On Finite Potential Games ⋆

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Abstract

A linear system, called the potential equation (PE), is presented. It is proved that a finite game is potential if and only if its potential equation has solution. Some properties of the potential equation are obtained. Based on these properties, a closed form solution of the PE is obtained. Moreover, a formula based on the solution of the PE is obtained to calculate the potential function. Finally, it is proved that a networked evolutionary game is potential if and only if its fundamental network game is potential. Some interesting examples are presented to illustrate the theoretical results.

Key words: Potential game; potential equation; networked evolutionary game; semi-tensor product of matrices.

1 Introduction

The concept of potential game was proposed by Rosenthal (Rosenthal, 1973). It was shown in Rosenthal (1973) that any congestion game is a potential game. Since then, the theory of potential games has been developed by many authors. For instance, Hart & Mas-Colell (1989) has applied potential theory to cooperative games. Blume (1993) and Milchtaich (1996) showed that ordinal potential games are naturally related to the evolutionary learning as well. Monderer & Shapley (1996a) proved that the fictitious play process converges to the equilibrium set in a class of games that contains the finite (weighted) potential games. It is worth noting that (Monderer & Shapley, 1996b) systematically investigated potential games and presented several fundamental results such as its relation with finite improvement property, its mixed extension etc. Particularly, it was also proved in Monderer & Shapley (1996b) that every finite potential game is isomorphic to a congestion game. The theory of potential games has been applied to many engineering problems, for instance, to distributed power control and scheduling (Heikkinen, 2006), to the consensus of noncooperative congestion games and road pricing (Wang et al., 2013), etc.

To apply the theory of potential games, the first question one faces is: how to verify whether a game is potential. To the author's best knowledge, the best result about this was given by Monderer & Shapley (1996b), and it is described in detail in subsection 2.2. It requires to verify all typical simple closed paths. This is a heavy load. In this paper we propose a linear system, called the potential equation (PE), and then prove that the game is potential if and only if the PE has solution. As a linear system, to verify whether the PE has solution is a straightforward work. Furthermore, we obtain some properties of the PE. Based on these properties a closed form solution of the PE is obtained. Using the solution of the PE, a formula is provided to calculate the potential. As we know, there is no known general formula for calculating potential.

Nowadays, the networked (evolutionary) game has become a hot topic (Nowak & May, 1992; Szabo & Toke, 1998; Santos et al., 2008). In a networked game there is a graph with the players as its nodes. Then player i plays with player j if (i, j) is an edge of the graph. Particularly, the graph gives an evolutionary game a topological structure. Since a potential game has many nice properties, we investigate whether a networked game is potential. Of course, theoretically, the general result, i.e., the PE, can be used to check this. But, as the size is not small, the computational complexity becomes a severe problem. We proved that a networked game is potential only if the games between each pair of edges (i, j) are potential. Moreover, the sum of the potentials becomes the potential of the overall networked game.

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The main tool for our approach is the semi-tensor product (STP) of matrices, which is a generalization of conventional matrix product (Cheng et al., 2012a). It has been successfully applied to the analysis and control of Boolean networks (Cheng et al., 2011; Fornasini & Valcher, 2013; Laschov & Margaliot, 2013), general logical networks (Cheng et al., 2012b), graph theory (Wang et al., 2012), etc. Recently, it has also been used to investigate the problems of NEGs (Guo et al., 2013; Cheng et al., Preprint2013a).

The paper is organized as follows. In Section 2 we introduce some basic preliminaries, including (i) the definition of potential games; (ii) the verification of potential games; (iii) a brief review of semi-tensor product of matrices; (iv) a brief introduction to pseudo-logical functions. Section 3 presents the PE. Moreover, the formula for calculating potential function using the solution of the PE is also obtained. The results are used to verify prisoner's dilemma. In Section 4 we discuss some properties of the PE. Based on these, a closed form solution of the PE is obtained. Some further illustrative examples are presented in Section 5. Section 6 investigates the potential NEGs. Section 7 is a conclusion.

2 Preliminaries

2.1 Potential Games

Definition 1 (Monderer & Shapley 1996b) *A normal (finite) game is a triple $G = (N, S, C)$, where*

- $N = \{1, 2, \cdots, n\}$ *are n players;*
- $S = \prod_{i=1}^{n} S_i$ *(where "$\prod$" is the Cartesian product), and $S_i = \{s_1^i, s_2^i, \cdots, s_{k_i}^i\}$, $i = 1, \cdots, n$ are strategies of player i.*
- $C = \{c_1, c_2, \cdots, c_n\}$ *is the set of payoff functions, where $c_i : S \to R$ is the payoff function of player i.*

Let $E \subset N$, we denote by $S^{-E} := \prod_{j \notin E} S_j$. Particularly, $S^{-i} := \prod_{j \neq i} S_j$, $S^{-\{i,j\}} := \prod_{k \neq i,\ k \neq j} S_k$.

Note that throughout this paper we assume the strategy sets for all players are the same, that is $S_i = \{1, 2, \cdots, k\}$, $i = 1, \cdots, n$. This assumption is for statement ease, it is not essential.

Definition 2 (Monderer & Shapley 1996b) *Consider a finite game $G = (N, S, C)$. G is a positive game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that for every $i \in N$ and for every $s^{-i} \in S^{-i}$ and $\forall x, y \in S_i$*

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = P(x, s^{-i}) - P(y, s^{-i}). \qquad (1)$$

The followings are some fundamental properties of potential.

Theorem 3 (Monderer & Shapley 1996b) *If G is a potential game, then the potential function P is unique up to a constant number. Precisely if P_1 and P_2 are two potential functions, then $P_1 - P_2 = c_0 \in \mathbb{R}$.*

Theorem 4 (Monderer & Shapley 1996b) *Let P be a potential function for G. Then $s \in S$ is an equilibrium point of G if and only if*

$$P(s) \geq P(s^{-i}, x), \quad \forall x \in S_i,\ i = 1, \cdots, n. \qquad (2)$$

Particularly, if P admits a maximal value in S, then G has a pure Nash equilibrium.

Corollary 5 *Every finite potential game possesses a pure Nash equilibrium.*

2.2 Verification of Potential Game

This subsection presents a method to verify whether a game is potential. To the author's best knowledge, this is the only one appeared in literature.

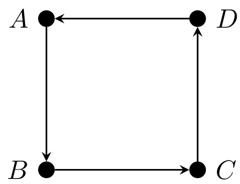

Fig. 1. A Closed Path of Length 4

Theorem 6 (Monderer & Shapley 1996b) *A game G is potential if and only if for every $i, j \in N$, and for every $a \in S^{-\{i,j\}}$ and for every $x_i, y_i \in S_i$, $x_j, y_j \in S_j$, we have*

$$\begin{aligned} & c_i(B) - c_i(A) + c_j(C) - c_j(B) + c_i(D) \\ & \quad - c_i(C) + c_j(A) - c_j(D) = 0, \end{aligned} \qquad (3)$$

where $A = (x_i, x_j, a)$, $B = (y_i, x_j, a)$, $C = (y_i, y_j, a)$, $D = (x_i, y_j, a)$ (refer to Fig. 1).

The advantage of this result is that it is applicable to both finite and infinite games. But its drawback is that according to Theorem 6, to verify whether a finite game G is potential we have to check (3) $N_o = \binom{n}{2} k^{n-2}$ times, where $|N| = n$ and $|S_i| = k$. In this paper we look for an alternative way to simplify the verification.

2.3 Semi-tensor Product of Matrices

This subsection provides a brief survey on semi-tensor product (STP) of matrices. We refer to Cheng et al. (2012a) for more details. We first introduce some notations.

(1) $\mathcal{M}_{m\times n}$: the set of $m\times n$ real matrices.
(2) $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the i-th column (row) of M.
(3) $\mathcal{D}_k := \{1,2,\cdots,k\}, \quad k\geq 2$.
(4) δ_n^i: the i-th column of the identity matrix I_n.
(5) $\Delta_n := \{\delta_n^i | i=1,\cdots,n\}$.
(6) $\mathbf{1}_\ell = (\underbrace{1,1,\cdots,1}_{\ell})^T$.
(7) A matrix $L\in\mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L, denoted by $\mathrm{Col}(L)$, are of the form of δ_m^k. That is, $\mathrm{Col}(L)\subset\Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m\times n$ logical matrixes.
(8) If $L\in\mathcal{L}_{n\times r}$, by definition it can be expressed as $L=[\delta_n^{i_1},\delta_n^{i_2},\cdots,\delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted as $L=\delta_n[i_1,i_2,\cdots,i_r]$.

Definition 7 (Cheng et al. 2011) *Let $M\in\mathcal{M}_{m\times n}$, $N\in\mathcal{M}_{p\times q}$, and $t=\mathrm{lcm}\{n,p\}$ be the least common multiple of n and p. The semi-tensor product of M and N is defined as*

$$M\ltimes N := \left(M\otimes I_{t/n}\right)\left(N\otimes I_{t/p}\right)\in\mathcal{M}_{mt/n\times qt/p}, \quad (4)$$

where $\otimes$ is the Kronecker product.

Remark 8 *(1) When $n=p$, $M\ltimes N=MN$. That is, the semi-tensor product is a generalization of conventional matrix product. Moreover, it keeps all the properties of conventional matrix product available (Cheng et al., 2012a).*
(2) Throughout this paper the matrix product is assumed to be the semi-tensor product and because of (1) the symbol "$\ltimes$" is always omitted.

Proposition 9 *Let $X\in\mathbb{R}^m$ be a column and M a matrix. Then*

$$XM=(I_m\otimes M)X. \quad (5)$$

2.4 Pseudo-logical Functions

Assume $x=i\in\mathcal{D}_k$. By identifying $i\sim\delta_k^i$, $i=1,2,\cdots,k$, we have $x=\delta_k^i\in\Delta_k$, which is called the vector form of x.

Definition 10 *(1) A function $f:\mathcal{D}_k^n\to\mathcal{D}_k$ is called a k-valued logical function. (If $k=2$ it is called a Boolean function.)*
(2) A function $c:\mathcal{D}_k^n\to\mathbb{R}$ is called a k-valued pseudo-logical function. (If $k=2$ it is called a pseudo-Boolean function (Boros & Hammer, 2002).)

Theorem 11 (Cheng et al. 2012a) *Let $f:\mathcal{D}_k^n\to\mathcal{D}_k$ be a k-valued logical function. Then there exists a unique $M_f\in\mathcal{L}_{k\times k^n}$, such that in vector form we have*

$$f(x_1,\cdots,x_n)=M_f\ltimes_{i=1}^n x_i. \quad (6)$$

M_f is called the structure matrix of f.

By similar argument as for Theorem 11, we can easily prove that for a k-valued pseudo-logical function, we have the following result.

Corollary 12 *Let $c:\mathcal{D}_k^n\to\mathbb{R}$ be a k-valued pseudo-logical function. Then there exists a unique row vector $V^c\in\mathbb{R}^{k^n}$, such that*

$$c(x_1,\cdots,x_n)=V^c\ltimes_{i=1}^n x_i. \quad (7)$$

V^c is called the structure vector of c.

2.5 Payoff Matrix

Assume $G=(N,S,C)$ is a finite game with $N=\{1,2,\cdots,n\}$, $S_i=\{1,2,\cdots,k\}$, $i=1,\cdots,n$, $C=(c_1,\cdots,c_n)$. We arrange the elements, called profiles, in S in the alphabetic order as

$$s^1=(1,1,\cdots,1,1); \quad s^2=(1,1,\cdots,1,2); \quad \cdots; \quad s^{k^n-1}=(k,k,\cdots,k,k-1); \quad s^{k^n}=(k,k,\cdots,k,k).$$

Corresponding to each s^j we have

$$C_j := \left(c_1(s^j),c_2(s^j),\cdots,c_n(s^j)\right)^T, \quad j=1,2,\cdots,k^n.$$

A matrix $P_G\in\mathcal{M}_{n\times k^n}$ is called the payoff matrix of G if

$$\mathrm{Col}_j(P_G)=C_j, \quad j=1,2,\cdots,k^n.$$

We use an example to depict this.

Example 13 *Three players stretch their hands out with either palm up ($U=1$) or palm down ($D=2$). The payoffs are shown in the Table 1, where rows (c) are payoffs and columns (p) are strategy profiles.*

Table 1
Payoff Matrix of Example 13

c\p	111	112	121	122	211	212	221	222
c_1	0	1	1	−2	−2	1	1	0
c_2	0	1	−2	1	1	−2	1	0
c_3	0	−2	1	1	1	1	−2	0

Consider a game G with payoff matrix P_G. Using Corollary 12, it is easy to verify that in vector form we have

$$c_i(x_1,\cdots,x_n)=V_i^c\ltimes_{i=1}^n x_i, \quad i=1,\cdots,n, \quad (8)$$

where

$$V_i^c = \mathrm{Row}_i(P_G), \quad i = 1, \cdots, n.$$

Example 14 *Recall Example 13. Then*

$$P_G = \begin{bmatrix} 0 & 1 & 1 & -2 & -2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 1 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} V_1^c &= [0, 1, 1, -2, -2, 1, 1, 0] \\ V_2^c &= [0, 1, -2, 1, 1, -2, 1, 0] \\ V_3^c &= [0, -2, 1, 1, 1, 1, -2, 0]. \end{aligned}$$

Instead of commonly used payoff bi-matrix (Gibbons, 1992), we prefer the payoff matrix. It is particularly convenient when $n > 2$.

3 Basic Formula for Potential Game

3.1 Algebraic Condition

Lemma 15 *G is a potential game if and only if there exist $d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n)$, $i = 1, \cdots, n$, where a caret is used to denote missing terms, (i.e., d_i is independent of x_i,) such that*

$$\begin{aligned} c_i(x_1, \cdots, x_n) &= P(x_1, \cdots, x_n) \\ &\quad + d_i(x_1, \cdots, \hat{x}_i, \cdots, x_n), \quad i = 1, \cdots, n, \end{aligned} \tag{9}$$

where P is the potential function.

Proof. (Sufficiency) Assume (9). Since d_i is independent of x_i, we have

$$\begin{aligned} & c_i(u, s^{-i}) - c_i(v, s^{-i}) = \left[P(u, s^{-i}) + d_i(s^{-i})\right] \\ & \quad - \left[P(v, s^{-i}) + d_i(s^{-i})\right] \\ & = P(u, s^{-i}) - P(v, s^{-i}), \quad u, v \in S_i, s^{-i} \in S^{-i}. \end{aligned}$$

(Necessity) Set

$$d_i(x_1, \cdots, x_n) := c_i(x_1, \cdots, x_n) - P(x_1, \cdots, x_n).$$

Let $u, v \in S_i$. Using (1), we have

$$\begin{aligned} & d_i(u, s^{-i}) - d_i(v, s^{-i}) = \left[c_i(u, s^{-i}) - c_i(v, s^{-i})\right] \\ & \quad - \left[P(u, s^{-i}) - P(v, s^{-i})\right] = 0. \end{aligned}$$

Since $u, v \in S_i$ are arbitrary, d_i is independent of x_i. □

Next, we express (9) in its vector form as

$$V_i^c \ltimes_{j=1}^n x_j = V_P \ltimes_{j=1}^n x_j + V_i^d \ltimes_{j \neq i} x_j, \quad i = 1, \cdots, n, \tag{10}$$

where $V_i^c, V_P \in \mathbb{R}^{k^n}$ and $V_i^d \in \mathbb{R}^{k^{n-1}}$ are row vectors.

Now verifying whether G is potential is equivalent to checking whether the solutions of (9) for unknowns P and d_i exist. Equivalently, whether the solutions of (10) for unknown vectors V_P and V_i^d exist.

We need some preliminaries. Define

$$D_f^{[p,q]} = \mathbf{1}_p^T \otimes I_q \in \mathcal{L}_{q \times pq}, \tag{11}$$

and

$$D_r^{[p,q]} = I_p \otimes \mathbf{1}_q^T \in \mathcal{L}_{p \times pq}. \tag{12}$$

Then we have the following lemma, which is easily verifiable.

Lemma 16 *Let $X \in \Delta_p$ and $Y \in \Delta_q$. Then*

$$D_f^{[p,q]} XY = Y, \tag{13}$$

and

$$D_r^{[p,q]} XY = X. \tag{14}$$

So the D_f (D_r) operator is used to delete front (rear) factor.

Using Lemma 16, we have

$$\ltimes_{j \neq i} x_j = \begin{cases} D_f^{[k,k]} \ltimes_{j=1}^n x_j, & i = 1 \\ D_r^{[k^{i-1},k]} \ltimes_{j=1}^n x_j, & 2 \leq i \leq n. \end{cases}$$

Since $x_1, \cdots, x_n \in \Delta_k$ are arbitrary, (10) can be rewritten as

$$V_i^c = V_P + V_i^d M_i, \quad i = 1, \cdots, n, \tag{15}$$

where

$$M_i = \begin{cases} D_f^{[k,k]}, & i = 1 \\ D_r^{[k^{i-1},k]}, & 2 \leq i \leq n. \end{cases} \tag{16}$$

Solving V_P from the first equation of (15) yields

$$V_P = V_1^c - V_1^d M_1.$$

Plugging it into the rest equations of (15) yields

$$V_i^c - V_1^c = V_i^d M_i - V_1^d M_1, \quad i = 2, \cdots, n. \tag{17}$$

Taking transpose, we have

$$[V_i^c - V_1^c]^T = (M_i)^T \left(V_i^d\right)^T - M_1^T \left(V_1^d\right)^T, \quad i = 2, \cdots, n. \tag{18}$$

Note that $M_1 \in \mathcal{L}_{k \times k^2}$, then $M_1^T \in \mathcal{M}_{k^2 \times k}$. Since $\left(V_i^d\right)^T \in \mathbb{R}^{k^{n-1}}$, to convert $M_1^T \left(V_1^d\right)^T$ into a conventional matrix product, by definition we have

$$M_1^T \ltimes \left(V_1^d\right)^T = \left[M_1^T \otimes I_{k^{n-2}}\right] \left(V_1^d\right)^T.$$

Then we define

$$\Psi_1 = M_1^T \otimes I_{k^{n-2}} = \left(D_f^{[k,k]}\right)^T \otimes I_{k^{n-2}}. \tag{19}$$

Similarly, we define

$$\begin{aligned} \Psi_2 &= M_2^T \otimes I_{k^{n-2}} = \left(D_r^{[k,k]}\right)^T \otimes I_{k^{n-2}} \\ \Psi_3 &= M_3^T \otimes I_{k^{n-3}} = \left(D_r^{[k^2,k]}\right)^T \otimes I_{k^{n-3}} \\ &\vdots \\ \Psi_n &= M_n^T \qquad\qquad = \left(D_r^{[k^{n-1},k]}\right)^T. \end{aligned} \tag{20}$$

Using (11) and (12), (19) and (20) can be expressed uniformly as

$$\Psi_i = I_{k^{i-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-i}}, \quad i = 1, \cdots, n, \tag{21}$$

where $\Psi_i \in \mathcal{M}_{k^n \times k^{n-1}}$, $i = 1, \cdots, n$. We also define some vectors as

$$\begin{aligned} \xi_i &:= \left(V_i^d\right)^T \in \mathbb{R}^{k^{n-1}}, \quad i = 1, \cdots, n; \\ b_i &:= (V_i^c - V_1^c)^T \in \mathbb{R}^{k^n}, \quad i = 2, \cdots, n. \end{aligned} \tag{22}$$

Then (18) can be expressed as a linear system:

$$\Psi \xi = b, \tag{23}$$

where

$$\Psi = \begin{bmatrix} -\Psi_1 & \Psi_2 & 0 & \cdots & 0 \\ -\Psi_1 & 0 & \Psi_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ -\Psi_1 & 0 & 0 & \cdots & \Psi_n \end{bmatrix}; \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}; \quad b = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}. \tag{24}$$

(23) is called the potential equation and Ψ is called the potential matrix.

Above argument leads to the following theorem.

Theorem 17 *A finite game G is potential if and only if the potential equation (23) has solution. Moreover, the potential P can be calculated by*

$$V_P = V_1^c - V_1^d M_1 = V_1^c - \xi_1^T D_f^{[k,k]}. \tag{25}$$

Remark 18 *Note that the potential matrix Ψ depends on $n = |N|$ and $k = |S_i|$ only. While b depends on the payoffs. So we may use n and k to classify the type of games, and each type has a common potential matrix. Then we can use the payoff matrix to verify whether a game is potential or not.*

3.2 Prisoner's Dilemma

As a direct application, we consider the game of prisoner's dilemma.

Example 19 *Consider a prisoner's dilemma with the payoff bi-matrix as in Table 2.*

Table 2
Payoff Bi-matrix of Prisoner's Dilemma

$P_1\backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

Table 2 can be rewritten into a payoff matrix form as

Table 3
Payoff Matrix of Prisoner's Dilemma

c\p	11	12	21	22
c_1	R	S	T	P
c_2	R	C	S	P

From Table 3 (equivalently, Table 2) we have

$$\begin{aligned} V_1^c &= (R, S, T, P) \\ V_2^c &= (R, T, S, P). \end{aligned}$$

Assume $V_1^d = (a, b)$ and $V_2^d = (c, d)$. It is easy to calculate that

$$\begin{aligned} M_1 &= D_f^{[2,2]} = \delta_2[1, 2, 1, 2], \\ M_2 &= D_r^{[2,2]} = \delta_2[1, 1, 2, 2]. \end{aligned}$$

$$\psi_1 = M_1^T; \quad \psi_2 = M_2^T.$$

$$b_2 = (V_2^c - V_1^c)^T = (0, T - S, S - T, 0)^T.$$

Then the equation (23) becomes

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ T-S \\ S-T \\ 0 \end{bmatrix}. \tag{26}$$

It is easy to solve it out as

$$\begin{cases} a = c = T - c_0 \\ b = d = S - c_0 \end{cases}$$

where $c_0 \in \mathbb{R}$ is an arbitrary number. We conclude that the general Prisoner's Dilemma is a potential game.

Using (25), the potential can be obtained as

$$\begin{aligned} V_P &= V_1^c - V_1^d D_f^{[2,2]} \\ &= (R-T, 0, 0, P-S) + c_0(1,1,1,1). \end{aligned} \tag{27}$$

Monderer & Shapley (1996b) considered the Prisoner's Dilemma with $R = 1$, $S = 9$, $T = 0$, $P = 6$, and $V_P = (4, 3, 3, 0)$. It is a special case of (27) with $c_0 = 3$.

Next, we consider a generalized (asymmetric) Prisoner's Dilemma G, where the payoff bi-matrix is as in Table 4.

Table 4
Asymmetric Payoff Bi-matrix of G

$P_1 \backslash P_2$	1	2
1	$(A,\ E)$	$(B,\ F)$
2	$(C,\ G)$	$(D,\ H)$

Let V_1^d and V_2^d be as in Example 19. Then we have equation (23), where Ψ and ξ are the same as in (26), and

$$b = \begin{bmatrix} E-A \\ F-B \\ G-C \\ H-D \end{bmatrix}.$$

Since $rank(\Psi) = 3$, it is easy to verify that (23) has solution if and only if

$$E - A - F + B - G + C + H - D = 0. \tag{28}$$

We conclude that

Proposition 20 *The asymmetric Prisoner's Dilemma is potential if and only if the entries of its payoff bi-matrix satisfy (28). Moreover, let (a, b, c, d) be a particular solution of (28). Then the potential is*

$$V_P = (A, B, C, D) - (a, a, b, b) + c_0(1, 1, 1, 1), \tag{29}$$

where $c_0 \in \mathbb{R}$ is an arbitrary number.

4 Solving the Potential Equation

In this section we explore some further properties of potential equation. Based on these properties, a closed form solution of the PE is obtained, provided the game is potential.

Lemma 21 $\mathbf{1}_{nk^{n-1}}$ *is a solution of $\Psi x = 0$, where Ψ is the potential matrix described in (23).*

Proof. Since both $D_r^{[p,q]}$ and $D_f^{[p,q]}$ are logical matrices, then each column of M_i has only one non-zero element, which is 1. Then it is easy to see that each row of Ψ_i has only one non-zero element, which is 1. According to the structure of Ψ, it is obvious that each row of Ψ has exactly two non-zero elements, which are -1 and 1 respectively. The conclusion follows. □

Lemma 22 *Let Ψ be the potential matrix in (23). Then*

$$\text{Span}\ \ \text{Row}(\Psi) = \mathbf{1}_{nk^{n-1}}^{\perp}. \tag{30}$$

Proof. From Lemma 21 one sees easily that

$$\text{Span}\ \ \text{Row}(\Psi) \subset \mathbf{1}_{nk^{n-1}}^{\perp}.$$

Next, if (30) is true, using Lemma 21 again, one sees easily that if ξ_0 is a solution of (23), then all the solutions can be expressed as $\xi = \xi_0 + c_0\mathbf{1}_{nk^{n-1}}$. Therefore, the corresponding P satisfies $P = P_0 + c_0$, where P_0 is a potential constructed from ξ_0.

But now if (30) is not true, then $\text{rank}(\Psi) < nk^{n-1} - 1$. Hence, we can find another nonzero solution, $\{\xi_i' | i = 1, \cdots, n\}$ of $\Psi x = 0$, which is linearly independent of $\mathbf{1}_{nk^{n-1}}$. Using this set of solutions, we can construct another potential P' such that $P' - P \neq c_0$, for any $c_0 \in \mathbb{R}$. This is a contradiction to Theorem 3. We, therefore, proved (30). □

Remark 23 *Theorem 3 can also be proved as follows. Let P and P' be two potential functions. That is*

$$\begin{aligned} &V_i^c(x_1, \cdots, x_n) = V_P(x_1, \cdots, x_n) \\ &+ V_i^d(x_1, \cdots, \hat{x}_i, \cdots, x_n); \\ &V_i^c(x_1, \cdots, x_n) = V_{P'}(x_1, \cdots, x_n) \\ &+ (V_i^d)'(x_1, \cdots, \hat{x}_i, \cdots, x_n), \quad i = 1, \cdots, n. \end{aligned}$$

Setting the error $d := V_{P'} - V_P$, then

$$d = \left(V_i^d\right)' - V_i^d.$$

Hence, d is independent of x_i. But i is arbitrary, so d is constant.

According to Theorem 17 and using Lemma 22, we have the following result.

Theorem 24 *Given a game G with $N = \{1, 2, \cdots, n\}$ and $S_i = \{1, 2, \cdots, k\}$, $i = 1, \cdots, n$, assume b is defined by (22)-(24), and Ψ is defined by (20)-(24), then the following four statements are equivalent.*

(1) G is potential.
(2)

$$\mathrm{rank}[\Psi, b] = nk^{n-1} - 1. \tag{31}$$

(3)

$$b \in \mathrm{Span}\ \ \mathrm{Col}(\Psi). \tag{32}$$

(4) For an arbitrarily chosen $i \in \{1, 2, \cdots, n\}$

$$b \in \mathrm{Span}\ \ \{\mathrm{Col}_j(\Psi) | j \neq i\}. \tag{33}$$

Proof. From Lemma 22 one sees that $\mathrm{rank}(\Psi) = nk^{n-1} - 1$.

$1 \Leftrightarrow 2$: From linear algebra we know that condition (31) is the necessary and sufficient condition for equation (23) to have solution. According to Theorem 17, the latter is equivalent to that G is potential. $2 \Leftrightarrow 3$: This is obvious. $3 \Leftrightarrow 4$: It is obvious that 4 implies 3. To see 3 also implies 4 we have only to show that any $nk^{n-1} - 1$ columns of Ψ form a basis of Span Col(Ψ). Using Lemma 21, we have

$$\sum_{j=1}^{nk^{n-1}} \mathrm{Col}_j(\Psi) = 0.$$

Hence for any $1 \leq i \leq nk^{n-1}$, we have

$$\mathrm{Col}_i(\Psi) = -\sum_{j \neq i} \mathrm{Col}_j(\Psi).$$

Hence any $nk^{n-1} - 1$ columns of Ψ are linearly independent. (33) follows immediately. □

Next, we assume that a game G is potential and search for an algorithm to calculate the potential function.

From Theorem 24 it is easy to see that any $nk^{n-1} - 1$ columns of Ψ are linearly independent. So to find a particular solution of (23), we can assume the last element of ξ is zero. This leads to the following algorithm.

Algorithm 1 • *Step 1. Construct Ψ_0 by deleting the last column of Ψ.*
• *Step 2. Let $\xi = \left[\xi^0, 0\right]$. Determine ξ^0 as*

$$\xi^0 := \left(\Psi_0^T \Psi_0\right)^{-1} \Psi_0^T b. \tag{34}$$

• *Step 3. Define $\xi_1 \in \mathbb{R}^{k^{n-1}}$ as the sub-vector of the first k^{n-1} elements of ξ^0. Use (25) to determine V_P.*

From the above statements and algorithm we have the following result:

Theorem 25 *(1) A game G is potential if and only if $\xi = \begin{bmatrix} \xi^0 \\ 0 \end{bmatrix}$, with ξ^0 defined in (34), is a solution of the potential equation (23).*
(2) If G is potential and V_P is a solution obtained from Algorithm 1, then

$$P(x) = V_P \ltimes_{i=1}^n x_i + c_0, \tag{35}$$

where $c_0 \in \mathbb{R}$ is arbitrary.

5 Illustrative Examples

Example 26 *Consider a symmetric game G with $n = 3$, $k = 2$, and payoff matrix as in Table 5.*

Table 5
Payoff Matrix for Example 26

c\p	111	112	121	122	111	212	221	222
c_1	a	b	b	d	c	e	e	f
c_2	a	b	c	e	b	d	e	f
c_3	a	c	b	e	b	e	d	f

We ask when G is potential? Using (19)-(20), we have

$$\begin{array}{rl} \Psi_1 = & \left(D_f^{[2,2]}\right)^T \otimes I_2 = \left(\delta_2[1,2,1,2]\right)^T \otimes I_2 \\ = & \begin{bmatrix} 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{bmatrix}^T; \end{array} \tag{36}$$

$$\begin{array}{rl} \Psi_2 = & \left(D_r^{[2,2]}\right)^T \otimes I_2 = \left(\delta_2[1,1,2,2]\right)^T \otimes I_2 \\ = & \begin{bmatrix} 1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1 \end{bmatrix}^T; \end{array} \tag{37}$$

$$\Psi_3 = \left(D_r^{[4,2]}\right)^T = (\delta_4[1,1,2,2,3,3,4,4])^T$$
$$= \begin{bmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1 \end{bmatrix}^T; \tag{38}$$

Hence, we have

$$\Psi = \begin{bmatrix}
-1&0&0&0&1&0&0&0&0&0&0&0\\
0&-1&0&0&0&1&0&0&0&0&0&0\\
0&0&-1&0&1&0&0&0&0&0&0&0\\
0&0&0&-1&0&1&0&0&0&0&0&0\\
-1&0&0&0&0&0&1&0&0&0&0&0\\
0&-1&0&0&0&0&0&1&0&0&0&0\\
0&0&-1&0&0&0&1&0&0&0&0&0\\
0&0&0&-1&0&0&0&1&0&0&0&0\\
-1&0&0&0&0&0&0&0&1&0&0&0\\
0&-1&0&0&0&0&0&0&1&0&0&0\\
0&0&-1&0&0&0&0&0&0&1&0&0\\
0&0&0&-1&0&0&0&0&0&1&0&0\\
-1&0&0&0&0&0&0&0&0&0&1&0\\
0&-1&0&0&0&0&0&0&0&0&1&0\\
0&0&-1&0&0&0&0&0&0&0&0&1\\
0&0&0&-1&0&0&0&0&0&0&0&1
\end{bmatrix}.$$

Next, we calculate

$$\begin{aligned} b_1 &= V_2^c - V_1^c = [0,0,c-b,e-d,b-c,d-e,0,0]^T \\ b_2 &= V_3^c - V_1^c = [0,c-b,0,e-d,b-c,0,d-e,0]^T. \end{aligned}$$

Hence,

$$\begin{aligned} b &= \left[b_1^T \; b_2^T\right]^T \\ &= [0,0,\alpha,\beta,-\alpha,-\beta,0,0,0,\alpha,0,\beta,-\alpha,0,-\beta,0]^T, \end{aligned}$$

where $\alpha = c-b$, $\beta = e-d$. Then it is easy to verify that

$$\begin{aligned} b &= (\alpha+\beta)\,\mathrm{Col}_1(\Psi) + \beta\,\mathrm{Col}_2(\Psi) + \beta\,\mathrm{Col}_3(\Psi) \\ &\quad +(\alpha+\beta)\,\mathrm{Col}_5(\Psi) + \beta\,\mathrm{Col}_6(\Psi) + \beta\,\mathrm{Col}_7(\Psi) \\ &\quad +(\alpha+\beta)\,\mathrm{Col}_9(\Psi) + \beta\,\mathrm{Col}_{10}(\Psi) + \beta\,\mathrm{Col}_{11}(\Psi). \end{aligned}$$

According to Theorem 24, we conclude that a symmetric game with $n = 2$ and $k = 3$ is potential.

Next, to demonstrate the calculation of potential we specify the payoff matrix. Assume $a = 1$, $b = 1$, $c = 2$, $d = -1$, $e = 1$, $f = -1$. Then it is easy to calculate that

$$\begin{aligned} b_1 &= [V_2^c - V_1^c]^T = [0,0,1,2,-1,-2,0,0]^T \\ b_2 &= [V_3^c - V_1^c]^T = [0,1,0,2,-1,0,-2,0]^T. \end{aligned}$$

Using (34), we have $\xi^0 = [3,2,2,0,3,2,2,0,3,2,2]^T$. It follows that $V_1^d = \xi_1^T = [3,2,2,0]$. Using (25), we have

$$\begin{aligned} V_P &= V_1^c - V_1^d D_r^{[2,2]} \\ &= [1,1,1,-1,2,1,1,-1] - [3,2,2,0]\delta_2[1,2,1,2] \\ &= [-2,-1,-1,-1,-1,-1,-1,-1]. \end{aligned}$$

We, therefore, have

$$P(x) = [-2,-1,-1,-1,-1,-1,-1,-1]x + c_0,$$

where $x = \ltimes_{i=1}^3 x_i \in \Delta_8$.

Next example is the rock-paper-scissors.

Example 27 *Consider the game of Rock-Paper-Scissors, which has the payoffs as in Table 6, where the strategies are denoted by Rock = 1, Paper = 2, Scissors = 3.*

Table 6
Payoff Matrix of Rock-Paper-Scissors Game

c\P	11	12	13	21	22	23	31	32	33
c_1	0	−1	1	1	0	−1	−1	1	0
c_2	0	1	−1	−1	0	1	1	−1	0

Is G potential? It is easy to verify that

$$\begin{aligned} \Psi_1 &= \left(D_f^{[3,3]}\right)^T = \delta_3[1,2,3,1,2,3,1,2,3]^T \\ \Psi_2 &= \left(D_r^{[3,3]}\right)^T = \delta_3[1,1,1,2,2,2,3,3,3]^T. \end{aligned}$$

Then

$$\Psi = [-\Psi_1 \; \Psi_2] = \begin{bmatrix}
-1&0&0&1&0&0\\
0&-1&0&1&0&0\\
0&0&-1&1&0&0\\
-1&0&0&0&1&0\\
0&-1&0&0&1&0\\
0&0&-1&0&1&0\\
-1&0&0&0&0&1\\
0&-1&0&0&0&1\\
0&0&-1&0&0&1
\end{bmatrix}$$

$$b = V_2^c - V_1^c = [0, 2, -2, -2, 0, 2, 2, -2, 0]^T.$$

Since $\mathrm{rank}(\Psi) = 5$ *and* $\mathrm{rank}[\Psi\ b] = 6$, *we conclude that the game Rock-Paper-Scissors is not potential.*

6 Potential Networked Evolutionary Games

Definition 28 (Cheng et al. Preprint2013a) *A networked evolutionary game (NEG), denoted by* $((N, E), G, \Pi)$, *consists of three items as:*

(i) a network graph: (N, E);
(ii) a fundamental network game (FNG): G.
(iii) a local information based strategy updating rule (SUR).

Remark 29 *(i) An FNG is a symmetric game with two players. For an NEG, if* $(i, j) \in E$, *then* i *and* j *play the FNG at each time* t *with strategies* $x_i(t)$ *and* $x_j(t)$ *respectively.*
(ii) SUR is a rule which determines the strategy of each player at next time $t + 1$. *Using local information means the rule depends on the neighborhood's strategies* (x_j) *and payoffs* (c_j) *at time* t. *Precisely,*

$$x_i(t+1) = f_i\left(x_j(t), c_j(t) \mid j \in U(i)\right), \quad i = 1, \cdots, n,$$

where

$$U(i) = \{j | (i, j) \in E\} \bigcup \{i\}$$

is the neighborhood of i.

Assume the SUR is the myopic learning process. That is, each player finds the best strategy among his neighbors at time t, and uses it as his strategy at time $(t + 1)$ (Monderer & Shapley, 1996b). To make the process uniquely determined we specify it as follows:

- At each time t only one player, say, i, is allowed to update his strategy either by leaving it unchanged, i.e., $x_i(t + 1) = x_i(t)$, if

  $$c_i(x^{-i}(t), x_i(t)) = max c_i(x^{-i}(t), j),$$

  or by choosing

  $$x_i(t + 1) = j^*,$$

  where

  $$j^* = argmax c_i(x^{-i}(t), j).$$

  If $|argmax c_i(x^{-i}(t), j)| > 1$, player i has to choose the smallest value of j^*.
- If

  $$x_i(t + 1) = x_i(t), \quad \forall i,$$

  the game stops.

This SUR is particularly suitable for potential games, because

Theorem 30 (Monderer & Shapley 1996b) *Using the myopic learning, a finite potential game converges to a (pure) Nash equilibrium.*

Next, we give an example to see when a networked game is potential.

Example 31 *Consider an NEG* $((N, E), G, \Pi)$, *where the network graph is described as in Fig. 2 or Fig. 3.*

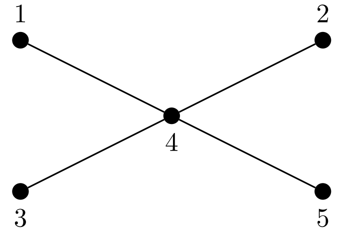

Fig. 2. Network Graph (a)

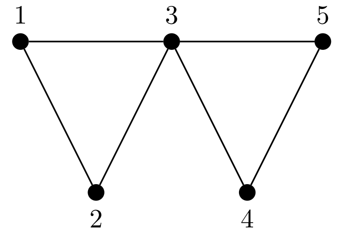

Fig. 3. Network Graph (b)

Assume the fundamental network game is the prisoner's dilemma, that is, it has the payoff bi-matrix as in Table 2, with $R = -1$, $S = -10$, $T = 0$, $P = -5$. *Since the potential matrix* Ψ *depends on* $|N|$ *and* k *only, it is independent of the network graph. So we can calculate the* Ψ *as*

$$\Psi = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{128 \times 80}.$$

(Because of the size, we omit the details.) Next, we can calculate V_i^c, $i = 1, \cdots, 5$, *and then using them to verify whether they are potential.*

(1) (Network Graph (a)): It is easy to check that

$$\begin{aligned} V_1^c = [&-1\ -1\ -10\ -10\ -1\ -1\ -10\ -10 \\ &-1\ -1\ -10\ -10\ -1\ -1\ -10\ -10 \\ &0\ \ 0\ \ -5\ \ -5\ \ 0\ \ 0\ \ -5\ \ -5 \\ &0\ \ 0\ \ -5\ \ -5\ \ 0\ \ 0\ \ -5\ \ 5]. \end{aligned}$$

$$V_2^c = [-1\ -1\ -10\ -10\ -1\ -1\ -10\ -10$$
$$0\ \ 0\ \ -5\ \ -5\ \ 0\ \ 0\ \ -5\ \ -5$$
$$-1\ -1\ -10\ -10\ -1\ -1\ -10\ -10$$
$$0\ \ 0\ \ -5\ \ -5\ \ 0\ \ 0\ \ -5\ \ -5].$$

$$V_3^c = [-1\ -1\ -10\ -10\ 0\ 0\ -5\ -5$$
$$-1\ -1\ -10\ -10\ 0\ 0\ -5\ -5$$
$$-1\ -1\ -10\ -10\ 0\ 0\ -5\ -5$$
$$-1\ -1\ -10\ -10\ 0\ 0\ -5\ -5].$$

$$V_4^c = [-4\ -13\ \ 0\ \ -5\ -13\ -22\ -5\ -10$$
$$-13\ -22\ -5\ -10\ -22\ -31\ -10\ -15$$
$$-13\ -22\ -5\ -10\ -22\ -31\ -10\ -15$$
$$-22\ -31\ -10\ -15\ -31\ -40\ -15\ -20].$$

$$V_5^c = [-1\ 0\ -10\ -5\ -1\ 0\ -10\ -5$$
$$-1\ 0\ -10\ -5\ -1\ 0\ -10\ -5$$
$$-1\ 0\ -10\ -5\ -1\ 0\ -10\ -5$$
$$-1\ 0\ -10\ -5\ -1\ 0\ -10\ -5].$$

Using Theorem 24 we can check that the networked game is potential. Then we can use Algorithm 1 to solve ξ out. We skip the details and give the result directly as

$$\xi_1 = [28\ 27\ 15\ 10\ 27\ 26\ 10\ 5$$
$$27\ 26\ 10\ 5\ 26\ 25\ 5\ 0].$$

Using (23), we have

$$V_P^a = [-29\ -28\ -25\ -20\ -28\ -27\ -20\ -15$$
$$-28\ -27\ -20\ -15\ -27\ -26\ -15\ -10$$
$$-28\ -27\ -20\ -15\ -27\ -26\ -15\ -10$$
$$-27\ -26\ -15\ -10\ -26\ -25\ -10\ -5].$$

(2) (Network Graph (b)): Similar verification shows that the networked game is also potential, and we have the potential as:

$$V_P^b = [-46\ -44\ -44\ -38\ -42\ -36\ -36\ -26$$
$$-44\ -42\ -42\ -36\ -36\ -30\ -30\ -20$$
$$-44\ -42\ -42\ -36\ -36\ -30\ -30\ -20$$
$$-38\ -36\ -36\ -30\ -26\ -20\ -20\ -10].$$

Motivated by the above Example, we can prove the following:

Theorem 32 *Consider a networked (evolutionary) game G with network graph (N, E). If the fundamental network game is potential, then G is also potential.*

Proof. Assume $(i,j) \in E$. Since the fundamental network game is potential, we have

$$c_i^j(u, s_j) - c_i^j(v, s_j) = P^{i,j}(u, s_j) - P^{i,j}(v, s_j),$$
$$\forall u, v \in S_i, \forall s_j \in S_j,$$

where c_i^j is the payoff of i from the game played with j, $P^{i,j}$ is the potential function for the game between i and j. As for the overall payoff of i, since player i plays only with his neighbors, we have

$$c_i(s) = c_i(U(i)) = \sum_{j \in U(i)^{-i}} c_i^j(s_i, s_j),$$
$$s \in S,\ s_i \in S_i,\ s_j \in S_j,$$

Now set

$$P(s) := \sum_{(i,j)\in E} P^{i,j}(s_i, s_j). \tag{39}$$

Then for any $i \in N$, we have

$$\begin{aligned}
&P(u, s^{-i}) - P(v, s^{-i}) \\
&= \sum_{(p,q)\in E,\ \text{and}\ i \notin \{p,q\}} [P^{p,q}(s_p, s_q) - P^{p,q}(s_p, s_q)] \\
&\quad + \sum_{\{j \mid (i,j)\in E\}} \left[P^{i,j}(u, s_j) - P^{i,j}(v, s_j)\right] \\
&= \sum_{j \in U(i)^{-i}} \left[P^{i,j}(u, s_j) - P^{i,j}(v, s_j)\right] \\
&= \sum_{j \in U(i)^{-i}} c_i^j(u, s_j) - \sum_{j \in U(i)^{-i}} c_i^j(v, s_j) \\
&= c_i(u, U(i)^{-i}) - c_i(v, U(i)^{-i}) \\
&= c_i(u, s^{-i}) - c_i(v, s^{-i}).
\end{aligned}$$

Hence, P is the potential function for the overall network. □

Remark 33 *(1) Theorem 32 is convenient when verifying whether a networked evolutionary is potential. Because when the size of the network is not small, verifying the overall network directly is very difficult. But now we need only check the fundamental network game.*

(2) In fact, (39) can be considered as a formula to calculate the potential of a networked game. It is much more convenient than using formula (25) directly. In the following we give another example to illustrate this.

Example 34 *Recall Example 31. One easily sees that the fundamental network game is the prisoner's dilemma, which has potential (refer to (27))*

$$V_0 = (R - T, 0, 0, P - S).$$

That is, for any $(i, j) \in E$ *we have*

$$P(x_i, x_j) = V_0 x_i x_j, \tag{40}$$

where

$$V_0 = (R - T, 0, 0, P - S) = (-1\ 0\ 0\ 5).$$

To get a universal expression, we have to convert (40) into a general form as

$$P(x_i, x_j) = V_0 x_i x_j = V_P^{i,j} x, \tag{41}$$

where $x = \ltimes_{i=1}^5 x_i$*. By adding dummy arguments, we have*

$$P(x_1, x_2) = V_0 x_1 x_2 = V_0 D_r^{[4,8]} x_1 x_2 x_3 x_4 x_5. \tag{42}$$

Hence we have

$$\begin{aligned} V_P^{1,2} &= V_0 D_r^{[4,8]} = V_0 \left(I_4 \otimes \mathbf{1}_8^T\right) \\ &= [-1\ -1\ -1\ -1\ -1\ -1\ -1\ -1 \\ &\quad\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ &\quad\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \\ &\quad\ 5\ \ 5\ \ 5\ \ 5\ \ 5\ \ 5\ \ 5\ \ 5]. \end{aligned}$$

Similarly, we can figure out all $V_P^{i,j}$ *as*

$$\begin{aligned} &V_P^{1,3} = V_0 D_r^{[2,2]} D_r^{[8,2]}, && V_P^{1,4} = V_0 D_r^{[2,4]} D_r^{[16,2]}, \\ &V_P^{1,5} = V_0 D_r^{[2,8]}, && V_P^{2,3} = V_0 D_f^{[2,2]} D_r^{[8,4]}, \\ &V_P^{2,4} = V_0 D_f^{[2,2]} D_r^{[4,2]} D_r^{[16,2]}, && V_P^{2,5} = V_0 D_f^{[2,2]} D_r^{[4,4]} \\ &V_P^{3,4} = V_0 D_f^{[4,2]} D_r^{[16,2]}, && V_P^{3,5} = V_0 D_f^{[4,2]} D_r^{[8,2]}, \\ &V_P^{4,5} = V_0 D_f^{[8,2]}. \end{aligned}$$

It is obvious that $V_P^{i,j} = V_P^{j,i}$*. Now we are ready to calculate the potentials, which are already obtained in Example 31. To distinguish them with those in Example 31, we denote the potentials obtained here by* $\tilde{P}$*.*

(1) Consider Figure 2. Using equation (39), we have

$$\begin{aligned} V_{\tilde{P}}^a &= V_P^{1,4} + V_P^{2,4} + V_P^{3,4} + V_P^{4,5} \\ &= [-4\ -3\ 0\ 5\ -3\ -2\ 5\ 10 \\ &\quad -3\ -2\ 5\ 10\ -2\ -1\ 10\ 15 \\ &\quad -3\ -2\ 5\ 10\ -2\ -1\ 10\ 15 \\ &\quad -2\ -1\ 10\ 15\ -1\ 0\ 15\ 20]. \end{aligned}$$

Comparing this result with V_P^a *in Example 31, since*

$$P^a(x) := V_P^a x; \quad \tilde{P}^a(x) = V_{\tilde{P}}^a x,$$

one sees easily that

$$\tilde{P}^a(x) = P^a(x) + 25.$$

So, up to a constant they are the same.

(2) Consider Figure 3. We have

$$\begin{aligned} V_{\tilde{P}}^b &= V_P^{1,2} + V_P^{1,3} + V_P^{2,3} + V_P^{3,4} + V_P^{3,5} + V_P^{4,5} \\ &= [-6\ -4\ -4\ 2\ -2\ 4\ 4\ 14 \\ &\quad -4\ -2\ -2\ 4\ 4\ 10\ 10\ 20 \\ &\quad -4\ -2\ -2\ 4\ 4\ 10\ 10\ 20 \\ &\quad 2\ 4\ 4\ 10\ 14\ 20\ 20\ 30]. \end{aligned}$$

Comparing this result with V_P^b *in Example 31, one sees easily that*

$$\tilde{P}^b(x) = P^b(x) + 40.$$

7 Conclusion

This paper investigates the computational aspects of the potential games. First, a linear system, called the potential equation, is presented. It is proved that a game is potential if and only if its PE has solution. Using the solution, a formula is obtained to calculate the potential function. Then some properties of the PE are obtained and used to provide a closed form solution for the PE. Some examples are presented to illustrate the results. Finally, the problem of whether an NEG is potential is considered. It is proved that an NEG is potential only if the fundamental network game is. That is, it is independent of the size and topological structure of the network. This property is very useful in investigating NEGs. Finally, we would like to mention some further results (Cheng et al., Preprint2013b):

(i) Equation (23) can be easily extended to weighted potential games, and players are allowed to have different numbers of strategies.
(ii) Using equation (23), we can prove that the set of potential games is a linear subspace of finite games. It is very useful in investigating the convergence of nearly potential games, etc.

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