I. Introduction

A doubly coprime factorization (DCF) plays an important role in investigating multivariable control problems by the factorization approach. Recently, Fujimori presented a parameterization of all proper stabilizing compensators using the DCF related to the minimal-order observer. However, the proper controller parameterization in the above paper[1] is incomplete as shown.

In this note, the transfer function matrix of the plant is denoted by

$$G(s) = C(sI - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{1}$$

$G(s)_o := C$ and $G(s)_i := B$ represent the output matrix and the input matrix of $G(s)$ respectively. Let $C_+$ denote the closed, complex right-half plane, and $C_{+e} := C_+ \bigcup \{\infty\}$. The assumptions and the remaining symbols are the same as those defined in the paper[1]. We now restate Theorems 1 and 2 from the above paper[1] as follows.

*Theorem 1:* The following stable rational function matrices:

$$\begin{aligned}\begin{pmatrix} Y_r & X_r \\ -N_l & D_l \end{pmatrix} &= \left[\begin{array}{c|cc} F & TB - HD & H \\ \hline KP & I_m - KVD & KV \\ -CAP & -CB - (sI_p - CAV)D & sI_p - CAV \end{array}\right]\end{aligned} \tag{2}$$

$$\begin{pmatrix} D_r & -X_l \\ N_r & Y_l \end{pmatrix} = \left[\begin{array}{c|cc} A_K & B & V \\ \hline -K & I_m & 0 \\ C - DK & D & 0 \end{array}\right] \tag{3}$$

or

$$\begin{pmatrix} Y_r & X_r \\ -N_l & D_l \end{pmatrix} = \left[\begin{array}{c|cc} A_L & B - LD & L \\ \hline N & 0 & 0 \\ -C & -D & I_p \end{array}\right] \tag{4}$$

$$\begin{aligned}\begin{pmatrix} D_r & -X_l \\ N_r & Y_l \end{pmatrix} &= \left[\begin{array}{c|cc} G & MAB & ML \\ \hline -J & sI_m - NAB & -NL \\ CS - DJ & CB + D(sI_m - NAB) & I_p - DNL \end{array}\right]\end{aligned} \tag{5}$$

satisfy the Bezout identity

$$\begin{bmatrix} Y_r & X_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -X_l \\ N_r & Y_l \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \tag{6}$$

*Theorem 2:* When using transfer function matrices given in (2) and (3) or (4) and (5), the set of all *proper* compensators $C(s)$ internally stabilizing $G(s)$ is parameterized by

$$\begin{aligned} C(s) &= (Y_r - QN_l)^{-1}(X_r + QD_l) \\ &= (X_l + D_rQ)(Y_l - N_rQ)^{-1} \end{aligned} \tag{7}$$

where $Q \in RH_2^{m\times p}$.

*Remark 1:* Since some elements in the DCF given by Theorem 1 are stable but nonproper, the free parameter $Q$ has to be restricted within the set of strictly proper and stable rational function matrices for obtaining resultant stabilizing compensators. It is noted that, the proof of Theorem 2 in the above mentioned paper[1] only proved that the compensators $C(s)$ in (7) are all compensators for stabilizing $G(s)$, however, the properness of these compensators was not discussed. Unfortunately, it will be shown that some of these compensators in (7) cannot be guaranteed to be proper even when $Q \in RH_2^{m\times p}$.

## II. A Counter-Example

Consider a controllable and observable system

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t) \\ y(t) = (\,1 \quad 0\,)x(t) + u(t). \end{cases} \tag{8}$$

a) We first consider the DCF defined in (2) and (3). Choose $F = -2$ and $H = 1$. From the restraint conditions of the minimal-order observer (13) in the above paper[1] (or see [1]), we have the solutions that $T = (2/5,\ -1/5)$, $V = \binom{1}{2}$, $P = \binom{0}{-5}$. Find

Manuscript received January 4, 2000; revised June 13, 2000, January 17, 2001, and June 15, 2001. Recommended by Associate Editor B. Chen. This work was supported by the Natural Science Foundation of China under Grant NSFC-69874027 and by the Hong Kong Research Grant Council (CityU 1073/97E).

Z. W. Gao is with the Department of Automation, Tianjin University, Tianjin 300072, P. R. China (e-mail: zhiwegac@public.tpt.tj.cn).

D. W. C. Ho is with the Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, P. R. China (e-mail: madaniel@cityu.edu.hk).

Publisher Item Identifier S 0018-9286(01)10348-X.

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