with F_s an adequately chosen matrix. Introducing (32) in (30a) we obtain that X satisfies (6). Relations (31) and (29) with (32) show that (7) and (8) are also satisfied.

Now some algebraic manipulations using known properties of the Moore–Penrose pseudo-inverse show that G_0 given by (9) satisfies (1) and that G_i given by (10) satisfies (3) and (4). A right inverse of G_0 in RH^∞ is given by

$$G_0^+(s) = \left[\begin{array}{c|c} A - B(D^TD)^+(D^TC + B^TX) + B(I - D^+D)F_s & BV^+ \\ \hline -(D^TD)^+(D^TC + B^TX) + (I - D^+D)F_s & V+ \end{array}\right] \tag{33}$$

which is stable by the choice of F_s, and the theorem is proved. ■

III. Conclusion

The above theorem shows that spectral factorization can be performed by computing the stabilizing solution of the constrained Riccati equation (6), (7). The proof of the theorem leads to a Schur-like method for computing this solution. The procedure is quite clear. An ordered QZ algorithm must be performed on the extended Hamiltonian pencil to obtain the stable reducing subspaces of the pencil and then a basis matrix for an n-dimensional stable proper deflating subspace. If the first n rows form a nonsingular matrix, then X and F can be computed with (28) and used for obtaining the realizations of G_0 and G_i.

The problem can be very ill-posed, so one has to be cautious about the implementation. For numerical computations on possibly singular pencils, we cite here [14] and [15], although there are also other papers on this subject.

One of the reviewers drew my attention to [16] whose main result has important connections with the present paper. In fact the problem considered in [16] is more general, but the result presented above suggests that the sequence of equivalent statements given there could be completed with one giving the existence of the stabilizing solution to the constrained Riccati equation. This has the important advantage of being easier to extend to more general factorizations such as the J-spectral factorization, for example, than the linear matrix inequality. Also, the proof given above gives further insight on the nonuniqueness of the spectral factor due to the use of the notion of proper deflating subspace for a singular pencil.

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Stability Analysis for Manufacturing Systems with Unreliable Machines and Random Inputs

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***Abstract*—The manufacturing system considered in the paper consists of possibly unreliable machines, whose inputs may be random and may be in batches. The key assumption on the system is that events of unsatisfactory processing independently occur at all machines. We derive a set of inequalities relating processing rates with probabilities of successful processing at unreliable machines. It is shown that the manufacturing system with a CFWL scheduling policy applied is stable if these inequalities are satisfied simultaneously; by CFWL policy we mean the policy that chooses any part-type with a work load exceeding a certain portion of the total one for the next run. This policy is motivated by CAF policy introduced in [7] and is the stochastic extension of CFW policy used in [9].**

I. Introduction

We consider a manufacturing system, which consists of N machines and processes L part-types from the outside of the system. In contrast to the static, open-loop approach to scheduling (e.g., see [2]), this paper takes the dynamic, closed-loop approach pioneered by Kimemia and Gershwin [3] and continued by Akella and Kumar [4], Sharifnia [5], Flemming, Sethi and Soner [6], and many others.

In recent years, the analysis for sample-path-based stability of distributed real-time scheduling of deterministic manufacturing systems has been studied in many papers [8]–[10]. The authors of these papers pointed out that the stochastic extension of the deterministic manufacturing system is of interest, and it may include uncertainties such

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as random processing time, unreliable machines, random machine failures, random inputs, and so on. In this paper we consider two kinds of randomness. In the first type, machines in the system are unreliable; by unreliable machine we mean that a part processed by it is of satisfactory quality only with some probability p less than or equal to one. With probability $1-p$, a part after processing should reenter the same machine for a rework. Thus, in the system parts of some type, say type l, entering the buffer of the ith machine from the jth machine form a random sequence. More precisely, with probability p_l there is a part of type l coming from the jth machine to the buffer of the ith machine if the jth machine is producing parts of this type. In the second type of randomness, parts from the outside of the system enter the manufacturing system in batches with random sizes; the case where parts are transferred by pellets or by automatically guided vehicles may serve as examples of batch inputs.

We will apply clear-a-fraction-of-work-load (CFWL) policy, by which we mean the policy that chooses for the next run any part-type with a work load exceeding a certain portion of the total load. This policy is motivated by CAF policy introduced in [7] and is the stochastic extension of CFW policy used in [9]. For the system containing uncertainties with CFWL policy applied to it, this paper gives sufficient conditions guaranteeing stability in the average sense. This paper is concerned about the stability analysis of the CFWL policy only, because with ϵ_0 appropriately chosen in (2.3), the CFWL policy implies other policies such as CAF, CLB, CLW policies [7], [8] and CFW policies [9]. The results of the paper are the extensions of Perkins' and Kumar's work [7] to the stochastic case. We hated but could not avoid using rather complicated notations in dealing with the system we now describe.

1) There are N machines labeled $1, 2, \cdots, N$ in the system, and there are L part-types labeled $1(1), 2(1), \cdots, L(1)$ which come from the outside to the system for processing. We might use l instead of $l(1)$, $l = 1, 2, \cdots, L$ to denote a part-type from the outside. The processing routes are given as a prerequisite. Parts of type $l(1)$ successively visit machines $M_{l,1}, M_{l,2}, \cdots, M_{l,n_l}$ and finally leave the system from machine M_{l,n_l} as products. A part may visit the same machine for many times, i.e., it may happen that $M_{l,i} = M_{l,j}$ for $i \neq j$. After a successful processing at a machine, parts are assumed to be of a new type, and are assumed to be of the same type after an unsatisfactory processing. Denote by $l(j+1)$ the new type formed by parts of type $l(j)$ after processing at machine $M_{l,j}$, $j = 1, 2, \cdots, n_l$; $l = 1, 2, \cdots, L$. Let $J(i)$ be the set of all part-types to be processed at machine i, i.e.,

$$J(i) = \{l(j)\colon M_{l,j} = i,\ 1 \leq j \leq n_l;\ 1 \leq l \leq L\}, \qquad \text{for } i = 1, 2, \cdots, N.$$

2) Parts of type $l(j)$ coming to machine $M_{l,j}$ for processing first arrive at the buffer denoted by $b_{l,j}$. A part of type $l(j)$ needs $(1/r_{l(j)})$, for processing at machine $M_{l,j}$.

3) From outside the system, parts of type $l(1)$ enter buffer $b_{l,1}$ of machine $M_{l,1}$ for processing in batches with size $\{W_k^{l(1)}, k \geq 1\}$ at the demand rate $r_{l(0)}$, i.e., a batch of type $l(1)$ is sent to the system per $(1/r_{l(0)})$ time-unit, and the size of the kth batch is written as a random variable (r.v.) $W_k^{l(1)}$. The random variables (r.v.s) $W_k^{l(1)}$, $l(1) = 1(1), 2(1), \cdots, L(1)$; $k = 1, 2, \cdots$, are mutually independent, $E(W_k^{l(1)})^{3m} < \infty$ for some $m \geq 1$, $1 \leq l \leq L$, and for fixed $l(1)$, $W_k^{l(1)}$, $k = 1, 2, \cdots$, are identically distributed with $EW_k^{l(1)} = p_{l(0)}$ and $\mu_l(m') \triangleq E(W_k^{l(1)})^{m'}$, i.e., $\{W_k^{l(1)}, k \geq 1\}$ is an i.i.d. sequence. In particular, $W_k^{l(1)}$ may identically equal 1, and this corresponds to the deterministic part flow.

4) Machines in the system are unreliable. The kth processing of parts of type $l(j)$ at machine $M_{l,j}$ yields a r.v. $W_k^{l(j+1)}$, which values at 1 if the processed part is of satisfactory quality, and 0 if unsatisfactory, $k = 1, 2, \cdots$. The events of unsatisfactory processing independently occur at machines, and they are mutually independent of the input sequences $\{W_k^{l(1)}, k \geq 1\}$ from the outside of the system. Thus, $\{W_k^{l(j+1)}, k \geq 1\}$ is an i.i.d. sequence with $P\{W_k^{l(j+1)} = 1\} = p_{l(j)}$, $P\{W_k^{l(j+1)} = 0\} = q_{l(j)} (= 1 - p_{l(j)})$, and the sequences $\{W_k^{l(j)}, k \geq 1\}$, $1 \leq j \leq n_l + 1$; $1 \leq l \leq L$ are mutually independent.

5) A part of type $l(j)$ after processing at machine $M_{l,j}$ either eventually leaves the system (when $j = n_l$) or visits a downstream machine (when $j < n_l$) in the case $W_k^{l(j+1)} = 1$, and reenters the buffer $b_{l,j}$ for a rework in the case $W_k^{l(j+1)} = 0$. Thus, the feedback is under consideration.

6) A setup time $\delta^i_{l(j),l'(j')}$ is required when the ith machine changes from processing parts of type $l(j)$ to type $l'(j')$ where $l(j), l'(j') \in J(i)$.

7) The setup time $\delta^i_{l(j),l'(j')}$, the demand rates $r_{l(0)}$, and processing rates $r_{l(j)}$, $1 \leq j \leq n_l$; $1 \leq l \leq L$; $l(j), l'(j') \in J(i)$; $1 \leq i \leq N$ all are deterministic.

II. Main Results

Denote by $X_{l(j)}(t)$ and $A_{l(j)}(t)$ the buffer level at $M_{l,j}$ and the work load of parts of type $l(j)$ at time t respectively, where

$$A_{l(j)}(t) = \frac{X_{l(j)}(t)}{r_{l(j)} p_{l(j)}}, \qquad \forall 1 \leq j \leq n_l,\ 1 \leq l \leq L. \tag{2.1}$$

For the ith machine, set

$$A^i(t) = \sum_{l(j) \in J(i)} A_{l(j)}(t), \quad X^i(t) = \sum_{l(j) \in J(i)} X_{l(j)}(t), \qquad i = 1, 2, \cdots, N \tag{2.2}$$

where and thereafter the superscript i always indicates that the quantity in question is connected with machine i; it should not be confused with the power of that quantity.

A scheduling policy is called a CFWL policy if the ith machine at time T_n^i commences a setup for processing parts of type $l_n(j_n) \in J(i)$ whenever

$$A_{l_n(j_n)}(T_n^i) \geq \epsilon_0 A^i(T_n^i), \quad \text{for some fixed} \quad \epsilon_0 \in (0, 1) \tag{2.3}$$

and the production of parts of type $l_n(j_n)$ is started at time $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)}$ and continued until the first time T_{n+1}^i at which the buffer level $X_{l_n(j_n)}(t)$ hits 0, i.e., $X_{l_n(j_n)}(T_{n+1}^i) = 0$, where $T_0^i = 0$ by setting $i = 1, 2, \cdots, N$ and n numbers setups. If there are many part-types satisfying (2.3), then we choose any of them for the next run.

Definition: The ith machine is said to be L_s-stable if

$$\sup_n E(X^i(T_n^i))^s \leq B_1(s) \qquad \text{or equivalently,} \qquad \sup_n E(A^i(T_n^i))^s \leq B_2(s) \tag{2.4}$$

where $B_i(s)$, $i = 1, 2$ are constants, depending on s only.

A manufacturing system is said to be L_s-stable if all machines in the system are L_s-stable.

When $W_k^{l(1)} \equiv 1$ and $p_{l(j)} = 1$ for all $1 \leq j \leq n_l$, $1 \leq l \leq L$, then the system considered in this paper degenerates to a deterministic one, and for some cases, results derived here directly lead to the corresponding ones obtained in [7]–[9] (e.g., Theorem 1 in [7], Theorem 2 in [9], and Example 2, condition (14) in [8]).

We now present main results of the paper; the proof will be given in Section III.

Theorem 1: For the manufacturing system described in the Introduction by 1)–7) with CFWL policy applied, if parameters $r_{l(j)}$ and $p_{l(j)}$ satisfy the following system of inequalities

$$\rho^i \triangleq \sum_{l(j)\in J(i)} \rho_{l(j)} < 1, \qquad \forall i = 1, 2, \cdots, N \tag{2.5}$$

with

$$\rho_{l(j)} \triangleq \frac{r_{l(j-1)}p_{l(j-1)}}{r_{l(j)}p_{l(j)}}, \qquad 1 \le j \le n_l,\ 1 \le l \le L \tag{2.6}$$

then the system is L_m-stable. Moreover,

$$\sup_n E\left(\sum_{l(j)\in J(i)} \bar{x}_{l(j)}(n)\right)^m \le B_3(m), \qquad i = 1, 2, \cdots, N \tag{2.7}$$

where $\bar{x}_{l(j)}(n) \triangleq \max_{t\in[T_n^i, T_{n+1}^i]} X_{l(j)}(t)$, $l(j) \in J(i)$ and $B_3(m)$ is a constant depending on m which is given in part 3 of the Introduction.

Example 1: Consider a manufacturing system composed of $N = 2$ machines and $L = 2$ part-types from the outside of the system. The processing routes are as follows: $M_{1,1} = 1$, $M_{1,2} = 2$; $M_{2,1} = 2$, $M_{2,2} = 1$; $n_1 = n_2 = 2$. Then (2.5) is expressed as

$$\begin{cases} \frac{r_{1(0)}p_{1(0)}}{r_{1(1)}p_{1(1)}} + \frac{r_{2(1)}p_{2(1)}}{r_{2(2)}p_{2(2)}} < 1, \\ \frac{r_{1(1)}p_{1(1)}}{r_{1(2)}p_{1(2)}} + \frac{r_{2(0)}p_{2(0)}}{r_{2(1)}p_{2(1)}} < 1. \end{cases} \tag{2.8}$$

Obviously, system (2.8) is satisfied if $r_{l(j)}p_{l(j)} > 2r_{l(j-1)}p_{l(j-1)}$, $l = 1, 2$; $j = 1, 2$. Further, if machines are reliable $p_{l(j)} = 1$, $l = 1, 2$; $j = 0, 1, 2$ and if $r_{1(0)} = r_{2(0)} = 1$, then condition (2.8) coincides with condition (14) in [8].

Theorem 2: Under the conditions of Theorem 1, the following estimates hold for the average work load of machine i ($\forall i \in [1, 2, \cdots, N]$):

$$\limsup_n E\ A^i(T_n^i) \le \frac{v_2^i}{\epsilon_0 \Phi_0^i}, \tag{2.9}$$

$$\sup_{n\ge 0} E\ A^i(T_n^i) \le \max\left\{A^i(0), \frac{v_2^i}{\epsilon_0\Phi_0^i}\right\} \tag{2.10}$$

where $A^i(0)$ is the initial work load, and

$$\Phi_0^i = \frac{1-\rho^i}{1-\min_{l(j)\in J(i)}\rho_{l(j)}} > 0, \tag{2.11}$$

$$v_2^i = \max_{l(j)\in J(i)} \frac{\rho^i - \rho_{l(j)}}{1-\rho_{l(j)}}\left\{\delta + \frac{1}{r_{l(j)}p_{l(j)}}\right\}. \tag{2.12}$$

III. Proof of Theorems

To prove theorems we need to explicitly express the buffer level $X_{l(j)}(T_n^i)$. For this we explain various quantities to appear.

Define

$$w_k^{l(j)} = 1 - W_k^{l(j+1)}, \qquad \forall 1 \le j \le n_l;\ 1 \le l \le L. \tag{3.1}$$

Let $\sigma_{n+1}^i(l_n(j_n))$ denote the number of parts of type $l_n(j_n) \in J(i)$ processed by machine i during the period $[T_n^i, T_{n+1}^i]$. Notice that starting from the time $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)}$ the $\sigma_{n+1}^i(l_n(j_n))$th processing of part-type $l_n(j_n)$ at machine i first empties buffer b_{l_n,j_n}. Mathematically, $\sigma_{n+1}^i(l_n(j_n))$ can be defined as follows:

$$\sigma_{n+1}^i(l_n(j_n)) = \inf\left\{k : k - X_{l_n(j_n)}(T_n^i) - \sum_{s=1}^{k} w^{l_n(j_n)}_{g_n^i(l_n(j_n))+s} - \sum_{s=1}^{N_{i,n}^{l_n(j_n)}(k)} W^{l_n(j_n)}_{f_n^i(l_n(j_n))+s} \ge 0\right\} \tag{3.2}$$

where $f_n^i(l(j))$ (for any $1 \le j \le n_l$, $1 \le l \le L$) denotes the number of $\{W_k^{l(j)}, k \ge 1\}$ (batches if $j = 1$, or random variables if $j > 1$) that have arrived at $b_{l,j}$ before time T_n^i, $g_n^i(l(j))$ denotes the number of parts of type $l(j) (\in J(i))$ that have been processed by machine i before time T_n^i, $N_{i,n}^{l(j)}(k)$ (for any $1 \le j \le n_l$, $1 \le l \le L$) denotes the number of $\{W_k^{l(j)}, k \ge 1\}$ entering $b_{l,j}$ from time T_n^i to $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)} + (k/r_{l_n(j_n)})$ during which machine i has processed k parts of type $l_n(j_n)$.

After the setup at time $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)}$, machine i has finished processing k parts of type $l_n(j_n)$ at time $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)} + (k/r_{l_n(j_n)})$. Among these k processed parts, $\sum_{s=1}^k w^{l_n(j_n)}_{g_n^i(l_n(j_n))+s}$ in (3.2) means the number of parts of unsatisfactory quality, while $\sum_{s=1}^{N_{i,n}^{l_n(j_n)}(k)} W^{l_n(j_n)}_{f_n^i(l_n(j_n))+s}$ is the number of parts of type $l_n(j_n)$ entering b_{l_n,j_n} during the time period $[T_n^i, T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)} + (k/r_{l_n(j_n)})]$. The last three terms at the left-hand side of (3.2) represent the total number of parts of type $l_n(j_n)$ that should be processed by machine i to clear the buffer b_{l_n,j_n} at time $T_n^i + \delta^i_{l_{n-1}(j_{n-1}),l_n(j_n)} + (k/r_{l_n(j_n)})$.

It is not difficult to understand that

$$f_n^i(l(j)) = f_{n-1}^i(l(j)) + N_{i,n-1}^{l(j)}(\sigma_n^i(l_{n-1}(j_{n-1}))) \quad \text{for all } 1 \le l \le L,\ 1 \le j \le n_l, \tag{3.3}$$

$$g_n^i(l(j)) = g_{n-1}^i(l(j)) + \sigma_n^i(l_{n-1}(j_{n-1}))I_{[l_{n-1}(j_{n-1})=l(j)]} \quad \text{for all } l(j) \in J(i) \tag{3.4}$$

and

$$\sigma_{n+1}^i(l_n(j_n)) = N_{i,n}^{l_n(j_n+1)}(\sigma_{n+1}^i(l_n(j_n))) \tag{3.5}$$

where I_A is the indicator function of set A, $f_0^i(l(j)) = 0$ ($\forall 1 \le j \le n_l$, $1 \le l \le L$), $g_0^i(l(j)) = 0$ ($\forall l(j) \in J(i)$).

For simplicity, we assume that the setup time is independent of both machine and type, i.e., $\delta^i_{l(j),l'(j')} = \delta$, $\forall l(j), l'(j') \in J(i)$, $1 \le i \le N$.

Applying the CFWL policy, we know that

$$T_{n+1}^i = T_n^i + \delta + \frac{\sigma_{n+1}^i(l_n(j_n))}{r_{l_n(j_n)}}, \tag{3.6}$$

$$X_{l(j)}(T_{n+1}^i) = \left(X_{l(j)}(T_n^i) + \sum_{s=1}^{N_{i,n}^{l(j)}(\sigma_{n+1}^i(l_n(j_n)))} W^{l(j)}_{f_n^i(l(j))+s}\right) \cdot I_{[l_n(j_n)\ne l(j)]}, \quad \text{for all } l(j) \in J(i). \tag{3.7}$$

The following lemma states that $\{W^{l(j)}_{f^i_n(l(j))+s},\ s \geq 1\}$ for fixed n, i and $l(j)$ is an i.i.d. sequence, and for fixed n and i but with $l(j)$ varying in J are mutually independent, where $n \geq 0$, $1 \leq i \leq N$, $l(j) \in J$ and

$$J = \bigcup_{i=1}^{N} J(i) \bigcup \bigcup_{l=1}^{L} \{l(n_l+1)\}. \tag{3.8}$$

The proof is essentially based on the following facts: i) an i.i.d. sequence with index starting from a stopping time remains i.i.d. and ii) events of unsatisfactory processing independently occur at all machines. For its detailed proof, refer to [12].

Lemma 1: Under the conditions of Theorem 1, the following assertions are true.

i) For any fixed i, n, and $l(j) \in J$, the random variable $N^{l(j)}_{i,n-1}(\sigma^i_n(l_{n-1}(j_{n-1})))$ is finite a.s., where $N^{l(j)}_{i,n}(k)$ has been defined for $j \leq n_l$, while for $j = n_l + 1$, $N^{l(n_l+1)}_{i,n}(k) = k\, I_{[l_n(j_n)=l(n_l)]}$.

ii) For fixed i, n, and $l(j) \in J$, $\{W^{l(j)}_{f^i_n(l(j))+s},\ s \geq 1\}$ is an i.i.d. sequence with the same distribution as $W^{l(j)}_1$, where $f^i_n(l(n_l+1))$ denotes the number of parts of type $l(n_l)$ that have been processed by machine M_{l,n_l} before T^i_n;

iii) For fixed i and n the sequences $\{W^{l(j)}_{f^i_n(l(j))+s},\ s \geq 1\}$ as $l(j)$ varying in J are mutually independent and independent of information contained in $\{W^{l(j)}_s,\ 1 \leq s \leq f^i_n(l(j)),\ \forall l(j) \in J\}$.

Fix $v \in (\frac{1}{2}, 1)$ and for $l(j) \in J(i)$ set

$$Z(l(j)) = p_{l(j)}(1 - \rho_{l(j)}) \tag{3.9}$$

$$G^i_{1n}(l(j)) = (N^{l(j)}_{i,n}(\sigma^i_{n+1}(l_n(j_n))))^{-v} \cdot \sum_{s=1}^{N^{l(j)}_{i,n}(\sigma^i_{n+1}(l_n(j_n)))} (W^{l(j)}_{f^i_n(l(j))+s} - p_{l(j-1)}) \tag{3.10}$$

$$G^i_{2n}(l_n(j_n)) = (\sigma^i_{n+1}(l_n(j_n)))^{-v} \cdot \sum_{s=1}^{\sigma^i_{n+1}(l_n(j_n))} (w^{l_n(j_n)}_{g^i_n(l_n(j_n))+s} - q_{l_n(j_n)}). \tag{3.11}$$

Lemma 2: For any $\gamma > 0$ the following estimate holds

$$\sigma^i_{n+1}(l_n(j_n)) \leq \frac{1+\gamma}{Z(l_n(j_n))} X_{l_n(j_n)}(T^i_n) + \beta + a^\eta \left(1 + \frac{1}{\gamma}\right)^{v\eta} \{|G^i_{1n}(l_n(j_n))| + |G^i_{2n}(l_n(j_n))|\}^\eta \tag{3.12}$$

where

$$\eta = \frac{1}{1-v}, \qquad \beta = (1+\gamma) \max_{l(j)\in J} \frac{1}{Z(l(j))} \{\delta r_{l(j-1)} p_{l(j-1)} + 1\},$$

$$a = \max_{l(j)\in J} \frac{1}{Z(l(j))} \max\left\{1,\ \left(\delta + \frac{1}{r_{l(j)}}\right)^v (r_{l(j-1)})^v\right\}.$$

Proof: From (3.2) by (2.6) we have

$$\sigma^i_{n+1}(l_n(j_n)) \leq \alpha_i(n) + \Delta_i(n) \tag{3.13}$$

where

$$\alpha_i(n) \triangleq \frac{1}{Z(l_n(j_n))} \{1 + \delta r_{l_n(j_n-1)} p_{l_n(j_n-1)} + X_{l_n(j_n)}(T^i_n)\} \tag{3.14}$$

$$\Delta_i(n) \triangleq \frac{1}{Z(l_n(j_n))} \{(\sigma^i_{n+1}(l_n(j_n)))^v |G^i_{2n}(l_n(j_n))| + (N^{l_n(j_n)}_{i,n}(\sigma^i_{n+1}(l_n(j_n))))^v |G^i_{1n}(l_n(j_n))|\}. \tag{3.15}$$

From (3.13), (3.15) it follows that

$$\Delta_i(n) \leq a(\alpha_i(n) + \Delta_i(n))^v \cdot \{|G^i_{1n}(l_n(j_n))| + |G^i_{2n}(l_n(j_n))|\}. \tag{3.16}$$

If $\Delta_i(n) \leq \gamma\alpha_i(n)$ for some $\gamma > 0$, then

$$\sigma^i_{n+1}(l_n(j_n)) \leq (1+\gamma)\alpha_i(n). \tag{3.17}$$

Otherwise, $\Delta_i(n) > \gamma\alpha_i(n)$, then from (3.16) we have

$$\Delta_i(n) \leq a\left(\frac{1}{\gamma}\Delta_i(n) + \Delta_i(n)\right)^v \{|G^i_{1n}(l_n(j_n))| + G^i_{2n}(l_n(j_n))|\}$$

and hence

$$\Delta_i(n) \leq a^\eta \left(1 + \frac{1}{\gamma}\right)^{v\eta} \{|G^i_{1n}(l_n(j_n))| + |G^i_{2n}(l_n(j_n))|\}^\eta. \tag{3.18}$$

Combining (3.13), (3.17), (3.18) leads to

$$\sigma^i_{n+1}(l_n(j_n)) \leq (1+\gamma)\alpha_i(n) + a^n \left(1 + \frac{1}{\gamma}\right)^{v\eta} \cdot \{|G^i_{1n}(l_n(j_n))| + |G^i_{2n}(l_n(j_n))|\}^\eta$$

which incorporating with (3.14) gives (3.12).

Let

$$\rho^i_0 = \min_{l(j)\in J(i)} \rho_{l(j)}. \tag{3.19}$$

By condition (2.5) we can take $\gamma > 0$ and $\epsilon > 0$ small enough so that

$$\xi^i \triangleq 1 - \frac{(1+\gamma)(1+v\epsilon)(\rho^i - \rho^i_0)}{1 - \rho^i_0} > 0. \tag{3.20}$$

Set

$$\Phi^i \triangleq 1 - \epsilon_0 \xi^i \tag{3.21}$$

where ϵ_0 is given in (2.3). It is clear that $\Phi^i \in (0, 1)$.

Lemma 3: If CFWL policy is used and condition (2.5) is satisfied, then

$$A^i(T^i_{n+1}) \leq \Phi^i A^i(T^i_n) + c_1 + \alpha |G^i_{2n}(l_n(j_n))|^\eta + \alpha \sum_{l(j)\in J(i)} |G^i_{1n}(l(j))|^\eta \tag{3.22}$$

where $\alpha > 0$, $c_1 > 0$ are constants.

Proof: By (3.6) and (3.7) it is known that

$$A^i(T_{n+1}^i) = \sum_{l(j)\in J(i), l(j)\neq l_n(j_n)} \frac{X_{l(j)}(T_{n+1}^i)}{r_{l(j)}p_{l(j)}}$$
$$\leq A^i(T_n^i) - A_{l_n(j_n)}(T_n^i) + (\rho^i - \rho_{l_n(j_n)})\delta$$
$$+ (\rho^i - \rho_{l_n(j_n)})\frac{\sigma_{n+1}^i(l_n(j_n))}{r_{l_n(j_n)}}$$
$$+ \sum_{l(j)\in J(i), l(j)\neq l_n(j_n)} \frac{1}{r_{l(j)}p_{l(j)}}(N_{i,n}^{l(j)}(\sigma_{n+1}^i(l_n(j_n))))^v$$
$$\cdot G_{1n}^i(l(j)). \tag{3.23}$$

By the Young's inequality $ab \leq pa^{\frac{1}{p}} - qb^{\frac{1}{q}}, \forall a \geq 0, b \geq 0, p+q = 1$, we have

$$(N_{i,n}^{l(j)}(\sigma_{n+1}^i(l_n(j_n))))^v|G_{1n}^i(l(j))|$$
$$\leq \epsilon v r_{l(j-1)}p_{l(j-1)}\frac{\sigma_{n+1}^i(l_n(j_n))}{r_{l_n(j_n)}}$$
$$+ (1-v)(1+\delta r_{l_n(j_n)})^{v\eta}(\epsilon p_{l(j-1)})^{-v\eta}|G_{1n}^i(l(j))|^\eta.$$

Hence we continue (3.23) as follows

$$A^i(T_{n+1}^i) \leq A^i(T_n^i) - A_{l_n(j_n)}^i(T_n^i)$$
$$+ (1+v\epsilon)(\rho^i - \rho_{l_n(j_n)})\frac{\sigma_{n+1}^i(l_n(j_n))}{r_{l_n(j_n)}}$$
$$+ \delta(\rho^i - \rho_{l_n(j_n)})$$
$$+ \zeta \sum_{l(j)\in J(i), l(j)\neq l_n(j_n)} |G_{1n}^i(l(j))|^\eta \tag{3.24}$$

where

$$\zeta = (1-v)\left(1 - \delta \max_{l(j)\in J} r_{l(j)}\right)^{v\eta} \max_{l(j)\in J} \frac{1}{r_{l(j)}p_{l(j)}}(\epsilon p_{l(j-1)})^{-v\eta}.$$

Substituting (3.12) into (3.24), by (2.1) we derive

$$A^i(T_{n+1}^i)$$
$$\leq A^i(T_n^i) - A_{l_n(j_n)}(T_n^i)\left(1 - \frac{(1+\gamma)(1+\epsilon v)(\rho^i - \rho_{l_n(j_n)})}{1-\rho_{l_n(j_n)}}\right)$$
$$+ \frac{(1+\epsilon v)(\rho^i - \rho_{l_n(j_n)})}{r_{l_n(j_n)}}\left\{\beta + a^\eta\left(1+\frac{1}{\gamma}\right)^{v\eta}\right.$$
$$\left.\cdot (|G_{1n}^i(l_n(j_n))| + |G_{2n}^i(l_n(j_n))|)^\eta\right\}$$
$$+ \zeta \sum_{l(j)\in J(i), l(j)\neq l_n(j_n)} |G_{1n}^i(l(j))|^\eta + \delta(\rho^i - \rho_{l_n(j_n)}).$$

From this by conditions (2.3) and (2.5) we find that there are constants $\alpha > 0$ and $c_1 > 0$ such that (3.22) holds.

Proof of Theorem 1: By Theorem 10.3.4 in [1] and Lemma 1, from (3.10) we see

$$E|G_{1n}^i(l(j))|^{m\eta}$$
$$= E\sum_{k=1}^{\infty} \frac{1}{k^{m\eta v}}\left|\sum_{s=1}^{k}(W_{f_n^i(l(j))+s}^{l(j)} - p_{l(j-1)})\right|^{m\eta} I_{[N_{i,n}^{l(j)}(\sigma_{n+1}^i(l_n(j_n)))=k]}$$
$$\leq E\sup_k \frac{1}{k^{m\eta v}}\left|\sum_{s=1}^{k}(W_{f_n^i(l(j))+s}^{l(j)} - p_{l(j-1)})\right|^{m\eta} < \infty \tag{3.25}$$

for all $l(j) \in J(i)$, if we choose $v \in (\frac{1}{2}, 1)$ and $2 < \eta = (1/1-v) < 3$. Since $\|G_{1n}^i(l(j))\|_k \triangleq \{E|G_{1n}^i(l(j))|^k\}^{\frac{1}{k}}$ is nondecreasing in $k > 0$, $E|G_{1n}^i(l(j))|^k < \infty$ holds also for $k \leq m\eta$, $\forall l(j) \in J(i)$.

From (3.25) we have

$$E|G_{1n}^i(l_n(j_n))|^{m\eta} \leq \sum_{l(j)\in J(i)} E|G_{1n}^i(l(j))|^{m\eta} < \infty. \tag{3.26}$$

Similarly

$$E|G_{2n}^i(l_n(j_n))|^{m\eta} < \infty. \tag{3.27}$$

Therefore, from (3.22) we know that there is a constant $c(m) > 0$ independent of n such that

$$\|A^i(T_{n+1}^i)\|_m \leq \Phi^i\|A^i(T_n^i)\|_m + c(m). \tag{3.28}$$

From (3.21) and (3.28) we conclude (2.4) with $s = m$ immediately.

Noticing that $\overline{x}_{l_n(j_n)}(n) \leq \sigma_{n+1}^i(l_n(j_n))$, we have

$$\sum_{l(j)\in J(i)} \overline{x}_{l(j)}(n) = \overline{x}_{l_n(j_n)}(n) + \sum_{l(j)\in J(i), l(j)\neq l_n(j_n)} X_{l(j)}(T_{n+1}^i)$$
$$\leq \sigma_{n+1}^i(l_n(j_n)) + X^i(T_{n+1}^i). \tag{3.29}$$

From (2.1), (3.9) and (3.12) it follows that

$$\sigma_{n+1}^i(l_n(j_n)) \leq \frac{(1+\gamma)r_{l_n(j_n)}}{1-\rho_{l_n(j_n)}}A_{l_n(j_n)}^i(T_n^i) + \beta$$
$$+ a^\eta\left(1+\frac{1}{\gamma}\right)^{v\eta}\{|G_{1n}^i(l_n(j_n))| + |G_{2n}^i(l_n(j_n))|\}^\eta. \tag{3.30}$$

By (3.26), (3.27) from this we find

$$\sup_{n\geq 0}\|\sigma_{n+1}^i(l_n(j_n))\|_m \leq c_1(m) \tag{3.31}$$

where $c_1(m)$ is a constant for fixed m. Finally (2.7) follows from (3.29) and (3.31).

Proof of Theorem 2: From (3.13) by (2.1) we have

$$\frac{\sigma_{n+1}^i(l_n(j_n))}{r_{l_n(j_n)}}$$
$$\leq \frac{1}{1-\rho_{l_n(j_n)}}\left\{A_{l_n(j_n)}^i(T_n^i)\right.$$
$$+ \frac{1}{r_{l_n(j_n)}p_{l_n(j_n)}}(1+\delta r_{l_n(j_n-1)}p_{l_n(j_n-1)})$$
$$+ \frac{1}{r_{l_n(j_n)}p_{l_n(j_n)}}(\sigma_{n+1}^i(l_n(j_n)))^v G_{2n}^i(l_n(j_n))$$
$$\left.+ \frac{1}{r_{l_n(j_n)}p_{l_n(j_n)}}(N_{i,n}^{l_n(j_n)}(\sigma_{n+1}^i(l_n(j_n))))^v G_{1n}^i(l_n(j_n))\right\}. \tag{3.32}$$

Noticing that by Lemma 1 and the conditional expectation version of the Wald's equation, which can be proved in a way similar to Theorem 5.3.1 in [1], we find

$$Eh(l_n(j_n))(\sigma_{n+1}^i(l_n(j_n)))^v G_{2n}^i(l_n(j_n))$$
$$= E\sum_{l(j)\in J(i)} h(l_n(j_n))I_{[l_n(j_n)=l(j)]}$$
$$\cdot E\left\{\sum_{s=1}^{\sigma_{n+1}^i(l_n(j_n))I_{[l_n(j_n)=l(j)]}} (w_{f_n^i(l(j))+s}^{l(j)} - q_{l(j)}) \;\middle|\; X_{l(j)}^i(T_n^i),\right.$$
$$\left.\forall l(j) \in J(i)\right\} = 0 \tag{3.33}$$

for any bounded measurable function $h(\cdot)$. Similarly, we have

$$Eh(l_n(j_n))(N_{i,n}^{l_n(j_n)}(\sigma_{n+1}^i(l_n(j_n))))^v G_{1n}^i(l_n(j_n)) = 0, \tag{3.34}$$

$$E\sum_{l(j)\in J(i)} h(l(j))(N_{i,n}^{l(j)}(\sigma_{n+1}^i(l_n(j_n))))^v G_{1n}^i(l(j)) = 0. \tag{3.35}$$

Taking expectation for both sides of (3.23) and putting (3.32)–(3.35) into (3.23) with policy (2.3) applied, we derive $E\,A^i(T_{n+1}^i) \leq (1-\epsilon_0\Phi_0^i)E\,A^i(T_n^i) + v_2^i$. Hence we obtain (2.9) and (2.10).

IV. Concluding Remarks

The manufacturing system considered in this paper consists of unreliable machines and random inputs. We have given conditions guaranteeing stability of the system, which means the boundedness of moments of buffer levels at stopping times at which machines change their processing from one part-type to another. The results are essentially based on the fact that for fixed n, i, and $l(j)$ the random sequence $\{W^{l(j)}_{f^i_n(l(j))+s}, s \geq 1\}$, representing the random part-flow, with random index is i.i.d. This important property is proved for the system with properties 1)–7) listed in Introduction, among which the key assumption is that events of unsatisfactory processing at all machines are mutually independent. It is of interest to consider the boundedness of buffer levels at any deterministic time.

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The Discrete-Time Riccati Equation Related to the H_∞ Control Problem

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***Abstract*— The H_∞ control problem has been solved recently with the use of discrete-time algebraic Riccati equations. In this paper, we investigate this Riccati equation. We derive recursive methods to find the stabilizing solution of this Riccati equation. Moreover, we derive several properties of the class of positive semi-definite solutions of this equation.**

I. Introduction

The discrete-time algebraic Riccati equation has been investigated extensively in the literature (see e.g., [4], [7], [9], [10], [16]). Most of this work, however, was based on the algebraic Riccati equation appearing in linear quadratic control. Recently, the H_∞ control problem for discrete time systems was solved (see e.g., [1], [2], [6], [8], [11]). This gave rise to a different kind of algebraic Riccati equation. This paper is concerned with this Riccati equation. It should be noted that this Riccati equation also arises in differential games (see e.g., [2]).

We show that we can always use a recursive method to find the stabilizing solution of this equation (as is done in [4] for the linear quadratic control algebraic Riccati equation). Moreover, we derive several properties of the solutions to this equation. It should be noted that this Riccati equation has less elegant properties, mainly because of an indefinite nonlinear term and because we cannot a priori guarantee existence of solutions.

Our research is partially based on [15]. After submission of this paper, another paper [5] was written which gives a nonrecursive method to determine stabilizing solutions of this Riccati equation via a connection to continuous time Riccati equations using a Möbius transformation.

The notation will be fairly standard. By $x_{u,w,\xi}$ and $z_{u,w,\xi}$ we denote the state x and the output z of a system after applying inputs u and w and with initial condition $x(0) = \xi$. Moreover l_2 denotes the class of square-summable functions and $\|\cdot\|_2$ denotes the standard l_2-norm. We will use in some of the proofs the subspaces $\mathcal{V}$ and $\mathcal{V}_g$ as defined in the appendix.

II. Problem Formulation

In this paper we investigate positive semi-definite matrices P such that

$$\begin{aligned} V(P) &:= B^T P B + D_1^T D_1 > 0, \\ R(P) &:= I - D_2^T D_2 - E^T P E + (E^T P B + D_2^T D_1) \\ &\quad \cdot V(P)^{-1}(B^T P E + D_1^T D_2) > 0 \end{aligned} \tag{2.1}$$

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