Simultaneous Stabilization for a Collection of Multi-input Nonlinear Systems

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(Received 7 September 2005; Revised 25 October 2005)


Abstract This paper considers the problem of designing a controller that simultaneously stabilizes a collection of multi-input nonlinear systems. Based on the technique of control Lyapunov function, a sufficient condition for the existence of time-invariant simultaneously stabilizing state feedback controller is obtained. A universal formula for constructing a continuous controller which simultaneously stabilizes the systems is derived. The result extends the existing one in Ref. [1] for the single-input case.

Key words simultaneous stabilization, affine nonlinear system, control Lyapunov function, state feedback

CLC O231.2

1 Introduction

The simultaneous stabilization problem is important in the field of robust control. The problem is designing a single controller which simultaneously stabilizes a finite collection of systems. The simultaneous stabilization problem was firstly introduced in Ref. [2] and Ref. [3]. It was showed that this problem reduces to a strong stabilization problem in the case of two plants. For linear systems, some necessary and sufficient conditions for the existence of simultaneously stabilizing state feedback and output feedback controllers have been obtained, see e.g., Refs. [4~6], and references therein. For nonlinear systems, the simultaneous stabilization problem is more difficult. Ref. [7] and Ref. [1] have presented some results for nonlinear systems. Ref. [7] designed a continuous state feedback controller which simultaneously stabilizes a countable family of nonlinear systems and provided a sufficient condition for the existence of simultaneously asymptotically stabilizing controller for a collection of nonlinear systems. However, the designed controller is not easy to implement for it is constructed as a sum of infinite time-varying functions. Moreover, the sufficient condition is difficult to verify because it requires firstly finding an asymptotically stabilizing state feedback controller for each system, then deriving a corresponding Lyapunov function for each closed-loop system, and finally determining whether some infinite sequences of time-varying functions exist that satisfy some specified conditions. Ref. [1] designed a more applicable controller for simultaneously stabilizing a collection of single-input affine nonlinear systems, and provided necessary and sufficient conditions for the existence of

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simultaneously stabilizing state feedback controller. Similar to Ref. [1], applying the control Lyapunov function (see e.g. Ref. [8] and Ref. [9]), this paper extends the results in Ref. [1] to multi-input affine nonlinear systems.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 presents sufficient conditions for the existence of simultaneously globally asymptotically stabilizing state feedback controller for a collection of multi-input affine nonlinear systems, and provides a universal formula for simultaneously stabilizing controller. Section 4 gives an illustrative example. Section 5 is the conclusion.

2 Preliminaries

Consider an affine nonlinear system
\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, f(0) = 0, \tag{1} \]
with the state \( x \in \mathbb{R}^n \) and the control input \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \), where vector fields \( f \) and \( g_i, i = 1, \ldots, m \), are smooth (i.e., \( f, g_i : \mathbb{R}^n \to \mathbb{R}^n \) for some suitable \( r > 0 \))

**Definition 2.1** A smooth, proper, and positive definite function \( V \) is a control Lyapunov function (CLF) for system (1), if for all \( x \in \mathbb{R}^n \setminus \{0\} \)
\[ \inf_{u \in \mathbb{R}^n} \left\{ \nabla V(x) \cdot \left[ f(x) + \sum_{i=1}^{m} g_i(x) u_i \right] \right\} < 0. \tag{2} \]
Denote
\[ a(x) := \nabla V(x) \cdot f(x), b_i(x) := \nabla V(x) \cdot g_i(x), i = 1, \ldots, m, \]
and set
\[ B(x) := (b_1(x), \ldots, b_m(x)) \quad \beta(x) := \left\| B(x) \right\|^2 = \sum_{i=1}^{m} b_i^2(x). \]
Then (2) is equivalent to the following expression in Ref. [8] \[ \beta(x) = 0 \Rightarrow a(x) < 0. \]

**Definition 2.2** A CLF \( V \) for system (1) is said to be satisfying small control property, if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that, if \( x \neq 0 \) satisfies \( \| x \| < \delta \), then there is some \( u \) with \( \| u \| < \epsilon \), such that
\[ \nabla V(x) \cdot \left[ f(x) + \sum_{i=1}^{m} g_i(x) u_i \right] < 0. \tag{3} \]
Denote
\[ \phi(a(x), \beta(x)) := \begin{cases} a(x) + \sqrt{a^2(x) + \beta^4(x)} & \text{if } \beta(x) \neq 0, \\ 0 & \text{if } \beta(x) = 0, \end{cases} \tag{4} \]
\[ p_i(x) := \begin{cases} -b_i(x) \phi(a(x), \beta(x)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{5} \]

**Theorem 2.3** \( p_i(x) \) is continuous if the CLF \( V \) for the system (1) satisfies the small control property, and the control law \( p = (p_1, \ldots, p_m) \) globally asymptotically stabilizes system (1).

3 Simultaneous stabilization

Consider a collection of affine nonlinear systems
\[ \dot{x} = f_i(x) + \sum_{i=1}^{m} g_i(x) u_i, f_i(0) = 0, i = 1, 2, \ldots, q, \tag{6} \]
where the state \( x \in \mathbb{R}^n \), the control input \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \), and for each \( i \in \{1, 2, \ldots, q\} \), vector fields \( f_i(x) \) and \( g_i(x) = (g_{i1}(x), \ldots, g_{im}(x)) \) are smooth.

Assume \( V_i(x) \) is a CLF for the system \( \Sigma_i, i = 1, \ldots, q \). For \( i = 1, \ldots, q \) and \( j = i, \ldots, m \), and set
\[ a_i(x) := \nabla V_i(x) \cdot f_i(x), b_i(x) := \nabla V_i(x) \cdot g_i(x), B_i(x) := (b_{i1}(x), \ldots, b_{im}(x)), \]
\[ \delta_i(x) := \sqrt{a_i^2(x) + \beta_i^4(x)} \]
\[ p_i(x) := -b_i(x) \delta_i(x) \]
For $j = 1, \ldots, m$, define
\[ I_j^0(x) := \{ i \in \{1, \ldots, q \} \mid b_i(x) > 0 \}, \quad I_j^0(x) := \{ i \in \{1, \ldots, q \} \mid b_i(x) < 0 \}, \quad I_j^0(x) := \{ i \in \{1, \ldots, q \} \mid b_i(x) = 0 \}. \]
and
\[ D_j^0 := \{ x \in \mathbb{R}^n \mid I_j^0(x) = \emptyset \} \quad \text{and} \quad D_j^0 := \{ x \in \mathbb{R}^n \mid I_j^0(x) \neq \emptyset \} \]
\[ D_j^0 := \{ x \in \mathbb{R}^n \mid I_j^0(x) \neq \emptyset \} \quad \text{and} \quad D_j^0 := \{ x \in \mathbb{R}^n \mid I_j^0(x) \neq \emptyset \}. \]
where $\emptyset$ denotes the empty set. Obviously,
\[ I_j^0(x) \cup I_j^0(x) \cup I_j^0(x) = \{1, \ldots, q\} \] for each $j \in \{1, \ldots, m\}$, and
\[ D_j^0 \cup D_j^0 \cup D_j^0 \cup D_j^0 = \mathbb{R}^n \] for each $j \in \{1, \ldots, m\}$.

Moreover, for each $j \in \{1, \ldots, m\}$, sets $I_j^0, I_j^0, I_j^0$ are disjoint, and sets $D_j^0, D_j^0, D_j^0, D_j^0$ are also disjoint. Function $V_i(x), i = 1, \ldots, q$, are smooth, so the origin $x = 0$ belongs to $D_j^0, j = 1, \ldots, m$.

For $i = 1, \ldots, q$ and $j = 1, \ldots, m$, take
\[ \Phi_i(a_i(x), \beta_i(x)) := \begin{cases} a_i(x) + \frac{\beta_i(x)}{\sqrt{a_i(x) + \beta_i(x)}} & \text{if } \beta_i(x) \neq 0, \\ 0 & \text{if } \beta_i(x) = 0. \end{cases} \]
\[ p_i(x) := \begin{cases} -b_i(x) \Phi_i(a_i(x), \beta_i(x)) & x \neq 0, \\ 0 & x = 0. \end{cases} \]

For each $j \in \{1, \ldots, m\}$, let
\[ u_j^0(x) := \begin{cases} \min_{i \in I_j^0} \frac{-b_i(x) a_i(x)}{\beta_i(x)} & \text{if } I_j^0(x) \neq \emptyset, \\ +\infty & \text{if } I_j^0(x) = \emptyset, \end{cases} \]
\[ u_j^0(x) := \begin{cases} \max_{i \in I_j^0} \frac{-b_i(x) a_i(x)}{\beta_i(x)} & \text{if } I_j^0(x) \neq \emptyset, \\ -\infty & \text{if } I_j^0(x) = \emptyset. \end{cases} \]
and
\[ u_j^0(x) := \begin{cases} \frac{1}{2} (u_j^0(x) + u_j^0(x)) & \text{if } x \in D_j^0, \\ \text{undefined} & \text{elsewhere.} \end{cases} \]

For each $i \in \{1, \ldots, q\}$, denote
\[ \beta_i^0 := \{ x \in \mathbb{R}^n \mid \beta_i(x) = 0 \}. \]

The following result gives a sufficient condition for the existence of simultaneously globally asymptotically stabilizing state feedback controller for a collection of multi-input affine nonlinear systems. It extends the corresponding result in Ref. [1].

**Theorem 3.1** Consider the collection of affine nonlinear systems (6). If there exists a collection of CLFs $V_i(x), i = 1, \ldots, q$, satisfying the small control property, such that for all $j \in \{1, \ldots, m\}$ and for all $x \in D_j^0$, the following conditions are satisfied.

1. \[ \max_{i \in I_j^0} \left\{ -b_i(x) \frac{a_i(x)}{\beta_i(x)} \right\} \leq \min_{i \in I_j^0} \left\{ -b_i(x) \frac{a_i(x)}{\beta_i(x)} \right\}. \]
2. If $i \in I_j^0(x)$, then $x \in \beta_i^0$ (i.e., $i \in \cap_{j=1}^m \beta_j^0(x)$).
3. For all $i \in \{1, \ldots, q\}$
\[
\lim_{t \to \infty} \frac{\beta_i(x)}{b_i(x)} = 0,
\]

for \( x \in \mathbb{R}^2 \).

Then there exists a continuous state feedback control law \( p(x) = \left( p_1(x), \ldots, p_n(x) \right) \) with

\[
p_i(x) = \begin{cases} 
p_i^z(x), & \text{if } \emptyset \not= I_i^z(x) \\
\text{undefined}, & \text{otherwise} \end{cases}
\]

\[
p_i^p(x) = \begin{cases} 
\min_{\beta_i(x) \in I_i^p(x)} p_i(x), & \text{if } \emptyset \not= I_i^p(x) \\
\text{undefined}, & \text{otherwise} \end{cases}
\]

\[
p_i^N(x) = \begin{cases} 
\max_{\beta_i(x) \in I_i^N(x)} p_i(x), & \text{if } \emptyset \not= I_i^N(x) \\
\text{undefined}, & \text{otherwise} \end{cases}
\]

\[
p_i^M(x) = \begin{cases} 
\min_{\beta_i(x) \in I_i^M(x)} (p_i^N(x), \max_{\beta_i(x) \in I_i^M(x)} (p_i^p(x), b_i^M(x))) , & \text{if } x \in D_i^M, \\
\text{undefined}, & \text{elsewhere} \end{cases}
\]

\( p_i^z(x) = 0 \),

which simultaneously globally asymptotically stabilizes the collection of affine nonlinear systems (6),

where

\[
p_i^z(x) = 0,
\]

\[
p_i^p(x) = \begin{cases} 
\min_{\beta_i(x) \in I_i^p(x)} p_i(x), & \text{if } \emptyset \not= I_i^p(x) \\
\text{undefined}, & \text{otherwise} \end{cases}
\]

\[
p_i^N(x) = \begin{cases} 
\max_{\beta_i(x) \in I_i^N(x)} p_i(x), & \text{if } \emptyset \not= I_i^N(x) \\
\text{undefined}, & \text{otherwise} \end{cases}
\]

and

\[
p_i^M(x) = \begin{cases} 
\min_{\beta_i(x) \in I_i^M(x)} (p_i^N(x), \max_{\beta_i(x) \in I_i^M(x)} (p_i^p(x), b_i^M(x))) , & \text{if } x \in D_i^M, \\
\text{undefined}, & \text{elsewhere} \end{cases}
\]

\( p_i^z(x) = 0 \),

We prove this theorem in two steps. First, we show the feedback control law \( u = p(x) \) simultaneously stabilizes the collection of affine nonlinear systems (6). That is, to show the following inequalities

\[
a_i(x) + \sum_{j=1}^m b_{ij}(x) p_j(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall i \in \{1, \ldots, q\}
\]

hold. Second, we show the continuity of the vector function \( p(x) \).

A. Simultaneous stabilization

1. If \( x \in D_i^2 \setminus \{0\} \), for each \( j \in \{1, \ldots, m\} \), then \( p_j(x) = p_i^z(x) = 0 \), \( b_i(x) = 0 \), \( \forall i \in \{1, \ldots, q\} \).

2. If \( x \in D_i^p \), for each \( j \in \{1, \ldots, m\} \), then \( p_j(x) = p_i^p(x) = \min_{\beta_i(x) \in I_i^p(x)} p_i(x) \).

3. If \( i \in I_i^p(x) \), then \( b_i(x) > 0 \), so \( b_i(x) p_i(x) < b_i(x) p_i(x) < -b_i(x) \frac{a_i(x)}{\beta_i(x)} \).

4. If \( i \in I_i^N(x) \), then \( b_i(x) = 0 \), so \( b_i(x) p_i(x) = 0 \).

5. If \( x \in D_i^N \), for each \( j \in \{1, \ldots, m\} \), then \( p_j(x) = p_i^N(x) = \max_{\beta_i(x) \in I_i^N(x)} p_i(x) \).

6. If \( i \in I_i^M(x) \), then \( b_i(x) < 0 \), so \( b_i(x) p_i(x) < b_i(x) p_i(x) < -b_i(x) \frac{a_i(x)}{\beta_i(x)} \).

7. If \( i \in I_i^M(x) \), then \( b_i(x) = 0 \), so \( b_i(x) p_i(x) = 0 \).

IV. If \( x \in D_i^M \), for each \( j \in \{1, \ldots, m\} \), then \( p_j(x) = p_i^M(x) \).

Note that for all \( x \in D_i^M \), the inequality (12) holds, that is

\[
- b_i(x) \frac{a_i(x)}{\beta_i(x)} < - b_i(x) \frac{a_i(x)}{\beta_i(x)} , \quad \forall i \in I_i^p(x) \text{ and } \forall k \in I_i^N(x).
\]

According to the definition of \( p_i^M(x) \), obviously, the inequality

\[
\max_{\beta_i(x) \in I_i^M(x)} \left\{ - b_i(x) \frac{a_i(x)}{\beta_i(x)} \right\} < p_i^M(x) \quad < \min_{\beta_i(x) \in I_i^p(x)} \left\{ - b_i(x) \frac{a_i(x)}{\beta_i(x)} \right\}
\]
holds. Then

\[ (1) \quad b_y(x) p_i(x) - b_y^0(x) \frac{a_i(x)}{\beta_i(x)} \leq 0, \quad \forall i \in \mathcal{I}_y^0(x). \]

\[ (2) \quad b_y(x) p_i(x) - b_y^0(x) \frac{a_i(x)}{\beta_i(x)} \leq 0, \quad \forall i \in \mathcal{I}_y^0(x). \]

Furthermore, note that \( b_y(x) p_i(x) = 0 \), \( \forall i \in \mathcal{I}_y^0(x) \). Hence, for each \( j \in \{1, \ldots, m\} \), we have

\[ (1) \quad \text{if } x \in D_y^j \setminus \{0\}, \text{ then } b_y(x) p_i(x) = 0, \forall i \in \{1, \ldots, q\}. \]

\[ (2) \quad \text{if } x \in \bigcup_{j=1}^m D_y^j \cup D_y^N, \text{ then} \]

\[ \cap_i b_y(x) p_i(x) - b_y^0(x) \frac{a_i(x)}{\beta_i(x)} \leq 0, \quad \forall i \in \{1, \ldots, q\} \setminus \mathcal{I}_y^0(x), \]

\[ \cap_i b_y(x) p_i(x) = 0, \quad \forall i \in \mathcal{I}_y^0(x). \]

Therefore, for any given \( x \in \mathbb{R}^n \setminus \{0\} \), according to (8), we have

\[ (1) \quad \text{if } x \in \bigcup_{j=1}^m D_y^j \cup D_y^N, \text{ then} \]

\[ a_i(x) + \sum_{j=1}^m b_y(x) p_j(x) = a_i(x) < 0, \quad \forall i \in \{1, \ldots, q\}. \]

\[ (2) \quad \text{if there exists at least a } j \in \{1, \ldots, m\} \text{ such that } x \notin D_y^j, \text{ then } x \in \bigcup_{j=1}^m D_y^j \cup D_y^N. \]

\[ \cap_i b_y(x) p_i(x) - b_y^0(x) \frac{a_i(x)}{\beta_i(x)} = 0, \quad \forall i \in \{1, \ldots, q\} \setminus \mathcal{I}_y^0(x). \]

\[ \cap_i b_y(x) p_i(x) = 0, \quad \forall i \in \mathcal{I}_y^0(x). \]

Summarizing the above, we see that for any given \( x \in \mathbb{R}^n \setminus \{0\} \)

\[ a_i(x) + \sum_{j=1}^m b_y(x) p_j(x) < 0, \quad \forall i \in \{1, \ldots, q\}. \]

Therefore, the feedback control law \( u = p(x) \) simultaneously globally asymptotically stabilizes the collection of affine nonlinear systems (6).

B. Continuity of vector function \( p(x) \)

To prove the continuity of vector function \( p(x) \) it is enough to show \( p_i(x) \), \( j = 1, \ldots, m \), the components of \( p(x) \), are continuous. The proof of the continuity of \( p_i(x) \), \( j = 1, \ldots, m \), is similar to that in Ref. [1].

According to Theorem 2.3, for each \( i \in \{1, \ldots, q\} \), the function \( p_i(x) \), \( j = 1, \ldots, m \), are continuous if the CLF \( V_i(x) \) for the affine nonlinear system (6) satisfies the small control property so with (8) and (14), it is enough to prove the continuity of \( p_i(x) \), \( j = 1, \ldots, m \), on the boundary between the set \( \{ x \in \mathbb{R}^n \mid b_y(x) = 0 \} \) and the set \( \{ x \in \mathbb{R}^n \mid b_y(x) \neq 0 \} \), for each \( i \in \{1, \ldots, q\} \). For convenience, we prove \( p_i(x) \), \( j = 1, \ldots, m \), are continuous in the interior of \( D_y^j, D_y^N \), and on the boundaries between them.

First, we show \( p_i(x) \), \( j = 1, \ldots, m \), are continuous in the interior of \( D_y^j, D_y^N \), and on the boundaries between them.

For \( j = 1, \ldots, m \), obviously \( p_i^j(x) = 0 \) is continuous in the interior of \( D_y^j \). Since for each \( i \in \{1, \ldots, q\} \), the function \( p_i(x) \) is continuous, \( p_i^j(x) \) is continuous in the interior of \( D_y^j \), and \( p_i^N(x) \) is continuous in the interior of \( D_y^N \). In the following, we prove \( p_i^N(x) \) is continuous in the interior of \( D_y^N \). Let \( D_y^{np} := \{ x \in D_y^N \mid b_y(x) > 0 \} \) for each \( i \in \mathcal{I}_y^N(x) \). For any \( x \in D_y^{np} \), let \( D_y^{np}(\xi) \), \( i.e., b_y(x) = 0 \), be a sequence of vector \( x_i \in D_y^{np} \), such that \( x_i \to x \), as \( k \to + \infty \). Since \( x \in D_y^N \), so \( \mathcal{I}_y^N(x) \neq \emptyset \), then there exists an \( i \neq n \) such that \( b_y(x_i) > 0 \) for sufficiently large \( k \). Since \( i \in \mathcal{I}_y^N(x) \), according to the Condition 2.3, \( x \in \mathcal{I}_y^N \), \( i.e., i \in \mathcal{I}_y^N(x) \) then \( a_i(x_i) \).
< 0 for sufficient large $k$, and with (13) \(-b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \to +\infty\) as $x_i \to x$. So

\[
\lim_{i \to +\infty} u_i^p(x_i) = \lim_{i \to +\infty} \min_{x_i \in i^p} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} = \lim_{i \to +\infty} \min_{x_i \in i^p} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} = \min_{x_i \in i^p} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} = u_i^p(x_i).
\]

This implies $u_i^p(x)$ is continuous on the boundary, contained in $D_j^i$ of $D_0^{M'}$. Similarly, $u_i^{M'}(x)$ is continuous on the boundary, contained in $D_j^i$ of $D_0^{M'} = \{ x \in D_j^i | b_i(x) < 0 \}$ for each $i \in i^p(x)$. Hence $u_i^{M'}(x)$ is continuous in the interior of $D_j^i$, and $I_j^i$ is continuous in the interior of $D_j^i$.

Next, we prove $p_i(x)$ is continuous on the boundaries between $D_j^i$, $D_j^{M'}$, $D_j^i$, and $D_j^M$.

(1) For any $\bar{x} \in D_j^i$, the boundary between $D_j^i$ and $D_j^{M'}$ (i.e., $b_i(\bar{x}) = 0$ for all $i \in \{1, \ldots, q\}$), take a sequence of vector $\{ x_i \} \in D_j^{M'}$, such that $x_i \to \bar{x}$ as $k \to +\infty$. Then

\[
\lim_{i \to +\infty} p_i^p(x_i) = \lim_{i \to +\infty} \min_{x_i \in i^p} p_i^p(x_i) = 0 = p_i^p(\bar{x}).
\]

This implies $p_i(x)$ is continuous on the boundary between $D_j^i$ and $D_j^{M'}$.

(2) Similarly, $p_i(x)$ is continuous on the boundary between $D_j^i$ and $D_j^M$.

(3) For any $\bar{x} \in D_j^{M'}$, the boundary between $D_j^i$ and $D_j^{M'}$ (i.e., $b_i(\bar{x}) = 0$, $\forall i \in \{1, \ldots, q\}$), take a sequence of vector $\{ x_i \} \in D_j^{M'}$, such that $x_i \to \bar{x}$ as $k \to +\infty$. Then

\[
\lim_{i \to +\infty} p_i^M(x_i) = \lim_{i \to +\infty} \min_{x_i \in i^M} p_i^M(x_i) = 0 = p_i^M(\bar{x}).
\]

This implies $p_i(x)$ is continuous on the boundary between $D_j^i$ and $D_j^{M'}$.

(4) For any $\bar{x} \in D_j^{M'}$, the boundary between $D_j^i$ and $D_j^{M'}$ (i.e., $b_i(\bar{x}) \geq 0$, $\forall i \in \{1, \ldots, q\}$), take a sequence of vector $\{ x_i \} \in D_j^{M'}$, such that $x_i \to \bar{x}$ as $k \to +\infty$. Since $\bar{x} \in D_j^i$, $I_j^i(\bar{x}) \neq \emptyset$ and $I_j^i(\bar{x}) = \emptyset$. Then with (13), we have

\[
u_i^p = \min_{x_i \in i^p} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} \text{ finite as } k \to +\infty,
\]

and

\[
u_i^{M'}(x_i) = \max_{x_i \in i^M} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} \to -\infty \text{ as } k \to +\infty.
\]

So $u_i^M(x_i) \to -\infty$ as $k \to +\infty$. Therefore

\[
\lim_{i \to +\infty} p_i^M(x_i) = \lim_{i \to +\infty} \min_{x_i \in i^M} (p_i^M(x_i), \max(p_i^M(x_i), u_i^M(x_i))) = \min(0, \max(0, \lim_{i \to +\infty} u_i^M(x_i))) = \lim_{i \to +\infty} p_i^M(x_i) = p_i^M(x).
\]

This implies $p_i(x)$ is continuous on the boundary between $D_j^i$ and $D_j^{M'}$.

(5) For any $\bar{x} \in D_j^{M'}$, the boundary of $D_j^M$ (i.e., $b_i(\bar{x}) \leq 0$, $\forall i \in \{1, \ldots, q\}$), take a sequence of vector $\{ x_i \} \in D_j^{M'}$, such that $x_i \to \bar{x}$ as $k \to +\infty$. Since $\bar{x} \in D_j^{M'}$, $I_j^i(\bar{x}) = \emptyset$ and $I_j^i(\bar{x}) \neq \emptyset$. Then with (13), we have

\[
u_i^M(x_i) = \max_{x_i \in i^M} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} \text{ finite as } k \to +\infty,
\]

and

\[
u_i^p(x_i) = \min_{x_i \in i^p} \left\{ -b_j(x_i) \frac{a_i(x_i)}{\beta_j(x_i)} \right\} \to +\infty \text{ as } k \to +\infty.
\]

So $u_i^M(x_i) \to +\infty$ as $k \to +\infty$. Therefore,
\[ \lim_{i \to 0} p_i^N(x) = \lim_{i \to 0} (p_i^N(x), \max(p_i^F(x), u_i^N(x))) \]

\[ = \min(\lim_{i \to 0} p_i^N(x), \max(0, +\infty)) = \lim_{i \to 0} p_i^N(x) = p_i^N(x \to x). \]

This implies \( p_i(x) \) is continuous on the boundary between \( D_i^R \) and \( D_i^N \).

(6) According to the definition, the boundary between \( D_i^R \) and \( D_i^N \) belongs to \( D_i^Z \). It has been proved that \( p_i(x) \) is continuous in both \( D_i^R \cup D_i^Z \) and \( D_i^N \cup D_i^Z \). So it must be continuous on the boundary between \( D_i^R \) and \( D_i^N \).

(7) According to equation (16) and equation (17), for any \( \bar{x} \in D_i^S \) belongs to the boundary between \( D_i^R \), \( D_i^Z \), and \( D_i^N \), \( p_i(x) \) is continuous at \( \bar{x} \). This implies \( p_i(x) \) is continuous on the boundary between \( D_i^R \), \( D_i^Z \) and \( D_i^N \). Similarly, \( p_i(x) \) is continuous on the boundary between \( D_i^N \), \( D_i^Z \) and \( D_i^N \), and continuous on the boundary between \( D_i^R \), \( D_i^Z \) and \( D_i^N \). Note that boundary between \( D_i^R \), \( D_i^N \) and \( D_i^M \) belongs to \( D_i^Z \). Since \( p_i(x) \) is continuous in both \( D_i^R \cup D_i^Z \cup D_i^M \) and \( D_i^N \cup D_i^Z \cup D_i^M \), it must be continuous on the boundary between \( D_i^R \), \( D_i^N \) and \( D_i^M \). Therefore, \( p_i(x) \) is also continuous on the boundary between \( D_i^R \), \( D_i^N \), \( D_i^Z \) and \( D_i^M \).

Thus, the proof is completed.

**Remark 3.2** Assume the Conditions (2) and (3) of Theorem 3.1 are respectively replaced by

\[ (2)' \quad p_i^N(x) = \emptyset. \]  

\[ (3)' \quad \text{For all } i \in \{1, \ldots, q\} \]

\[ \lim_{i \to \infty} \frac{\theta_i(x)}{b_i(x)} < +\infty, \]  

where \( \bar{x} \in B_i^Z \).

Then the feedback control law \( p(x) \) can also simultaneously globally asymptotically stabilize the collection of the systems (6). In fact, if the Condition (2)' holds, then \( p_i^N \) is obviously continuous in the interior of \( D_i^N \), and if the Condition (3)' holds, then equation (18) and equation (19) are still valid.

### 4 An illustrative example

Consider the following three affine nonlinear systems:

\[ S_1: \begin{cases} x_1 &= x_1 x_2, \\ x_2 &= -x_2 - u_1 - u_2, \end{cases} \]

\[ S_2: \begin{cases} x_1 &= x_1 - 2x_1 x_2 (x_1^2 + x_2^2) + x_1 u_1 + x_1 u_2, \\ x_2 &= x_2 u_1 + x_2 u_2, \end{cases} \]

\[ S_3: \begin{cases} x_1 &= x_1 x_2, \\ x_2 &= -x_2 - u_1 - u_2. \end{cases} \]

Let \( x = [x_1 \ x_2]^T \), and take \( V_1(x) = \frac{1}{2} \left[ x_1^2 + (x_2 + x_1^2)^2 \right], \)

\( V_2(x) = \frac{1}{2} (x_1^2 + x_2^2), \)

\( V_3(x) = \frac{1}{2} \left[ x_1^2 + (x_2 + x_1^2)^2 \right]. \)

Then

\[ V_1(x) \big|_{S_1} = -x_2^2 + 2x_2^2 x_2 (x_1 + x_1^2) - (x_2 + x_1^2) u_1 - (x_2 + x_1^2) u_2, \]

\[ V_2(x) \big|_{S_2} = -x_1^2 - 2x_1^2 x_2 (x_1^2 + x_2^2) + (x_1^2 + x_2^2) u_1 + (x_1^2 + x_2^2) u_2, \]

\[ V_3(x) \big|_{S_3} = -x_2^2 + 2x_2^2 x_2 (x_1 - x_1^2) - (x_2 - x_1^2) u_1 - (x_2 - x_1^2) u_2. \]

Take \( u_1 = u_2 = x_1^2 x_2 \), then
Additionally, it is easy to see that the Condition (3) of Theorem 3.1 also holds. Let

\[
\begin{align*}
\Phi_1(\alpha_i(\beta_1(x))) & : = \begin{cases} 
\frac{a_1(x)}{\beta_1(x)} + \frac{a^2_i(x)}{\beta_1^2(x)} + \frac{b_1(x)}{\beta_1(x)} & \text{if } x_2 < -x_1^2, \\
0 & \text{if } x_2 = -x_1^2
\end{cases}, \\
\Phi_2(\alpha_i(\beta_2(x))) & : = \begin{cases} 
\frac{a_2(x)}{\beta_2(x)} + \frac{a^2_i(x)}{\beta_2^2(x)} + \frac{b_2(x)}{\beta_2(x)} & \text{if } x_2 < 0, \\
0 & \text{if } x_2 = 0
\end{cases}, \\
\Phi_3(\alpha_i(\beta_3(x))) & : = \begin{cases} 
\frac{a_3(x)}{\beta_3(x)} + \frac{a^2_i(x)}{\beta_3^2(x)} + \frac{b_3(x)}{\beta_3(x)} & \text{if } x_2 < x_1^2, \\
0 & \text{if } x_2 = x_1^2
\end{cases}
\end{align*}
\]
and for \( i = 1,2,3, j = 1,2 \), let

\[
p_j(x) = \begin{cases} \begin{align*}
- b_j (x) \Phi(a_j (x) \beta_j (x)) , & \text{if } x \neq 0 , \\
0 , & \text{if } x = 0 .
\end{align*} \end{cases}
\]

For \( j = 1,2 \), set \( p_j^2 (x) = 0 \), if \( x = 0 \),

\[
p_j^p (x) = \begin{cases} \begin{align*}
\min (p_{j1} (x) , p_{j2} (x) , p_{j3} (x)) , & \text{if } x_2 < - x_1^2 , \\
\min (p_{j2} (x) , p_{j3} (x)) , & \text{if } - x_1^2 \leq x_2 < x_1^2 , \\
p_{j2} (x) , & \text{if } x_2 \geq x_1^2 \text{ and } x \neq 0 , \\
\text{undefined} , & \text{if } x = 0 ,
\end{align*} \end{cases}
\]

\[
p_j^N (x) = \begin{cases} \begin{align*}
\max (p_{j1} (x) , p_{j3} (x)) , & \text{if } x_2 > x_1^2 , \\
p_{j1} (x) , & \text{if } x_2 < x_1^2 \text{ and } x \neq 0 , \\
\text{undefined} , & \text{if } x_2 \leq x_1^2 ,
\end{align*} \end{cases}
\]

\[
p_j^M (x) = \begin{cases} \begin{align*}
\min (p_j^N (x) , \max (p_j^p (x) , u_j^M (x))) , & \text{if } x_2 > - x_1^2 , \\
\text{undefined} , & \text{if } x_2 \leq - x_1^2 ,
\end{align*} \end{cases}
\]

where

\[
u_j^M (x) = \begin{cases} \begin{align*}
\frac{1}{2} & \left( - b_j \frac{a_j (x)}{\beta_j (x)} + \max \left( - b_j \frac{a_j (x)}{\beta_j (x)} , - b_j \frac{a_j (x)}{\beta_j (x)} \right) \right) , & \text{if } x_2 > x_1^2 , \\
\frac{1}{2} & \left( - b_j \frac{a_j (x)}{\beta_j (x)} - b_j \frac{a_j (x)}{\beta_j (x)} - b_j \frac{a_j (x)}{\beta_j (x)} - b_j \frac{a_j (x)}{\beta_j (x)} \right) , & \text{if } - x_1^2 < x_2 < x_1^2 , \\
\frac{1}{2} & \left( - b_j \frac{a_j (x)}{\beta_j (x)} - b_j \frac{a_j (x)}{\beta_j (x)} \right) - b_j \frac{a_j (x)}{\beta_j (x)} , & \text{if } x_2 = x_1^2 \text{ and } x \neq 0 , \\
\text{undefined} , & \text{if } x_2 \leq x_1^2 .
\end{align*} \end{cases}
\]

Therefore, according to Theorem 3.1, there exists a continuous feedback control law \( p(x) = (p_1 (x) , p_2 (x)) \), with

\[
p_j (x) = \begin{cases} \begin{align*}
p_j^p (x) , & \text{if } x = 0 , \\
p_j^p (x) , & \text{if } x_2 \leq x_1^2 \text{ and } x \neq 0 , \\
p_j^M (x) , & \text{if } x_2 > - x_1^2 ,
\end{align*} \end{cases}
\]

simultaneously globally asymptotically stabilizes the systems \( S_1 , S_2 \), and \( S_3 \).

**Remark 3.3** If we just consider simultaneous stabilization for systems \( S_1 \) and \( S_2 \), it is easy to see that sets \( D_j^p , D_j^N , D_j^M \) are the same as the above, and for all \( x \in D_j^M \), Conditions (1), (2)', and (3)' are satisfied. So according to Remark 3.2, there exists a continuous feedback control law, which simultaneously globally asymptotically stabilizes the systems \( S_1 \) and \( S_2 \).

In fact, for \( j = 1,2 \), let

\[
p_j^p (x) = \begin{cases} \begin{align*}
\min (p_{j1} (x) , p_{j2} (x)) , & \text{if } x_2 < - x_1^2 , \\
p_{j2} (x) , & \text{if } x_2 \geq - x_1^2 \text{ and } x \neq 0 , \\
\text{undefined} , & \text{if } x = 0 ,
\end{align*} \end{cases}
\]

\[
p_j^N (x) = \begin{cases} \begin{align*}
p_{j1} (x) , & \text{if } x_2 > x_1^2 , \\
\text{undefined} , & \text{if } x_2 \leq x_1^2 ,
\end{align*} \end{cases}
\]

\[
p_j^M (x) = \begin{cases} \begin{align*}
\min (p_j^N (x) , \max (p_j^p (x) , u_j^M (x))) , & \text{if } x_2 > - x_1^2 , \\
\text{undefined} , & \text{if } x_2 \leq - x_1^2 ,
\end{align*} \end{cases}
\]

where

\[
u_j^M (x) = \begin{cases} \begin{align*}
\frac{1}{2} & \left( - b_j (x) \frac{a_j (x)}{\beta_j (x)} - b_j (x) \frac{a_j (x)}{\beta_j (x)} \right) , & \text{if } x_2 > - x_1^2 , \\
\text{undefined} , & \text{if } x_2 \leq - x_1^2 .
\end{align*} \end{cases}
\]
According to Remark 3.2, there exists a continuous feedback control law \( p' (x) = (p'_{1}(x), p'_{2}(x)) \), with

\[
p'_{j}(x) = \begin{cases} 
p_{j}^{p}(x), & \text{if } x = 0, \\
p_{j}^{p}(x), & \text{if } x_{2} \leq -x_{1}^{2}, \text{and } x \neq 0, \\
p_{j}^{M}(x), & \text{if } x_{2} > x_{1}^{2},
\end{cases}
\]

simultaneously globally asymptotically stabilizes the systems \( S_{1} \) and \( S_{2} \).

### 5 Conclusion

The paper considered the problem of simultaneous stabilization for a collection of multi-input affine nonlinear systems. Sufficient conditions for the existence of the controller have been obtained, and a feedback controller has been designed. An illustrative example was included to demonstrate the design procedure.

### References


