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Stabilization of nonlinear systems via the center manifold approach ☆

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Abstract

This paper considers the problem of the stabilization of affine nonlinear control systems. First, we assume that the systems under investigation are of the generalized Byrnes–Isidori normal form. A new way to approximate the center manifold is proposed, which can reduce the error degree of the center manifold approximation. A new matrix product, called the semi-tensor product, is introduced to obtain the approximation of the center manifold. Then the Lyapunov function with homogeneous derivative (LFHD) is used to design a stable center manifold by state feedback control. Finally, the method is applied to general affine nonlinear control systems.

Keywords: Stabilization; Center manifold; Byrnes–Isidori normal form; Semi-tensor product

1. Preliminary

The center manifold theory emerged in the sixties of the last century, and soon became a powerful tool for the investigation of the stability of dynamic systems [4,14]. Later it was used for the stabilization of nonlinear control systems [1,2].

Consider an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f(x)$ and $g_i(x)$ are smooth vector fields on $\mathbb{R}^n$ with $f(0) = 0$. The normal form of an affine nonlinear control system was proposed in [3]. Under the normal form and with a minimum phase assumption, a stabilization technique via the center manifold, proposed by Byrnes and Isidori, has been used to stabilize general affine nonlinear control systems [3,13].

To use the Byrnes–Isidori approach, two things are essential: 1. normal form (called Byrnes–Isidori normal form in some literatures); 2. minimum phase. Recently, certain efforts have been made to weaken these two restrictions.

First of all, for the case of the non-minimum phase, recent results have been obtained by introducing some new tools such as a Lyapunov function with homogeneous derivative (LFHD) to design the center manifold. [6,5,12]. Roughly speaking, we use higher degree (deg ≥ 2) state feedback to design a center manifold. We then use a LFHD to verify the stability of the approximated dynamics on an approximated center manifold, which assures us that the stability of the approximated dynamics implies the stability of the real dynamics on the center manifold.

Secondly, in recent work [9] we intended to relax the first constraint. For system (1.1) we can adopt any m smooth functions:

$$y_i = h_i(x), \quad i = 1, \ldots, m, \tag{1.2}$$

as outputs and define the relative degree vector and decoupling matrix at a fixed point. They are called the point relative degree vector and point decoupling matrix. For those outputs, that reach the largest norm of the relative degree vector, the corresponding point relative degree vector and point decoupling matrix are called the essential point relative degree vector and the essential decoupling matrix. In the light of the essential point relative degree vector and the essential point decoupling matrix, a generalized Byrnes–Isidori normal form can be obtained, which is cited here:

Theorem 1.1 (*[9]*). *For system* (1.1) *and* (1.2)*, if* (1) *the essential point relative degree vector is* $(\rho_1, \ldots, \rho_m)$*;* (2) *the corresponding essential point decoupling matrix is nonsingu-*

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lar; (3) $G = \text{Span}\{g_1, \ldots, g_m\}$ *is involutive, then the system* (1.1) *can be expressed into the following form, which is called the generalized Byrnes–Isidori normal form:*

$$\begin{cases} \dot{x}^i = A_i x^i + b_i u_i + \begin{pmatrix} 0 \\ \alpha_i(x,z) \end{pmatrix} + p^i(x,z)u, \\ \qquad x^i \in \mathbb{R}^{\rho_i} \; i = 1, \ldots, m \\ \dot{z} = q(x,z), \quad z \in \mathbb{R}^r, \\ y_i = x_1^i, \quad i = 1, \ldots, m, \end{cases} \tag{1.3}$$

where $r + \sum_{i=1}^m \rho_i = n$, (A_i, b_i) *is in the Brunovsky canonical form as*

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad i = 1, \ldots, m,$$

$p^i(0,0) = 0$, *and* $\alpha_i(x,z)$ *is of dimension* 1.

Based on this generalized normal form, the center manifold approach developed for standard normal form has been extended to a much larger class of systems. A description of this was given in [10].

The main purpose of this paper is to provide a new design technique of the center manifold to stabilize system (1.3), which is then extended to general affine nonlinear systems.

The paper is organized as follows: Section 2 explains the motivation of this work. The main result for the stabilization of systems in the generalized Byrnes–Isidori normal form is in Section 3. Section 4 provides the method for calculating the approximation of the center manifold. Section 5 extends the result to general affine nonlinear control systems. Some illustrative examples are presented in Section 6. Section 7 deals with some concluding remarks.

2. Motivating examples

This section gives a sequence of simple examples to expose the weakness of the approach proposed in [6], (used also in [9] for generalized normal form), and then to reveal the idea for a new more powerful design technique. Meanwhile, the examples can also be implemented as a clear description of the technique for stabilization via the center manifold approach.

Example 2.1. Consider the stabilization problem of the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 \sin(z) + u \\ \dot{z} = x_1 z. \end{cases} \tag{2.1}$$

The system is in Byrnes–Isidori normal form. First, we can choose a linear feedback to stabilize the linearly controllable states x. Say, set the eigenvalues as $\{-1, -2, -3\}$; to this end we use

$$\begin{aligned} u &= -x_1 \sin(z) - 6x_1 - 11x_2 - 6x_3 \\ &:= -x_1 \sin(z) + a_1 x_1 + a_2 x_2 + a_3 x_3. \end{aligned}$$

Let the center manifold to be designed be $x = h(z)$. We may choose

$$\begin{cases} x_1 = \phi(z) := -z^2 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

to approximate the center manifold equation $h(z)$. Then the control can be chosen as [6]

$$\begin{aligned} u &= -x_1 \sin(z) + a_1 x_1 + a_2 x_2 + a_3 x_3 - a_1(\phi(z)) \\ &= -x_1 \sin(z) - 6x_1 - x_2 - 7x_3 - 6z^2. \end{aligned} \tag{2.2}$$

It follows that the error degree of the approximation (EDA) is [4]

$$\begin{aligned} \text{EDA} &= \frac{\partial x}{\partial z}(x_1(z)z) - (Ax(z) + bu(x(z), z)) \\ &= (-2z)(-z^2 z) - 0 = O(\|z\|^4). \end{aligned} \tag{2.3}$$

So the center manifold can be expressed as

$$h(z) = \begin{pmatrix} -z^2 \\ 0 \\ 0 \end{pmatrix} + O(\|z\|^4). \tag{2.4}$$

The dynamics on the center manifold of the closed-loop system are given by

$$\dot{z} = h(z)z = \left(-z^2 + O(\|z\|^4)\right) z = -z^3 + O(\|z\|^5). \tag{2.5}$$

Obviously, it is asymptotically stable and so is the closed-loop system. Therefore, the state feedback control (2.2) stabilizes system (2.1). □

Example 1 describes the basic idea in [6]. Next, we modify the system a little.

Example 2.2. Consider the stabilization problem for the following system

$$\begin{cases} \dot{x}_1 = x_2 + x_1 z u \\ \dot{x}_2 = x_3 + z u \\ \dot{x}_3 = x_1 \sin(z) + u \\ \dot{z} = x_1 z. \end{cases} \tag{2.6}$$

The system is in a generalized Byrnes–Isidori normal form. As proposed in [9], we may still use the same method as in Example 2.1 to stabilize it. The only difference that has to be verified is the degree of approximation. In fact, the error degree can be calculated by the Eqs. (36) and (37) of [9]. For the Eq. (36), we have

$$\begin{pmatrix} dx_1(z) \\ dx_2(z) \\ dx_3(z) \end{pmatrix} x_1(z)z = \begin{pmatrix} -2z \\ 0 \\ 0 \end{pmatrix} x_1(z)z = O(\|z\|^4).$$

As for the Eq. (37), we have

$$\begin{pmatrix} x_1(z)z(x_1(z)\sin(z)) \\ z(x_1(z)\sin(z)) \end{pmatrix} = \begin{pmatrix} -z^2 z(-z^2 \sin(z)) \\ z(-z^2 \sin(z)) \end{pmatrix} = O(\|z\|^4).$$

Then (2.4) remains true. We conclude that the control (2.2) can also stabilize system (2.6). □

The method used above is what is proposed in [9]. Now, we modify system (2.6) to see that the method developed in [6,9] can fail.

Example 2.3. Consider the stabilization problem of the following system

$$\begin{cases} \dot{x}_1 = x_2 + x_1 z u \\ \dot{x}_2 = x_3 + z u \\ \dot{x}_3 = \sin(z) + u \\ \dot{z} = x_1 z. \end{cases} \tag{2.7}$$

The system is still in a generalized Byrnes–Isidori normal form. We can still use the Eqs. (36) and (37) of [9] to check the EDA. (36) is the same as in Example 2.2. For (37) we have

$$\text{EDA} = \begin{pmatrix} x_1(z)\sin(z) \\ z\sin(z) \end{pmatrix} = \begin{pmatrix} -z^2\sin(z) \\ z\sin(z) \end{pmatrix} = O(\|z\|^2).$$

So we can only claim that the center manifold has the form of

$$h(x) = \begin{pmatrix} -z^2 \\ 0 \\ 0 \end{pmatrix} + O(\|z\|^2) = O(\|z\|^2). \tag{2.8}$$

The dynamics on the center manifold of the closed-loop system is

$$\dot{z} = h(z)z = \left(-z^2 + O(\|z\|^2)\right) z = O(\|z\|^3). \tag{2.9}$$

Clearly we can say nothing about the stability of (2.9). □

It is obvious that the method developed in [6] and [9] fails in Example 2.3. But it does not mean that the system (2.7) is not stabilizable by state feedback. In fact, the problem is that the tool is not subtle enough.

Let's try to sharpen our tool. Say, choose

$$\begin{cases} x_1 = \phi_1(z) \\ x_2 = \phi_2(z) \\ x_3 = \phi_3(z) \end{cases}$$

to approximate the center manifold, and as before set $x_1(z) = \phi_1(z) = -z^2$. Then choose x_2, x_3 and u in such a way that turns the right-hand side of (2.7) to zero. That is

$$\begin{cases} x_2 + x_1 z u = 0 \\ x_3 + z u = 0 \\ \sin(z) + u = 0. \end{cases}$$

Note that the linear part of u has to be chosen such that the linearly controllable state variables are stable. So the control can be chosen as

$$\begin{aligned} u = &-\sin(z) + a_1 x_1 + a_2 x_2 + a_3 x_3 - a_1(\phi_1(z)) \\ &- a_2(\phi_2(z)) - a_3(\phi_3(z)). \end{aligned} \tag{2.10}$$

Accordingly, we have

$$\begin{aligned} x_2(z) &= \phi_2(z) = -x_1(z) z u(x(z), z) = -z^3 \sin(z) \\ x_3(z) &= \phi_3(z) = -z u(x(z), z) = -z\sin(z). \end{aligned}$$

Checking the EDA, we have

$$\begin{aligned} \text{EDA} &= \begin{pmatrix} \dfrac{\partial \phi_1(z)}{\partial z} \\ \dfrac{\partial \phi_2(z)}{\partial z} \\ \dfrac{\partial \phi_3(z)}{\partial z} \end{pmatrix} \phi_1(z) z \\ &\quad - \begin{pmatrix} \phi_2(z) + \phi_1(z) z u(z, \phi_1(z), \phi_2(z), \phi_3(z)) \\ \phi_3(z) + z u(z, \phi_1(z), \phi_2(z), \phi_3(z)) \\ \sin(z) + u(z, \phi_1(z), \phi_2(z), \phi_3(z)) \end{pmatrix} \\ &= \begin{pmatrix} -2z \\ -3z^2\sin(z) - z^3\cos(z) \\ -\sin(z) - 2\cos(z) \end{pmatrix} (-z^2) z = O(\|z\|^4). \end{aligned}$$

Then (2.5) remains true. That is, control (2.10) stabilizes the system (2.7). □

We would like the emphasize the idea in the above argument: In [6], we use $x_1^i = \phi_1^i(z)$ to design the center manifold to assure the stability of the dynamics on the center manifold and set $x_j^i = 0$, $j > 1$ to assure the required accuracy (degree) of the approximation error. In [9], the method is used in the context of a generalized Byrnes–Isidori normal form. It was shown in the above examples that unlike the standard case, $x_j^i = 0$, $j > 1$ may not be enough to assure the required accuracy (degree) of the approximation error. But we may also design $x_j^i = \phi_j^i(z)$, $j > 1$ to improve the accuracy (degree) of the approximation error to meet our requirement on the accuracy of the dynamics on the center manifold.

In the rest of this paper, we will develop this idea into a systematic treatment for the design of an approximate center manifold, $x_j^i = \phi_j^i(z)$, $j \geq 1$.

3. Stabilization under the generalized normal form

Consider the stabilization problem of the generalized Byrnes–Isidori normal form (1.3). First, assume the linear feedbacks are chosen such that the linearly controllable variables x_j^i, $i = 1, \ldots, m$; $j = 1, \ldots, \rho_i$ are stabilized by

$$u_i = a_1^i x_1^i + \cdots + a_{\rho_i}^i x_{\rho_i}^i, \quad i = 1, \ldots, m.$$

Assume that $\phi_j^i(z)$ are used to approximate the center manifold. Then following the design idea in Section 2, we construct controls as

$$\begin{aligned} u_i(x, z) = &-\alpha_i(x, z) + a_1^i x_1^1 + \cdots + a_{\rho_i}^i x_{\rho_i}^i - a_1^i \phi_1^i(z) \\ &- \cdots - a_{\rho_i}^i \phi_{\rho_i}^i(z), \quad i = 1, \ldots, m. \end{aligned} \tag{3.1}$$

Now $\phi_1^i(z)$ can be chosen freely as a polynomial with lowest degree ≥ 2. Then we can solve $\phi_j^i = x_j^i$, $j > 1$ from the following equations.

$$\begin{aligned} &F_1^i(x, z) := x_2^i + p_1^i(x, z)\alpha_i(x, z) = 0, \\ &\ldots \\ &F_{\rho_i - 1}^i(x, z) := x_{\rho_i}^i + p_{\rho_i - 1}^i(x, z)\alpha_i(x, z) = 0, \\ &\qquad i = 1, \ldots, m. \end{aligned} \tag{3.2}$$

Lemma 3.1. *Locally on a neighborhood U of the origin, the x^i_j, $i = 1, \ldots, m$, $j = 2, \ldots, \rho_i$ can be uniquely solved from Eq. (3.2) as functions of z (denoted by $x^i_j = \phi^i_j(z)$, $j \geq 2$), where $x^i_1 = \phi^i_1(z)$ are known.*

Proof. Note that since $f(0) = 0$, $\alpha^i(0,0) = 0$. Moreover, $p^i_j(0,0) = 0$, $i = 1, \ldots, m$, $j = 1, \ldots, \rho_i - 1$. It follows that the Jacobian matrix

$$J = \left[\left. \frac{\partial F^i_j(x,z)}{\partial x^i_{j+1}} \right| i = 1, \ldots, m, j = 1, \ldots, \rho_i - 1 \right] \Bigg|_{(0,0)} = I_m.$$

By the implicit function theorem, x^i_j, $j \geq 2$ can be solved from (3.2) locally. □

Using the above notations and Lemma 3.1, the following theorem is an immediate consequence of the center manifold theory.

Theorem 3.2. *Assume there exist $\phi^i_j(z)$, $i = 1, \ldots, m$, $j = 1, \ldots, \rho_i$, such that* (1) *the error degree of approximation is*

$$\mathrm{EDA} = \left[\left. \frac{\partial \phi^i_j(z)}{\partial z} \right| i = 1, \ldots, m, j = 1, \ldots, \rho_i \right] \times q(\phi(z), z) = O(\|z\|^{d+1}); \tag{3.3}$$

(2) *the errors in the dynamics on the center manifold, caused by the approximation error, are*

$$q_k(\phi(z) + O(\|z\|^{d+1}), z) = q_k(\phi(z), z) + O(\|z\|^{t_k+1}), \quad k = 1, \ldots, r; \tag{3.4}$$

(3) *the approximated dynamics of the center manifold*

$$\dot{z}_k = q_k(\phi(z), z), \quad k = 1, \ldots, r, \tag{3.5}$$

are $(t_1, \ldots, t_r)$-degree approximately stable, i.e.,

$$\dot{z}_k = q_k(\phi(z), z) + O(\|z\|^{t_k+1}), \quad k = 1, \ldots, r, \tag{3.6}$$

are asymptotically stable.

Then the system (1.3) is asymptotically stabilizable by control (3.1).

Now let $\xi_k(z)$ be an approximation of $q_k(\phi(z), z)$, consisting of its lowest degree non-vanishing terms, with the degrees of $\xi_k(z)$ being $\deg(\xi_k(z)) = s_k$, $k = 1, \ldots, m$. Then we can construct the approximate system of (3.5) as [6]

$$\dot{z}_k = \xi_k(z), \quad k = 1, \ldots, r. \tag{3.7}$$

Using the result about LFHD in [6], we have

Theorem 3.3. *Assume* (1) $s_k \leq t_k$, $k = 1, \ldots, r$; (2) *there exists a LFHD $V > 0$ such that*

$$\dot{V}|_{(3.7)} < 0. \tag{3.8}$$

Then the system (1.3) is asymptotically stabilizable by control (3.1).

Proof. An LFHD $V > 0$ satisfying (3.8) assures the asymptotical stability of

$$\dot{z}_k = \xi_k(z) + O(\|z\|^{s_k+1}), \quad k = 1, \ldots, r.$$

Now the true dynamics on the center manifold are

$$\begin{aligned} \dot{z}_k &= q_k(\phi(z), z) + O(\|z\|^{t_k+1}) \\ &= \xi_k(z) + O(\|z\|^{s_k+1}) + O(\|z\|^{t_k+1}) \\ &= \xi_k(z) + O(\|z\|^{s_k+1}), \quad k = 1, \ldots, r. \end{aligned}$$

The conclusion follows. □

Remark. About the theory of LFHD, including testing the negativity of the homogeneous derivatives, we refer to [6], or [8].

4. Solving for an approximate center manifold

In Section 3, we defined $x^i_j(z) = \phi^i_j(z)$, $i = 1, \ldots, m$, $j = 1, \ldots, \rho_i$ to approximate the center manifold, and proved by the implicit function theorem that they do exist. But unless we can solve $\phi^i_j(z)$, we are not able to go any further with the design of the stabilizer. We introduce a new matrix product, called the left semi-tensor product of matrices, which makes the multiplication of multi-variable polynomials exactly the same as that of a single-variable case. The following concepts and properties of the left semi-tensor product can be found in [8] or [11].

Definition 4.1. 1. Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then we split X into p equal-size blocks as $X^1, \ldots, X^p$, which are $1 \times n$ rows. Define the left semi-tensor product, denoted by $\ltimes$, as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in \mathbb{R}^n, \\ Y^{\mathrm{T}} \ltimes X^{\mathrm{T}} = \sum_{i=1}^{p} y_i (X^i)^{\mathrm{T}} \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

2. Let $A \in M_{m \times n}$ and $B \in M_{p \times q}$. If either n is a factor of p, say $nt = p$, or p is a factor of n, say $n = pt$, then we define the left semi-tensor product of A and B, denoted by $C = A \ltimes B$, as the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, \ldots, m, j = 1, \ldots, q,$$

where A^i is the i-th row of A and B_j is the j-th column of B.

Example 4.2. 1. Let $X = (a\ b\ c\ d)$ and $Y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then

$$X \ltimes Y = (a\ b) \cdot \alpha + (c\ d) \cdot \beta = \begin{pmatrix} a\alpha + c\beta & b\alpha + d\beta \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix}.$$

Then, see the equation given in Box I.

$$A \ltimes B = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} \\ b_{21}\end{pmatrix} + a_{12}\begin{pmatrix} b_{31} \\ b_{41}\end{pmatrix} & a_{11}\begin{pmatrix} b_{12} \\ b_{22}\end{pmatrix} + a_{12}\begin{pmatrix} b_{32} \\ b_{42}\end{pmatrix} & a_{11}\begin{pmatrix} b_{13} \\ b_{23}\end{pmatrix} + a_{12}\begin{pmatrix} b_{33} \\ b_{43}\end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} \\ b_{21}\end{pmatrix} + a_{22}\begin{pmatrix} b_{31} \\ b_{41}\end{pmatrix} & a_{21}\begin{pmatrix} b_{12} \\ b_{22}\end{pmatrix} + a_{22}\begin{pmatrix} b_{32} \\ b_{42}\end{pmatrix} & a_{21}\begin{pmatrix} b_{13} \\ b_{23}\end{pmatrix} + a_{22}\begin{pmatrix} b_{33} \\ b_{43}\end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{31} & a_{11}b_{12} + a_{12}b_{32} & a_{11}b_{13} + a_{12}b_{33} \\ a_{11}b_{21} + a_{12}b_{41} & a_{11}b_{22} + a_{12}b_{42} & a_{11}b_{23} + a_{12}b_{43} \\ a_{21}b_{11} + a_{22}b_{31} & a_{21}b_{12} + a_{22}b_{32} & a_{21}b_{13} + a_{22}b_{33} \\ a_{21}b_{21} + a_{22}b_{41} & a_{21}b_{22} + a_{22}b_{42} & a_{21}b_{23} + a_{22}b_{43} \end{pmatrix}.$$

Box I.

Remark. Note that when $n = p$, the left semi-tensor product coincides with the conventional matrix product. Therefore, the left semi-tensor product is only a generalization of the conventional product. For convenience, we may omit the product symbol $\ltimes$. □

Some fundamental properties of the left semi-tensor product, which are related to our approach are collected in the following:

Proposition 4.3. *The left semi-tensor product satisfies (as long as the related products are well defined)*

1. *(Distributive rule)*

$$\begin{aligned} &A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C; \\ &(\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{4.2}$$

2. *(Associative rule)*

$$\begin{aligned} &A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C; \\ &(B \ltimes C) \ltimes A = B \ltimes (C \ltimes A). \end{aligned} \tag{4.3}$$

Proposition 4.4. 1. *Assume A and B are of proper dimensions such that $A \ltimes B$ is well defined. Then*

$$(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}. \tag{4.4}$$

2. *In addition, assume both A and B are invertible; then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{4.5}$$

Proposition 4.5. *Let $X \in \mathbb{R}^t$ be a column and $A \in M_{m\times n}$ be an $m \times n$ matrix. Then*

$$X \ltimes A = (I_t \otimes A) \ltimes X. \tag{4.6}$$

Note that when $\xi \in \mathbb{R}^n$ is a column or a row, then $\underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}$ is well defined. We, therefore, denote

$$\xi^k := \underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}.$$

In the sequel, we need the tensor expression of polynomials. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then $x^k := \underbrace{x \ltimes \cdots \ltimes x}_{k}$, which is a basis of k-th degree polynomials. A k-th degree homogeneous polynomial can be expressed as $\alpha \ltimes x^k$, where the coefficient vector α is a $1 \times n^k$ row, briefly, $\alpha x^k := \alpha \ltimes x^k$. Similarly, a vector field of k-th degree homogeneous polynomials can be expressed as $F \ltimes x^k$, where the coefficient vector F is an $n \times n^k$ matrix, briefly, $F x^k := F \ltimes x^k$.

Proposition 4.6. *Let αx^m and βx^n be m-th and n-th homogeneous polynomials respectively. Then the product is*

$$(\alpha x^m)(\beta x^n) = \alpha \ltimes \beta \ltimes x^{m+n}. \tag{4.7}$$

The following example shows how the above propositions may be used to form the product of vector fields, Lie derivatives of forms etc.

Example 4.7. Let X and Y be k-th and s-th degree homogeneous polynomial vector fields. Then we can express X and Y as

$$X = F x^k, \qquad Y = G x^s,$$

where F and G are $n \times n^k$ and $n \times n^s$ matrices respectively. Then

$$\begin{aligned} X \ltimes Y &= F x^k \ltimes G x^s = F \ltimes (x^k \ltimes G) \ltimes x^s \\ &= F(I_{n^k} \otimes G) x^{k+s}. \quad \square \end{aligned} \tag{4.8}$$

Return back to the calculation of the approximation of the center manifold. Set

$$X = \left(\phi_2^1(z), \ldots, \phi_{\rho_1}^1(z), \ldots, \phi_2^m(z), \ldots, \phi_{\rho_m}^m(z)\right)^{\mathrm{T}}.$$

Using the Taylor expansion on (3.2), we can express it as

$$A_0 X + A_1(z) X^2 + A_2(z) X^3 + \cdots = B_0 z^2 + B_1 z^3 + \cdots, \tag{4.9}$$

where A_0 is a constant invertible matrix.

Note that it is almost impossible to solve X from (4.9). Fortunately, we don't need to solve X precisely. Let the error degree of approximation be $d + 1$, i.e.,

$$\mathrm{EDA} = \|\phi(z) - h(z)\| = O(\|z\|^{d+1}).$$

Then it is enough to solve X as

$$X = C_0 z^2 + C_1 z^3 + \cdots + C_{d-2} z^d + O(\|z\|^{d+1}). \tag{4.10}$$

To solve X from (4.9), we have only to compare the coefficients on both sides. To start we have

$$C_0 = A_0^{-1} B_0.$$

We don't need all the terms of $A_i(z)$. In fact, ignoring higher degree (deg $> d$) terms, we may assume

$$\begin{aligned} &A_1(z) = A_0^1 + A_1^1 z + \cdots + A_{d-4}^1 z^{d-4}\\ &A_2(z) = A_0^2 + A_1^2 z + \cdots + A_{d-6}^2 z^{d-6}\\ &\ldots\\ &A_s(z) = 0, \quad 2s > d-2. \end{aligned} \tag{4.11}$$

Now (4.9) can be expressed as

$$\begin{aligned} &A_0 X + A_1(z)X^2 + A_2(z)X^3 + \cdots + A_{t-1}(z)X^t\\ &\quad = B_0 z^2 + B_1 z^3 + \cdots + B_{d-2} z^d + O(\|z\|^{d+1}), \end{aligned} \tag{4.12}$$

where $t = \left[\frac{d}{2}\right]$, which is the integral part of $\frac{d}{2}$.

Next, consider X^k. Using (4.8), we have

$$\begin{aligned} &X^2 = \sum_{\mu=4}^{d} \sum_{i+j=\mu} C_i (I_{n^i} \otimes C_j) z^\mu\\ &X^3 = \sum_{\mu=6}^{d} \sum_{i+j+k=\mu} C_i (I_{n^i} \otimes C_j)(I_{n^{i+j}} \otimes C_k) z^\mu\\ &\ldots. \end{aligned} \tag{4.13}$$

Plugging (4.11) and (4.13) into (4.9), we have a set of linear algebraic equations for C_i. If (4.9) is too complicated, we refer to [8], or [7] for a systematic treatment.

5. Stabilization of general affine nonlinear systems

In section 3, as well as in [9], when the stabilization of systems under the (generalized) Byrnes–Isidori normal form is considered, the nonlinear part z is considered as being of zero center (i.e., with zero linear part). If the dynamics of z have some unstable linear part (i.e., its Jacobian matrix at the origin has positive real part eigenvalues), the system is not stabilizable. So we can assume

A1. For system (1.1) let $A = J_f(0)$ and $B = (g_1(0), \ldots, g_m(0))$. The uncontrollable subspace of (A, B) does not contain any eigenvalues with a positive real part.

Under A1, the system should be allowed to have a partly stable linear part. In this case we can combine the linearly stable part (subspace associated with the negative real part eigenvalues) with the linearly controllable part (x). We propose the following general form for affine nonlinear systems.

$$\begin{cases} \dot{x} = Ax + \xi(x, z) + (B + \eta(x, z))u, & x \in \mathbb{R}^t\\ \dot{z} = q(x, z), & z \in \mathbb{R}^{n-t} \end{cases} \tag{5.1}$$

where $\xi(x, z)$ vanish with their first derivatives at zero, $\eta(0, 0) = 0$, (A, B) is assumed to be stabilizable.

For ease of exposition, assume A is a Hurwitz matrix, which can be assured by a linear pre-feedback control.

In this general case, we can choose higher degree (deg ≥ 2) polynomials as controls, say,

$$u(z) = B_0 z^2 + B_1 z^3 + \cdots + B_{d-2} z^d \in \mathbb{R}^m. \tag{5.2}$$

Choose $x = \phi(z)$ to approximate the center manifold, where $x_i = \phi_i(z)$, $i = 1, \ldots, t$ are solved from

$$Ax + \xi(x, z) + (B + \eta(x, z))u(z) = O(\|z\|^{d+1}). \tag{5.3}$$

The solution of (5.3) exists, because if we set the right hand side to be zero, a particular solution is assured by the implicit function theorem.

Now the algorithm described in Section 4 can be used to solve $\phi_i(z)$. For the approximation of the center manifold, using the same argument as for Theorem 3.2, we have the result:

Theorem 5.1. *Assume that* (1) *the error degree of the approximation is*

$$EDA = \frac{\partial \phi(z)}{\partial z} q(\phi(z), z) = O(\|z\|^{d+1});$$

(2) *the errors in the dynamics of the center manifold, caused by the approximation error, are*

$$q_k(\phi(z) + O(\|z\|^{d+1}), z) = q_k(\phi(z), z) + O(\|z\|^{t_k+1}), \quad k = 1, \ldots, r;$$

(3) *The approximated dynamics of the center manifold*

$$\dot{z}_k = q_k(\phi(z), z), \quad k = 1, \ldots, r,$$

are $(t_1, \ldots, t_r)$*-degrees approximately stable.*

Then the system (5.1) *is asymptotically stabilizable by control* (5.2) *satisfying* (5.3).

Remark. As for the case of the generalized Byrnes–Isidori normal form, the LFHD approach can also be used for the center manifold design of system (5.1).

6. Some examples

Consider a system in the generalized Byrnes–Isidori normal form.

Example 6.1.

$$\begin{cases} \dot{x}_1 = x_2 + x_3 u_1\\ \dot{x}_2 = \sin(z_1 + z_2) + u_1\\ \dot{x}_3 = x_4 - z_2 u_2\\ \dot{x}_4 = x_2 + u_2\\ \dot{z}_1 = -x_1 \sin(z_1 + z_2)\\ \dot{z}_2 = (x_1 + x_3) z_2^2 + z_1^4 z_2. \end{cases} \tag{6.1}$$

Since the equations of states z involve neither u nor the nonzero linear part, it is easy to verify that x is the largest approximately linearizable part. Following the procedure described in Section 3, we construct the stabilizer as follows. First, we choose the approximation for the center manifold as

$$\begin{aligned} &x_1 = \phi_1(z) \in O(\|z\|^2), \qquad x_2 = \phi_2(z),\\ &x_3 = \phi_3(z) \in O(\|z\|^2), \qquad x_4 = \phi_4(z). \end{aligned} \tag{6.2}$$

Note that $\phi_1(z)$ and $\phi_3(z)$ can be chosen freely, while $\phi_2(z)$ and $\phi_4(z)$ have to be solved from the restriction equation (3.2). Let the eigenvalues of the linear part be $\{-1,-1,-1,-1\}$. Then the controls are

$$\begin{cases} u_1 = -\sin(z_1+z_2) - x_1 - 2x_2 + \phi_1(z) + 2\phi_2(z) \\ u_2 = -x_2 - x_3 - 2x_4 + \phi_3(z) + 2\phi_4(z). \end{cases} \tag{6.3}$$

Then $x_2 = \phi_2(z)$ and $x_4 = \phi_4(z)$ are assumed to satisfy the following

$$\begin{cases} x_2(z) + \phi_3(z)[-\sin(z_1+z_2)] = 0 \\ x_4(z) - z_2(-x_2(z)) = 0. \end{cases} \tag{6.4}$$

That is,

$$\begin{cases} x_2(z) = \phi_2(z) = \sin(z_1+z_2)\phi_3(z) \\ x_4(z) = \phi_4(z) = -z_2\sin(z_1+z_2)\phi_3(z). \end{cases} \tag{6.5}$$

Check the error degree of approximation:

$$\begin{aligned} \text{EDA} &= \begin{pmatrix} \frac{\partial\phi_1}{\partial z_1} & \frac{\partial\phi_1}{\partial z_2} \\ \frac{\partial\phi_2}{\partial z_1} & \frac{\partial\phi_2}{\partial z_2} \\ \frac{\partial\phi_3}{\partial z_1} & \frac{\partial\phi_3}{\partial z_2} \\ \frac{\partial\phi_4}{\partial z_1} & \frac{\partial\phi_4}{\partial z_2} \end{pmatrix} \begin{pmatrix} -\phi_2(z)\sin(z_1+z_2) \\ [\phi_1(z)+\phi_3(z)]z_2^2 + z_1^2 z_2^3 \end{pmatrix} \\ &= O(\|z\|^4). \end{aligned} \tag{6.6}$$

Now we consider the dynamics on the center manifold.

$$\begin{cases} \dot z_1 = -[\phi_1(z) + O(\|z\|^4)]\sin(z_1+z_2) \\ \quad = -\phi_1(z)\sin(z_1+z_2) + O(\|z\|^5) \\ \dot z_2 = [\phi_1(z)+\phi_3(z)+O(\|z\|^4)]z_2^2 + z_1^4 z_2 \\ \quad = [\phi_1(z)+\phi_3(z)]z_2^2 + z_1^4 z_2 + O(\|z\|^6). \end{cases} \tag{6.7}$$

If we can choose $\phi_1(z)$ and $\phi_3(z)$ such that the approximated system of (6.7)

$$\begin{cases} \dot z_1 = -\phi_1(z)(z_1+z_2) \\ \dot z_2 = [\phi_1(z)+\phi_3(z)]z_2^2 + z_1^4 z_2 \end{cases} \tag{6.8}$$

is $(3,5)$-degree approximately stable, we are done. Choosing $\phi_1(z) = \alpha z_1^2$ and $\phi_3(z) = -\alpha z_1^2 + \beta z_2^3$, (6.8) becomes

$$\begin{cases} \dot z_1 = -\alpha z_1^3 - \alpha z_1^2 z_2 \\ \dot z_2 = \beta z_2^5 + z_1^4 z_2. \end{cases} \tag{6.9}$$

Setting a LFHD as

$$V = z_1^4 + z_2^2,$$

and choosing $\alpha = 3$, $\beta = -2$, we then have

$$\begin{aligned} \dot V_{(6.9)} &= -12z_1^6 - 12z_1^5 z_2 - 4z_2^6 + 2z_1^4 z_2^2 \\ &\le -12z_1^6 + (10z_1^6 + 2z_2^6) - 4z_2^6 + \left(\frac{4}{3}z_1^6 + \frac{2}{3}z_2^6\right) \\ &= -\frac{2}{3}z_1^6 - \frac{4}{3}z_2^6 < 0. \end{aligned}$$

The inequality in the above follows from the inequality (3.1) of [6].

Summarizing the above argument, we conclude that the system (6.1) can be stabilized by controls (6.3) with

$$\begin{cases} \phi_1(z) = 3z_1^2 \\ \phi_2(z) = -\sin(z_1+z_2)(3z_1^2 + 2z_2^3) \\ \phi_3(z) = -3z_1^2 - 2z_2^3 \\ \phi_4(z) = -3z_1^2 z_2 \sin(z_1+z_2). \quad \square \end{cases}$$

The second example is a general affine nonlinear system. In this Case, the approximation of the center manifold cannot be solved from (3.2) precisely. It has to be solved as a certain degree approximation.

Example 6.2. Consider the stabilization of the following system:

$$\begin{cases} \dot x_1 = -x_1 - \sin(x_2 z_1) \\ \dot x_2 = 2x_1 - x_2 + x_1^2 + \cos(x_2 z_1)u \\ \dot z_1 = x_1 x_2 \\ \dot z_2 = (x_1+x_2)z_2. \end{cases} \tag{6.10}$$

Let $x_1(z) = \phi_1(z)$ and $x_2(z) = \phi_2(z)$, solved from (3.2) for (6.10), be of degree $O(\|z\|^2)$. This is fundamental for defining its center manifold. Then the error degree of approximation is

$$\text{EDA} = \frac{\partial\phi(z)}{\partial z}\begin{pmatrix} \phi_1(z)\phi_2(z) \\ [\phi_1(z)+\phi_2(z)]z_2 \end{pmatrix} = O(\|z\|^4).$$

Now if we can find $\phi_1(z)$ and $\phi_2(z)$ such that the dynamics on the center manifold

$$\begin{cases} \dot z_1 = [\phi_1(z)+O(\|z\|^4)][\phi_2(z)+O(\|z\|^4)] \\ \quad = \phi_1(z)\phi_2(z) + O(\|z\|^6) \\ \dot z_2 = [\phi_1(z)+\phi_2(z)+O(\|z\|^4)]z_2 \\ \quad = [\phi_1(z)+\phi_2(z)]z_2 + O(\|z\|^5) \end{cases} \tag{6.11}$$

is asymptotically stable we are done. It suffices that

$$\begin{cases} \dot z_1 = \phi_1(z)\phi_2(z) \\ \dot z_2 = [\phi_1(z)+\phi_2(z)]z_2 \end{cases} \tag{6.12}$$

are $(5,3)$-degree approximately stable.

Choosing $u = \alpha z_1^2 + \beta z_1 z_2 + \gamma z_2^2$, (3.2) becomes

$$\begin{cases} -x_1 - \sin(x_2 z_1) = 0 \\ 2x_1 - x_2 + \cos(x_2 z_1)(\alpha z_1^2 + \beta z_1 z_2 + \gamma z_2^2) = 0. \end{cases} \tag{6.13}$$

Now x_1 and x_2 cannot be solved from (6.13) accurately. Due to the approximation error, we know that they have to be solved precisely up to all third degree terms. After some tedious algebraic computations, using the tool developed in Section 4, we have

$$\begin{aligned} \phi_1(z) &= x_1(z) = -\left[\alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2\right] + O(\|z\|^4) \\ \phi_2(z) &= x_2(z) \\ &= \left[\alpha z_1^2 + \beta z_1 z_2 + \gamma z_2^2\right](1+2z_1) + O(\|z\|^4). \end{aligned} \tag{6.14}$$

Choosing $\beta = 0$, (6.12) leads to

$$\begin{cases} \dot z_1 = -\alpha^2 z_1^5 - \gamma^2 z_1 z_4 \\ \dot z_2 = \gamma z_2^3 + \alpha z_1^2 z_2. \end{cases} \tag{6.15}$$

Setting $V = z_1^2 + z_2^4$ and choosing $\alpha = 1, \gamma = -1$, we then have

$$\dot{V}_{(6.15)} = -2z_1^6 - 2z_1^2 z_2^4 - 4z_2^6 + 4z_1^2 z_2^4$$
$$\leq -2z_1^6 - 4z_2^6 + 4\left(\frac{1}{3}z_1^6 + \frac{2}{3}z_2^6\right) = -\frac{2}{3}z_1^6 - \frac{4}{3}z_2^6 < 0.$$

So the system (6.10) can be stabilized by the control

$u = z_1^2 - z_2^2$. □

Remark. From the above two examples, one sees that under the (generalized) normal form, the design of the center manifold is much easier than starting from the general form, but the "trade off" is the additional normal form transformation.

7. Conclusion

This paper considered the problem of the stabilization of affine nonlinear control systems via the design of a center manifold. The basic technique has been developed in [6] for systems in the Byrnes–Isidori normal form. Then in [9] we proposed the generalized Byrnes–Isidori normal form, which covers a much larger class of affine nonlinear systems (almost all). Meanwhile, it was also proved that the design technique of the stabilizer via the center manifold approach, developed in [6], can also be used for systems of generalized normal form. Motivated by some simple examples, it was revealed that the design technique for standard normal form is not subtle enough for the generalized normal form. So in this paper we proposed a new design technique to design the whole approximate center manifold $x_j^i = \phi_j^i(z)$, $j \geq 1$, so as to improve (raise) the EDA and to meet the accuracy requirement of the dynamics on the center manifold. It significantly improved the existing technique, which designs $x_1^i = \phi_1^i(z)$ only. It is easy to prove that in the standard normal form our new technique degenerates to the classic one.

It has been proved that the method can also be used for a more general case: when the linear part (linear approximation, or Jacobian linearization—as it is called in some literatures) consists of two parts: the stabilizable part and the center part (with zero real part eigenvalues). Some examples were presented to demonstrate the design procedure.

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