IV. CONCLUSION

The sliding-mode control of nonlinear uncertain systems with unmodeled first-order actuator dynamics has been considered. A 2-SMC scheme with adaptive switching rule has been proposed, and its effectiveness has been shown for a class of systems encompassing non zero-input-stable (ZIS) systems and non-BIBS stable plants. The proposed algorithm is easy to implement and therefore suited to being used in practice; it is also effective in counteracting the transient peaking phenomenon.

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Stability of Adaptively Stabilized Stochastic Systems

Han-Fu Chen, Xian-Bing Cao, and Hai-Tao Fang

Abstract—**Stochastic adaptive stabilization usually leads to the boundedness of the average of squared output of the stabilized system, but gives no conclusion on stability of the resulting system. This note proves that adaptively stabilized stochastic system in its steady state is truly stable in the conventional sense.**

Index Terms—**Adaptive stabilization, ARMA, stability.**

I. INTRODUCTION

Adaptive control for ARMAX systems has extensively been studied in the literature. Normally, in addition to conclusions concerning performance indices, the resulting systems are adaptively stabilized in the sense that the average of squared input and output is bounded, i.e.,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\|u_k\|^2 + \|y_k\|^2) < \infty \tag{1}$$

where u_k and y_k denote the system input and output, respectively (see, e.g., [1]–[3]).

In particular, for stochastic adaptive stabilization the following single-input–single-output (SISO) system is considered in [4]–[6] among others:

$$A^0(z)y_k = B^0(z)u_k + C^0(z)w_k \tag{2}$$

where $A^0(z)$, $B^0(z)$ and $C^0(z)$ are polynomials in backward shift operator z: $zy_k = y_{k-1}$ with unknown coefficients, and $\{w_k\}$ is a sequence of martingale differences or independent random variables.

It is normally assumed that polynomials $A^0(z)$ and $B^0(z)$ are coprime, but both may be unstable. The problem of adaptive stabilization is to design feedback control so that the closed-loop system is stabilized in the sense of (1). As a matter of fact, (2) is adaptively stabilized in [4]–[6], and the resulting system given in [5] and [6] in a finite number of steps becomes an ARMA system with constant coefficients

$$A(z)y_k = C(z)w_k. \tag{3}$$

Although

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} y_k^2 < \infty$$

for (3), it is not clear if $A(z)$ is stable. We say $A(z)$ is stable if all roots of $A(z)$ are outside the closed unit disk, i.e., $A(z) \neq 0$, $\forall z: |z| \leq 1$.

Just recently, it was shown in [7] that for the steady-state system (3), after adaptive stabilization, $A(z) \neq 0$, $\forall z: |z| < 1$. However, the possibility of having roots on the unit circle is not excluded in [7]. To prove that $A(z)$ is truly stable is the topic of this note.

II. MAIN RESULT

Let

$$A(z)y_k = C(z)w_k, \quad y_k = w_k = 0 \text{ for } k \leq 0 \tag{4}$$

be a one-dimensional (1-D) ARMA process, where

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \qquad a_p \neq 0$$
$$C(z) = 1 + c_1 z + \cdots + c_r z^r, \qquad c_r \neq 0. \tag{5}$$

Assume

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} y_k^2 < \infty. \tag{6}$$

Problem: Under which conditions is $A(z)$ stable, i.e., $A(z) \neq 0$, $\forall z: |z| \leq 1$?

To this problem, a partial answer is given in [7]. For convenience of reading we present this result here as a lemma, and prove it in the Appendix.

Lemma 1: For ARMA process (4) assume $\{w_k, \mathcal{F}_k\}$ is a martingale difference sequence with

$$\sup_{k} E\left(w_{k+1}^2 | \mathcal{F}_k\right) < \infty \text{ a.s.}$$

and (6) holds. Then $A(z)$ has no zeros inside the open unit disk, i.e., $A(z) \neq 0$, $\forall z: |z| < 1$.

The gap between the complete answer to our problem and Lemma 1 is to show that $A(z) \neq 0$, $\forall z: |z| = 1$. For this we will use the law of iterated logarithm given by Wittmann, which is formulated here as Lemma 2. For its proof we refer to [8].

Lemma 2: Let a sequence of independent random variables $\{\xi_n\}$ satisfy the following conditions:

1)
$$\lim_{n \to \infty} s_n^2 = \infty;$$

2)
$$\limsup_{n \to \infty} \frac{s_{n+1}^2}{s_n^2} < \infty;$$

3)
$$\sum_{k=1}^{\infty} \left(2s_k^2 \log\log s_k^2\right)^{-\gamma/2} E\left[|\xi_k|^\gamma\right] < \infty;$$

for $\gamma: 2 < \gamma \leq 3$, where

$$s_n^2 = \sum_{k=1}^{n} E(\xi_k^2).$$

Then

$$\limsup_{n \to \infty} \frac{1}{(2s_n^2 \log\log s_n^2)^{1/2}} \sum_{k=1}^{n} \xi_k = 1 \text{ a.s.}$$

Theorem 1: Assume (6) and the following conditions for ARMA process (4):

A1) $\{w_k\}$ is a sequence of independent random variables with zero mean, $\sup_k E(|w_k|^\gamma) < \infty$ for some $\gamma: 2 < \gamma \leq 3$, and $\liminf_{k \to \infty} E(w_k^2) = 2\sigma^2 > 0$;

A2) $A(z)$ and $C(z)$ are coprime.

Then, $A(z)$ is stable, i.e., all roots of $A(z)$ are outside the closed-unit disk.

The proof of the theorem is given in Sections III and IV. It is worth noting that the converse conclusion is a well-known fact, i.e., for (4), if A1) holds, then stability of $A(z)$ implies (6).

III. PROOF OF THEOREM: DISTINCT EIGENVALUES CASE

By Lemma 1, we need only to prove that $A(z) \neq 0$, $\forall z$ with $|z| = 1$. Set

$$A \triangleq \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{p-1} & 0 & 0 & \cdots & 1 \\ -a_p & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$H \triangleq [\underbrace{1\ 0\ \cdots\ 0}_{p}], \quad x_k \triangleq \begin{bmatrix} x_k^1 & x_k^2 & \cdots & x_k^p \end{bmatrix}^T. \tag{7}$$

Then we present (4) in the state-space form as follows:

$$x_{k+1} = Ax_k + H^T C(z)w_{k+1}, \quad y_k = Hx_k. \tag{8}$$

Since $x_k^1 = y_k$, by (6) we have

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left(x_k^1\right)^2 < \infty.$$

From (7) and (8), we see $x_{k+1}^p = -a_p x_k^1$, $x_{k+1}^i = -a_i x_k^1 + x_k^{i+1}$, for $i: 1 < i < p$, and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left(x_k^i\right)^2 < \infty, \quad \forall i, 1 \leq i \leq p$$

i.e.,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k\|^2 < \infty. \tag{9}$$

Since eigenvalues of A are reciprocals of roots of $A(z)$ (see, for example, [9, Lemma 2.5, p. 45]), by Lemma 1 we only need to prove that no eigenvalue of A is on the unit circle.

Assume the theorem is not true; A has at least one eigenvalue on the unit circle. Without loss of generality, we may assume that $\lambda_1 = e^{j\lambda}$, $\lambda \in [0, 2\pi)$, $j = \sqrt{-1}$.

We first prove the theorem for the simple case, where all p eigenvalues of A are distinct. In this case A can be diagonized: there exists an invertible matrix P such that $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, where $\lambda_1, \ldots, \lambda_p$ are eigenvalues of A and $\text{diag}(\lambda_1, \ldots, \lambda_p)$ denotes the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_p$.

From (8), we obtain that

$$P^{-1}x_{k+1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)P^{-1}x_k$$
$$\qquad + P^{-1}H^T C(z)w_{k+1}. \tag{10}$$

Define

$$\eta_{k+1} = \begin{bmatrix} \eta_{k+1}^1 & \eta_{k+1}^2 & \cdots & \eta_{k+1}^p \end{bmatrix}^T \triangleq P^{-1}x_{k+1}$$
$$d = [d_1\ d_2\ \cdots\ d_p]^T \triangleq P^{-1}H^T. \tag{11}$$

Since $y_0 = y_{-1} = \cdots = y_{-p+1} = 0$, we have $x_0 = 0$, and hence $\eta_0 = 0$. From (10) and (11), it follows that:

$$\eta_{k+1}^1 = \lambda_1 \eta_k^1 + d_1 C(z)w_{k+1} = \cdots$$
$$= \sum_{i=1}^{k+1} d_1 \lambda_1^{k+1-i} C(z)w_i. \tag{12}$$

Noticing $w_k = 0$ for $k \leq 0$ and setting $c_0 = 1$ we have

$$\eta_k^1$$

$$= d_1 \lambda_1^k \sum_{i=1}^{k} \lambda_1^{-i} C(z) w_i = d_1 \lambda_1^k \left[\sum_{s=0}^{r} \sum_{i=1}^{k-s} \lambda_1^{-i-s} c_s w_i \right]$$

$$= d_1 \lambda_1^k \left[\sum_{i=1}^{k-r} \lambda_1^{-i} w_i \left(\sum_{s=0}^{r} c_s \lambda_1^{-s} \right) + \sum_{s=0}^{r-1} \sum_{i=k-r+1}^{k-s} \lambda_1^{-i-s} c_s w_i \right]$$

$$= d_1 \lambda_1^k C\left(\lambda_1^{-1}\right) \sum_{i=1}^{k-r} \lambda_1^{-i} w_i + d_1 \lambda_1^k \sum_{s=0}^{r-1} \sum_{i=k-r+1}^{k-s} \lambda_1^{-i-s} c_s w_i$$

$$= d_1 \lambda_1^k C\left(\lambda_1^{-1}\right)$$

$$\cdot \left[\sum_{i=1}^{k-r} \lambda_1^{-i} w_i + \frac{1}{C\left(\lambda_1^{-1}\right)} \sum_{i=k-r+1}^{k} \lambda_1^{-i} \left(\sum_{s=0}^{k-i} \lambda_1^{-s} c_s \right) w_i \right]. \tag{13}$$

Let

$$g_i = \begin{cases} \cos(i\lambda), & 1 \leq i \leq k-r \\ \mathrm{Re}\left[\dfrac{\lambda_1^{-i}}{C\left(\lambda_1^{-1}\right)} \sum_{s=0}^{k-i} c_s \lambda_1^{-s} \right], & k-r+1 \leq i \leq k \end{cases} \tag{14}$$

where $\mathrm{Re}(z)$ denotes the real part of z. Then, we have that

$$|\eta_k^1|^2 \geq |d_1|^2 |\lambda_1|^{2k} \left| C\left(\lambda_1^{-1}\right) \right|^2 \left(\sum_{i=1}^{k} g_i w_i \right)^2$$

$$= |d_1|^2 \left| C\left(\lambda_1^{-1}\right) \right|^2 \left(\sum_{i=1}^{k} g_i w_i \right)^2. \tag{15}$$

By A1) $\{g_i w_i\}$ is a sequence of independent random variables with zero mean, and

$$s_n^2 \triangleq \sum_{k=1}^{n} E(g_k w_k)^2 = \sum_{i=1}^{n} g_k^2 E\left(w_k^2\right)$$

$$= \sum_{k=1}^{n-r} \cos^2(k\lambda) E\left(w_k^2\right) + \sum_{k=n-r+1}^{n} g_k^2 E\left(w_k^2\right). \tag{16}$$

Since

$$\liminf_{k \to \infty} E\left(w_k^2\right) = 2\sigma^2 > 0$$

there is k_0 such that $E(w_k^2) \geq \sigma^2$, $\forall k \geq k_0$.

Noticing that

$$2\sin\lambda \sum_{k=1}^{n} \cos(2k\lambda) = \sum_{k=1}^{n} [\sin(2k+1)\lambda - \sin(2k-1)\lambda]$$

$$= (\sin(2n+1)\lambda - \sin\lambda)$$

we have

$$s_n^2 \geq \sigma^2 \sum_{k=k_0}^{n-r} \cos^2(k\lambda) = \frac{\sigma^2}{2} \sum_{k=k_0}^{n-r} (1 + \cos(2k\lambda))$$

$$= \begin{cases} \dfrac{\sigma^2}{2}(n-r-k_0+1)(1+o(1)), & \lambda \neq 0, \\ \sigma^2(n-r-k_0+1), & \lambda = 0 \end{cases} \tag{17}$$

where by $o(1)$ we mean a quantity tending to zero as $n \to \infty$. Therefore, we have $s_n^2 \to \infty$, as $n \to \infty$. Further, paying attention to the boundedness of $\{g_i\}$ and $\{E(w_k^2)\}$ from (16), (17) it is easy to see

$$\lim_{n \to \infty} \frac{s_{n+1}^2}{s_n^2} = 1.$$

Since by A1) and the boundedness of $\{g_k\}$

$$\sup_k E|g_k w_k|^\gamma < \infty$$

for some $\gamma \in (2, 3]$, by (17) it follows that:

$$\sum_{k=1}^{\infty} \left(2 s_n^2 \log\log s_n^2 \right)^{-\gamma/2} E\left[|g_k w_k|^\gamma \right] < \infty.$$

Consequently, by Lemma 2 there exists a subsequence $\{n_k\}$, which may depend on sample path, such that

$$\lim_{k \to \infty} \frac{1}{\left(2 s_{n_k}^2 \log\log s_{n_k}^2 \right)^{1/2}} \sum_{i=1}^{n_k} g_i w_i = 1 \ \text{ a.s.} \tag{18}$$

i.e.,

$$\sum_{i=1}^{n_k} g_i w_i \geq \left(\sigma^2 n_k \log\log \frac{\sigma^2}{2} n_k \right)^{1/2} (1 + o(1)) \ \text{ a.s.} \tag{19}$$

where and hereafter by $o(1)$ we mean a quantity which tends to zero as $k \to \infty$. From (15) and (19), we obtain

$$\left| \eta_{n_k}^1 \right|^2 \geq |d_1|^2 \left| C\left(\lambda_1^{-1}\right) \right|^2 \sigma^2 n_k \left(\log\log \frac{\sigma^2}{2} n_k \right)$$

$$\cdot (1 + o(1)) \ \text{ a.s.} \tag{20}$$

Since $\eta_{k+1} = P^{-1} x_{k+1}$, we immediately obtain that $\|x_{k+1}\|^2 \geq \mu_{\min} \|\eta_{k+1}\|^2$, $\forall k$, where $\mu_{\min} > 0$ denotes the minimum eigenvalue of the positive definite matrix $P^* P$.

From (20) it follows that

$$\frac{1}{n_k} \sum_{i=1}^{n_k} \|x_i\|^2 \geq \frac{1}{n_k} \|x_{n_k}\|^2 \geq \frac{1}{n_k} \mu_{\min} \left| \eta_{n_k}^1 \right|^2$$

$$\geq |d_1|^2 \left| C\left(\lambda_1^{-1}\right) \right|^2 \mu_{\min} \sigma^2 \left(\log\log \frac{\sigma^2}{2} n_k \right)$$

$$\cdot (1 + o(1)). \tag{21}$$

We now prove $d_1 \neq 0$. Writing P^{-1} in the form of column vectors p_i, $i = 1, \ldots, p$, i.e., $P^{-1} = [p_1 \ p_2 \ \cdots \ p_p]$, we have $d = P^{-1} H^T = p_1$.

On the other hand, since

$$P^{-1} A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) P^{-1}$$

we have

$$A^T (P^{-1})^* = (P^{-1})^* \mathrm{diag}\left(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_p \right) \tag{22}$$

where $*$ denotes the conjugate transpose operation and $\overline{\lambda}_i$ denotes the complex conjugate of λ_i.

Let $q_1 = [b_1 \ b_2 \ \cdots \ b_p]^T$ be the first column of $(P^{-1})^*$. Then from (22) we have $A^T q_1 = \overline{\lambda}_1 q_1$.

Notice that b_1, serving as the left-upper corner element of $(P^{-1})^*$, can be derived from $d = P^{-1} H^T$ by taking transpose and complex conjugate. This yields that $\overline{d}_1 = b_1$.

Assume the converse: $d_1 = 0$. Then from $A^T q_1 = \overline{\lambda}_1 q_1$, we have $b_2 = 0, \ldots, b_p = 0$, i.e., $q_1 = 0$. This contradicts the invertibility of P. The obtained contradiction shows $d_1 \neq 0$. Since λ_1^{-1} is the root of $A(z)$, by A2), $C(\lambda_1^{-1}) \neq 0$. Consequently, the right-hand side of (21) tends to infinity a.s. as $k \to \infty$, or

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k\|^2 = \infty \quad \text{a.s.}$$

This contradicts (9) and proves the theorem for the case of distinct eigenvalues of A. □

IV. PROOF OF THEOREM: GENERAL CASE

We now prove the Theorem for the general case where A has at least one eigenvalue with multiplicity greater than one. Without loss of generality, let $\lambda_1 = e^{j\lambda}$ be the eigenvalue of A with multiplicity $s_1 > 1$, and let the order of the first block in its Jordan form be r_1. Let P be invertible such that

$$P^{-1} D P = \mathrm{diag}(J_1, \ldots, J_d) \tag{23}$$

where $J_1, J_2, \ldots, J_d$ are Jordan blocks.

Similar to (10) we have

$$P^{-1} x_{k+1} = \mathrm{diag}(J_1, \ldots, J_d) P^{-1} x_k + P^{-1} H^T C(z) w_{k+1}. \tag{24}$$

Using the notations as those introduced in (11), we then have

$$\eta_{k+1}^{r_1} = \lambda_1 \eta_k^{r_1} + d_{r_1} C(z) w_{k+1} = \cdots$$
$$= \sum_{i=1}^{k+1} d_{r_1} \lambda_1^{k+1-i} C(z) w_i$$
$$= d_{r_1} \lambda_1^{k+1} \sum_{i=1}^{k+1} \lambda_1^{-i} C(z) w_i.$$

Comparing (25) with (12), we are convinced of that (25) implies

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k\|^2 = \infty \tag{25}$$

if we can show that $d_{r_1} \neq 0$. So, to complete the proof it suffices to show $d_{r_1} \neq 0$.

Corresponding to (22), we now have

$$A^T (P^{-1})^* = (P^{-1})^* \mathrm{diag}(J_1^*, J_2^*, \ldots, J_d^*). \tag{26}$$

Let $q_{r_1} = [b_1 \cdots b_{r_1} \cdots b_p]^T$ be the r_1th column of $(P^{-1})^*$. From (26), it follows that: $A^T q_{r_1} = \overline{\lambda}_1 q_{r_1}$.

The element b_1 located at the first row and r_1th column of $(P^{-1})^*$ can also be derived from $d = P^{-1} H^T$. Consequently, we have $b_1 = \overline{d}_{r_1}$.

Assume the converse: $d_{r_1} = 0$. Then $b_1 = 0$, and we have that $b_2 = 0, \ldots, b_p = 0$, i.e., $q_{r_1} = 0$. This contradicts the invertibility of P. Therefore, $d_{r_1} \neq 0$. □

V. CONCLUSION

The purpose of stochastic adaptive control may be to optimize some performance index, but the basic requirement which should always be met, is to stabilize the system in the sense that the average of squared output is bounded. An adaptively controlled ARMAX system normally is nonlinear and time-varying, but the closed-loop adaptive control system may tend to a steady-state system if adaptive control

is successfully designed. The closed-loop system in its steady state form is an ARMA system. We have shown in this paper that for the 1-D ARMA system, the boundedness of average of the squared output implies stability of the system indeed. So, the result of the paper may be used to judge if the steady state closed-loop system is stable or not. Extension of the result to multidimensional systems is of interest.

APPENDIX
PROOF OF LEMMA 1

Assume the converse, i.e., $A(z)$ has explosive root(s).
Let us factorize

$$A(z) A(z) = A_1(z) A_2(z) \tag{27}$$
$$A_1(z) = 1 + a_1^{(1)} z + \cdots + a_\alpha^{(1)} z^\alpha \tag{28}$$
$$A_2(z) = 1 + a_1^{(2)} z + \cdots + a_\beta^{(2)} z^\beta \tag{29}$$

where $\alpha + \beta = p$, all roots of $A_1(z)$ are outside or on the unit circle and all roots of $A_2(z)$ are inside the unit circle. By the converse assumption $|a_\beta^{(2)}| \neq 0$ with $\beta \geq 1$.

Define the $p \times p$ matrix M

$$M \triangleq \begin{bmatrix} 1 & a_1^{(1)} & \cdots & a_\alpha^{(1)} & 0 & \cdots & 0 \\ 0 & 1 & a_1^{(1)} & \cdots & a_\alpha^{(1)} & \cdots & o \\ \vdots & \ddots & \ddots & & & & \\ 0 & \cdots & 0 & 1 & a_1^{(1)} & \cdots & a_\alpha^{(1)} \\ 1 & a_1^{(2)} & \cdots & & a_\beta^{(2)} & \cdots & 0 \\ 0 & \ddots & & & & & \\ \vdots & \ddots & \ddots & & & & \\ 0 & \cdots & 0 & 1 & a_1^{(2)} & \cdots & a_\beta^{(2)} \end{bmatrix} \tag{30}$$

which is nonsingular because $A_1(z)$ and $A_2(z)$ are coprime.

Define the p-dimensional vector π_k by

$$\pi_k \triangleq [y_k, \ldots, y_{k-p+1}]^T$$

and

$$M \pi_k = [\xi_k, \xi_{k-1}, \ldots, \xi_{k-\beta+1}, \eta_k, \eta_{k-1}, \ldots, \eta_{k-\alpha+1}]^T \tag{31}$$

where $A_2(z) \xi_k = C(z) w_k$, $A_1(z) \eta_k = C(z) w_k$.

From (6), it follows that:

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \xi_i^2 < \infty, \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \eta_i^2 < \infty \quad \text{a.s.} \tag{32}$$

Setting

$$D \triangleq \begin{bmatrix} -a_1^{(2)} & 1 & 0 & & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \\ & & & & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ & & & & & 1 \\ -a_\beta^{(2)} & 0 & \cdots & & & 0 \end{bmatrix}$$

and

$$G \triangleq \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & -a_2^{(2)} & \cdots & & -a_\beta^{(2)} \\ 0 & -a_3^{(2)} & & & 0 \\ \vdots & & & & \vdots \\ 0 & -a_\beta^{(2)} & 0 & \cdots & 0 \end{bmatrix}$$

and recursively defining β-dimensional

$$x_k \triangleq [x_{k,1} \; \cdots \; x_{k,\beta}]^T$$

by

$$x_{k+1} = D x_k + H^T C(z) w_{k+1} \tag{33}$$

where $H = \underbrace{[1 \; 0 \; \cdots \; 0]}_{\beta}$, we have

$$x_k = G[\xi_k \; \xi_{k-1} \; \cdots \; \xi_{k-\beta+1}]^T$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 < \infty \quad \text{a.s.}$$

From (33), we have

$$D^{-(k+1)} x_{k+1} = x_0 + \sum_{i=1}^{k+1} D^{-i} H^T C(z) w_i \tag{34}$$

where x_0 is a deterministic vector defined by initial values y_0, y_{-1}, $\ldots$, y_{-p}. The right-hand side of (34) converges a.s. to a nonzero random vector.

On the other side, however, $\{\|x_k\| / \sqrt{k}\}$ is a bounded sequence. This means that

$$\left\| D^{-(k+1)} x_{k+1} \right\| \le c \lambda^{k+1} \frac{\|x_{k+1}\|}{\sqrt{k+1}} \sqrt{k+1} \xrightarrow[k \to \infty]{} 0 \quad \text{a.s.}$$

where $\lambda \in (0, 1)$ and c is a constant.

The obtained contradiction shows that no root of $A(z)$ can be explosive. □

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Adaptive Regulator for Discrete-Time Nonlinear Nonparametric Systems

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Abstract—A direct adaptive regulator for nonlinear nonparametric systems with measurement corrupted by noise is proposed. Under reasonable conditions the state of the closed-loop system is adaptively regulated so that it converges to zero as time tends to infinity. An illustrative example, being an affine nonlinear system, with all imposed conditions satisfied is given. The method of proof is based on stochastic approximation techniques.

Index Terms—Adaptive regulator, nonlinear nonparametric systems, stochastic approximation.

I. INTRODUCTION

For most of practical systems the linear model is merely an approximation to the true system dynamics. This probably is the reason why much research attention has been paid to the nonlinear systems for recent years. Various typical nonlinear models are considered in literature, for example, the nonlinear ARX model is considered in [12], bilinear model in [14] and the Hammerstein model in [17]. The common feature for all these models is that the system is parameterized and the parameters linearly enter the models. Therefore, when the parameters are unknown in these models, they may recursively be estimated by conventional methods, for example, the least-squares (LS) method, and the parameter estimates may be used to form adaptive controls [7], [6], [15], [13], [8], [9]. Although parameterization of system uncertainties simplifies forming adaptive control laws, it is not an easy task to analyze the resulting nonlinear adaptive control systems (see [12]).

To design and to analyze adaptive control for nonparametric nonlinear systems in a random environment is the topic of the present note. To the authors' knowledge this is the first attempt to make a rigorous analysis for this difficult problem. As a first step, we have to restrict ourselves to consider the relatively simple case, adaptive regulation, rather than the general adaptive control problem. The purpose of regulation is to control a system in order its state or output to reach a desired value. Since the system is unknown, one may intend to realize regulation adaptively. The resulting adaptive control system is then called adaptive regulator. Even for this rather simple task, we have to impose rather restrictive but reasonable conditions on the nonlinear dynamics of the system. The system state is observed with additive noise. By noticing the inherent connection between adaptive regulation and the problem of searching zero of an unknown nonlinear function, we will apply the stochastic approximation method to propose an adaptive regulator and prove the regulation error asymptotically tending to zero.

To solve the stated problem under general conditions is beyond the target of this note. This note aims at stimulating research on nonlinear stochastic adaptive control, pointing out the possibility of shifting from the parametrization framework to more natural nonparametric approach. It is worth noting that stochastic approximation only serves as a tool to solve the stated problem rather than a research topic in this note.

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