On Decomposed Subspaces of Finite Games

Daizhan Cheng, *Fellow, IEEE,* Ting Liu, Kuize Zhang, *Member, IEEE,* and Hongsheng Qi, *Member, IEEE*

***Abstract*—It was shown in [2] that the vector space of finite noncooperative games can be decomposed into three orthogonal subspaces: the pure potential games ($\mathcal{P}$), non-strategic games ($\mathcal{N}$), and pure harmonic games ($\mathcal{H}$). This paper considers the detailed description of these three subspaces by providing their bases. We first provide the basis of potential games $\mathcal{G}_P=\mathcal{P}\oplus\mathcal{N}$ and the basis of $\mathcal{N}$. Then the bases of $\mathcal{P}$ and $\mathcal{H}$ are also obtained. These bases make the decomposition numerically easier. Meanwhile, they provide a convenient tool for investigating the properties of the corresponding subspaces. As an application, we consider the dynamics of (networked) evolutionary games (NEGs). Three problems are considered: (1) the dynamic equivalence of evolutionary games; (2) the dynamics of near potential games; (3) the decomposition of NEGs.**

***Index Terms*—Potential game, Harmonic game, Non-strategic game, Decomposition, Semi-tensor product of matrices.**

I. Preliminaries

The potential game was firstly introduced by Rosenthal [18]. It not only plays an important role in game-theoretic analysis but also becomes a powerful tool in several control problems. We refer to [15] for the concept and general properties of potential games. Some of its applications to control problems are: (i) consensus of multi-agent systems [12]; (ii) optimization of distributed coverage of graphs [23], [25]; (iii) congestion control [22]; (iv) control of power networks [11], just to name a few.

Partly because of the importance of the potential games, people are interested in the topological structure of the set of finite noncooperative games, as well as their neighboring games. To this end, a vector space structure has been posed to the set of finite noncooperative games. This was firstly presented by Candogan, Menache, *et al* in their novel paper [2]. In this follow-up work, we consider the structure of decomposed subspaces.

For statement ease, we first introduce some notations:

1) $\mathcal{M}_{m\times n}$: the set of $m\times n$ real matrices.
2) $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$): the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$): the i-th column (row) of M.
3) $\mathcal{D}_k:=\{1,2,\cdots,k\}, \quad k\geq 2$.
4) δ_n^i: the i-th column of the identity matrix I_n.
5) $\Delta_n:=\left\{\delta_n^i | i=1,\cdots,n\right\}$.
6) $\mathbf{1}_\ell=(\underbrace{1,1,\cdots,1}_{\ell})^T$.
7) $\mathbf{0}_{p\times q}$: a $p\times q$ matrix with zero entries.
8) A matrix $L\in\mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L are of the form of δ_m^k. That is, $\mathrm{Col}(L)\subset\Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m\times n$ logical matrices.
9) If $L\in\mathcal{L}_{n\times r}$, by definition it can be expressed as $L=[\delta_n^{i_1},\delta_n^{i_2},\cdots,\delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted as $L=\delta_n[i_1,i_2,\cdots,i_r]$.
10) $\mathrm{Span}\{A_1,\cdots,A_s\}$: The subspace spanned by $\{\mathrm{Col}(A_i)\mid i=1,\cdots,s\}$.
11) $U\uplus V$: direct sum of two vector spaces, *i.e.*, $U\cap V=\{0\}$.
12) $U\oplus V$: orthogonal sum of two vector spaces, *i.e.*, $u\perp v$, $\forall\, u\in U,\ v\in V$.

It is well known that a noncooperative strategic form finite game can be described as a triple (N,S,c), where

1) $N=\{1,2,\cdots,n\}$ is the set of players;
2) $S_i=\mathcal{D}_{k_i}$ is the set of strategies of player i, $i=1,\cdots,n$; $S=\prod_{i=1}^n S_i$ is called the set of strategy profiles;
3) $c=\{c_1,c_2,\cdots,c_n\}$, where $c_i:\ S\to\mathbb{R}$ is the payoff function of player i.

The semi-tensor product of matrices is defined as follows [4], [5]:

Definition 1.1: Let $M\in\mathcal{M}_{m\times n}$, $N\in\mathcal{M}_{p\times q}$, and $t=\mathrm{lcm}\{n,p\}$ be the least common multiple of n and p. The semi-tensor product (STP) of M and N is defined as

$$M\ltimes N:=\left(M\otimes I_{t/n}\right)\left(N\otimes I_{t/p}\right)\in\mathcal{M}_{mt/n\times qt/p},\qquad(1)$$

where $\otimes$ is the Kronecker product.

The STP of matrices is a generalization of conventional matrix product, and all the computational properties of conventional matrix product remain available. Throughout this paper, the default matrix product is STP, so the product of two arbitrary matrices is well defined, and the symbol $\ltimes$ is mostly omitted.

To use matrix expression to games, we identify

$$j\sim\delta_{k_i}^j,\quad j=1,\cdots,k_i,$$

then the set of strategies $S_i\sim\Delta_{k_i}$, $i=1,\cdots,n$. It follows that the payoff functions can be expressed as

$$c_i(x_1,\cdots,x_n)=V_i^c\ltimes_{j=1}^n x_j,\quad i=1,\cdots,n,\qquad(2)$$

where $V_i^c\in\mathbb{R}^k$ is a row vector called the structure vector of c_i, ($k=\prod_{i=1}^n k_i$). Define the structure vector of the given game G by

$$V_G^c=(V_1^c,V_2^c,\cdots,V_n^c)\in\mathbb{R}^{nk}.\qquad(3)$$

A brief version has been submitted to IEEE CDC 2015. This work is supported partly by the National Natural Science Foundation of China (NSFC) under Grants 61074114 and 61273013.

Daizhan Cheng, Ting Liu and Hongsheng Qi are with Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China. E-mail: dcheng@iss.ac.cn, tliu@amss.ac.cn, qihongsh@amss.ac.cn.

Kuize Zhang is with College of Automation, Harbin Engineering University, Harbin 150001, P.R.China e-mail: zkz0017@163.com.

Corresponding author: Daizhan Cheng. Tel.: +86 10 62651445; fax.: +86 10 62587343.

Then it is clear that the set of strategic form finite games with $|N| = n$, and $|S_i| = k_i$, $i = 1, \cdots, n$, denoted by $\mathcal{G}_{[n;k_1,\cdots,k_n]}$, has a natural vector space structure as

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} \sim \mathbb{R}^{nk}. \tag{4}$$

Note that for a given game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, its structure vector V_G^c completely determines G. So the vector space structure (4) is very natural and reasonable.

The vector space structure of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ was firstly proposed in [2], our statement is a little bit different from theirs. Moreover, [2] proposes three vector subspaces of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$: (i) the pure potential subspace $\mathcal{P}$; (ii) the non-strategic subspace $\mathcal{N}$; and (iii) the pure harmonic subspaces $\mathcal{H}$, such that $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ is decomposed as an orthogomal sum of $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{H}$:

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{Potential\quad games}\!\!\overbrace{\quad \oplus \quad \mathcal{H}}^{Harmonic\quad games} \quad . \tag{5}$$

It is also demonstrated in (5) that the pure potential subspace plus the non-strategic subspace is the subspace of potential games, denoted as $\mathcal{G}_P = \mathcal{P} \oplus \mathcal{N}$; and the pure harmonic subspace plus the non-strategic subspace is the subspace of harmonic games, denoted as $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$.

The main technical tool used in [2] is the Helmholtz decomposition theorem, a classical result from algebraic topology [20]. Then the three subspaces are defined as follows [2]

$$\begin{array}{lll} \mathcal{P} & := & \left\{u \in \mathbb{R}^{kn} \mid u = \Pi u, \text{ and } Du \in \operatorname{im} \delta_0\right\}; \\ \mathcal{H} & := & \left\{u \in \mathbb{R}^{kn} \mid u = \Pi u, \text{ and } Du \in \ker \delta_0^*\right\}; \\ \mathcal{N} & := & \left\{u \in \mathbb{R}^{kn} \mid u \in \ker D\right\}, \end{array} \tag{6}$$

where δ_0, Π, D are certain combinatorial operators. [2] is very well written and it can be understood without any knowledge of algebraic topology. But it may still not straightforward to engineering-background readers.

In this paper we define these subspaces in a linear algebraic framework, which shows their physical meanings clearly. Our approach also constructs the bases of $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{H}$ respectively. They are used to decompose any $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ numerically, and provide a tool for investigating the properties of subspaces. To show the usefulness of the algebraic framework, we consider evolutionary games. Three problems are investigated: First, the dynamic equivalence of evolutionary games is introduced. Then, we show that when a game is near a subspace, it may dynamically equivalent to a game in the subspace. Then the game has the same properties of the games in the subspaces. Particularly, the near potential games are investigated. It is shown that when a near potential game and its nearest potential game are dynamically equivalent, the near potential game can also converge to a pure Nash equilibrium, using asynchronous myopic best response adjustment, *etc*. Finally, the decomposition of an NEG is considered: We show that the decomposition of an NEG can be obtained by summarizing the decompositions of pairwise network games.

The rest of this paper is built up as follows: In Section 2 we discuss the potential subspace $\mathcal{G}_P$, its basis is provided. Section 3 considers the non-strategic subspace $\mathcal{N}$. It is defined alternatively and its basis is also revealed. The pure potential subspace $\mathcal{P}$ and the pure harmonic subspace are discussed in Sections 4 and 5 respectively, and their bases are also obtained. Section 6 is devoted to the orthogonal decomposition of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$. Section 7 considers the dynamic of evolutionary games (EGs). The dynamic equivalence of EGs is introduced first. The properties of near potential games are then discussed. and finally, the decomposition of NEGs is investigated.

II. Subspace of Potential Games

We first review some results in [6] with a mild and straightforward generalization. We need some notations:

- Let $|S_i| = k_i$, $i = 1, \cdots, n$. Then
$$k^{[p,q]} := \begin{cases} \prod_{j=p}^{q} k_j, & q \geq p \\ 1, & q < p. \end{cases}$$
- $E_i := I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \in \mathcal{M}_{k \times k/k_i}$, $i = 1, \cdots, n$. Note that $\mathbf{1}_k \in \mathbb{R}^k$ is a column vector with all entries equal 1; $I_s \in \mathcal{M}_{s \times s}$ is the identity matrix and $I_1 := 1$.

Construct a linear equation, called the potential equation, as

$$\begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix}, \tag{7}$$

where $\xi_i \in \mathbb{R}^{k/k_i}$. Then we have the following result:

Theorem 2.1 ([6]): A finite game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is a potential game, if and only if the potential equation (7) has solution. Moreover, if ξ is a solution then the potential function can be expressed as

$$P(x_1, \cdots, x_n) = V^P \ltimes_{j=1}^n x_j, \tag{8}$$

where V^P, the structure vector of the potential function, is

$$V^P = V_1^c - \xi_1^T E_1^T.$$

Denote

$$E := \begin{bmatrix} -E_1 & E_2 & 0 & \cdots & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \\ -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{9}$$

Theorem 2.1 tells us that G is potential if and only if

$$\begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \operatorname{Span}(E). \tag{10}$$

Observing that in (10) we have freedom to arbitrarily choose V_1^c, (10) can be rewritten as

$$\begin{bmatrix} (V_1^c)^T \\ (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix} \in \operatorname{Span}(E^e), \tag{11}$$

where

$$E^e = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.$$

Equivalently, we have

$$\begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix} \begin{bmatrix} (V_1^c)^T \\ (V_2^c)^T \\ (V_3^c)^T \\ \vdots \\ (V_n^c)^T \end{bmatrix} \in \mathrm{Span}(E^e). \tag{12}$$

That is

$$V_G^T \in \mathrm{Span}(E_P), \tag{13}$$

where

$$\begin{aligned} E_P \quad &:= \quad \begin{bmatrix} I_k & 0 & \cdots & 0 \\ -I_k & I_k & \cdots & 0 \\ \vdots & & \ddots & \\ -I_k & 0 & \cdots & I_k \end{bmatrix}^{-1} E^e \\ &= \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n \end{bmatrix}. \end{aligned} \tag{14}$$

As discussed in [6], deleting any one column of E the remaining columns form a basis of $\mathrm{Span}(E)$. Comparing E_P with E, we may delete the last column of E_n and denote the remaining part of E_n by E_n^0, and define

$$E_P^0 := \begin{bmatrix} I_k & 0 & 0 & 0 & \cdots & 0 \\ I_k & -E_1 & E_2 & 0 & \cdots & 0 \\ I_k & -E_1 & 0 & E_3 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ I_k & -E_1 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}.$$

Then we have

$$\mathrm{Span}(E_P) = \mathrm{Span}(E_P^0).$$

Moreover, it is easy to see that the columns of E_P^0 are linearly independent.

Summarizing the above argument, we have the following result:

Theorem 2.2: The subspace of potential games is

$$\mathcal{G}_P = \mathrm{Span}(E_P), \tag{15}$$

which has $\mathrm{Col}(E_P^0)$ as its basis.

According to the construction of E_P^0 it is clear that

Corollary 2.3: 1) The dimension of the subspace of potential games of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ is

$$\dim(\mathcal{G}_P) = k + \sum_{j=1}^{n} \frac{k}{k_j} - 1. \tag{16}$$

2) The dimension of the subspace $\mathcal{H}$ is

$$\dim(\mathcal{H}) = (n-1)k - \sum_{j=1}^{n} \frac{k}{k_j} + 1. \tag{17}$$

Remark 2.4: Equations (16) and (17) coincide with the result in [2].

III. Non-strategic Subspace

Definition 3.1: Let $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. G and $\tilde{G}$ are said to be strategically equivalent, if for any $i \in N$, any $x_i,\ y_i \in S_i$, and any $x^{-i} \in S^{-i}$, (where $S^{-i} = \prod_{j\neq i} S_j$), we have

$$c_i(x_i, x^{-i}) - c_i(y_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) - \tilde{c}_i(y_i, x^{-i}). \tag{18}$$

Note that the physical meaning of this definition is very clear: In G, no matter what policy a player will take for his strategy selection, the same strategy will be selected by his corresponding player in $\tilde{G}$, as long as the same policy is implemented.

Lemma 3.2: Two games $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ are strategically equivalent, if and only if for each $x^{-i} \in S^{-i}$ there exists $d_i(x^{-i})$ such that

$$\begin{aligned} &c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i}) = d_i(x^{-i}), \\ &\forall\, x_i \in S_i,\ \forall\, x^{-i} \in S^{-i},\ i = 1, \cdots, n. \end{aligned} \tag{19}$$

Proof: (Necessity) Assume (19) fails. Then there exist an i and an $x^{-i} \in S^{-i}$, such that $c_i(x_i, x^{-i}) - \tilde{c}_i(x_i, x^{-i})$ depends on x_i. That is, there exist $a_i, b_i \in S_i$ such that

$$c_i(a_i, x^{-i}) - \tilde{c}_i(a_i, x^{-i}) \neq c_i(b_i, x^{-i}) - \tilde{c}_i(b_i, x^{-i}).$$

Then

$$c_i(a_i, x^{-i}) - c_i(b_i, x^{-i}) \neq \tilde{c}_i(a_i, x^{-i}) - \tilde{c}_i(b_i, x^{-i}),$$

which violates (19).

(Sufficiency) From (19) we have

$$c_i(x_i, x^{-i}) = \tilde{c}_i(x_i, x^{-i}) + d_i(x^{-i}), \quad \forall\, x_i \in S_i.$$

Plugging it into left-hand side of (18) yields the equality. ■

Next, denote the structure vectors of c_i, $\tilde{c}_i$, and d_i by V_i^c, $\tilde{V}_i^c$, and V_i^d respectively, we express (19) into a matrix form as

$$\begin{aligned} V_i^c \ltimes_{j=1}^n x_j - \tilde{V}_i^c \ltimes_{j=1}^n x_j &= V_i^d \ltimes_{j\neq i}^n x_j \\ &= V_i^d \left(I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i}^T \otimes I_{k^{[i+1,n]}} \right) \ltimes_{j=1}^n x_j. \end{aligned}$$

Finally, we have

$$B_N^i (V_i^d)^T = (V_i^c - \tilde{V}_i^c)^T, \tag{20}$$

where

$$\begin{aligned} B_N^i &:= I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \\ &= E_i, \quad i = 1, \cdots, n. \end{aligned} \tag{21}$$

We conclude that

Theorem 3.3: G and $\tilde{G}$ are strategically equivalent if and only if

$$\left(V_G^c - V_{\tilde{G}}^c \right)^T \in \mathrm{Span}\left(B_N \right), \tag{22}$$

where

$$B_N = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{23}$$

Definition 3.4: The subspace

$$\mathcal{N} := \mathrm{Span}(B_N)$$

is called the non-strategic subspace.

From Theorem 3.3 one sees easily that G and $\tilde{G}$ are strategically equivalent if and only if there exists an $\eta \in \mathcal{N}$, such that

$$\left(V_{\tilde{G}}^c\right)^T = (V_G^c)^T + \eta. \tag{24}$$

Since E_i has k/k_i columns, which are linearly independent, $i = 1, \cdots, n$, we conclude that

Corollary 3.5: 1) The dimension of $\mathcal{N}$ is

$$\dim(\mathcal{N}) = \sum_{i=1}^{n} \frac{k}{k_i}. \tag{25}$$

2) The dimension of $\mathcal{P}$ is

$$\dim(\mathcal{P}) = k - 1. \tag{26}$$

Remark 3.6: Equations (25) and (26) also coincide with the result in [2].

Define

$$\tilde{E}_P := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}. \tag{27}$$

Comparing (27) with (14), it is ready to verify that

$$\mathcal{G}_P = \mathrm{Span}\left(\tilde{E}_P\right) = \mathrm{Span}\left(E_P\right). \tag{28}$$

Deleting the last column of $\tilde{E}_P$, (equivalently, replacing the E_n in $\tilde{E}_P$ by E_n^0), the remaining matrix is denoted as

$$\tilde{E}_P^0 := \begin{bmatrix} I_k & E_1 & 0 & 0 & \cdots & 0 \\ I_k & 0 & E_2 & 0 & \cdots & 0 \\ I_k & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ I_k & 0 & 0 & 0 & \cdots & E_n^0 \end{bmatrix}. \tag{29}$$

Then it is clear that $\mathrm{Col}\left(\tilde{E}_P^0\right)$ is a basis of $\mathcal{G}_P$.

Observing (27) again, it follows immediately that

Corollary 3.7: The subspace $\mathcal{N}$ is a linear subspace of $\mathcal{G}_P$. That is,

$$\mathcal{N} \subset \mathcal{G}_P.$$

For a $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, if its payoff vector $(V_G^c)^T \in \mathcal{N}$ then G is called a non-strategic game. According to the construction of $B_{\mathcal{N}}$, the following is obvious.

Theorem 3.8: If G is a non-strategic game, then for each given $x^{-i} \in S^{-i}$

$$c_i(x_i, x^{-i}) = \text{const.}, \quad \forall x_i \in S_i. \tag{30}$$

Example 3.9: Given a finite game with $N = \{1, 2\}$, $S_1 = \{1, 2\}$, $S_2 = \{1, 2, 3\}$. G is a non-strategic game, if and only if

$$(V_G^c)^T \in \mathcal{N}.$$

Equivalently,

$$(V_i^c)^T \in \mathrm{Span}(E_i), \quad i = 1, 2.$$

Since

$$E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} V_1^c &= a[1\ 0\ 0\ 1\ 0\ 0] + b[0\ 1\ 0\ 0\ 1\ 0] + c[0\ 0\ 1\ 0\ 0\ 1] \\ &= [a\ b\ c\ a\ b\ c]. \end{aligned}$$

Similarly, we have

$$E_2 = I_2 \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} V_2^c &= d[1\ 1\ 1\ 0\ 0\ 0] + e[0\ 0\ 0\ 1\ 1\ 1] \\ &= [d\ d\ d\ e\ e\ e]. \end{aligned}$$

Putting them into a payoff bi-matrix, we have

TABLE I
PAYOFF BI-MATRIX OF EXAMPLE 3.9

$P_1 \backslash P_2$	1	2	3
1	$a,\ d$	$b,\ d$	$c,\ d$
2	$a,\ e$	$b,\ e$	$c,\ e$

From Table I one sees easily that

(i) $\dim(\mathcal{N}) = 5$, which verifies (25);

(ii) As long as $x_2 \in S_2$ ($x_1 \in S_1$) is fixed, the payoff of P_1 (P_2) is constant, no matter what strategy is chosen.

IV. PURE POTENTIAL SUBSPACE $\mathcal{P}$

Using (28) we have

$$\begin{aligned} \mathcal{G}_P &= \mathrm{Span}(\tilde{E}_P) \\ &= \mathrm{Span} \begin{bmatrix} I_k - \frac{1}{k_1}E_1E_1^T & E_1 & 0 & 0 & \cdots & 0 \\ I_k - \frac{1}{k_2}E_2E_2^T & 0 & E_2 & 0 & \cdots & 0 \\ I_k - \frac{1}{k_3}E_3E_3^T & 0 & 0 & E_3 & \cdots & 0 \\ \vdots & & & & \ddots & \\ I_k - \frac{1}{k_n}E_nE_n^T & 0 & 0 & 0 & \cdots & E_n \end{bmatrix}. \end{aligned} \tag{31}$$

Define an auxiliary space as

$$\mathcal{V} := \mathrm{Span}\left(B_P\right),$$

where

$$B_P = \begin{bmatrix} I_k - \frac{1}{k_1}E_1E_1^T \\ I_k - \frac{1}{k_2}E_2E_2^T \\ \vdots \\ I_k - \frac{1}{k_n}E_nE_n^T \end{bmatrix} \in \mathcal{M}_{nk \times k}. \tag{32}$$

Then it is obvious that

$$B_P^T B_N = 0.$$

Hence we have $\mathcal{G}_P = \mathcal{V} \oplus \mathcal{N}$. That is,

$$\mathcal{P} = \mathcal{V} = \operatorname{Span}(B_P). \tag{33}$$

Since $\dim(\mathcal{P}) = k-1$, to find the basis of $\mathcal{P}$ one column of V needs to be removed. Note that

$$\begin{aligned}
&\left(I_k - \frac{1}{k_i} E_i E_i^T\right) \mathbf{1}_k \\
=&\left(I_{k^{[1,i-1]}} \mathbf{1}_{k^{[1,i-1]}}\right) \left[\left(I_{k_i} - \frac{1}{k_i} \mathbf{1}_{k_i \times k_i}\right) \mathbf{1}_{k_i}\right] \\
&\left(I_{k^{[i+1,n]}} \mathbf{1}_{k^{[i+1,n]}}\right) \\
=&0, \quad i = 1, \cdots, n.
\end{aligned}$$

It follows that

$$V \mathbf{1}_{nk} = 0.$$

Deleting any one column of B_P, say, the last column, and denoting the remaining matrix by B_P^0, then we know that

$$\mathcal{P} = \operatorname{Span}(B_P) = \operatorname{Span}\left(B_P^0\right),$$

where B_P^0 is a basis of $\mathcal{P}$.

V. Pure Harmonic Subspace $\mathcal{H}$

Observing (28) again, and denoting

$$\psi_n := \tilde{E}_P^T,$$

it is clear that

$$\mathcal{H} = \mathcal{G}_P^{\perp} = \ker(\psi_n).$$

First, assume $n = 2$. Then we have

$$\psi_2 = \begin{bmatrix} I_k & I_k \\ E_1^T & 0 \\ 0 & E_2^T \end{bmatrix}.$$

Setting

$$\begin{aligned}
x_{i_1,i_2} := &\begin{bmatrix} \left(\delta_{k_1}^1 - \delta_{k_1}^{i_1}\right) \left(\delta_{k_2}^1 - \delta_{k_2}^{i_2}\right) \\ \left(\delta_{k_1}^{i_1} - \delta_{k_1}^1\right) \left(\delta_{k_2}^1 - \delta_{k_2}^{i_2}\right) \end{bmatrix} \\
& i_1 = 2, 3, \cdots, k_1; i_2 = 2, 3, \cdots, k_2.
\end{aligned} \tag{34}$$

Then it is easy to see that

$$x_{i_1,i_2} \in \ker(\psi_2), \quad i_1 = 2, 3, \cdots, k_1; i_2 = 2, 3, \cdots, k_2,$$

and $\{x_{i_1,i_2} \mid i_1 = 2, 3, \cdots, k_1; i_2 = 2, 3, \cdots, k_2\}$ are linearly independent. Hence, taking the dimension $\dim(\mathcal{H}_2) = (k_1 - 1)(k_2 - 1)$ into consideration, they form a basis of $\mathcal{H}_2$.

Next we would like to construct $\mathcal{H}$ inductively with respect to n. So we consider $n = \mu$, and $k = k^{[1,\mu]} = \prod_{i=1}^{\mu} k_i$. It is easy to have the following recursive form of ψ_s as

$$\psi_s = \begin{pmatrix} \psi_{s-1} \otimes I_{k_s} & \beta \\ \mathbf{0}_{k^{[1,s-1]} \times (s-1)k^{[1,s]}} & I_{k^{[1,s-1]}} \otimes \mathbf{1}_{k_s}^T \end{pmatrix}, \tag{35}$$

where

$$\beta = [I_{k \times k}, 0_{k \times \frac{k}{k_1}}, \cdots, 0_{k \times \frac{k}{k_{s-1}}}]^T.$$

It is ready to verify the following claims by straightforward computations:

Lemma 5.1: If $x \in \ker(\psi_{s-1})$, then

$$\begin{bmatrix} x \otimes \delta_{k_s}^{i_s} \\ \mathbf{0}_{k^{[1,s]}} \end{bmatrix} \in \ker(\psi_s), \quad i_s = 1, \cdots, k_s; \tag{36}$$

and

$$\begin{aligned}
&\begin{bmatrix}
(\delta_{k_1}^1 - \delta_{k_1}^{i_1}) \delta_{k_2}^1 \delta_{k_3}^1 \cdots \delta_{k_{s-1}}^1 \\
\delta_{k_1}^{i_1} (\delta_{k_2}^1 - \delta_{k_2}^{i_2}) \delta_{k_3}^1 \cdots \delta_{k_{s-1}}^1 \\
\delta_{k_1}^{i_1} \delta_{k_2}^{i_2} (\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \cdots \delta_{k_{s-1}}^1 \\
\vdots \\
\delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \delta_{k_3}^{i_3} \cdots (\delta_{k_{s-1}}^1 - \delta_{k_{s-1}}^{i_{s-1}}) \\
\delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \cdots \delta_{k_{s-1}}^{i_{s-1}} - \delta_{k_1}^1 \delta_{k_2}^1 \cdots \delta_{k_{s-1}}^1
\end{bmatrix} \otimes (\delta_{k_s}^1 - \delta_{k_s}^{i_s}) \in \ker(\psi_s) \\
&\qquad i_j = 1, \cdots, k_j; \ j = 1, 2, \cdots, s.
\end{aligned} \tag{37}$$

According to Lemma 5.1 and starting from (34), we can construct a set of vectors, which are in $\ker(\psi_n)$ as

$$J_1 := \left\{ \begin{matrix} \begin{bmatrix} (\delta_{k_1}^1 - \delta_{k_1}^{i_1})(\delta_{k_2}^1 - \delta_{k_2}^{i_2}) \delta_{k_3}^{i_3} \cdots \delta_{k_n}^{i_n} \\ -(\delta_{k_1}^1 - \delta_{k_1}^{i_1})(\delta_{k_2}^1 - \delta_{k_2}^{i_2}) \delta_{k_3}^{i_3} \cdots \delta_{k_n}^{i_n} \\ \mathbf{0}_{(n-2)k} \end{bmatrix} \\ i_1 \neq 1, i_2 \neq 1 \end{matrix} \right\};$$

$$J_2 := \left\{ \begin{matrix} \begin{bmatrix} (\delta_{k_1}^1 - \delta_{k_1}^{i_1}) \delta_{k_2}^1 (\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \delta_{k_4}^{i_4} \cdots \delta_{k_n}^{i_n} \\ \delta_{k_1}^{i_1} (\delta_{k_2}^1 - \delta_{k_2}^{i_2})(\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \delta_{k_4}^{i_4} \cdots \delta_{k_n}^{i_n} \\ -(\delta_{k_1}^1 \delta_{k_2}^1 - \delta_{k_1}^{i_1} \delta_{k_2}^{i_2})(\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \delta_{k_4}^{i_4} \cdots \delta_{k_n}^{i_n} \\ \mathbf{0}_{(n-3)k} \end{bmatrix} \\ (i_1, i_2) \neq \mathbf{1}_2^T; i_3 \neq 1 \end{matrix} \right\};$$

$$\vdots$$

$$J_s := \left\{ \begin{matrix} \begin{bmatrix} (\delta_{k_1}^1 - \delta_{k_1}^{i_1}) \delta_{k_2}^1 \delta_{k_3}^1 \cdots \delta_{k_s}^1 (\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}) \delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \delta_{k_1}^{i_1} (\delta_{k_2}^1 - \delta_{k_2}^{i_2}) \delta_{k_3}^1 \cdots \delta_{k_s}^1 (\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}) \delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} (\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \cdots \delta_{k_s}^1 (\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}) \delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \vdots \\ \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \delta_{k_3}^{i_3} \cdots (\delta_{k_s}^1 - \delta_{k_s}^{i_s})(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}) \cdots \delta_{k_n}^{i_n} \\ -(\delta_{k_1}^1 \cdots \delta_{k_s}^1 - \delta_{k_1}^{i_1} \cdots \delta_{k_s}^{i_s})(\delta_{k_{s+1}}^1 - \delta_{k_{s+1}}^{i_{s+1}}) \delta_{k_{s+2}}^{i_{s+2}} \cdots \delta_{k_n}^{i_n} \\ \mathbf{0}_{(n-1-s)k} \end{bmatrix} \\ (i_1, \cdots, i_s) \neq \mathbf{1}_{i_s}^T; i_{s+1} \neq 1 \end{matrix} \right\};$$

$$\vdots$$

$$J_{n-1} := \left\{ \begin{matrix} \begin{bmatrix} (\delta_{k_1}^1 - \delta_{k_1}^{i_1}) \delta_{k_2}^1 \delta_{k_3}^1 \delta_{k_4}^1 \cdots \delta_{k_{n-1}}^1 (\delta_{k_n}^1 - \delta_{k_n}^{i_n}) \\ \delta_{k_1}^{i_1} (\delta_{k_2}^1 - \delta_{k_2}^{i_2}) \delta_{k_3}^1 \delta_{k_4}^1 \cdots \delta_{k_{n-1}}^1 (\delta_{k_n}^1 - \delta_{k_n}^{i_n}) \\ \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} (\delta_{k_3}^1 - \delta_{k_3}^{i_3}) \delta_{k_4}^1 \cdots \delta_{k_{n-1}}^1 (\delta_{k_n}^1 - \delta_{k_n}^{i_n}) \\ \vdots \\ \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \delta_{k_3}^{i_3} \delta_{k_4}^{i_4} \cdots (\delta_{k_{n-1}}^1 - \delta_{k_{n-1}}^{i_{n-1}})(\delta_{k_n}^1 - \delta_{k_n}^{i_n}) \\ -(\delta_{k_1}^1 \delta_{k_2}^1 \cdots \delta_{k_{n-1}}^1 - \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \cdots \delta_{k_{n-1}}^{i_{n-1}})(\delta_{k_n}^1 - \delta_{k_n}^{i_n}) \end{bmatrix} \\ (i_1, \cdots, i_{n-1}) \neq \mathbf{1}_{n-1}^T; i_n \neq 1 \end{matrix} \right\}.$$

Define

$$B_H := [J_1, J_2, \cdots, J_{n-1}]. \tag{38}$$

Then we can show B_H is the basis of $\mathcal{H}$:

Theorem 5.2: B_H has full column rank and

$$\mathcal{H} = \operatorname{Span}(B_H). \tag{39}$$

To prove Theorem 5.2, we first give the following lemma.

Lemma 5.3: Given positive integers $p, q, r > 1$. Then the vectors $(\delta_p^1 - \delta_p^{i_p})(\delta_q^1 - \delta_q^{i_q})\delta_s^{i_s}$, $1 < i_p \leq p$, $1 < i_q \leq q$, $1 \leq i_s \leq s$, are linearly independent.

Proof: Let

$$\sum_{1<i_p\leq p, 1<i_q\leq q, 1\leq i_s\leq s} a_{i_p,i_q,i_s}(\delta_p^1 - \delta_p^{i_p})(\delta_q^1 - \delta_q^{i_q})\delta_s^{i_s} = 0,$$

where for all $1 < i_p \leq p, 1 < i_q \leq q, 1 \leq i_s \leq s$, a_{i_p,i_q,i_s} are real numbers.

Note that for all $1 \leq j \leq pq$, all distinct $1 \leq j_1, j_2 \leq s$, if the l-th entry of $\delta_{pq}^j \delta_s^{j_1}$ does not equal 0, then the l-th entry of $\delta_{pq}^j \delta_s^{j_2}$ must equal 0. Hence for all $1 \leq i_s \leq s$,

$$\begin{aligned}&\sum_{1<i_p\leq p, 1<i_q\leq q} a_{i_p,i_q,i_s}(\delta_p^1 - \delta_p^{i_p})(\delta_q^1 - \delta_q^{i_q}) = \\ &\sum_{1<i_p\leq p, 1<i_q\leq q} a_{i_p,i_q,i_s}(\delta_p^1\delta_q^1 - \delta_p^1\delta_q^{i_q} - \delta_p^{i_p}\delta_q^1 + \delta_p^{i_p}\delta_q^{i_q}) = 0.\end{aligned}$$

It is obviously that $\delta_p^1\delta_q^1, \delta_p^1\delta_q^{i_q}, \delta_p^{i_p}\delta_q^1, \delta_p^{i_p}\delta_q^{i_q}$, $1 < i_p \leq p$, $1 < i_q \leq q$, are exactly all columns of I_{pq}, hence linearly independent. Then for all $1 < i_p \leq p$, all $1 < i_q \leq q$, all $1 \leq i_s \leq s$, $a_{i_p,i_q,i_s} = 0$. ∎

Using Lemma 5.3, we prove Theorem 5.2.

Proof of Theorem 5.2: In J_1 only when $i_1 = 1$ or $i_2 = 1$ we get zero vectors. From Lemma 5.3, J_1 are linearly independent vectors. The cardinality of J_1 is

$$|J_1| = \frac{k}{k_1k_2}(k_1 - 1)(k_2 - 1).$$

In J_2 only when $(i_1, i_2) = (1, 1)$ or $i_3 = 1$ we have the third block is zero. And when the third block is zero, the first two blocks must be zero. From Lemma 5.3, the third block of J_2 are linearly independent vectors. Then J_2 are linearly independent vectors, and J_1, J_2 are also linearly independent vectors.

The cardinality of J_2

$$|J_2| = \frac{k}{k_1k_2k_3}(k_1k_2 - 1)(k_3 - 1).$$

In general, for all $s = 1, \ldots, n-1$,

$$|J_s| \quad = \quad \tfrac{k}{k_1k_2\cdots k_{s+1}}(k_1k_2\cdots k_s - 1)(k_{s+1} - 1),$$

from Lemma 5.3, the s-th block of J_{s-1} are linearly independent vectors, then J_{s-1} are linearly independent vectors, and $J_1, J_2, \ldots, J_{s-1}$ are also linearly independent vectors.

Hence $J_1, J_2, \ldots, J_{n-1}$ are $\sum_{s=1}^{n-1}|J_s|$ linearly independent vectors.

Since

$$\sum_{i=1}^{n-1}|J_s| = (n-1)k - \sum_{i=1}^{n}\frac{k}{k_i} + 1,$$

which is the dimension of $\mathcal{H}$, we conclude that $\mathcal{H} = \mathrm{Span}(B_H)$. ∎

VI. Orthogonal Decomposition of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$

This section considers the numerical computation of decomposition (5).

A. Cascading Decomposition

The following lemma is well known from linear algebra:

Lemma 6.1: Let $S \subset V$ be a subspace of V, with $\mathrm{Col}\,(B_s)$ as its basis. Then for any $w \in V$,

$$\pi_S(w) = B_s(B_s^TB_s)^{-1}B_s^Tw \in S, \tag{40}$$

is the projection of w on S. Moreover, $w = [w - P_S(w)] \oplus P_s(w)$ is the orthogonal decomposition of w.

In previous section, the bases of $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{H}$ are obtained as B_P^0, B_N, and B_H respectively. Using Lemma 6.1, for any $w \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, its projections $\pi_{\mathcal{P}}(w)$, $\pi_{\mathcal{N}}(w)$, and $\pi_{\mathcal{H}}(w)$ can be obtained easily.

In fact, to get an orthogonal decomposition $V = S_1 \oplus S_2 \oplus \cdots \oplus S_k$, we need only a set of cascading bases: B^i, the basis of $S_1 \oplus S_2 \oplus \cdots \oplus S_i$, $i = 1, \cdots, k-1$. Hence, to get all the decomposed components of $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, we need only the basis of $\mathcal{G}_P$ (which is $\tilde{E}_P^0$) and the basis of $\mathcal{N}$ (which is B_N). That is,

Theorem 6.2: Assume the structure vector of $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is

$$V_G^c = [V_1^c, V_2^c, \cdots, V_n^c].$$

Then

1) its $\mathcal{G}_P$ projection is:

$$\pi_{\mathcal{G}_P}(G) = \tilde{E}_P^0\left((\tilde{E}_P^0)^T\tilde{E}_P^0\right)^{-1}(\tilde{E}_P^0)^T(V_G^c)^T; \tag{41}$$

2) its $\mathcal{N}$ projection is:

$$\pi_{\mathcal{N}}(G) = B_N\left(B_N^TB_N\right)^{-1}B_N^T(V_G^c)^T; \tag{42}$$

3) its $\mathcal{H}$ projection is:

$$\pi_{\mathcal{H}}(G) = (V_G^c)^T - \pi_{\mathcal{G}_P}(G); \tag{43}$$

4) its $\mathcal{P}$ projection is:

$$\pi_{\mathcal{P}}(G) = \pi_{\mathcal{G}_P}(G) - \pi_{\mathcal{N}}(G); \tag{44}$$

5) its $\mathcal{G}_H$ projection is:

$$\pi_{\mathcal{G}_H}(G) = \pi_{\mathcal{N}}(G) + \pi_{\mathcal{H}}(G). \tag{45}$$

B. Parallel Decomposition

Construct a matrix

$$B := \left[B_P^0, B_N, B_H\right], \tag{46}$$

and set

$$\begin{aligned}&d_1 = \dim(\mathcal{P}) = k - 1,\\ &d_2 = \dim(\mathcal{N}) = \sum_{j=1}^{n}k/k_j,\\ &d_3 = \dim(\mathcal{H}) = (n-1)k - \sum_{j=1}^{n}k/k_j + 1.\end{aligned}$$

Then the following result is obvious:

Proposition 6.3: Let

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} := B^{-1}(V_G^c)^T, \tag{47}$$

where $x_i \in \mathbb{R}^{d_i}$, $i = 1, 2, 3$. Then

$$\pi_{\mathcal{P}}(G) = B \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}; \pi_{\mathcal{N}}(G) = B \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}; \pi_{\mathcal{H}}(G) = B \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix};$$

$$\pi_{\mathcal{G}_P}(G) = B \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}; \pi_{\mathcal{G}_H}(G) = B \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}.$$

C. Decomposition with Weights

If the inner product of $\mathbb{R}^{nk}$ is defined by

$$\langle X, Y \rangle := X^T Q Y, \tag{48}$$

where $Q \in \mathcal{M}_{nk \times nk}$ is a positive definite matrix, the formula (40) becomes

$$\pi_S(w) = B_s (B_s^T Q B_s)^{-1} B_s^T Q w \in S. \tag{49}$$

Therefore (41) becomes

$$V_{\mathcal{G}_P}^T = \tilde{E}_P^0 \left((\tilde{E}_P^0)^T Q \tilde{E}_P^0 \right)^{-1} (\tilde{E}_P^0)^T Q \left(V_G^c\right)^T; \tag{50}$$

formula (42) becomes

$$V_{\mathcal{N}}^T = B_N \left(B_N^T Q B_N\right)^{-1} B_N^T Q \left(V_G^c\right)^T. \tag{51}$$

Formulas (43)-(45) remain available.

In [2] the inner product is defined as in (48) with

$$Q = \operatorname{diag}\left(\underbrace{k_1, \cdots, k_1}_{k}, \underbrace{k_2, \cdots, k_2}_{k}, \cdots, \underbrace{k_n, \cdots, k_n}_{k} \right).$$

In this case, it is easy to verify that we can keep $\mathcal{G}_p$, $\mathcal{P}$ and $\mathcal{N}$ as before, i.e., with $\tilde{E}_P^0$, B_P^0, and B_N as their bases respectively. But to assure Q-orthogonal, the basis of the harmonic subspace, denoted as $\mathcal{H}_Q$, should be

$$B_{H_Q} = Q^{-1} B_H.$$

Hence, the harmonic subspace $\mathcal{H}_Q$, defined in [2], is not the same as $\mathcal{H}$, defined in this paper.

D. Some Examples

This subsection presents some examples.

Example 6.4: 1) Consider a game $G \in \mathcal{G}_{[2;2,2]}$. Its payoffs are as in Table II.

TABLE II
PAYOFF BI-MATRIX FOR 1 OF EXAMPLE 6.4

$P_1 \backslash P_2$	1	2
1	$a_1,\ b_1$	$a_2,\ b_2$
2	$a_3,\ b_3$	$a_4,\ b_4$

Then we have

$$V_G^c = [a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4].$$

$$E_1 = \mathbf{1}_2 \otimes I_2; \qquad E_2 = I_2 \otimes \mathbf{1}_2.$$

$$\tilde{E}_P^0 = \begin{bmatrix} I_4 & E_1 & 0 \\ I_4 & 0 & E_2^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$B_N = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the Battle of Sex, where $a_1 = 2$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, $b_1 = 1$, $b_2 = 0$, $b_3 = 0$, $b_4 = 2$ [10]. Using formulas (41)-(45), we have

$$\begin{aligned} \pi_{\mathcal{H}}(G) &= [0, 0, 0, 0, 0, 0, 0, 0]; \\ \pi_{\mathcal{N}}(G) &= [1, 0.5, 1, 0.5, 0.5, 0.5, 1, 1]; \\ \pi_{\mathcal{P}}(G) &= [1, -0.5, -1, 0.5, 0.5, -0.5, -1, 1]; \\ \pi_{\mathcal{G}_P}(G) &= [2, 0, 0, 1, 1, 0, 0, 2]; \\ \pi_{\mathcal{G}_H}(G) &= [1, 0.5, 1, 0.5, 0.5, 0.5, 1, 1]. \end{aligned}$$

It is clear that this is a potential game.

2) Consider the Rock-Scissors-Paper game $G \in \mathcal{G}_{[2;3,3]}$, which has the payoff bi-matrix as in Table III.

TABLE III
PAYOFF BI-MATRIX FOR (2) OF EXAMPLE 6.4

$P_1 \backslash P_2$	R	S	P
R	$0,\ 0$	$1, -1$	$-1,\ 1$
S	$-1,\ 1$	$0,\ 0$	$1, -1$
P	$1, -1$	$-1,\ 1$	$0,\ 0$

Then we have

$$\begin{aligned} V_G^c = [&0, 1, -1, -1, 0, 1, 1, -1, 0, \\ &0, -1, 1, 1, 0, -1, -1, 1, 0]; \end{aligned}$$

$$E_1 = \mathbf{1}_3 \otimes I_3 = \begin{bmatrix} I_3 \\ I_3 \\ I_3 \end{bmatrix};$$

$$E_2 = I_3 \otimes \mathbf{1}_3 = \begin{bmatrix} \mathbf{1}_3 & 0 & 0 \\ 0 & \mathbf{1}_3 & 0 \\ 0 & 0 & \mathbf{1}_3 \end{bmatrix}.$$

It follows that

$$B_N = \begin{bmatrix} I_3 & 0 & 0 & 0 \\ I_3 & 0 & 0 & 0 \\ I_3 & 0 & 0 & 0 \\ 0 & \mathbf{1}_3 & 0 & 0 \\ 0 & 0 & \mathbf{1}_3 & 0 \\ 0 & 0 & 0 & \mathbf{1}_3 \end{bmatrix}; \ B_N^0 = \begin{bmatrix} I_3 & 0 & 0 \\ I_3 & 0 & 0 \\ I_3 & 0 & 0 \\ 0 & \mathbf{1}_3 & 0 \\ 0 & 0 & \mathbf{1}_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\tilde{E}_P^0 = \begin{bmatrix} \begin{pmatrix} I_9 \\ I_9 \end{pmatrix} & B_{\mathcal{N}}^0 \end{bmatrix}.$$

Using formulas (41)-(45), we have

$$\begin{aligned}\pi_{\mathcal{G}_P}(G) &= \pi_{\mathcal{N}}(G) = \pi_{\mathcal{P}}(G) = 0;\\ \pi_{\mathcal{H}}(G) &= \pi_{\mathcal{G}_H}(G) = [0,1,-1,-1,0,1,1,-1,0,\\ &\qquad 0,-1,1,1,0,-1,-1,1,0].\end{aligned}$$

This is a pure harmonic game.

The following example is from [9].

Example 6.5: Consider a finite game $G \in \mathcal{G}_{[2;3,2]}$. $S_1 = \{1(U), 2(M), 3(D)\}$ and $S_2 = \{1(L), 2(R)\}$ are the strategies of player 1 and 2 respectively. Its payoff is as Table IV.

TABLE IV
PAYOFF BI-MATRIX OF EXAMPLE 6.5.

$P_1\backslash P_2$	$L=1$	$R=2$
$U=1$	$1,\ 3$	$-3,\ 0$
$M=2$	$-2,\ 0$	$1,\ 3$
$D=3$	$0,\ 1$	$0,\ 1$

Then we have

$$V_G = [1,-3,-2,1,0,0,3,0,0,3,1,1].$$

Using formulas (23), (32), (38), the bases of pure potential subspace $\mathcal{P}$, non-strategic subspace $\mathcal{N}$, and the harmonic subspace $\mathcal{H}$ are calculated as

$$B_P^0 = \begin{pmatrix} 0.67 & 0.00 & -0.33 & 0.00 & 0.00\\ 0.00 & 0.67 & 0.00 & -0.33 & -0.33\\ -0.33 & 0.00 & 0.67 & 0.00 & 0.00\\ 0.00 & -0.33 & 0.00 & 0.67 & 0.67\\ -0.33 & 0.00 & -0.33 & 0.00 & 0.00\\ 0.00 & -0.33 & 0.00 & -0.33 & -0.33\\ 0.50 & -0.50 & 0.00 & 0.00 & 0.00\\ -0.50 & 0.50 & 0.00 & 0.00 & 0.00\\ 0.00 & 0.00 & 0.50 & -0.50 & -0.50\\ 0.00 & 0.00 & -0.50 & 0.50 & 0.50\\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00\\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{pmatrix};$$

$$B_N = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&1 \end{pmatrix}; B_H = \begin{pmatrix} 1&1\\ -1&-1\\ -1&0\\ 1&0\\ 0&-1\\ 0&1\\ -1&-1\\ 1&1\\ 1&0\\ -1&0\\ 0&1\\ 0&-1 \end{pmatrix}.$$

According to Proposition 6.3, we can conclude the following decomposition

$$(V_G^c)^T = B_P^0 x_1 + B_N x_2 + B_H x_3$$

where the coefficients can be calculated by (47) as

$$\begin{aligned} x_1 &= [0.58, -2.75, -2.17, 1.00, -0.17]^T,\\ x_2 &= [-0.33, -0.67, 1.50, 1.50, 1.00]^T,\\ x_3 &= [0.08, 0.08]^T. \end{aligned}$$

Hence,

$$\begin{aligned}\pi_{\mathcal{P}}(G) = [&1.17, -2.17, -1.58, 1.58, 0.42, 0.58\\ &1.67, -1.67, -1.58, 1.58, -0.08, -0.08],\\ \pi_{\mathcal{N}}(G) = [&-0.33, -0.67, -0.33, -0.67, -0.33, -0.67\\ &1.50, 1.50, 1.50, 1.50, 1.00, 1.00],\\ \pi_{\mathcal{H}}(G) = [&0.17, -0.17, -0.08, 0.08, -0.08, 0.08\\ &-0.17, 0.17, 0.08, -0.08, 0.08, 0.08].\end{aligned}$$

We can get the same results using Theorem 6.2.

VII. Application to Evolutionary Games

A. Evolutionary Games

Assume a noncooperative finite game in strategic form is repeated infinitely. Then each player can update his strategy by using the game historical knowledge. Assume the strategy updating rule (SUR) can be expressed as

$$x_i(t+1) = f_i\left(x_j(s), c_j(s) \mid j \in N, s = 0, 1, \cdots, t\right), \quad i = 1, \cdots, n. \tag{52}$$

There are some commonly used SURs, including (i) Unconditional imitation [16]; (ii) Fermi rule [19], [21]; (iii) Myopic best response adjustment (MBRA) [24]. These three SURs are mostly useful for networked evolutionary games, because they have an advantage that only the previous moment information is used to update the strategies. That is, using such SURs (52) becomes [7], [8]

$$x_i(t+1) = f_i(x_1(t)), \cdots, x_n(t)), \quad i = 1, \cdots, n. \tag{53}$$

(53) (or (52)) is called the profile dynamics of an evolutionary game.

For our purpose, we introduce MBRA in detail: Assume

$$\begin{aligned} S_i^* &:= \left\{ z^* \mid c_i\left(x_i = z^*, x^{-i} = x^{-i}(t)\right)\right.\\ &\qquad \left. = \max_{z\in S_i} c_i\left(x_i = z, x^{-i} = x^{-i}(t)\right)\right\} \\ &:= \{z_1^*, \cdots, z_r^*\}. \end{aligned} \tag{54}$$

We may use the following 2 options:

(i) Deterministic MBRA (D-MBRA): Choose one corresponding to a priority. For instance, (as a default)

$$x_i(t+1) = \min\{z_j^* \in S_i^*\}. \tag{55}$$

This method leads to a deterministic multi-valued logical dynamics.

(ii) Stochastic MBRA (S-MBRA): Choose any $z_j^* \in S_i^*$ with equal probability. That is,

$$x_i(t+1) = z_j^*(t), \quad \text{with probability } p_\mu^i = \tfrac{1}{r}, \quad \mu = 1, \cdots, r. \tag{56}$$

This method leads to a probabilistic multi-valued logical dynamics.

Then the SUR can determine the profile dynamics of evolutionary games.

We use an example to illustrate it. The following example is from [3]:

TABLE V
PAYOFF MATRIX OF EXAMPLE 7.1

c\s	11	12	13	21	22	23	31	32	33
c_1	90	−12	48	−12	1	24	48	24	1
c_2	90	−12	48	−12	−1	24	48	24	−1

Example 7.1: A game $G \in \mathcal{G}_{[2;3,3]}$, where $S_1 = S_2 = \{1, 2, 3\}$: 1: work; 2: shirk at office; 3: shirk at home. The payoffs are described in the payoff matrix (Table V).

Using MBRA, we can get the best responding strategies, which are shown in Table VI. (Since we have $|S_1^*| = |S_2^*| = 1$, D-MBRA and S-MBRA lead to the same result.)

TABLE VI
RESPONDING STRATEGIES OF EXAMPLE 7.1

s(t+1)\s(t)	11	12	13	21	22	23	31	32	33
f_1	1	3	1	1	3	1	1	3	1
f_2	1	1	1	3	3	3	1	1	1

That is,

$$x_i(t+1) = f_i(x_1(t), x_2(t)) = M_i x(t), \quad i = 1, 2, \tag{57}$$

where $x(t) = \ltimes_{i=1}^2 x_i(t)$, M_i, $i = 1, 2$ are the structure matrices of f_i, which are

$$\begin{aligned} M_1 &= \delta_3[1, 3, 1, 1, 3, 1, 1, 3, 1]; \\ M_2 &= \delta_3[1, 1, 1, 3, 3, 3, 1, 1, 1]. \end{aligned} \tag{58}$$

B. Convergence of Near Potential Games

It is well known that a potential game has many nice dynamical properties. For instance, we are particularly interested in the following convergence property: If at each time a single player is chosen at random for updating his strategy, the MBRA, called the asynchronous MBRA, will lead to a pure Nash equilibrium. The near potential games were firstly investigated in detail in [3]. Their basic idea is: "Intuitively, dynamics in potential games and dynamics in games that are 'close' (in terms of the payoffs of the players) to potential games should be related". Their main result is: a near potential game will converge to an ϵ equilibrium, where ϵ is estimated by the distance between the game with it closest potential game.

Definition 7.2: Two evolutionary games are said to be dynamically equivalent if they have the same strategy profile dynamics.

The following proposition is straightforward verifiable.

Proposition 7.3: If two games are strategically equivalent then they are dynamically equivalent.

Proposition 7.4: If a game G and its closest potential game $\pi_P(G)$ are dynamically equivalent, then the asynchronous MBRA will lead G to a pure Nash equilibrium.

Remark 7.5: There are many SURs which lead a potential game to a pure Nash equilibrium. Then it is obvious that they will also lead a game G to a pure Nash equilibrium provided G is dynamically equivalent to its closest potential game $\pi_P(G)$. For instance, it is easy to prove that for a cascading potential game:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ x_2(t+1) = f_2(x_1(t+1), x_2(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_1(x_1(t+1), \cdots, x_{n-1}(t+1), x_n(t)) \end{cases}$$

MBRA will lead it to a pure Nash equilibrium.

Most of the learning algorithms for potential games guarantee convergence to a pure Nash equilibrium. Say, Fictitious Play [14], Log-linear learning [1] and its relaxed version [13], *etc*. They also guarantee the convergence of a near potential game G to a pure Nash equilibrium, provided G and $\pi_P(G)$ are dynamically equivalent.

Example 7.6: Recall Example 7.1. Using the same calculating matrices as in the second part of Example 6.4, we can calculate that

$$\begin{aligned} V_{\pi_P(G)} = [&89.7778, -11.8889, 48.1111, -11.8889, \\ &0.4444, 24.4444, 48.1111, 24.4444, 0.4444, \\ &90.2222, -12.1111, 47.8889, -12.1111, -0.4444, \\ &23.5556, 47.8889, 23.5556, -0.4444]. \end{aligned}$$

It is easy to verify that for this $\pi_P(G)$ the profile dynamics is the same as (57)-(58). According to Proposition 7.4, $\mathcal{G}$ will also converge to a pure Nash equilibrium using asynchronous MBRA (or any other SURs mentioned in Remark 7.5).

Note that [3] shows that $\mathcal{G}$ will converge to an ϵ-equilibrium and we prove that it will converge to a pure Nash equilibrium.

In general, it is expected that when a game is close enough to its certain projection, it has the same dynamic behaviors as the games in the subspaces as long as it is dynamically equivalent to its projection.

C. Decomposition of Networked Evolutionary Games

Definition 7.7 ([8], [7]): A networked evolutionary game (NEG), denoted by $\mathcal{G} = ((N, E), G, \Pi)$, consists of three factors:

(i) a network graph: (N, E);
(ii) a fundamental network game (FNG): G with two players. Players i and j play this game provided $(i, j) \in E$.
(iii) a local information based strategy updating rule (SUR):

$$x_i(t+1) = f_i(x_j(t), c_j(t) \mid j \in U(i)), \quad i = 1, \cdots, n. \tag{59}$$

We use

$$U_s(i) := \{k \mid \text{there is a path connecting } i \text{ and } k, \text{ which has length} \le s\}.$$

Then (59) can be expressed as

$$x_i(t+1) = f_i(x_j(t) \mid j \in U_2(i)), \quad i = 1, \cdots, n. \tag{60}$$

It was proved that

Theorem 7.8 ([6]): A networked evolutionary game $\mathcal{G} = ((N, E), G, \Pi)$ is potential, if the fundamental network game G is potential. Moreover, let $e = (i, j) \in E$ be an edge with potential function of the game between i and j being P_e. Then the overall network potential is:

$$P_{\mathcal{G}} = \sum_{e\in E} P_e. \tag{61}$$

Proposition 7.9: $G \in \mathcal{N}$ if and only if G is potential and its potential function is zero (or equivalently, is constant).

Proof: (Necessity) Since $\mathcal{N} \subset \mathcal{G}_P$, G is potential. Moreover, by definition

$$(V_G^c)^T \in \mathrm{Span}\,(B_N)\,. \tag{62}$$

Using (23), (62) is equivalent to that there exist $\xi_i \in \mathbb{R}^{k/k_i}$, $i = 1, \cdots, n$ such that

$$V_i^c = \xi_i^T E_i^T. \tag{63}$$

Then

$$\begin{aligned} c_i(x_1, \cdots, x_n) &= V_i^c \ltimes_{j=1}^n x_j \\ &= \xi_i^T \left(I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i}^T \otimes I_{k^{[i+1,n]}}\right) \ltimes_{j=1}^n x_j \\ &= \xi_i^T \ltimes_{j\neq i} x_j, \quad i = 1, \cdots, n. \end{aligned}$$

Hence, c_i is independent of x_i. It follows that the potential function $P(x_i, \cdots, x_n)$ is also independent of x_i. Since i is arbitrary, $P(x_i, \cdots, x_n) = \text{const}$.

(Sufficiency) Assume G has a zero potential function. Then c_i is independent of x_i, $\forall\, i$. It follows that there exists $\xi_i \in \mathbb{R}^{k/k_i}$ such that

$$\begin{aligned} c_i(x_1, \cdots, x_n) &= \xi_i^T \ltimes_{j\neq i} x_j \\ &= \xi_i^T \left(I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i}^T \otimes I_{k^{[i+1,n]}}\right) \ltimes_{j=1}^n x_j \\ &= \xi_i^T E_i^T \ltimes_{j=1}^n x_j \\ &= V_i^c \ltimes_{j=1}^n x_j, \quad i = 1, \cdots, n. \end{aligned}$$

That is,

$$(V_i^c)^T = E_i \xi_i, \quad i = 1, \cdots, n.$$

Hence,

$$B_N \xi = (V_G^c)^T,$$

which means $G \in \mathcal{N}$. ■

Proposition 7.10: Consider a networked evolutionary game $\mathcal{G} = ((N, E), G, \Pi)$. If $G \in \mathcal{N}$, then $\mathcal{G} \in \mathcal{N}$, *i.e.*, it is also non-strategic.

Proof: Since G has constant potential function, using (61), the potential function of $\mathcal{G}$ is also constant. According to Proposition 7.9, $\mathcal{G} \in \mathcal{N}$. ■

Consider an NEG $\mathcal{G} = ((N, E), G, \Pi)$. Let G_e, $e = (i, j) \in E$ be the fundamental game performed over edge e. Define a natural inclusion mapping: $\psi : G_e \hookrightarrow \mathcal{G}$ as:

$$c_\ell(x_1, \cdots, x_n) := \begin{cases} c_\ell(x_i, x_j), & \ell \in \{i, j\} \\ 0, & \text{Otherwise.} \end{cases}$$

Then it is easy to prove the following:

Lemma 7.11: Figure 1 is commutative. That is,

$$\psi \circ \pi(G_e) = \pi \circ \psi(G_e), \quad e \in E, \tag{64}$$

where

$$\pi \in \{\pi_P, \pi_N, \pi_H, \pi_{\mathcal{G}_P}, \pi_{\mathcal{G}_H}\}.$$

Proof: We prove Lemma 7.11 for $\pi = \pi_N$ only. The proof for other $\pi's$ is similar. Without loss of generality, set $e = (1, 2)$.

$$\begin{tikzcd} G_e \arrow[r, "\pi"] \arrow[d, "\psi"'] & \pi(G_e) \arrow[d, "\psi"] \\ \mathcal{G} \arrow[r, "\pi"'] & \pi(\mathcal{G}) \end{tikzcd}$$

Fig. 1. Commutative Mappings

In G_e, all players share the same set of strategies. The number of strategies is denoted by k.

For G_e, regarding $x_1, \cdots, x_n$ as vector forms, then

$$\begin{aligned} c_1(x_1, \cdots, x_n) &= c_1(x_1, x_2) = \bar{V}_1^c x_1 x_2 \\ &= \bar{V}_1^c x_1 x_2 (1_{k^{n-2}}^T x_3 \ltimes \cdots \ltimes x_n) \\ &= \bar{V}_1^c (I_{k^2} \otimes 1_{k^{n-2}}^T) x_1 \ltimes \cdots \ltimes x_n, \\ &:= V_1^c x_1 \ltimes \cdots \ltimes x_n, \\ c_2(x_1, \cdots, x_n) &= c_2(x_1, x_2) = \bar{V}_2^c x_1 x_2 \\ &= \bar{V}_2^c x_1 x_2 (1_{k^{n-2}}^T x_3 \ltimes \cdots \ltimes x_n) \\ &= \bar{V}_2^c (I_{k^2} \otimes 1_{k^{n-2}}^T) x_1 \ltimes \cdots \ltimes x_n, \\ &:= V_2^c x_1 \ltimes \cdots \ltimes x_n, \\ c_i(x_1, \cdots, x_n) &= \mathbf{0}_{1\times k^n} x_1 \ltimes \cdots \ltimes x_n, \qquad 2 < i \le n, \end{aligned} \tag{65}$$

where $\bar{V}_1^c, \bar{V}_2^c \in \mathbb{R}^{k^2}$, $V_1^c, V_2^c \in \mathbb{R}^{k^n}$.

From (42), the orthogonal projection operator onto $\mathcal{N}$ is

$$\begin{bmatrix} \frac{1}{k} E_1^T E_1 & 0 & \cdots & 0 \\ 0 & \frac{1}{k} E_2^T E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{k} E_n^T E_n \end{bmatrix} := \Pi_{\mathcal{N}}.$$

Then

$$\begin{aligned} &\psi \circ \pi(G_e) \\ =& \begin{bmatrix} I_{k^2} \otimes \mathbf{1}_{k^{n-2}} & & \\ & I_{k^2} \otimes \mathbf{1}_{k^{n-2}} & \\ & & \mathbf{0}_{(n-2)k^n \times (n-2)k^2} \end{bmatrix} \\ & \begin{bmatrix} \begin{bmatrix} \frac{1}{k}(\mathbf{1}_{k\times k} \otimes I_k) & \\ & \frac{1}{k}(I_k \otimes \mathbf{1}_{k\times k}) \end{bmatrix} \begin{bmatrix} (\bar{V}_1^c)^T \\ (\bar{V}_2^c)^T \end{bmatrix} \\ \mathbf{0}_{(n-2)k^2 \times 1} \end{bmatrix} \\ =& \begin{bmatrix} \begin{bmatrix} I_{k^2} \otimes \mathbf{1}_{k^{n-2}} & \\ & I_{k^2} \otimes \mathbf{1}_{k^{n-2}} \end{bmatrix} \begin{bmatrix} \frac{1}{k}(\mathbf{1}_{k\times k} \otimes I_k)(\bar{V}_1^c)^T \\ \frac{1}{k}(I_k \otimes \mathbf{1}_{k\times k})(\bar{V}_2^c)^T \end{bmatrix} \\ \mathbf{0}_{(n-2)k^n \times 1} \end{bmatrix} \\ =& \begin{bmatrix} \frac{1}{k}(\mathbf{1}_{k\times k} \otimes I_k \otimes \mathbf{1}_{k^{n-2}})(\bar{V}_1^c)^T \\ \frac{1}{k}(I_k \otimes \mathbf{1}_{k\times k} \otimes \mathbf{1}_{k^{n-2}})(\bar{V}_1^c)^T \\ \mathbf{0}_{(n-2)k^n \times 1} \end{bmatrix} \\ =& \begin{bmatrix} \frac{1}{k}(\mathbf{1}_{k\times k} \otimes I_k \otimes I_{k^{n-2}})(I_{k^2} \otimes \mathbf{1}_{k^{n-2}})(\bar{V}_1^c)^T \\ \frac{1}{k}(I_k \otimes \mathbf{1}_{k\times k} \otimes I_{k^{n-2}})(I_{k^2} \otimes \mathbf{1}_{k^{n-2}})(\bar{V}_1^c)^T \\ \mathbf{0}_{(n-2)k^n \times 1} \end{bmatrix} \\ =& \Pi_{\mathcal{N}} \begin{bmatrix} (V_1^c)^T \\ (V_2^c)^T \\ \mathbf{0}_{(n-2)k^n \times 1} \end{bmatrix} \\ =& \pi \circ \psi(G_e). \end{aligned} \tag{66}$$

■

In the light of Lemma 7.11 the following decomposition of NEGs is obvious.

Proposition 7.12: Consider a networked evolutionary game $\mathcal{G} = ((N,E),G,\Pi)$. It has the orthogonal decomposition as

$$\mathcal{G} = \underbrace{\mathcal{P} \oplus \rlap{\overbrace{\phantom{\mathcal{N} \oplus \mathcal{H}}}^{Harmonic\quad games}}\mathcal{N}}_{Potential\quad games} \oplus \mathcal{H}, \tag{67}$$

where $\mathcal{P} = \sum_{e\in E} \mathcal{P}^e$, $\mathcal{N} = \sum_{e\in E} \mathcal{N}^e$, $\mathcal{H} = \sum_{e\in E} \mathcal{H}^e$, $\mathcal{G}_P = \sum_{e\in E} \mathcal{G}_P^e$, and $\mathcal{G}_H = \sum_{e\in E} \mathcal{G}_H^e$, and $\mathcal{P}^e$, $\mathcal{N}^e$, $\mathcal{H}^e$, $\mathcal{G}_P^e$, and $\mathcal{G}_H^e$ are the corresponding subspaces of the game G over edge $e \in E$.

We give an example to illustrate this.

Example 7.13: Consider a game $\mathcal{G} = ((N,E),G,\Pi)$, where (i) the network graph is shown in Fig. 2; (ii) G is the Benoit-Krishna Game [17]. Recall that the Benoit-Krishna Game has

$$S_0 = \{1(D): \text{ Deny}, \quad 2(W): \text{ Waffle}, \quad 3(C): \text{ Confess}\}.$$

The payoff bi-matrix is shown in Table VII; (iii) for this problem Π is ignored.

TABLE VII
PAYOFF TABLE (BENOIT-KRISHNA)

$P_1 \backslash P_2$	$D=1$	$W=2$	$C=3$
$D=1$	$10,\ 10$	$-1,-12$	$-1,\ 15$
$W=2$	$-12,-1$	$8,\ 8$	$-1,\ -1$
$C=3$	$15,-1$	$8,\ 1$	$0,\ 0$

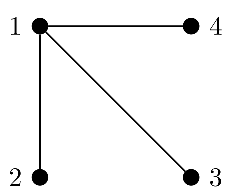

Fig. 2. Network Graph of Example 7.13

We use two ways to calculate the decomposition of $\mathcal{G}$.

1) Edge by Edge Calculation (Calculate the decomposition of G over pair of players on each $e \in E$, then sum up.) Note that the Benoit-Krishna Game $G \in \mathcal{G}_{[2;3,3]}$ is in the same space as the Rock-Scissors-Paper game in Example 6.4, Same B_P (B_P^0) and B_N can be used to calculate the decomposition. The payoff vector of G is:

$$\begin{aligned} V_G^c = [&10,-1,-1,-12,8,-1,15,8,0,\\ &10,-12,15,-1,8,-1,-1,1,0]. \end{aligned}$$

Similar to the calculation for Rock-Scissors-Paper game, we can have

$$\begin{aligned} \pi_{\mathcal{H}}(G) = [&0.389,3.722,-4.111,-4.111,-0.778,4.889,\\ &3.722,-2.944,-0.778,-0.389,-3.722,4.111,\\ &4.111,0.778,-4.889,-3.722,2.944,0.778]; \end{aligned}$$

$$\begin{aligned} \pi_{\mathcal{N}}(G) = [&4.333,5,-0.667,4.333,5,-0.667,4.333,5,\\ &-0.667,4.333,4.333,4.333,2,2,2,0,0,0]; \end{aligned}$$

$$\begin{aligned} \pi_{\mathcal{P}}(G) = [&5.278,-9.722,3.778,-12.222,3.778,-5.222,\\ &6.944,\ 5.944,\ 1.444,\ 6.056,-12.611,\ 6.556,\\ &-7.111,5.222,1.889,2.722,-1.944,-0.778]; \end{aligned}$$

$$\begin{aligned} \pi_{\mathcal{G}_P}(G) = [&9.611,-4.722,3.111,-7.889,8.778,-5.889,\\ &11.278,10.944,0.778,10.389,-8.278,10.889,\\ &-5.111,7.222,3.889,2.722,-1.944,-0.778]; \end{aligned}$$

$$\begin{aligned} \pi_{\mathcal{G}_H}(G) = [&4.722,8.722,-4.778,\ 0.222,\ 4.222,\ 4.222,\\ &8.056,2.056,-1.444,\ 3.944,\ 0.611,\ 8.444,\\ &6.111,2.778,-2.889,-3.722,2.944,0.778]. \end{aligned}$$

Now split $\pi_{\mathcal{H}}(G)$ into two equal parts as

$$\pi_{\mathcal{H}}(G) = \left[V_{\mathcal{H}}^1, V_{\mathcal{H}}^2\right],$$

where $V_{\mathcal{H}}^1$ and $V_{\mathcal{H}}^2$ are the projections of the payoffs V_1^c of players 1 and V_2^c of players 2 respectively.
Now consider the game G over edge $(1,2)$. We have the projection of V_1^c and V_2^c on $\mathcal{H}$ as

$$\begin{aligned} H_1^1(x_1,x_2) &= V_{\mathcal{H}}^1 x_1 x_2 = V_{\mathcal{H}}^1 \left(I_9 \otimes \mathbf{1}_9^T\right) \ltimes_{i=1}^4 x_i;\\ H_2^1(x_1,x_2) &= V_{\mathcal{H}}^2 x_1 x_2 = V_{\mathcal{H}}^2 \left(I_9 \otimes \mathbf{1}_9^T\right) \ltimes_{i=1}^4 x_i. \end{aligned}$$

Similarly, for the game G over $(1,3)$, we have

$$\begin{aligned} H_1^2(x_1,x_3) &= V_{\mathcal{H}}^1 x_1 x_3\\ &= V_{\mathcal{H}}^1 \left(I_3 \otimes \mathbf{1}_3^T \otimes I_3 \otimes \mathbf{1}_3^T\right) \ltimes_{i=1}^4 x_i;\\ H_3^2(x_1,x_3) &= V_{\mathcal{H}}^2 x_1 x_3\\ &= V_{\mathcal{H}}^2 \left(I_3 \otimes \mathbf{1}_3^T \otimes I_3 \otimes \mathbf{1}_3^T\right) \ltimes_{i=1}^4 x_i. \end{aligned}$$

For the game G over $(1,4)$, we have

$$\begin{aligned} H_1^3(x_1,x_4) &= V_{\mathcal{H}}^1 x_1 x_4 = V_{\mathcal{H}}^1 \left(I_3 \otimes \mathbf{1}_9^T \otimes I_3\right) \ltimes_{i=1}^4 x_i;\\ H_4^3(x_1,x_4) &= V_{\mathcal{H}}^2 x_1 x_4 = V_{\mathcal{H}}^2 \left(I_3 \otimes \mathbf{1}_9^T \otimes I_3\right) \ltimes_{i=1}^4 x_i. \end{aligned}$$

Overall, the projection of $\mathcal{G}$ on the pure harmonic subspace $\mathcal{H}$ is

$$V_{\mathcal{H}} = [H_1, H_2, H_3, H_4],$$

where

$$H_1 = H_1^1 + H_1^2 + H_1^3;\ H_2 = H_2^1;\ H_3 = H_3^2;\ H_4 = H_4^3.$$

Finally, the numerical result is:

$$\begin{aligned} \pi_{\mathcal{H}}(G) = [&1.167,4.5,-3.333,4.5,7.833,\cdots,\\ &2.944,0.778,-3.722,2.944,0.778] \in \mathbb{R}^{324}. \end{aligned} \tag{68}$$

Similar calculations yield the projections on $\mathcal{N}$, $\mathcal{P}$, $\mathcal{G}_P$, and $\mathcal{G}_H$ as

$$\begin{aligned} \pi_{\mathcal{N}}(G) = [&13,13.667,8,13.667,14.333,\cdots,\\ &0,0,0,0,0] \in \mathbb{R}^{324}. \end{aligned} \tag{69}$$

$$\begin{aligned} \pi_{\mathcal{P}}(G) = [&15.833,0.833,14.333,0.833,-14.167,\cdots,\\ &-1.944,-0.778,2.722,-1.944,-0.778] \in \mathbb{R}^{324}. \end{aligned} \tag{70}$$

$$\begin{aligned} \pi_{\mathcal{G}_P}(G) = [&28.833,14.5,22.333,14.5,0.167,\cdots\\ &-1.944,-0.778,2.722,-1.944,-0.778] \in \mathbb{R}^{324}. \end{aligned} \tag{71}$$

$$\pi_{\mathcal{G}_H}(G) = [14.167, 18.167, 4.667, 18.167, 22.167, \cdots \\ 2.944, 0.778, -3.722, 2.944, 0.778] \in \mathbb{R}^{324}. \tag{72}$$

2) Global Calculation (Consider the networked game as an integrated game and calculated the projections of $\mathcal{G}$ directly.) First, it is easy to calculate the payoff structure vector of the overall game. It is

$$V_G^c = [30, 19, 19, 19, 8, \cdots, 1, 0, -1, 1, 0] \in \mathbb{R}^{324}.$$

Using

$$\begin{aligned} E_1 &= \mathbf{1}_3 \otimes I_{27} \\ E_2 &= I_3 \otimes \mathbf{1}_3 \otimes I_9 \\ E_3 &= I_9 \otimes \mathbf{1}_3 \otimes I_3 \\ E_4 &= I_{27} \otimes \mathbf{1}_3, \end{aligned}$$

Then it is easy to calculate B_N and B_P^0. Finally, using formulas (41)-(45), the projections can be calculated. It is easy to show that the projections obtained in this way are exactly the same as (68)-(72).

VIII. Conclusion

In this paper we investigated the decomposition and the decomposed subspaces of non-cooperative strategic form finite games, $\mathcal{G}_{[n;k_1,\cdots,k_n]}$. First, $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ was given a vector space structure as $\mathbb{R}^{nk}$ in a very natural way. Then the subspace of potential games, $\mathcal{G}_P$, and the non-strategic subspace, $\mathcal{N}$, defined in a straightforward way, are investigated and their subspace bases were obtained. Using them, the bases of the pure potential subspace $\mathcal{P}$ and the pure harmonic subspace $\mathcal{H}$ were obtained. Then the orthogonal decomposition was investigated and straightforward numerical formulas were presented. The mild difference between the decompositions in this paper and in [2] was explained. Finally, the decomposition results were applied to investigating evolutionary games. Two results were obtained: (i) the convergence of near-potential games to a pure Nash equilibrium was revealed; (ii) the decomposition of networked evolutionary games were obtained via the decomposition of their fundamental network game.

References

[1] L. Blume, The statistical mechanics of strategic interaction, *Games Econ. Behav.*, Vol. 5, 387-424, 1993.

[2] O. Candogan, I. Menache, A. Ozdaglar, P.A. Parrilo, Flows and decompositions of games: Harmonic and potential games, *Mathematcs of Operations Research*, Vol. 36, No. 3, 474-503, 2011.

[3] O. Candogan, A. Ozdaglar, P.A. Parrilo, Dynamics in near-potential games, *Games Econ. Behav.*, Vol. 82, 66-90, 2013.

[4] D. Cheng, H. Qi, Z. Li, *Analysis and Control of Boolean Networks - A Semi-tensor Product Approach*, Springer, London, 2011.

[5] D. Cheng, H. Qi, Y. Zhao, *An Introduction to Semi-tensor Product of Matrices and Its Applications*, World Scientific, Singapore, 2012.

[6] D. Cheng, On finite potential games, *Automatica*, Vol. 50, No. 7, 1793-1801, 2014.

[7] D. Cheng, H. Qi, F. He, T. Xu, H. Dong, Semi-tensor product approach to networked evolutionary games, *Contr. Theory Tech.*, Vol. 12, No. 2, 198-214, 2014.

[8] D. Cheng, F. He, H. Qi, T. Xu. Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Aut. Contr.*, (in print), On line: http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=7042754, DOI:10.1109/TAC.2015.2404471.

[9] D. Fudenberg, J. Tirole, Game Theorey, *China Renmin University Press*, pp. 6-7, 2010. (Simplified Chinese Version)

[10] R. Gibbons, A Primer in Game Theory, Prentice Hall, Harlow, 1992.

[11] T. Heikkinen, A potential game approach to distributed power control and scheduling, *Computer Networks*, Vol. 50, 2295-2311, 2006.

[12] J.R. Marden, G. Arslan, J. S. Shamma, Cooperative control and potential games, *IEEE Trans. Sys., Man, Cybernetcs, Part B*, Vol. 39, No. 6, 1393-1407, 2009.

[13] J.R. Marden, J. S. Shamma, Revisiting log-linear learning: Asynchrony, completeness and bayoff-based implementation, *Games Econ. Behav.*, Vol. 75, 788-808, 2012.

[14] D. Monderer, L.S. Shapley, Fictitious play property for games with identical interests, *J. Econ. Theory*, Vol. 1, 258-265, 1996.

[15] D. Monderer, L.S. Shapley, Potential Games, *Games Econ. Behav.*, Vol. 14, 124-143, 1996.

[16] M.A. Nowak, R.M. May. Evolutionary games and spatial chaos, *Nature*, 359(6357), 826-829, 1992.

[17] E. Rasmusen, *Games and Information: An Introduction to Game Theory*, 4th Ed., Basil Blackwell, Oxford, 2007.

[18] R.W. Rosenthal, A class of games possessing pure-strategy Nash equilibria, *Int. J. Game Theory*, Vol. 2, 65-67, 1973.

[19] G. Szabo, C. Toke. Evolutionary prisoner's dilemma game on a square lattice, *Phys. Rev. E*, Vol. 58, No. 1, 69-73, 1998.

[20] Y. Tong, S. Lombeyda, A.N. Hirani, M. Desbrun, Discrete multiscale vector field decomposition, *ACM Trans. Graphics*, Vol. 22, No. 3, 445-452, 2003.

[21] A. Traulsen, M.A. Nowak, J.M. Pacheco. Stochastic dynamics of invasion and fixation, *Phys. Rev., E*, vol. 74, No. 1, 74.011909, 2006.

[22] X. Wang, N. Xiao, T. Wongpiromsarn, L. Xie, E. Frazzoli, D. Rus, Distributed consensus in noncooperative congestion games: an application to road pricing, *Proc. 10th IEEE Int. Conf. Contr. Aut.*, Hangzhou, China, 1668-1673, 2013.

[23] A.Y. Yazicioglu, M. Egerstedt, J.S. Shamma, A game theoretic approach to distributed coverage of graphs by heterogeneous mobile agents, *Est. Contr. Netw. Sys.*, Vol. 4, 309-315, 2013.

[24] H.P. Young. The evolution of conventions, *Econometrica*, Vol. 61, No. 1, 57-84, 1993.

[25] M. Zhu, S. Martinez, Distributed coverage games for energy-aware mobile sensor networks, *SIAM J. Cont. Opt.*, Vol. 51, No. 1, 1-27, 2013.