Robust Adaptive Pole Placement for Linear Time-Varying Systems

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Abstract— **This paper studies the robustness properties of a basic adaptive control algorithm with respect to plant parameter variation as well as modeling errors and bounded disturbances. The algorithm consists of a projected gradient estimator and a pole assignment controller. A unified method of proof is presented for robust stability of both discrete and continuous-time adaptively controlled time-varying systems. The robust performance of an adaptive pole assignment controller is also discussed.**

I. Introduction

From the beginning of the 1980's, as a result of the discovery that unmodeled dynamics or even small bounded disturbances can cause instability in adaptively controlled systems, robustness studies in adaptive control have been drawing much attention from researchers.

Several modifications, such as parameter projection together with normalization [2], normalized dead-zone [3], and σ-modification with normalization [4] etc., have been suggested to cope with unmodeled dynamics and bounded disturbances. In [5], Ydstie (1989) showed that parameter projection in a gradient update law is sufficient for ensuring the boundedness of closed-loop signals for a nominally minimum phase linear time-invariant (LTI) discrete-time plant with some type of unmodeled dynamics as well as bounded disturbances. This work has been extended to continuous-time by Naik *et al.* [6]. For discrete-time LTI systems, Wen and Hill [7] established robustness of an adaptive control algorithm consisting of a gradient estimator, subject to parameter projection as the only modification, plus a pole assignment controller.

Parallel attention has also been devoted to the problem of robust adaptive controller for linear time-varying (LTV) systems. In [8], de Larminat and Raynaud considered such robustness for an indirect adaptive control law which uses two parallel estimators as well as a specially constructed "normalization" signal. Middleton and Goodwin [9] also incorporated a normalization signal and additionally assumed that the constant by which it overbounds the unmodeled dynamics is known. This constant is then used to set up a normalized dead-zone which is used in the estimation algorithm when they establish robust stability. Giri *et al.* [10], by using an internally generated excitation, showed the robustness of direct adaptive pole-placement controllers with respect to plant parameter variation and modeling errors. In [11], they discussed the robustness of an adaptive regulator for a plant with small-in-the-mean parameter variations and unmodeled dynamics by using only the knowledge of the order of a nominal plant model. However, some circulation in their argument was noted in [12]. Furthermore, in [11] arbitrarily large bounded disturbances cannot be handled, and the regulation objective is not achieved even in the "ideal" case. Recently, for a class of nominally minimum phase discrete time plants with slow-in-the-mean parameter variations, Naik and Kumar [13] established the robustness results as the extension of [6], while Wen [14] extended results in [7] to discrete-time systems with slowly time-varying parameters.

In this paper, for a class of LTV systems with unmodeled dynamics and bounded disturbances, we study the robustness properties of an indirect adaptive control algorithm consisting of a projected parameter estimator and the pole assignment control law. It is shown that this modification can actually counteract the instability caused by system modeling errors as well as bounded disturbances such that the global stability can be ensured. The method of the proof presented in this paper for robust stability of the adaptive control systems is suitable for both discrete and continuous-time systems.

II. Description of the Systems

Consider the continuous-time single-input–single-output time-varying system modeled as follows:

$$y(t) = \varphi^\tau(t)\theta(t) + \eta(t) + w(t) \tag{1}$$

where $y(t)$, $u(t)$ are system output and input, respectively, and $\varphi(t)$ is the regressor vector

$$\varphi(t) = \left[\frac{D^{n-1}}{F(D)}y(t), \ldots, \frac{1}{F(D)}y(t)\right. \\ \left.\frac{D^{n-1}}{F(D)}u(t), \ldots, \frac{1}{F(D)}u(t)\right]^\tau \tag{2}$$

where $F(D) = D^n + \sum_{i=0}^{n-1} f_i D^i$ is a known Hurwitz polynomial in the differential operator D.

In (1), $\eta(t)$ and $w(t)$ denote the system uncertain dynamic errors and disturbances, respectively, and $\theta(t) \in R^{2n}$ denotes a vector of unknown time-varying parameters. The order n of the nominal part of the system is assumed to be known.

Several examples of time-varying linear systems which belong to the above class are examined in [9] to illustrate how this model might arise.

Concerning (1), we assume the following conditions.

A.1: The parameter time variations are uniformly small in mean, i.e., there exist nonnegative constants v_0 and v_1 such that

$$\int_t^{t+h} \|\dot{\theta}(s)\| ds \le v_0 + v_1 h \tag{3}$$

for all $t, h \ge 0$.

A.2: There exists a known, closed bounded convex subset C of R^{2n} such that $\theta(t) \in C$ for all t.

A.3: For some constants $\alpha \ge 0$

$$|\eta(t)| \le \alpha \sup_{0 \le s \le t} \|\varphi(s)\|. \tag{4}$$

A.4: For an arbitrary (nonzero) parameter vector

$$\theta = [\theta_1, \theta_2, \ldots, \theta_{2n}]^\tau \in C$$

the polynomials $A(D,\theta) = D^n + \sum_{i=1}^n (f_{n-i} - \theta_i) D^{n-i}$, $B(D,\theta) = \sum_{i=1}^n \theta_{n+i} D^{n-i}$ are coprime.

A.5:

$$\sup_{t\ge 0} |w(t)| < \infty.$$

Remark 2.1: System (1) can be written in terms $A(D)$ and $B(D)$ as follows:

$$\frac{A(D)}{F(D)}y(t) = \frac{B(D)}{F(D)}u(t) + \eta(t) + w(t). \tag{5}$$

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Remark 2.2: Assumption A.1 clearly covers the case of slowly drifting parameters with

$$\|\dot{\theta}(t)\| \leq v_1. \tag{6}$$

Remark 2.3: There are mainly two possible sources of dynamic errors contributing to the term $\eta(t)$, namely undermodeling the system order and commuting time-varying operators as illustrated in the discussion of examples in [9].

III. Parameter Estimation and Adaptive Control

To assure the existence and uniqueness of solution to the parameter estimation algorithm, we impose the following assumption as in [17].

A.6: There exists a continuously differentiable function g: $R^{2n} \to R$ such that $C^0 := \{\theta: g(\theta) < 0\}$ and $\Sigma := \{\theta: g(\theta) = 0\}$, where C^0 is the interior of C, and Σ, denoting the boundary of C, is a smooth hypersurface of dimension $2n - 1$.

We will use the following projected version of the gradient algorithm to estimate the unknown process $\theta(t)$ (see [17]):

$$\dot{\hat{\theta}}(t) = \begin{cases} \frac{\varphi(t)}{1+\|\varphi(t)\|^2}(y(t) - \varphi^\tau(t)\hat{\theta}(t)), & \text{if } \hat{\theta}(t) \in C^0 \cup \Sigma'; \\ \frac{\varphi(t)}{1+\|\varphi(t)\|^2}(y(t) - \varphi^\tau(t)\hat{\theta}(t)) \\ \quad - \frac{\nabla g(\hat{\theta}(t))\nabla g^\tau(\hat{\theta}(t))}{\nabla g^\tau(\hat{\theta}(t))\nabla g(\hat{\theta}(t))} \\ \quad \cdot \left[\frac{\varphi(t)}{1+\|\varphi(t)\|^2}(y(t) - \varphi^\tau(t)\hat{\theta}(t))\right], & \text{otherwise} \end{cases} \tag{7}$$

with arbitrary initial condition $\hat{\theta}(0) \in C$, where

$$\Sigma' = \left\{\theta \in \Sigma, \nabla g^\tau(\theta)\left[\frac{\varphi(t)}{1+\|\varphi(t)\|^2}(y(t) - \varphi^\tau(t)\theta)\right] \leq 0\right\}$$

and

$$\nabla g(\theta) = \left[\frac{\partial g(\theta)}{\partial \theta_1}, \frac{\partial g(\theta)}{\partial \theta_2}, \ldots, \frac{\partial g(\theta)}{\partial \theta_{2n}}\right]^\tau.$$

Algorithm (7) guarantees that $\hat{\theta}(t) \in C$ for all $t \geq 0$ (see [17]) and hence

$$\|\hat{\theta}(t) - \theta(t)\| \leq d := \sup_{\theta', \theta'' \in C} \|\theta' - \theta''\|.$$

For any $\theta \in C$, from the definition of C^0 and Σ we see that if $\hat{\theta}(t) \in \Sigma$, then

$$(\hat{\theta}(t) - \theta)^\tau \nabla g(\hat{\theta}(t)) \geq 0.$$

From this and the definition of Σ', we can readily prove that

$$\|\dot{\hat{\theta}}(t)\| \leq \left\|\frac{\varphi(t)}{1+\|\varphi(t)\|^2}[y(t) - \varphi^\tau(t)\hat{\theta}(t)]\right\|$$

and

$$(\hat{\theta}(t) - \theta)^\tau \dot{\hat{\theta}}(t) \leq (\hat{\theta}(t) - \theta)^\tau \frac{\varphi(t)}{1+\|\varphi(t)\|^2}[y(t) - \varphi^\tau(t)\hat{\theta}(t)].$$

Define the output prediction error

$$\begin{aligned} e(t) &= y(t) - \varphi^\tau(t)\hat{\theta}(t) \\ &= \varphi^\tau(t)(\theta(t) - \hat{\theta}(t)) + \eta(t) + w(t) \end{aligned} \tag{8}$$

and denote $\bar{x}(t)$ as the normalization of $x(t)$

$$\bar{x}(t) = \frac{x(t)}{\sqrt{1+\|\varphi(t)\|^2}}. \tag{9}$$

Lemma 3.1: For (1), under Assumptions A.1–A.3, A.5, and A.6, if there exists a positive constant M_1 such that for some $T > 0$

$$\sup_{0 \leq t \leq T} \|\varphi(t)\| \leq M_1 \tag{10}$$

then

$$\sup_{0 \leq t \leq T} |\bar{e}(t)| \leq d + \alpha M_1 + W \tag{11}$$

where

$$W = \sup_{t \geq 0} |w(t)|.$$

Furthermore, if there exist positive constants M and $\tau \in [0, T]$, such that for $t \in [\tau, T]$

$$M \leq \|\varphi(t)\| \leq M_1 \tag{12}$$

then

$$|\bar{e}(t)| \leq d + (\alpha M_1 + W)/M, \quad t \in [\tau, T] \tag{13}$$

and

$$\int_t^{t+h} \bar{e}^2(s)ds \leq b_1 + b_2 h, \quad \tau \leq t < t+h \leq T \tag{14}$$

where

$$b_1 = \frac{1}{2}d^2 + dv_0 \tag{15}$$

$$b_2 = 2\left(\alpha\frac{M_1}{M}\right)^2 + d\alpha\frac{M_1}{M} + dv_1 + 2\left(\frac{W}{M}\right)^2 + d\frac{W}{M}. \tag{16}$$

Proof: Set

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t).$$

From (8) and (9), using (3), (4), and the property that

$$\|\tilde{\theta}(t)\| \leq d$$

for $0 \leq t \leq T$, we have

$$\begin{aligned} |\bar{e}(t)| &\leq \frac{1}{\sqrt{1+\|\varphi(t)\|^2}}(\|\varphi(t)\|\|\tilde{\theta}(t)\| + |\eta(t)| + |w(t)|) \\ &\leq d + \alpha M_1 + W. \end{aligned} \tag{17}$$

Thus, (11) is true. Similarly, we obtain (13).

Set

$$V(t) = \frac{1}{2}\tilde{\theta}^\tau(t)\tilde{\theta}(t). \tag{18}$$

Then, for $\tau \leq t \leq T$, we have

$$\begin{aligned} \dot{V}(t) &= \dot{\tilde{\theta}}^\tau(t)\tilde{\theta}(t) = \dot{\hat{\theta}}^\tau(t)\tilde{\theta}(t) - \dot{\theta}^\tau(t)\tilde{\theta}(t) \\ &\leq \left(\frac{\varphi(t)e(t)}{1+\|\varphi(t)\|^2}\right)^\tau \tilde{\theta}(t) - \dot{\theta}^\tau(t)\tilde{\theta}(t) \\ &= \bar{e}(t)\bar{\varphi}^\tau(t)\tilde{\theta}(t) - \dot{\theta}^\tau(t)\tilde{\theta}(t) \\ &= \bar{e}(t)(-\bar{e}(t) + \bar{\eta}(t) + \bar{w}(t)) - \dot{\theta}^\tau(t)\tilde{\theta}(t) \\ &= -\bar{e}^2(t) + (-\bar{\varphi}^\tau(t)\tilde{\theta}(t) + \bar{\eta}(t) + \bar{w}(t)) \\ &\quad \cdot (\bar{\eta}(t) + \bar{w}(t)) - \dot{\theta}^\tau(t)\tilde{\theta}(t) \end{aligned} \tag{19}$$

where $\bar{\varphi}(t)$, $\bar{\eta}(t)$, $\bar{w}(t)$ are normalized regressor, normalized unmodeled dynamics, and normalized disturbance, respectively.

For $\tau \leq t < t+h \leq T$, integrating both sides of (19) and using the property that

$$\|\tilde{\theta}(t)\| \leq d, \quad \|\bar{\varphi}(t)\| < 1$$

we obtain

$$\begin{aligned}\int_t^{t+h} \bar{e}^2(s)ds &\le -\int_t^{t+h} \dot{V}(s)ds + d\int_t^{t+h} \|\dot{\theta}(s)\|ds \\ &\quad + \int_t^{t+h} 2(\bar{\eta}^2(s) + \bar{w}^2(s)) + d|\bar{\eta}(s) + \bar{w}(s)|ds \\ &\le \frac{1}{2}d^2 + d(v_0 + v_1 h) + 2\left[\left(\alpha\frac{M_1}{M}\right)^2 + \left(\frac{W}{M}\right)^2\right]h \\ &\quad + d\left(\alpha\frac{M_1}{M} + \frac{W}{M}\right)h \\ &= b_1 + b_2 h.\end{aligned}$$

This verifies (14). □

In the sequel, the norm $\|\cdot\|$ of a polynomial $X(D) = \sum_{i=0}^m x_i D^i$ is defined by

$$\|X(D)\| = \left(\sum_{i=0}^m |x_i|^2\right)^{1/2}.$$

In this paper, let us assume the goal of adaptive control is to assign poles of the closed-loop system and let the certainty equivalence control be used to solve the problem. For many other adaptive control problems including linear quadratic (LQ), model reference and tracking, the analysis is essentially the same. The control $u(t)$ is given by the equation

$$L(D,t)\left(\frac{1}{F(D)}u(t)\right) = P(D,t)\left(\frac{1}{F(D)}y^*(t) - \frac{1}{F(D)}y(t)\right) \tag{20}$$

where $y^*(t)$ is a bounded regular reference signal and

$$L(D,t) = D^n + l_{n-1}(t)D^{n-1} + \cdots + l_0(t) \tag{21}$$

$$P(D,t) = p_{n-1}(t)D^{n-1} + \cdots + p_0(t) \tag{22}$$

are obtained by solving the following Diophantine equation

$$A(D,\hat{\theta}(t))L(D,t) + B(D,\hat{\theta}(t))P(D,t) = A^*(D,\hat{\theta}(t)). \tag{23}$$

The polynomial $A^*(D,\theta)$ is of degree $2n$ with coefficients depending on θ such that $\|\frac{\partial A^*(D,\theta)}{\partial\theta}\|$ is uniformly bounded for all $\theta \in C$ and $A^*(D,\theta)$ has a uniform stability margin, i.e.,

$$\mathrm{Re}\{\rho_i[A^*(D,\theta)]\} \le -\sigma < 0, \quad \forall\theta \in C \tag{24}$$

where $\rho_i[\cdot]$ denotes the ith-zero of the polynomial stated in the bracket, and $\mathrm{Re}\{z\}$ denotes the real part of z. From Assumption A.4, we see that the solution to (23) gives uniformly bounded $L(D,\hat{\theta}(t))$ and $P(D,\hat{\theta}(t))$.

IV. Robust Stability Analysis

For proving the robust stability result, we need the following lemma which follows from [18] and [19, p. 86, Theorem 2.3].

Lemma 4.1: For the LTV system

$$\dot{x}(t) = (A(t) + B(t))x(t) \tag{25}$$

assume that for all t

$$\|A(t)\| \le M_a, \quad \mathrm{Re}\{\lambda_i[A(t)]\} \le -\sigma_a < 0 \tag{26}$$

and for all $t \ge s$

$$\int_s^t \|\dot{A}(\tau)\|^2 d\tau \le \alpha_0 + \alpha_1(t-s) \tag{27}$$

$$\int_s^t \|B(\tau)\|^2 d\tau \le \beta_0 + \beta_1(t-s) \tag{28}$$

where $\lambda_i[A]$ denotes the ith-eigenvalue of the matrix A, and M_a, σ_a and α_0, α_1, β_0, β_1, are positive constants.

Then there exist some positive constants $\varepsilon(M_a,\sigma_a)$, $M_\Psi(M_a,\sigma_a,\alpha_0,\beta_0)$ and $\mu(M_a,\sigma_a)$ such that

$$\|\Psi(t,s)\| \le M_\Psi(M_a,\sigma_a,\alpha_0,\beta_0)e^{-\mu(M_a,\sigma_a)(t-s)}, \quad \forall t \ge s \tag{29}$$

for $\alpha_1 \le \varepsilon(M_a,\sigma_a)$ and $\beta_1 \le \varepsilon(M_a,\sigma_a)$, where $\Psi(t,s)$ is the state transition matrix of (25).

We now prove the main result of the paper.

Theorem 4.2: Under Assumptions A.1–A.6, there exist real numbers $v_1^* > 0$ and $\alpha^* > 0$ such that the closed-loop system consisting of parameter estimator (7) and control law (20), is globally bounded input–bounded output (BIBO) stable provided $0 \le v_1 \le v_1^*$, $0 \le \alpha \le \alpha^*$.

Proof: From (1) and (20), we see that the closed-loop system can be written in the following state space form:

$$D\varphi(t) = Z(t)\varphi(t) + ge(t) + br(t) \tag{30}$$

where we have (31), shown at the bottom of the page

$$b = [\overbrace{0 \ \ 0 \ \ \cdots \ \ 0}^{n} \ \ 1 \ \ 0 \ \ \cdots \ \ 0]^\tau$$
$$g = [1 \ \ 0 \ \ \cdots \ \ 0 \ \ 0 \ \ 0 \ \ \cdots \ \ 0]^\tau$$

and

$$r(t) = P(D,\hat{\theta}(t))\frac{1}{F(D)}y^*(t). \tag{32}$$

Since $P(D,t)$ is bounded, $r(t)$ is bounded by some constant M_r.

From the boundedness of $\hat{\theta}(t)$, $P(D,t)$, and $L(D,t)$, we see that $Z(t)$ is bounded by a positive constant W_z. It is well known [9], [15] that eigenvalues of $Z(t)$ coincide with zeros of $A^*(D,\hat{\theta}(t))$, so we have

$$\mathrm{Re}\{\rho_i[Z(t)]\} \le -\sigma < 0, \quad \forall t \le 0. \tag{33}$$

$$Z(t) = \begin{bmatrix} \hat{\theta}_1(t) - f_{n-1} & \cdots & \cdots & \hat{\theta}_n(t) - f_0 & \hat{\theta}_{n+1}(t) & \cdots & \cdots & \hat{\theta}_{2n}(t) \\ 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & 0 & & & & \\ -p_{n-1}(t) & \cdots & \cdots & -p_0(t) & -l_{n-1}(t) & \cdots & \cdots & -l_0(t) \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 0 \end{bmatrix} \tag{31}$$

Differentiating both sides of (23) gives

$$A(D,\hat{\theta}(t))\frac{\partial L(D,t)}{\partial t}+B(D,\hat{\theta}(t))\frac{\partial P(D,t)}{\partial t} = \frac{\partial A^*(D,\hat{\theta}(t))}{\partial\hat{\theta}(t)}\dot{\hat{\theta}}(t)-\frac{\partial A(D,\hat{\theta}(t))}{\partial t}L(D,t) - \frac{\partial B(D,\hat{\theta}(t))}{\partial t}P(D,t). \quad (34)$$

By Assumption A.4, the boundedness of $\|L(D,t)\|$, $\|P(D,t)\|$, and $\|\frac{\partial A^*(D,\theta)}{\partial\theta}\|$, we derive from (34) that there exists a positive constant k_2 such that [20]

$$\left\|\frac{\partial L(D,t)}{\partial t}\right\|^2+\left\|\frac{\partial P(D,t)}{\partial t}\right\|^2 \le k_2\|\dot{\hat{\theta}}(t)\|^2, \quad \forall t\ge 0. \quad (35)$$

From (9), we have

$$e(t)=\frac{\bar{\varepsilon}(t)}{\sqrt{1+\|\varphi(t)\|^2}}+\bar{\varepsilon}(t)\bar{\varphi}^\tau(t)\varphi(t). \quad (36)$$

Inserting (36) into (30) gives

$$D\varphi(t)=(Z(t)+Q(t))\varphi(t)+q(t)+br(t) \quad (37)$$

where

$$Q(t)=g\bar{\varphi}^\tau(t)\bar{\varepsilon}(t), \quad q(t)=g\frac{\bar{\varepsilon}(t)}{\sqrt{1+\|\varphi(t)\|^2}}. \quad (38)$$

Denote

$$\alpha_0'=(1+k_2)\left(\frac{1}{2}d^2+dv_0\right), \quad \beta_0'=\frac{1}{2}d^2+dv_0$$
$$\varepsilon'=\varepsilon(W_z,\sigma), \quad \mu'=\mu(W_z,\sigma), \quad M_\Psi'=M_\Psi(W_z,\sigma,\alpha_0',\beta_0')$$

where $\varepsilon(.,.)$, $\mu(.,.)$ and $M_\Psi(.,.,.,.)$ are constants defined in Lemma 4.1.

Now, we define

$$M^*=\max\left\{2\frac{1+k_2}{\varepsilon'}(2W^2+2+dW+d),1\right\}$$
$$v_1^*=\frac{\varepsilon'}{2d(1+k_2)}$$
$$M=\max\left\{\|\varphi(0)\|+1, M^*, 2\frac{M_\Psi'}{\mu'} \times\left[M_r+\frac{1}{M^*}\left(d+\frac{1}{M^*}+\frac{W}{M^*}\right)\right]\right\} \quad (39)$$
$$T'=\frac{1}{\mu'}\log 2M_\Psi' \quad (40)$$
$$M_1=\left(M+\frac{M_r+d+W+1}{W_z+d+W+1}\right)\cdot\exp\{(W_z+d+W+1)T_1\} \quad (41)$$
$$\alpha^*=1/M_1. \quad (42)$$

From (30), we see $\|\varphi(\cdot)\|\in C[0,+\infty)$, so by (39) we can divide the interval $T^+=[0,\infty)$ into two subsequences $T_i^+=[\sigma_i,\tau_i]$ and $T_i^-=(\tau_i,\sigma_{i+1})$ with $\tau_0=0$ such that

$$T^+=\left(\bigcup_{i=1}^\infty T_i^+\right)\cup\left(\bigcup_{i=0}^\infty T_i^-\right) \quad (43)$$
$$\|\varphi(t)\|\ge M, \quad t\in T_i^+; \quad \|\varphi(t)\|<M, \quad t\in T_i^-. \quad (44)$$

Clearly, the assertion of the theorem will be verified if we can show that

$$\sup_{i\ge 1}\sup_{t\in T_i^+}\|\varphi(t)\|<M_1 \quad (45)$$

for $0\le v_1\le v_1^*$ and $0\le\alpha\le\alpha^*$.

In fact, if

$$\sup_{t\in T_1^+}\|\varphi(t)\|\ge M_1 \quad (46)$$

then this and (44) imply existence of a real number t_m so that

$$t_m=\min\{t\in T_1^+, \quad \|\varphi(t)\|=M_1\}. \quad (47)$$

Hence, we have

$$\|\varphi(t)\|\le M_1, \quad \forall t\in[0,t_m] \quad (48)$$

and

$$M\le\|\varphi(t)\|\le M_1, \quad \forall t\in[\sigma_1,t_m]. \quad (49)$$

From this and (39), (42), by using Lemma 3.1, we obtain that for $v_1\le v_1^*$ and $\alpha\le\alpha^*$

$$\sup_{0\le t\le t_m}|\bar{\varepsilon}(t)|\le d+1+W$$
$$\sup_{\sigma_1\le t\le t_m}|\bar{\varepsilon}(t)|\le d+\frac{1}{M^*}+\frac{W}{M^*} \quad (50)$$

and for $\sigma_1\le t<t+h\le t_m$

$$\int_t^{t+h}\bar{\varepsilon}^2(s)ds\le b_1+b_2'h \quad (51)$$

where b_1 is defined in (15) and

$$b_2'=2\left(\frac{1}{M^*}\right)^2+d\frac{1}{M^*}+dv_1^*+2\left(\frac{W}{M^*}\right)^2+d\frac{W}{M^*}. \quad (52)$$

From (31) and (35), we see that

$$\|\dot{Z}(t)\|^2\le(1+k_2)\|\dot{\hat{\theta}}(t)\|^2\le(1+k_2)\|\bar{\varphi}(t)\bar{\varepsilon}(t)\|^2 \le(1+k_2)|\bar{\varepsilon}(t)|^2. \quad (53)$$

For $\sigma_1\le t<t+h\le t_m$, integrating both sides of (53) from t to $t+h$, we obtain

$$\int_t^{t+h}\|\dot{Z}(s)\|^2ds\le(1+k_2)\int_t^{t+h}|\bar{\varepsilon}(s)|^2ds \le(1+k_2)(b_1+b_2'h). \quad (54)$$

Also, from (38), it follows:

$$\int_t^{t+h}\|Q(s)\|^2ds\le\int_t^{t+h}|\bar{\varepsilon}(s)|^2ds\le b_1+b_2'h. \quad (55)$$

From the definition of M^* and v_1^*, we see

$$(1+k_2)b_2'<\varepsilon'. \quad (56)$$

By (33), (54)–(56), and the boundedness of $Z(t)$, applying Lemma 4.1 to (37), we derive

$$\|\Psi_z(t,s)\|\le M_\Psi' e^{-\mu'(t-s)}, \quad \forall t_m\ge t\ge s\ge\sigma_1 \quad (57)$$

where $\Psi_z(t,s)$ is the state transition matrix for (37) and μ_z', M_Ψ' have been defined above.

Therefore, from (37)–(38) and (57), using Lemma 3.1, we derive

$$\begin{aligned}\|\varphi(t_m)\| &\le\|\Psi_z(t_m,\sigma_1)\|\cdot\|\varphi(\sigma_1)\| +\int_{\sigma_1}^{t_m}\|\Psi_z(t_m,\tau)\|(\|q(\tau)\|+\|br(\tau)\|)d\tau\\ &\le M_\Psi' e^{-\mu'(t_m-\sigma_1)}\|\varphi(\sigma_1)\| +\int_{\sigma_1}^{t_m}M_\Psi' e^{-\mu'(t_m-\tau)}(\|q(\tau)\|+\|br(\tau)\|)d\tau\\ &\le M_\Psi' e^{-\mu'(t_m-\sigma_1)}M +\int_{\sigma_1}^{t_m}M_\Psi' e^{-\mu'(t_m-\tau)}\left[M_r+\frac{1}{M^*}\left(d+\frac{1}{M^*}+\frac{W}{M^*}\right)\right]d\tau\\ &\le M_\Psi' e^{-\mu'(t_m-\sigma_1)}M+M/2\end{aligned} \quad (58)$$

and (47) yields

$$M'_\Psi e^{-\mu'(t_m-\sigma_1)}M > M/2. \tag{59}$$

This implies that

$$t_m - \sigma_1 < T'. \tag{60}$$

Thus, again from (37) and (38) and by using Lemma 3.1, (60), and (41), we have

$$\begin{aligned}&\|\varphi(t_m)\|\\ &\le \exp\left\{\sup_{0\le t\le t_m}\{\|Z(t)+Q(t)\|\}(t_m-\sigma_1)\right\}\|\varphi(\sigma_1)\|\\ &\quad+\int_{\sigma_1}^{t_m}\exp\left\{\sup_{0\le t\le t_m}\{\|Z(t)+Q(t)\|\}(t_m-\tau)\right\}\\ &\quad\cdot(\|q(\tau)\|+\|br(\tau)\|d\tau\\ &\le \exp\{(W_z+d+W+1)(t_m-\sigma_1)\}M\\ &\quad+\frac{M_r+d+W+1}{W_z+d+W+1}(e^{(W_z+d+W+1)(t_m-\sigma_1)}-1)\\ &< M_1\end{aligned} \tag{61}$$

which contradicts (47). Therefore, (46) is impossible, i.e.,

$$\sup_{t\in T_1^+}\|\varphi(t)\| < M_1. \tag{62}$$

Similarly, we can prove that

$$\sup_{1\le i\le n+1}\ \sup_{t\in T_i^+}\|\varphi(t)\| < M_1$$

provided

$$\sup_{1\le i\le n}\ \sup_{t\in T_i^+}\|\varphi(t)\| < M_1.$$

Thus, (45) is true and BIBO stability follows.

Remark 4.1: From (39), (41), and (42), we see that the size of the allowable unmodeled dynamics depends on the initial condition of the plant.

Remark 4.2: The method of the proof above is applicable to discrete time systems for which t_m should be defined as an integer such that

$$t_m = \min\{t \in T_1^+,\quad \|\varphi(t)\| \ge M_1\}.$$

V. Robust Performance Analysis

As a direct consequence of the parameter properties, the boundedness of signals implies the following performance results.

If there are no modeling errors and disturbance in the system, and if $v_1 = 0$, then from Lemma 3.1 and Theorem 4.2 we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^2(t)dt = 0 \tag{63}$$

and from [15] we see that the closed-loop characteristic polynomial tends to $A^*(D,\hat\theta(t))$ in the sense that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |A^*(D,\hat\theta(t))y(t) - B(D,\hat\theta(t))P(D,t)y^*(t)|dt = 0. \tag{64}$$

If disturbances and unmodeled dynamics are present for the LTV system with v_1, α sufficiently small, then again from Lemma 3.1, Theorem 4.2, and [15], we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^2(t)dt \le c_1(\alpha^2+\alpha+v_1+W^2+W) \tag{65}$$

and

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T |A^*(D,\hat\theta(t))y(t) - B(D,\hat\theta(t))P(D,t)y^*(t)|^2dt\\ &\qquad\le c_2(\alpha^2+\alpha+v_1+W^2+W)\end{aligned} \tag{66}$$

where c_1, c_2 are some constants.

Suppose that the reference signal $y^*(t)$ is generated by a linear system

$$S(D,t)y^*(t) = 0 \tag{67}$$

where $S(D,t)$ and $B(D,\hat\theta(t))$ are relatively prime. If the control synthesis strategy uses the internal model principle (see [15]), then for sufficiently small α and v_1, the tracking error $|y(t)-y^*(t)|$ satisfies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |y(t)-y^*(t)|^2dt \le c_3(\alpha^2+\alpha+v_1+W^2+W) \tag{68}$$

where c_3 is a constant.

VI. Conclusion

In this paper, we have studied a basic adaptive control algorithm consisting of a gradient estimator, modified only by parameter projection, plus a pole assignment controller for LTV systems. No special signal normalization is employed to ensure robustness. We have presented a unified method of proof for the robust stability of discrete-time and continuous-time adaptive control systems. We have also analyzed the robust performance of the adaptive pole assignment controller.

It is straightforward to extend this analysis to least-squares based update laws with monitoring the boundedness of the condition number of the matrix consisting of regressors.

Several issues still need to be explored. A major restriction is that we assume the true time varying parameter vector to lie in a set with specified properties. Transient performance, and the precise sizes of unmodeled dynamics and parameter variations tolerated, are issues requiring further study.

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Minimum Variance Bounds for Overparameterized Models

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***Abstract*—In this paper it is shown that the generalized Cramér–Rao lower bound on the estimate of invariants of (over)parameterized models is independent of the particular (over)parameterization chosen and equals that of the identifiable form.**

I. Introduction

Overparameterized models are models which contain redundant parameters. They result in singular Fisher information matrices [1]. This situation is often encountered in practical parameter estimation problems. Consider, for example, the identification of a state-space model representation of a proper linear time invariant multiple-input–multiple-output (MIMO) system

$$X_{k+1} = AX_k + BU_k$$
$$Y_k = CX_k + DU_k \tag{1}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$, and $D \in R^{p\times m}$, from measurements of the input $U_k \in R^{m\times 1}$ and output $Y_k \in R^{p\times 1}$ signals using subspace identification techniques [2]–[4]. Model (1) contains $(n^2 + n(p+m) + pm)$ parameters (A, B, C, and D are unknown). Since all possible realizations (1) of a particular MIMO system are related through a similarity transform $T \in R^{n\times n}$ $(\det(T) \neq 0)$ [5], n^2 dependencies exist between these parameters. Assuming that the true model order is n, the dimension of the null space of the Fisher information matrix equals n^2. Invariants of model (1) are for example the eigenvalues of A.

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A second example is the identification of the numerator and denominator coefficients of a rational transfer function model of a single-input–single-output (SISO) system

$$H(\theta,\omega) = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{d} b_k x^k} \tag{2}$$

where $\theta = (a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_d)^T$ and x equals $j\omega$ or $\exp(j\omega T_s)$ for, respectively, continuous and discrete time systems. Transfer function model (2) contains a parameter ambiguity ($\forall\lambda \in R_0$: $H(\lambda\theta,\omega) = H(\theta,\omega)$) which is removed by fixing one coefficient of the numerator or denominator (e.g., monic denominator polynomial: $b_d = 1$). To avoid that, a zero coefficient is put equal to one; it is often numerically more stable to estimate all the coefficients θ and to scale them afterwards such that $\|\theta\|_2 = 1$. Some examples of estimators using this approach are the total least squares [6], the generalized total least squares [7], the bootstrapped total least squares, and maximum likelihood [8]. Assuming that the true model order is (n,d), the dimension of the null space of the Fisher information matrix equals one. Invariants of model (2) are, for example, the roots of the numerator and denominator polynomials.

II. Minimum Variance Bounds for Invariants of (Over)Parameterized Models

Consider the identification of a particular model out of a model set $\mathcal{M}$ using noisy measurements $y \in R^m$. Assume that the model set can be parameterized in an identifiable parameter set $\psi \in R^q$ with corresponding regular Fisher information matrix

$$F_\psi = E\left\{\left(\frac{\partial}{\partial\psi}\ln(f(y \mid \psi))\right)^T \left(\frac{\partial}{\partial\psi}\ln(f(y \mid \psi))\right)\right\}. \tag{3}$$

($f(y \mid \psi)$ is the joint probability function). Assume that it can also be overparameterized in $\theta \in R^p$ $(p > q)$ with corresponding singular Fisher information matrix

$$F_\theta = E\left\{\left(\frac{\partial}{\partial\theta}\ln(f(y \mid \psi(\theta)))\right)^T \left(\frac{\partial}{\partial\theta}\ln(f(y \mid \psi(\theta)))\right)\right\} \tag{4}$$

such that $\psi(\theta)$ is a continuous function of θ with full rank partial derivative with respect to θ

$$\operatorname{rank}\left(\frac{\partial}{\partial\theta}\psi(\theta)\right) = q. \tag{5}$$

Define an invariant of the model set $\mathcal{M}$ as any model related quantity G which is invariant with respect to all possible parameterizations of $\mathcal{M}$

$$G(\chi) = G(\phi) \qquad \forall\chi,\phi \in \mathcal{M}. \tag{6}$$

Theorem: The Cramér–Rao lower bound on the covariance matrix of any estimator $\hat{G}(y)$ of an invariant $G \in C^r$ of the model set $\mathcal{M}$ is independent of the particular (over)parameterization chosen.

Proof: The Cramér–Rao lower bounds on the covariance matrix of an estimator $\hat{G}(y)$ using parameterization ψ or θ are, respectively, given by

$$\operatorname{cov}(\hat{G}(\hat{\psi}(y))) \geq \left(\frac{\partial G}{\partial\psi} + \frac{\partial b_G}{\partial\psi}\right) F_\psi^{-1} \left(\frac{\partial G}{\partial\psi} + \frac{\partial b_G}{\partial\psi}\right)^H \tag{7}$$

and

$$\operatorname{cov}(\hat{G}(\hat{\theta}(y))) \geq \left(\frac{\partial G}{\partial\theta} + \frac{\partial b_G}{\partial\theta}\right) F_\theta^{+} \left(\frac{\partial G}{\partial\theta} + \frac{\partial b_G}{\partial\theta}\right)^H \tag{8}$$