The Åström–Wittenmark Self-Tuning Regulator Revisited and ELS-Based Adaptive Trackers

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***Abstract*—Although there has been made a considerable progress in stochastic adaptive control, the problem concerning the convergence of the original self-tuning regulator proposed by Åström and Wittenmark in 1973 is still open until now. Since it is attractive in theory and important in applications, we return to this problem and give a rigorous proof for its stability and optimality. Related problems such as convergence of the extended-least-squares (ELS)-based adaptive tracker are also considered in this paper.**

I. A Long Standing Open Problem

Let us consider the following ARMAX model:

$$A(z)y_n = B(z)u_{n-1} + C(z)w_n, \qquad n \geq 0, \quad (1.1)$$

$$A(z) = 1 + A_1 z + \cdots + A_p z^p, \qquad p \geq 1,$$

$$B(z) = B_1 + B_2 z + \cdots + B_q z^{q-1}, \quad q \geq 1,$$

$$C(z) = 1 + C_1 z + \cdots + C_r z^r, \qquad r \geq 0$$

where y_n, u_n, and w_n are the m-dimensional system output, input, and random disturbance, respectively, $y_n = 0$, $u_n = 0$, $w_n = 0$ for $n < 0$, $A(z)$, $B(z)$, and $C(z)$ are polynomials in backward-shift operator z with unknown matrix coefficients A_i, B_j, and C_k and with known upper bounds p, q, and r for orders.

Let us denote

$$\theta = \left[-A_1 \cdots -A_p \quad B_1 \cdots B_q \quad C_1 \cdots C_r\right]^\tau.$$

The most commonly used method for estimating θ is the following extended least-squares (ELS) algorithm:

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - \theta_n^\tau \varphi_n)^\tau \qquad n \geq 0, \quad (1.2)$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^\tau P_n, \; a_n = \left(1 + \varphi_n^\tau P_n \varphi_n\right)^{-1}, \quad (1.3)$$

$$\varphi_n = \left[y_n^\tau \cdots y_{n-p+1}^\tau \quad u_n^\tau \cdots u_{n-q+1}^\tau \quad \hat{w}_n^\tau \cdots \hat{w}_{n-r+1}^\tau\right]^\tau, \quad (1.4)$$

$$\hat{w}_n = y_n - \theta_n^\tau \varphi_{n-1}, \; n \geq 0; \qquad \hat{w}_n = 0, \; n < 0 \quad (1.5)$$

with arbitrary initial values θ_0 and $P_0 > 0$.

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The assumptions made on system (1.1) are as follows:

A1: $\{w_n, \mathcal{F}_n\}$ is a Martingale difference sequence satisfying the following conditions:

$$\sup_{n \geq 0} E\left[\|w_{n+1}\|^\beta \mid \mathcal{F}_n\right] < \infty, \quad \text{a.s., for some } \beta > 2 \quad (1.6)$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n w_i w_i^\tau = R > 0, \qquad \text{a.s.} \qquad (1.7)$$

A2: $C^{-1}(e^{i\lambda}) + C^{-\tau}(e^{-i\lambda}) - I > 0, \qquad \forall \lambda \in [0, 2\pi].$

A3: $\det B(z) \neq 0, \qquad \forall z\colon |z| \leq 1.$

We recall that A2 is known as the strictly positive-real condition which is automatically satisfied if $C(z) = I$, and A3 is known as the minimum phase condition.

Let us formulate the basic problem discussed in the paper.

Basic problem: Let $\{y_n^*\}$ be a given almost surely (a.s.) bounded reference signal and let y_{n+1}^* be $\mathcal{F}_n$-measurable. Under conditions A1–A3 it is required to design an adaptive control u_n purely based on the ELS algorithm (1.2)–(1.5) in order that

1) the closed-loop system is globally stable, i.e.,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left(\|u_i\|^2 + \|y_i\|^2\right) < \infty, \qquad \text{a.s.} \quad (1.8)$$

2) the tracking error is minimized

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (y_i - y_i^*)(y_i - y_i^*)^\tau = R, \qquad \text{a.s.} \quad (1.9)$$

3) the estimate θ_n given by (1.2)–(1.5) is strongly consistent.

An early reference on the basic problem is the self-tuning regulator proposed by Åström and Wittenmark [1] in 1973. The goal of the adaptive control in [1] is to minimize the output variance or the tracking error with $y_n^* \equiv 0$ for a single-input and single-output system with $C(z) \equiv 1$, and with B_1 known and different from zero.

Since its appearance, the self-tuning regulator has got a great success in applications and naturally has drawn much attention from control theorists in an attempt to establish its convergence. The first significant progress in this direction was made by Goodwin, Ramadge, and Caines [2]. They have shown (1.8) and (1.9) for an adaptive tracker that is not based on ELS but on the stochastic approximation (SA)

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algorithm, a modification of (1.2)–(1.5). Later, Becker, Kumar, and Wei [3] have shown that in general the SA estimate θ_n is not strongly consistent in the adaptive scheme of [2] unless the reference signal $\{y_n^*\}$ is sufficiently rich in some sense (see [4, p. 569] and [5]). To get strong consistency of θ_n when $\{y_n^*\}$ is arbitrary but bounded signal, in [6] and [7] by invoking a "continuously disturbed controller," the strong consistency of parameter estimate and the global stability of the closed-loop system have been achieved simultaneously under Conditions A1–A3 and an identifiability condition. However, in these papers the estimate is carried out not by the ELS algorithm and the tracking is no longer optimal but suboptimal. In order that the external excitation does not worsen the tracking accuracy, a diminishing excitation technique is applied in [4], where requirements 1)–3) of our basic problem are met, however, again they are established for the SA algorithm.

As pointed out by Sin and Goodwin [8], "it seems that in practically all applications of stochastic adaptive control, a least squares iteration is used," since "it generally has much superior rates of convergence compared with stochastic approximation." Consequently, research on the ELS-based adaptive tracker is continuing. With modifications of Åström–Wittenmark (Å–W) self-tuning regulator, Lai and Wei [9] have a sharp convergence rate for the tracking error. They assume that the open-loop system is stable, the random noise is uniformly bounded, and that a certain identifiability condition is satisfied. Under similar assumptions (but without requiring boundedness of the noise), the present authors get the convergence rates for both tracking and parameter estimation in [10]. Again, the algorithm used there is a modified version of the Å–W self-tuning regulator. The main restriction of [9] and [10] is that there the stability and optimality are established only for open-loop stable systems. Certainly, if the parallel algorithm introduced in [11] is used, then the assumption of open-loop stability can easily be removed [12]. The idea is that besides the ELS algorithm, a parallel SA algorithm aiming at slowing down the growth rate of the system input and output, is used for a finite period of operation. A similar idea is used also in [13]. However, such parallel algorithms are complicated, the heavy computation burden may prevent such algorithms from being applied in any applications. Moreover, in [9], [10], [12], and [13], stability of the closed-loop system is established via consistency of parameter estimates, for which identifiability conditions are required. Such conditions are not necessary for achieving stability and optimality of adaptive tracking systems (see, e.g., [2]).

Recently, assuming independence and Gaussianity of $\{w_n\}$ with $C(z) = I$, and having noticed the connection between the least-squares estimate and the conditional expectation for an unknown parameter, Kumar [14] has shown that the least-squares based adaptive tracker converges outside an exceptional set of Lebesgue measure zeros in the parameter space of θ. In this approach, it is necessary to preclude θ corresponding to the system under consideration from being in the exceptional set, in which, convergence is not guaranteed for almost all sampling points. Moreover, the exceptional set may vary with various selections of initial values for the estimation algorithm. So, in his conclusions, Kumar [14] points out, "it would be of considerable interest to remove at least the Gaussianity restriction on the disturbance w_n. The whiteness restriction cannot be removed so easily . . . . It would also be of considerable interest to show that the exceptional set of Theorem 1 is really an empty set."

Thus, in summary, as noted by Kumar [14], even "the convergence of the original self-tuning regulator of Åström and Wittenmark which uses a reclusive least-squares parameter estimator followed by a minimum variance certainty equivalent control law, has been an open question for more than fifteen years." In this paper, we will rigorously prove the convergence of the original Å–W self-tuning regulator, and at the same time give a solution to the basic problem stated previously. Some preliminary results on this topic are presented in [15].

II. Convergence of Å–W Self-Tuning Tracker

The Å–W self-tuning tracker discussed in this section is an extension of the self-tuning regulator introduced by Åström–Wittenmark [1] based on the least-squares estimation for single-input and single-output systems with B_1 known. The extension consists of the reference signal $\{y_n^*\}$ which is an arbitrary bounded sequence not necessarily equal to zero, and the system may be multidimensional.

Definition: Let B_1 be known and nondegenerate, $\{y_n^*\}$ be an a.s. bounded sequence of m-dimensional vectors with y_{n+1}^* being $\mathscr{F}_n$-measurable, and let u_n be defined from

$$u_n = B_1^{-1}(y_{n+1}^* - \theta_n^\tau \varphi_n) \tag{2.1}$$

where θ_n is the ELS estimate for

$$\theta = [-A_1 \cdots -A_p \quad B_2 \cdots B_q \quad C_1 \cdots C_r]^\tau \tag{2.2}$$

calculated according to the following ELS recursion:

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - B_1 u_n - \theta_n^\tau \varphi_n)^\tau, \quad n \ge 0 \tag{2.3}$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^\tau P_n, \quad a_n = (1 + \varphi_n^\tau P_n \varphi_n)^{-1}, \tag{2.4}$$

$$\varphi_n = [y_n^\tau \cdots y_{n-p+1}^\tau \quad u_{n-1}^\tau \cdots u_{n-q+1}^\tau \quad \hat{w}_n^\tau \cdots \hat{w}_{n-r+1}^\tau]^\tau, \tag{2.5}$$

$$\hat{w}_n = y_n - B_1 u_{n-1} - \theta_n^\tau \varphi_{n-1}, \quad n \ge 0; \quad \hat{w}_n = 0, \; n < 0 \tag{2.6}$$

with arbitrary initial values θ_0 and $P_0 > 0$.

The adaptive control system (1.1), (2.1)–(2.6) is called the Å–W self-tuning tracker.

We note that the proposed recursion (2.3)–(2.6) is the same as that used in Åström–Wittenmark [1] in the white noise case. However, in the general colored-noise case the widely used *a posteriori* errors, which are slightly different from the *a priori* errors appearing in Åström-Wittenmark [1], are applied here. The following theorem shows the stability and optimality of the Å–W self-tuning tracker (2.1)–(2.6).

Theorem 1: If Conditions A1–A3 are satisfied, then the Å–W self-tuning tracker (1.1), (2.1)–(2.6) is stable and optimal in the sense that (1.8) and (1.9) hold. Furthermore, let $\{d_n\}$ be a nondecreasing positive sequence satisfying

$$\sup_{n\geq 0}\frac{d_{n+1}}{d_n}<\infty,\ \|w_n\|^2=O(d_n),\qquad \text{a.s.}\qquad (2.7)$$

Then the following convergence rate holds:

$$\sum_{i=1}^{n}\|y_i-y_i^*-w_i\|^2=O(n^{\epsilon}d_n),\qquad \text{a.s., }\forall\epsilon>0.\qquad (2.8)$$

The proof is given in Section IV.

Remark: We note at once that the sequence $\{d_n\}$ defined in Theorem 1, in fact, can be taken as

$$d_n=n^{\delta},\ \delta\in\left(\frac{2}{\beta},1\right).\qquad (2.9)$$

where β is given by (1.6). To see this, by (1.6) and the Markov inequality we have

$$\sum_{n=1}^{\infty}P(\|w_{n+1}\|^2\geq n^{\delta}\mid\mathscr{F}_n)\leq\sum_{n=1}^{\infty}\frac{E[\|w_{n+1}\|^{\beta}\mid\mathscr{F}_n]}{n^{\beta\delta/2}}<\infty,\qquad \text{a.s.}$$

and by the conditional Borel–Cantelli lemma (e.g., [16, p. 55]) we know that

$$\|w_{n+1}\|^2=O(n^{\delta})\qquad \text{a.s. }\forall\delta\in\left(\frac{2}{\beta},1\right).\qquad (2.10)$$

Hence (2.9) is true. Moreover, if there are further assumptions on the noise sequence $\{w_n\}$, the convergence rate in (2.8) can be improved. For example, if $\{w_n\}$ is a Gaussian white noise sequence, then again by Borel–Cantelli lemma and the Gaussian density function it is easily shown that d_n can be taken as $d_n=\log n$ (see also [17]); if $\{w_n\}$ is a bounded sequence, then $d_n=1$.

III. Convergence of ELS-Based Adaptive Tracker (with B_1 Unknown)

In the last section, we have claimed the stability and optimality of an adaptive tracker when the leading matrix coefficient in $B(z)$ is known. Here, we shall no longer impose the availability of B_1 and will use (1.2)–(1.5) to estimate the whole θ including B_1.

We first give a solution to our basic problem but with the consistency of parameter estimate ignored.

Let us write the estimate θ_n given by (1.2)–(1.5) in the block form

$$\theta_n=[-A_{1n}\cdots-A_{pn}\quad B_{1n}\cdots B_{qn}\quad C_{1n}\cdots C_{rn}]^{\tau}\qquad (3.1)$$

and define

$$r_n=e+\sum_{i=0}^{n}\|\varphi_i\|^2,\qquad n\geq 0.\qquad (3.2)$$

The certainty equivalence principle suggests us to define adaptive control from

$$\theta_n^{\tau}\varphi_n=y_{n+1}^*,\qquad n\geq 1\qquad (3.3)$$

or

$$u_n=B_{1n}^{-1}\{y_{n+1}^*+(B_{1n}u_n-\theta_n^{\tau}\varphi_n)\},\qquad \text{if }\det[B_{1n}]\neq 0.\qquad (3.4)$$

The first problem arising here is that u_n may not be well-defined because the set $\{\det[B_{1n}]=0\}$ may have a positive probability unless some sort of continuity assumption is imposed on the distribution of w_n (see [4] and [18] for related discussions). However, we do not intend to make such a restriction on distributions of w_n, instead, we will slightly modify B_{1n} when we define u_n, so that it is kept from being zero or being too small.

As a matter of fact, we are replacing "B_{1n}^{-1}" in (3.4) by any $\mathscr{F}_n$-measurable $\hat{B}_{1n}^{-1}$ that satisfies the following conditions (3.5) and (3.6):

$$\hat{B}_{1n}^{\tau}\hat{B}_{1n}\geq\frac{1}{\log r_{n-1}}I,\qquad n\geq 1\qquad (3.5)$$

$$\|\hat{B}_{1n}-B_{1n}\|\leq\frac{1}{(\log r_{n-1})^{1/2}},\qquad n\geq 1\qquad (3.6)$$

when defining the adaptive control, where r_n is given by (3.2).

We note at once that 1) $\hat{B}_{1n}$ is asymptotically equivalent to B_{1n} since as will be shown later $r_n\to\infty$ as $n\to\infty$; 2) for parameter estimation the ELS algorithm is not modified.

For single-input and single-output systems ($m=1$), it is immediately verified that $\hat{B}_{1n}$ given by the following simple modification from B_{1n} satisfies (3.5) and (3.6)

$$\hat{B}_{1n}=\begin{cases}B_{1n}, & \text{if }|B_{1n}|\geq\dfrac{1}{(\log r_{n-1})^{1/2}};\\ B_{1n}+\dfrac{1}{(\log r_{n-1})^{1/2}}\operatorname{sgn}(B_{1n}), & \text{otherwise}\end{cases}\qquad (3.7)$$

where

$$\operatorname{sgn}(x)=\begin{cases}1, & x\geq 0;\\ -1, & x<0.\end{cases}\qquad (3.8)$$

For the multidimensional case, one way of defining $\hat{B}_{1n}$, which is an analog of (3.7), is as follows. Let the singular value decomposition of B_{1n} be (see, e.g., [19, p. 318])

$$B_{1n}=V_n\begin{bmatrix}\Sigma_n & 0\\ 0 & 0\end{bmatrix}U_n^{\tau}\qquad (3.9)$$

where U_n and V_n are orthogonal matrices and Σ_n is a positive definite diagonal matrix. The following choice corresponds to (3.7) and satisfies (3.5) and (3.6)

$$\hat{B}_{1n} = \begin{cases} B_{1n}, & \text{if } B_{1n}^{\tau} B_{1n} \geq \dfrac{1}{\log r_{n-1}} I; \\ B_{1n} + V_n U_n^{\tau} \dfrac{1}{(\log r_{n-1})^{1/2}}, \\ \text{otherwise}. \end{cases} \tag{3.10}$$

In accordance with (3.4) we define u_n by

$$u_n = \hat{B}_{1n}^{-1}\{y_{n+1}^* + (B_{1n}u_n - \theta_n^{\tau}\varphi_n)\}. \tag{3.11}$$

Definition: The adaptive control system (1.1)–(1.5) and (3.11) with (3.5) and (3.6) satisfied for an a.s. bounded $\{y_n^*\}$ with y_{n+1}^* being $\mathscr{F}_n$-measurable is called the ELS-based adaptive tracker.

Theorem 2: Under conditions A1–A3 the ELS-based adaptive tracker is stable and optimal in the sense that (1.8) and (1.9) hold. Moreover

$$\|y_n\|^2 + \|u_n\|^2 = o(n^{\epsilon}d_n), \qquad \text{a.s. } \forall \epsilon > 0 \tag{3.12}$$

where d_n is defined in Theorem 1.

The proof is given in Section IV.

Theorems 1 and 2 have established the convergence of ELS-based adaptive trackers without paying attention to the consistence issue of the estimates. We now give a solution to the basic problem by using the diminishing excitation technique developed in [4] and [20].

We first define the excitation source. Let $\{\epsilon_i\}$ be the sequence of m-dimensional i.i.d. random vectors independent of $\{w_i, y_i^*\}$ with $E\epsilon_i = 0$

$$E\epsilon_k\epsilon_k^{\tau} = I, \qquad \|\epsilon_k\| \leq \sigma$$

where σ is a constant.

Replacing (3.11), we define a vector u_n^o as

$$u_n^o \triangleq \hat{B}_{1n}^{-1}\{y_{n+1}^* + (B_{1n}u_n - \theta_n^{\tau}\varphi_n)\} \tag{3.13}$$

and the diminishingly excited input u_n as

$$u_n = u_n^o + v_n \tag{3.14}$$

where

$$v_n = \frac{\epsilon_n}{r_{n-1}^{\bar{\epsilon}/2}}, \ \bar{\epsilon} \in \left(0, \frac{1}{2(t+1)}\right),$$
$$t = \max(p, q, r) + mp - 1. \tag{3.15}$$

Theorem 3: Assume that conditions A1–A3 hold, $A(z)$, $B(z)$, and $C(z)$ have no common left factor and $[A_p B_q C_r]$ is of full-row rank. Then the adaptive tracker consisting of (1.1)–(1.5), (3.10), (3.13), and (3.14) solves the basic problem. To be precise, (1.8) and (1.9) are fulfilled and

$$\|\theta_n - \theta\|^2 = O\left(\frac{\log n}{n^{1-(t+1)\bar{\epsilon}}}\right), \qquad \text{a.s.} \tag{3.16}$$

and

$$\sum_{i=1}^{n} \|y_i - y_i^* - w_i\|^2 = O(n^{1-\bar{\epsilon}}) + O(d_n), \qquad \text{a.s.} \tag{3.17}$$

where $\bar{\epsilon}$ is given by (3.15) and d_n is defined in Theorem 1.

Remark: Theorems 1–3 remain valid if the boundedness of $\{y_n^*\}$ is replaced by

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|y_i^*\|^2 < \infty \text{ and } \|y_n^*\| = O(n^b), \qquad \text{a.s.}, \forall b > 0.$$

IV. Proof of the Theorems

We start with lemmas.

Lemma 1: For $\{\theta_n\}$ generated by either the algorithm (1.2)–(1.5) or the algorithm (2.3)–(2.6) , if conditions A1 and A2 hold, and u_n is $\mathscr{F}_n$-measurable, then

1) $$\|\theta_{n+1} - \theta\|^2 = O\left(\frac{\log r_n}{\lambda_{\min}(n)}\right), \qquad \text{a.s.} \tag{4.1}$$

2) $$\sum_{i=1}^{n+1} \|\hat{w}_i - w_i\|^2 = O(\log r_n), \qquad \text{a.s.} \tag{4.2}$$

3) $$\sum_{i=1}^{n} \frac{\|\tilde{\theta}_i^{\tau}\varphi_i\|^2}{1 + \varphi_i^{\tau}P_i\varphi_i} = O(\log r_n), \qquad \text{a.s.} \tag{4.3}$$

where $\tilde{\theta}_n = \theta - \theta_n$ and $\lambda_{\min}(n)$ is the minimum eigenvalue of

$$P_{n+1}^{-1} = \sum_{i=1}^{n} \varphi_i\varphi_i^{\tau} + P_o^{-1}. \tag{4.4}$$

Except (4.3), whose proof is given in the Appendix, this lemma is not new. We note that Lai and Wei [21] are the first to use the condition $\log r_n/\lambda_{\min}(n) \to 0$ in guaranteeing strong consistency of the LS estimates.

Remark: It should not be confused that we use the same notations for algorithms (1.2)–(1.5) and (2.3)–(2.6). For example, r_n is defined by (3.2) with φ_i given by (1.4) for the algorithm (1.2)–(1.5), but when the algorithm (2.3)–(2.6) is under consideration, φ_i in (3.2) should be understood as that given by (2.5).

The following Lemma 2 is crucial in establishing our results. The proof of it benefits from some ideas in [22].

Lemma 2: Under Conditions A1–A3, for the ELS-based adaptive tracker (1.1)–(1.5) and (3.11) with $\hat{B}_{1k}$ satisfying (3.5) and (3.6), and for the Å–W self-tuning tracker (1.1), (2.1)–(2.6) the sequence $\{\varphi_k\}$ has the following estimation:

$$\|\varphi_k\|^2 = O(r_k^{\epsilon}d_k), \qquad \text{a.s. } \forall \epsilon > 0 \tag{4.5}$$

where d_k is defined in Theorem 1.

Before proceeding to prove Lemma 2, we first show that given (4.5) it is a rather easy task to conclude Theorems 1 and 2.

Proof of Theorems 1 and 2: By (1.1), (1.7), and (3.2), it is easy to show that (see, for example, [17, Eq. (35), p. 1097])

$$\liminf_{n\to\infty} \frac{r_n}{n} > 0 \qquad \text{a.s.} \tag{4.6}$$

So, by (2.9)

$$d_n = O(r_n^{\delta}), \qquad \text{a.s. } \forall\delta \in \left(\frac{2}{\beta}, 1\right). \tag{4.7}$$

From Lemma 1-3) and (4.5) it follows that

$$\sum_{i=0}^{n} \|\varphi_i^\tau \tilde{\theta}_i\|^2 = \sum_{i=0}^{n} \frac{\|\varphi_i^\tau \tilde{\theta}_i\|^2}{1+\varphi_i^\tau P_i \varphi_i}(1+\varphi_i^\tau P_i \varphi_i)$$

$$= O(\log r_n) + O\left(r_n^\epsilon d_n \sum_{i=0}^{n} \frac{\|\varphi_i^\tau \tilde{\theta}_i\|^2}{1+\varphi_i^\tau P_i \varphi_i}\right)$$

$$= O(\log r_n) + O(r_n^\epsilon d_n \log r_n), \qquad \forall \epsilon > 0$$

hence by the arbitrariness of ϵ

$$\sum_{i=0}^{n} \|\varphi_i^\tau \tilde{\theta}_i\|^2 = O(r_n^\epsilon d_n), \qquad \text{a.s. } \forall \epsilon > 0. \tag{4.8}$$

Consequently, by the arbitrariness of δ and ϵ in (4.7) and (4.8)

$$\sum_{i=0}^{n} \|\varphi_i^\tau \tilde{\theta}_i\|^2 = O(r_n^\delta), \qquad \text{a.s. } \forall \delta \in \left(\frac{2}{\beta}, 1\right). \tag{4.9}$$

Set

$$\varphi_n^0 = [y_n^\tau \cdots y_{n-p+1}^\tau \quad u_s^\tau u_{s-1}^\tau \cdots u_{n-q+1}^\tau \quad w_n^\tau \cdots w_{n-r+1}^\tau]^\tau \tag{4.10}$$

where $s = n-1$ for Theorem 1 and $s = n$ for Theorem 2.

In the case of Theorem 2, by (3.11) we have

$$\begin{aligned} y_{k+1} &= \theta^\tau \varphi_k + \theta^\tau(\varphi_k^0 - \varphi_k) + w_{k+1} \\ &= \tilde{\theta}_k^\tau \varphi_k - \Delta \hat{B}_{1k} u_k + y_{k+1}^* \\ &\quad + \theta^\tau(\varphi_k^0 - \varphi_k) + w_{k+1}, \\ \Delta \hat{B}_{1k} &= \hat{B}_{1k} - B_{1k} \end{aligned} \tag{4.11}$$

while in the case of Theorem 1, (4.11) holds with the term $\Delta \hat{B}_{1k} u_k$ removed. Here, we should keep in mind that the definitions of θ, θ_n, φ_n in Theorem 1 are different from those in Theorem 2. Noticing (4.2) and (4.9) from (4.11) we see

$$\begin{aligned} \sum_{k=0}^{n} \|y_{k+1}\|^2 &= O(r_n^\delta) + o\left(\sum_{k=0}^{n} \|u_k\|^2\right) \\ &\quad + O(\log r_n) + O(n) \\ &= o(r_n) + O(n). \end{aligned} \tag{4.12}$$

From this and condition A3, it is evident that

$$\begin{aligned} \sum_{k=0}^{n} \|u_k\|^2 &= O\left(\sum_{i=0}^{n+1} \|y_i\|^2\right) + O\left(\sum_{i=0}^{n+1} \|w_i\|^2\right) \\ &= o(r_n) + O(n). \end{aligned} \tag{4.13}$$

By using Lemma 1-2) and (1.7), we have

$$\begin{aligned} \sum_{i=1}^{n} \|\hat{w}_i\|^2 &\le 2\sum_{i=1}^{n} [\|\hat{w}_i - w_i\|^2 + \|w_i\|^2] \\ &= O(\log r_n) + O(n), \qquad \text{a.s.} \end{aligned} \tag{4.14}$$

Consequently, by the definition of r_n, we see from (4.12)–(4.14) that $r_n = o(r_n) + O(n)$ a.s.. From this, it follows that

$$r_n = O(n), \quad \text{a.s.} \tag{4.15}$$

Hence it is easy to see that (1.8) holds, while (3.12) follows from Lemma 2 and (4.15).

By (3.6), (4.2), (4.9), and (4.15), it is seen that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|\tilde{\theta}_k^\tau \varphi_k - \Delta \hat{B}_{1k} u_k + \theta^\tau(\varphi_k^0 - \varphi_k)\|^2 = 0 \text{ a.s.}$$

for the case of Theorem 2 or with the term $\Delta \hat{B}_{1k} u_k$ removed in the case of Theorem 1. From this, (4.11), and (1.7), it is easy to see that (1.9) is true. This completes the proof of Theorem 2, while for Theorem 1 it remains to prove (2.8).

By (4.2) and (4.8), it is clear that

$$\sum_{k=1}^{n} \|\tilde{\theta}_k^\tau \varphi_k + \theta^\tau(\varphi_k^0 - \varphi_k)\|^2 = O(n^\epsilon d_n), \qquad \forall \epsilon > 0$$

which in conjunction with (4.11) (with $\Delta \hat{B}_{1k} u_k$ removed) implies (2.8). The proof is completed.

Proof of Lemma 2: We here prove the lemma for (1.1), (2.1)–(2.6) (i.e., for the case of Theorem 1) and put the proof for (1.1)–(1.5) and (3.11) (i.e., for the case of Theorem 2) in the Appendix.

We first show that there are constants $c > 0$ and $\lambda \in (0, 1)$ such that

$$\|y_{n+1}\|^2 \le c\alpha_n \delta_n L_n + \xi_n \tag{4.16}$$

where

$$\alpha_n = \frac{\|\tilde{\theta}_n^\tau \varphi_n\|^2}{1+\varphi_n P_n \varphi_n}, \; \delta_n = \operatorname{tr}(P_n - P_{n+1}) \tag{4.17}$$

$$L_n = \sum_{i=0}^{n} \lambda^{n-i} \|y_i\|^2 \tag{4.18}$$

and $\{\xi_n\}$ is a nondecreasing positive sequence satisfying

$$\xi_n = O(d_n \log r_n + \log^2 r_n). \tag{4.19}$$

Let φ_n^0 be defined by (4.10) with $s = n-1$. Then in view of (2.2), it follows from (1.1) that

$$y_{n+1} = \theta^\tau \varphi_n^0 + B_1 u_n + w_{n+1}. \tag{4.20}$$

So taking account of (2.1), we have

$$\begin{aligned} y_{n+1} &= B_1 u_n + \theta^\tau \varphi_n + w_{n+1} + \theta^\tau(\varphi_n^0 - \varphi_n) \\ &= y_{n+1}^* - \theta_n^\tau \varphi_n + \theta^\tau \varphi_n + w_{n+1} + \theta^\tau(\varphi_n^0 - \varphi_n) \\ &= \tilde{\theta}_n^\tau \varphi_n + y_{n+1}^* + w_{n+1} + \theta^\tau(\varphi_n^0 - \varphi_n). \end{aligned} \tag{4.21}$$

By (2.7), (4.2), and the boundedness of $\{y_n^*\}$ from (4.21), it follows that

$$\begin{aligned} \|y_{n+1}\|^2 &\le 2\|\tilde{\theta}_n^\tau \varphi_n\|^2 + O(\log r_n) + O(d_n) \\ &= 2\alpha_n[1 + \varphi_n^\tau P_{n+1} \varphi_n + \varphi_n^\tau (P_n - P_{n+1})\varphi_n] \\ &\quad + O(d_n + \log r_n) \\ &\le 2\alpha_n[2 + \delta_n \|\varphi_n\|^2] + O(d_n + \log r_n) \end{aligned} \tag{4.22}$$

where for the last inequality we have used the fact that $\varphi_n^\tau P_{n+1}\varphi_n \le 1$.

From condition A3, it follows that there exists a constant $\lambda \in (0, 1)$ such that

$$\|u_{n-1}\|^2 = O(L_n) + O\left(\sum_{i=0}^{n} \lambda^{n-i}\|w_i\|^2\right) = O(L_n) + O(d_n). \tag{4.23}$$

Combining (4.23) with (4.2) yields

$$\begin{aligned}\|\varphi_n\|^2 &= \sum_{i=0}^{p-1}\|y_{n-i}\|^2 + \sum_{i=1}^{q-1}\|u_{n-i}\|^2 + \sum_{i=1}^{r-1}\|w_{n-i}\|^2\\ &= O(L_n) + O(d_n) + O\left(\sum_{i=0}^{n}\lambda^{n-i}(\|\hat{w}_{n-i} - w_{n-i}\|^2 + \|w_{n-i}\|^2)\right)\\ &= O(L_n) + O(d_n + \log r_n).\end{aligned} \tag{4.24}$$

Substituting (4.24) into (4.22) and noticing $\alpha_n\delta_n = O(\alpha_n) = O(\log r_n)$ (a consequence of (4.3)) we immediately derive (4.16).

From (4.18) and (4.16) it follows that

$$L_{n+1} = \sum_{i=0}^{n+1}\lambda^{n+1-i}\|y_i\|^2 = \|y_{n+1}\|^2 + \lambda L_n \le (\lambda + c\alpha_n\delta_n)L_n + \xi_n \tag{4.25}$$

and hence

$$\begin{aligned}L_{n+1} &\le \prod_{j=0}^{n}(\lambda + c\alpha_j\delta_j)L_0 + \sum_{i=0}^{n}\prod_{j=i+1}^{n}(\lambda + c\alpha_j\delta_j)\xi_i\\ &= \lambda^{n+1}\prod_{j=0}^{n}(1 + \lambda^{-1}c\alpha_j\delta_j)L_0 + \sum_{i=0}^{n}\lambda^{n-i} \cdot \prod_{j=i+1}^{n}(1 + \lambda^{-1}\alpha_j\delta_j)\xi_i\end{aligned} \tag{4.26}$$

where as usual $\Pi_{n+1}^{n}(\cdot) \triangleq 1$.

We now proceed to analyze the product in (4.26).

We note that $\delta_j \xrightarrow[j\to\infty]{} 0$ because

$$\sum_{j=0}^{\infty}\delta_j = \sum_{j=0}^{\infty}(\operatorname{tr} P_j - \operatorname{tr} P_{j+1}) \le \operatorname{tr} P_0 < \infty. \tag{4.27}$$

Consequently, for any $\epsilon > 0$ by (4.3) there exists i_0 such that

$$\lambda^{-1}c\sum_{j=i}^{n}\alpha_j\delta_j \le \epsilon \log r_n, \qquad \forall n \ge i \ge i_0. \tag{4.28}$$

Using this and the inequality $1 + x \le e^x$, $x \ge 0$ one readily obtains

$$\prod_{j=i}^{n}(1 + \lambda^{-1}\alpha_j\delta_j) \le \exp\left\{\lambda^{-1}c\sum_{j=i}^{n}\alpha_j\delta_j\right\} \le \exp\{\epsilon \log r_n\} = r_n^\epsilon, \qquad \forall n \ge i \ge i_0. \tag{4.29}$$

Putting this into (4.26) and taking account of (4.19) yields

$$L_{n+1} = O(r_n^\epsilon(d_n \log r_n + \log^2 r_n)) \qquad \text{a.s. } \forall \epsilon > 0$$

which implies by the arbitrariness of ϵ

$$L_{n+1} = O(r_n^\epsilon d_n) \text{ and } \|y_{n+1}\|^2 = O(r_n^\epsilon d_n) \qquad \text{a.s. } \forall \epsilon > 0. \tag{4.30}$$

Thus by noting (4.23) and (4.30), we know that (4.5) holds in the case of Theorem 1.

Lemma 3: Assume that conditions A1 and A2 hold, $A(z)$, $B(z)$, and $C(z)$ have no common left factor and $[A_p, B_q, C_r]$ is of full-row rank, and that the output of system (1.1) under control (3.14) and (3.15) has growth rate

$$\frac{1}{n}\sum_{i=0}^{n}\|y_i\|^2 = O(1) \qquad \text{a.s.} \tag{4.31}$$

where u_n^o is any $\mathscr{F}_n'$-measurable vector (not necessarily defined by (3.13)) with

$$\frac{1}{n}\sum_{i=0}^{n}\|u_i^o\|^2 = O(1) \qquad \text{a.s.} \tag{4.32}$$

where $\{\mathscr{F}_n'\}$ is a family of nondecreasing σ-algebras such that $\mathscr{F}_n'$ is a sub σ-algebra of $\mathscr{F}_n$ and ϵ_n independent of $\mathscr{F}_n'$. Then θ_n given by (1.2)–(1.5) has the convergence rate indicated in (3.16) and

$$P_n^{-1} \ge c_o n^{1-\bar{\epsilon}(t+1)} I \tag{4.33}$$

for some constant $c_o > 0$, where $\bar{\epsilon}$ is defined in (3.15).

The proof is given in the Appendix.

Proof of Theorem 3: Without loss of generality assume that

$$\mathscr{F}_n = \sigma\{w_i, y_{i+1}^*, \epsilon_i, i \le n\}$$

and

$$\mathscr{F}_n' = \sigma\{w_i, y_{i+1}^*, \epsilon_{i-1}, i \le n\}.$$

Set

$$\bar{y}_{n+1}^* = y_{n+1}^* + \frac{1}{r_{n-1}^{\bar{\epsilon}/2}}\hat{B}_{1n}\epsilon_n.$$

Combining this with (3.13) and (3.14) we have

$$u_n = \hat{B}_{1n}^{-1}\{\bar{y}_{n+1}^* + (B_{1n}u_n - \theta_n^\tau\varphi_n)\}. \tag{4.34}$$

Clearly, $\bar{y}_{n+1}^*$ is $\mathscr{F}_n$-measurable. Next, by (3.10) and Lemma 1-1), we know that $\|\hat{B}_{1n}\| = O(\{\log r_{n-1}\}^{1/2})$, hence $\{\bar{y}_n^*\}$ in a.s. bounded. Thus $\{\bar{y}_n^*\}$ may serve as a new reference signal that satisfies requirements in Theorem 2, and so (1.1)–(1.5), (3.10), and (4.34) form an ELS-based adaptive tracker. Consequently, by Theorem 2, (1.8) and (1.9) hold. Hence, (4.31) and (4.32) are satisfied because $\{\|v_n\|\}$ is bounded. Moreover, u_n^o defined by (3.13) is $\mathscr{F}_n'$-measurable and $\mathscr{F}_n'$ is independent of ϵ_n by definition. Then Lemma 3 is applicable and (3.16) follows from Lemma 1-1) and (4.33). It remains to show (3.17).

By Lemma 2 and (4.33), it follows that

$$\varphi_n^\tau P_n\varphi_n = O(r_n^{-\delta}d_n), \qquad \text{a.s. } \forall \delta \in (0, 1 - \bar{\epsilon}(t+1)). \tag{4.35}$$

Set $S_i = \sum_{j=1}^{i}\alpha_j$, $S_o = 0$. Then by Lemma 1-3), $S_i =$

$O(\log r_i) = O(\log i)$, so by (4.35) we have

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \varphi_i^\tau P_i \varphi_i &= O\left(\sum_{i=1}^{n} \alpha_i r^{-\delta} d_i\right) \\ &= O\left(d_n \sum_{i=1}^{n} [S_i - S_{i-1}] i^{-\delta}\right) \\ &= O\left(d_n \left\{\sum_{i=1}^{n-1} S_i [i^{-\delta} - (i+1)^{-\delta}] + S_n n^{-\delta}\right\}\right) \\ &= O\left(d_n \left\{\sum_{i=1}^{n-1} \log(1+i) [i^{-\delta} - (i+1)^{-\delta}]\right\}\right) \\ &= O(d_n), \qquad \forall \delta \in (0, 1 - \bar{\epsilon}(t+1)).\end{aligned}$$

Consequently by Lemma 1-3) again, we get

$$\begin{aligned}\sum_{i=0}^{n} \|\varphi_i^\tau \tilde{\theta}_i\|^2 &= \sum_{i=0}^{n} \alpha_i (1 + \varphi_i^\tau P_i \varphi_i) \\ &= O(\log n) + O(d_n), \qquad \text{a.s.} \quad (4.36)\end{aligned}$$

Note that by (3.16), $B_{1n} \underset{n\to\infty}{\to} B_1$, and B_1 is nondegenerate, we see from (3.10) that $\Delta \hat{B}_{1n} = 0$ for all sufficiently large n. Hence, by (3.16), (4.11) (with y_{k+1}^* replaced by $\bar{y}_{k+1}^*$), (4.2) and (3.6) we see that

$$\begin{aligned}\sum_{k=1}^{n} \| y_{k+1} - y_{k+1}^* - w_{k+1} \|^2 &= \sum_{k=1}^{n} \| \tilde{\theta}_k^\tau \varphi_k + \theta^\tau (\varphi_k^o - \varphi_k) + \Delta \hat{B}_{1k} u_k \\ &\quad + \frac{1}{r_{k-1}^{\bar{\epsilon}/2}} \hat{B}_{1k} \epsilon_k \|^2 \\ &= O(d_n) + O(\log n) + O(n^{1-\bar{\epsilon}}) \\ &= O(d_n) + O(n^{1-\bar{\epsilon}}), \qquad \text{a.s.,}\end{aligned}$$

where $\bar{\epsilon}$ is defined in (3.15). Hence, the desired result (3.17) holds.

V. Concluding Remarks

In this paper, we have proved the stability and optimality of the Å-W self-tuning regulator and an ELS-based adaptive tracker. However, several problems are still left open. We do not know if the ELS-based adaptive tracker (1.1)–(1.5) and (3.4) is still stable and optimal if no modification is made on B_{1n} when $\det B_{1n} \neq 0$, a.s.. In (3.7) or (3.10), we have modified B_{1n}. However, we conjecture that in Theorem 2 not only the modification may happen at most for a finite number of steps, but also $B_{1n}^\tau B_{1n}$ is asymptotically bounded from below.

Appendix

In this section, we give the proofs for Lemmas 1 and 3 and complete the proof for Lemma 2 in the Theorem 2 case.

Proof of Lemma 1: We need only to consider the ELS algorithm (1.2)–(1.5), since the algorithm (2.2)–(2.6) can be analyzed in completely the same way. Note that conclusions 1) and 2) are known results, see, for example, (9), (29), and (31) in reference [20] (note that r_{n+1} of [20] equals $(r_n - e + 1)$ of the present paper). Similar results may also be found in [9]. Hence, we need only to prove conclusion 3) for the algorithm (1.2)–(1.5).

Set

$$\bar{w}_{k+1} = w_{k+1} + \theta^\tau(\varphi_k^o - \varphi_k). \qquad (A.1)$$

It is easy to see that $y_{k+1} = \theta^\tau \varphi_k + \bar{w}_{k+1}$. Substituting this into (1.2) we have

$$\tilde{\theta}_{k+1} = (I - a_k P_k \varphi_k \varphi_k^\tau)\tilde{\theta}_k - a_k P_k \varphi_k \bar{w}_{k+1}^\tau. \qquad (A.2)$$

From (1.3), it is clear that

$$P_{k+1} P_k^{-1} = I - a_k P_k \varphi_k \varphi_k^\tau, \quad P_{k+1}^{-1} P_k = I + \varphi_k \varphi_k^\tau P_k.$$

From this and (A.2) it follows that

$$\begin{aligned}\operatorname{tr}\left[\tilde{\theta}_{k+1}^\tau P_{k+1}^{-1} \tilde{\theta}_{k+1}\right] &= \operatorname{tr}\left[\tilde{\theta}_k^\tau (I - a_k \varphi_k \varphi_k^\tau P_k) - a_k \bar{w}_{k+1} \varphi_k^\tau P_k\right] \\ &\quad \cdot \left[P_k^{-1} \tilde{\theta}_k - P_{k+1}^{-1} a_k P_k \varphi_k \bar{w}_{k+1}^\tau\right] \\ &= \operatorname{tr} \tilde{\theta}_k^\tau P_k^{-1} \tilde{\theta}_k - a_k \|\varphi_k^\tau \tilde{\theta}_k\|^2 - 2 a_k \varphi_k^\tau \tilde{\theta}_k \bar{w}_{k+1} \\ &\quad + a_k \varphi_k^\tau P_k \varphi_k \|\bar{w}_{k+1}\|^2. \qquad (A.3)\end{aligned}$$

We now proceed to estimate the last two terms on the right-hand side of (A.3). For this we need the following fact (see, e.g., [20, Eq. (21)]: for any Martingale difference sequence $\{w_n, \mathscr{F}_n\}$ satisfying (1.6) and any adapted matrix sequence $\{M_n, \mathscr{F}_n\}$

$$\sum_{i=1}^{n} M_i w_{i+1} = O\left(\left\{\sum_{i=1}^{n} \|M_i\|^2\right\}^{\frac{1}{2}+\delta}\right), \qquad \text{a.s. } \forall \delta > 0. \qquad (A.4)$$

By this, A.1, conclusion 2) and the inequality

$$2xy \le \delta x^2 + \delta^{-1} y^2, \qquad x \ge 0,\ y \ge 0,\ \delta > 0$$

it is not difficult to see that

$$\begin{aligned}2\sum_{k=1}^{n} a_k \varphi_k^\tau \tilde{\theta}_k \bar{w}_{k+1} &= 2\sum_{k=1}^{n} a_k \varphi_k^\tau \tilde{\theta}_k w_{k+1} + 2\sum_{k=1}^{n} a_k \varphi_k^\tau \tilde{\theta}_k \theta^\tau (\varphi_k^o - \varphi_k) \\ &= O\left(\left\{\sum_{k=1}^{n} a_k \|\varphi_k^\tau \tilde{\theta}_k\|^2\right\}^{1/2+\delta}\right) + \delta \sum_{k=1}^{n} a_k \|\varphi_k^\tau \tilde{\theta}_k\|^2 \\ &\quad + \delta^{-1} \sum_{k=1}^{n} \|\theta^\tau(\varphi_k^o - \varphi_k)\|^2, \qquad 0 < \delta < \tfrac{1}{2} \\ &\le 2\delta \sum_{k=1}^{n} a_k \|\varphi_k^\tau \tilde{\theta}_k\|^2 + O(\log r_n), \qquad 0 < \delta < \tfrac{1}{2}. \qquad (A.5)\end{aligned}$$

For the last term of (A.3), we need the following result (see [20, Eq. (29)]:

$$\sum_{k=1}^{n} a_k \varphi_k^\tau P_k \varphi_k \| w_{k+1} \|^2 = O(\log r_n), \qquad \text{a.s.}$$

From this, (A.1) and conclusion 2), we get

$$\begin{aligned} &\sum_{k=1}^{n} a_k \varphi_k^\tau P_k \varphi_k \| \overline{w}_{k+1} \|^2 \\ &\le 2 \sum_{k=1}^{n} a_k \varphi_k^\tau P_k \varphi_k \left[\| w_{k+1} \|^2 + \| \theta^\tau (\varphi_k^o - \varphi_k) \|^2 \right] \\ &\le O(\log r_n) + 2 \sum_{k=1}^{n} \| \theta^\tau (\varphi_k^o - \varphi_k) \|^2 \\ &= O(\log r_n), \qquad \text{a.s.} \end{aligned} \tag{A.6}$$

Finally, summing up both sides of (A.3) from 1 to n, and using (A.5) and (A.6) we see that

$$(1 - 2\delta) \sum_{k=1}^{n} a_k \| \varphi_k^\tau \tilde{\theta}_k \|^2 \le \operatorname{tr}\left[\tilde{\theta}_1^\tau P_1^{-1} \tilde{\theta}_1 \right] + O(\log r_n)$$

which yields conclusion 3) because $1 - 2\delta > 0$. This completes the proof of Lemma 1.

Proof of Lemma 2 in the Theorem 2 case: Similar to (4.16), we first show that there exist constants $c > 0$ and $\lambda \in (0, 1)$ such that

$$\| y_{n+1} \|^2 \le c f_n L_n + \xi_n \tag{A.7}$$

where L_n is defined as in (4.18), f_n is defined as

$$f_n = (\alpha_n \delta_n \log r_{n-1})^2 + \alpha_n \delta_n + \frac{1}{\log r_{n-1}} \tag{A.8}$$

with α_n, δ_n defined as in (4.17), and where $\{\xi_k\}$ is a nondecreasing positive sequence satisfying

$$\xi_k = O(d_k \log^4 r_k + \log^5 r_k). \tag{A.9}$$

By (3.11) we have

$$y_{k+1}^* = \Delta \hat{B}_{1k} u_k + \theta_k^\tau \varphi_k \tag{A.10}$$

and from this

$$\begin{aligned} B_1 u_k &= \theta^\tau \varphi_k - y_{k+1}^* + y_{k+1}^* + (B_1 u_k - \theta^\tau \varphi_k) \\ &= \tilde{\theta}_k^\tau \varphi_k - \Delta \hat{B}_{1n} u_k + y_{k+1}^* + (B_1 u_k - \theta^\tau \varphi_k). \end{aligned} \tag{A.11}$$

Noting that condition A3 implies the nondegeneracy of B_1, so from (3.6) and (4.6) $\| B_1^{-1} \Delta \hat{B}_{1k} \| < \frac{1}{2}$, for all suitably large k, we then see from (A.11) that

$$\| u_k \| \le 2 \| B_1^{-1} \| (\| \tilde{\theta}_k^\tau \varphi_k \| + \| y_{k+1}^* \| + \| B_1 u_k - \theta^\tau \varphi_k \|)$$

and consequently

$$\| u_k \|^2 \le 4 \| B_1^{-1} \|^2 (3 \| \tilde{\theta}_k^\tau \varphi_k \|^2 + 3 \| y_{k+1}^* \|^2 + 3 \| B_1 u_k - \theta^\tau \varphi_k \|^2). \tag{A.12}$$

By (2.7), (4.2), and the fact that $\varphi_k^\tau P_{k+1} \varphi_k \le 1$, we see from (4.11) that

$$\begin{aligned} \| y_{k+1} \|^2 &\le 3 \| \tilde{\theta}_k^\tau \varphi_k \|^2 + 3 \| \Delta \hat{B}_{1k} \|^2 \| u_k \|^2 \\ &\quad + O(\log r_k) + O(d_k) \\ &\le 3\alpha_k \{ 1 + \varphi_k^\tau P_{k+1} \varphi_k + \varphi_k^\tau (P_k - P_{k+1}) \varphi_k \} \\ &\quad + 3 \| \Delta \hat{B}_{1k} \|^2 \| u_k \|^2 + O(\log r_k) + O(d_k) \\ &\le 3\alpha_k (2 + \delta_k \| \varphi_k \|^2) + 3 \| \Delta \hat{B}_{1k} \|^2 \| u_k \|^2 \\ &\quad + O(d_k + \log r_k) \\ &= 3\alpha_k \delta_k \| \varphi_k \|^2 + 3 \| \Delta \hat{B}_{1k} \|^2 \| u_k \|^2 \\ &\quad + O(d_k + \log r_k) \end{aligned} \tag{A.13}$$

where for the last equality we have used the estimate $\alpha_k = O(\log r_k)$, a consequence of (4.3).

We now proceed to estimate $\| u_k \|^2$. By condition A3, we know from (1.1) that there exists $\lambda \in (0, 1)$ such that

$$\| u_k \|^2 = O\left(\sum_{i=0}^{k+1} \lambda^{k-i} \| y_i \|^2 \right) + O\left(\sum_{i=0}^{k+1} \lambda^{k-i} \| w_i \|^2 \right). \tag{A.14}$$

Note that $(B_1 u_k - \theta^\tau \varphi_k)$ is free of u_k, it is easy to see from (A.14) that

$$\begin{aligned} \| B_1 u_k - \theta^\tau \varphi_k \|^2 &= O(L_k) + O\left(\sum_{i=0}^{r} \| \hat{w}_{k-i} - w_{k-i} \|^2 \right) \\ &\quad + O\left(\sum_{i=0}^{k} \lambda^{k-i} \| w_i \|^2 \right) \\ &= O(L_k) + O(\log r_k) + O(d_k) \end{aligned}$$

where for the last relationship Lemma 1-2) is invoked and L_k is defined by (4.18). Substituting this into (A.12), we have

$$\| u_k \|^2 \le 12 \| B_1^{-1} \|^2 \| \tilde{\theta}_k^\tau \varphi_k \|^2 + O(L_k) + O(d_k + \log r_k). \tag{A.15}$$

Consequently, similar to the treatment used above, by (A.14) and (A.15) we derive

$$\begin{aligned} \| \varphi_k \|^2 &= \| u_k \|^2 + \left[\| \varphi_k \|^2 - \| u_k \|^2 \right] \\ &\le 12 \| B_1^{-1} \|^2 \| \tilde{\theta}_k^\tau \varphi_k \|^2 + O(L_k) \\ &\quad + O(d_k + \log r_k). \end{aligned} \tag{A.16}$$

Now, substituting (A.14) and (A.16) into (A.13), we obtain

$$\begin{aligned} \| y_{k+1} \|^2 &\le 36 \| B_1^{-1} \|^2 \alpha_k \delta_k \| \tilde{\theta}_k^\tau \varphi_k \|^2 \\ &\quad + O(\| \Delta \hat{B}_{1k} \|^2 \| y_{k+1} \|^2) \\ &\quad + O(\alpha_k \delta_k + \| \Delta \hat{B}_{1k} \|^2) \sum_{i=0}^{k} \lambda^{k-i} \| y_i \|^2 \\ &\quad + O([d_k + \log r_k] \log r_k). \end{aligned}$$

Then by the fact that $\| \Delta \hat{B}_{1n} \| \underset{n \to \infty}{\to} 0$, it follows that for some constant c and all suitably large k

$$\begin{aligned}\|y_{k+1}\|^2 &\le c\alpha_k\delta_k\|\tilde{\theta}_k^\tau\varphi_k\|^2 \\ &\quad + O(\alpha_k\delta_k + \|\Delta\hat{B}_{1k}\|^2)L_k \\ &\quad + O(d_k \log r_k + \log^2 r_k) \\ &\le c\alpha_k^2\delta_k(1 + \varphi_k^\tau P_{k+1}\varphi_k + \delta_k\|\varphi_k\|^2) \\ &\quad + O(\alpha_k\delta_k + \|\Delta\hat{B}_{1k}\|^2)L_k \\ &\quad + O(d_k \log r_k + \log^2 r_k) \\ &\le c(\alpha_k\delta_k)^2\|\varphi_k\|^2 \\ &\quad + O(\alpha_k\delta_k + \|\Delta\hat{B}_{1k}\|^2)L_k \\ &\quad + O(d_k \log r_k + \log^2 r_k). \qquad (A.17)\end{aligned}$$

To complete the proof of (A.7) we need to derive an upper bound for $\|\varphi_k\|^2$ in terms of L_k.

By (3.5) and Lemma 1-1), it is easy to see from(3.11) that

$$\|u_k\|^2 \le O\left(\log^2 r_{k-1}\left[\sum_{i=0}^{p-1}\|y_{k-i}\|^2 + \sum_{i=1}^{q-1}\|u_{k-i}\|^2 + \sum_{i=0}^{k}\lambda^{k-i}\|\hat{w}_i\|^2\right]\right) + O(\log r_{k-1}) \qquad (A.18)$$

hence, by (A.14) again

$$\begin{aligned}\|\varphi_k\|^2 &= [\|\varphi_k\|^2 - \|u_k\|^2] + \|u_k\|^2 \\ &= O\left(\log^2 r_{k-1}\sum_{i=0}^{k}\lambda^{k-i}[\|y_i\|^2 + \|\hat{w}_i\|^2 + \|w_i\|^2]\right) \\ &\quad + O(\log r_{k-1}) \\ &= O(\log^2 r_{k-1})L_k + O(\log^3 r_k + d_k\log^2 r_k).\end{aligned}$$

Substituting this into (A.17) and noting (3.6), we finally get

$$\begin{aligned}\|y_{k+1}\|^2 &\le c[(\alpha_k\delta_k\log r_{k-1})^2 + \alpha_k\delta_k + \|\Delta\hat{B}_{1k}\|^2]L_k \\ &\quad + O(\log^5 r_k + d_k\log^4 r_k) \\ &\le c\left[(\alpha_k\delta_k\log r_{k-1})^2 + \alpha_k\delta_k + \frac{1}{\log r_{k-1}}\right]L_k \\ &\quad + O(\log^5 r_k + d_k\log^4 r_k)\end{aligned}$$

where c is some constant. Hence (A.7) is verified.

Corresponding to (4.25) and (4.26), we have from (A.7)

$$L_{n+1} \le (\lambda + cf_n)L_n + \xi_n$$

and

$$\begin{aligned}L_{n+1} &\le \lambda^{n+1}\left[\prod_{j=0}^{n}(1 + \lambda^{-1}cf_j)\right]L_0 \\ &\quad + \sum_{i=0}^{n}\lambda^{n-i}\left[\prod_{j=i+1}^{n}(1 + \lambda^{-1}cf_j)\right]\xi_i. \qquad (A.19)\end{aligned}$$

We now estimate the product $\Pi_{j=i+1}^{n}(1 + \lambda^{-1}cf_j)$.

For any $\epsilon > 0$, by Lemma 1-3), there exists a small, possibly random $\delta > 0$ such that

$$\delta\sum_{j=0}^{n}\alpha_j \le \epsilon(\log r_n), \qquad \text{a.s. } \forall n \qquad (A.20)$$

and by (4.27) there exists a (random) integer i_o such that

$$\frac{4}{\delta}\left(\frac{c}{\lambda}\right)^{1/2}\sum_{j=i}^{\infty}\delta_j \le \epsilon, \qquad \text{a.s.}, \forall i \ge i_0. \qquad (A.21)$$

Thus, by the inequalities $1 + xy \le (1+x)(1+y)$, $x \ge 0$, $y \ge 0$ and $1 + x^2 \le e^{2x}$, $x \ge 0$, we get for all $n \ge i \ge i_o$

$$\begin{aligned}&\prod_{j=i+1}^{n}\left[1 + \lambda^{-1}c(\alpha_j\delta_j\log r_{j-1})^2\right] \\ &\le \prod_{j=i+1}^{n}\left[1 + \left(\frac{\delta}{2}\alpha_j\right)^2\right] \cdot \prod_{j=i}^{n}\left[1 + \lambda^{-1}c\left(\frac{2}{\delta}\delta_j\log r_{j-1}\right)^2\right] \\ &\le \exp\left\{\delta\sum_{j=i+1}^{n}\alpha_j\right\}\exp\left\{\frac{4}{\delta}\left(\frac{c}{\lambda}\right)^{1/2}\sum_{j=i+1}^{n}\delta_j\log r_{j-1}\right\} \\ &\le \exp\{\epsilon\log r_n\}\exp\left\{(\log r_n)\left[\frac{4}{\delta}\left(\frac{c}{\lambda}\right)^{1/2}\sum_{j=i}^{n}\delta_j\right]\right\} \\ &\le r_n^\epsilon\exp\{(\log r_n)\epsilon\} = r_n^{2\epsilon}, \qquad \text{a.s.} \qquad (A.22)\end{aligned}$$

and by the inequality $1 + x \le e^x$, $x \ge 0$, we have for $n \ge i \ge i_0$

$$\begin{aligned}\prod_{j=i}^{n}(1 + \lambda^{-1}c\alpha_j\delta_j) &\le \exp\left\{\delta\sum_{j=i}^{n}\alpha_j\right\}\exp\left\{\frac{c}{\lambda\delta}\sum_{j=i}^{n}\delta_j\right\} \\ &\le O(r_n^\epsilon), \qquad \text{a.s.} \qquad (A.23)\end{aligned}$$

Since $r_n \to \infty$ and $\lambda < 1$, we know that i_0 can be taken large enough such that $\sup_{j \ge i_0}\{1 + (c\lambda^{-1}/\log r_j)\} < 2 - \lambda$ and hence for all $n > i \ge i_0$

$$\prod_{j=i+1}^{n}\left(1 + \frac{c\lambda^{-1}}{\log r_j}\right) \le (2-\lambda)^{n-i}. \qquad (A.24)$$

Finally, from the definition of f_j and (A.22)–(A.24) it follows that for any $\epsilon > 0$

$$\begin{aligned}&\prod_{j=i+1}^{n}(1 + c\lambda^{-1}f_j) \\ &\le \prod_{j=i+1}^{n}\left(1 + c\lambda^{-1}(\alpha_j\delta_j\log r_{j-1})^2\right) \cdot \prod_{j=i+1}^{n}(1 + c\lambda^{-1}\alpha_j\delta_j)\prod_{j=i+1}^{n}\left(1 + \frac{c\lambda^{-1}}{\log r_j}\right) \\ &\le O(r_n^{3\epsilon}[2-\lambda]^{n-i}), \qquad \text{a.s.} \qquad \forall n > i \ge i_0.\end{aligned}$$

Substituting this into (A.19) and noting that $2\lambda - \lambda^2 < 1$, we get

$$\begin{aligned}L_{n+1} &\le O(r_n^{3\epsilon}[2\lambda - \lambda^2]^n) + O\left(r_n^{3\epsilon}\sum_{i=0}^{n}(2\lambda - \lambda^2)^{n-i}\xi_i\right) \\ &= O(r_n^{3\epsilon}[\log^5 r_n + d_n\log^4 r_n]), \qquad \text{a.s.} \quad \forall\epsilon > 0.\end{aligned}$$

By the arbitrariness of ϵ, this implies $L_{n+1} = O(r_n^\epsilon d_n)$, $\forall \epsilon > 0$, which in turn implies the desired result (4.5) since by (A.14)

$$\begin{aligned} \|y_k\|^2 + \|u_k\|^2 + \|\hat{w}_k\|^2 &\le L_k + O(L_{k+1}) + O(d_k) \\ &\quad + 2\|\hat{w}_k - w_k\|^2 + 2\|w_k\|^2 \\ &= O(r_k^\epsilon d_k) + O(d_k) + O(\log r_k) = O(r_k^\epsilon d_k). \end{aligned}$$

Proof of Lemma 3: Comparing conditions in this theorem with those in [20, Theorem 3 ($\beta > 2$, $\delta = 0$)], we find that $n^{\bar{\epsilon}/2}$ is replaced by $r_{n-1}^{\bar{\epsilon}/2}$ in the denominator of v_n and the full-rank requirement for A_p is weakened to that of $[A_p, B_q, C_r]$.

From the proof of [20, Theorem 3], we see that only the following two properties of $\{v_n\}$ are used: 1) $\{v_n, \mathscr{F}_n\}$ is a Martingale difference sequence (for [20, (50)–(52)] and 2)

$$\frac{1}{n^{1-\bar{\epsilon}}} \sum_{i=1}^{n} v_i v_i^\tau \ge c_1 I, \qquad c_1 > 0, \qquad \text{a.s.} \quad (A.25)$$

for sufficiently large n.

The property (A.25) is a consequence of (41) in [20], and is used in deriving (56) and (59) of [20]. Since without loss of generality we may assume that ϵ_n is $\mathscr{F}_n$-measurable and is independent of $\mathscr{F}_{n-1}$, $\{v_n, \mathscr{F}_n\}$ is obviously a Martingale difference sequence because r_{n-1} is $\mathscr{F}_{n-1}$-measurable.

Further, by (4.31) and (4.32) and the boundedness of $\{v_n\}$ we have

$$r_n = O(n) + O\left(\sum_{i=1}^{n} \|\hat{w}_i\|^2\right). \quad (A.26)$$

Hence by Lemma 1-2), (1.7), and (A.26), it is easy to see that for some $c > 0$ $r_n \le cn$, a.s. $\forall n \ge 0$. Thus, by [20, Eq. (41)] we have for sufficiently large n

$$\frac{1}{n^{1-\bar{\epsilon}}} \sum_{i=1}^{n} v_i v_i^\tau \ge \frac{1}{c^{\bar{\epsilon}} n^{1-\bar{\epsilon}}} \sum_{i=1}^{n} \frac{\epsilon_i \epsilon_i^\tau}{i^{\bar{\epsilon}}} \ge \frac{1}{2c^{\bar{\epsilon}}} I$$

which verifies (A.25).

We now justify the relaxation of replacing the rank condition on A_p by that of $[A_p, B_q, C_r]$. Under the assumptions converse to [20, Eq. (46)], following the proof there, we find a unit vector

$$\left[\alpha^{(0)\tau} \cdots \alpha^{(p-1)\tau} \beta^{(0)\tau} \cdots \beta^{(q-1)\tau} \gamma^{(0)\tau} \cdots \gamma^{(r-1)\tau}\right]^\tau$$

such that

$$\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i [\operatorname{adj}(z)] B(z) = -\sum_{i=0}^{q-1} \beta^{(i)\tau} z^i [\det A(z)] I$$

$$\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i [\operatorname{adj}(z)] C(z) = -\sum_{i=0}^{r-1} \gamma^{(i)\tau} z^i [\det A(z)] I.$$

From this and using the fact that $A(z)$, $B(z)$, and $C(z)$ have no common left factor we find that there are m-dimensional vectors $\mu^{(i)}$, $i = 1, \cdots, \lambda$, for some λ, such that

$$\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i = \sum_{i=0}^{\lambda} \mu^{(i)\tau} z^i A(z), \; \sum_{i=0}^{q-1} \beta^{(i)\tau} z^i = \sum_{i=0}^{\lambda} \mu^{(i)\tau} z^i b(z)$$

and

$$\sum_{i=0}^{r-1} \gamma^{(i)\tau} z^i = \sum_{i=0}^{\lambda} \mu^{(i)\tau} z^i C(z)$$

which imply

$$\mu^{(i)} = 0, \qquad i = 0, \cdots, \lambda. \quad (A.27)$$

since $[A_p, B_q, C_r]$ is of full-row rank.

Clearly, (A.27) leads to a contradictory result $\alpha^{(i)} = 0$, $\beta^{(j)} = 0$, $\gamma^{(k)} = 0$, $i = 1, \cdots, p-1$, $j = 0, \cdots, q-1$, $k = 0, \cdots, r-1$. Thus [20, Theorem 3 remain valid. Hence (3.16) follows, while (4.33) can be seen from [20, Eq. (46)] is true, and all results of [20, Eq. (44)]. The proof is completed.

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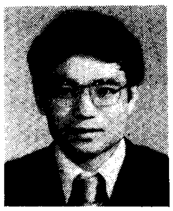

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