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Lyapunov-Based Approach to Multiagent Systems With Switching Jointly Connected Interconnection

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***Abstract*—This note addresses a coordination problem of a multiagent system with jointly connected interconnection topologies. Neighbor-based rules are adopted to realize local control strategies for these continuous-time autonomous agents described by double integrators. Although the interagent connection structures vary over time and related graphs may not be connected, a sufficient condition to make all the agents converge to a common value is given for the problem by a proposed Lyapunov-based approach and related space decomposition technique.**

***Index Terms*—Common Lyapunov function, double integrator, joint connection, multiagent system.**

I. INTRODUCTION

Collective behaviors of large numbers of autonomous agents have drawn a broad interest from various research communities in the last decades. With applications in different disciplines including physics, ecology, and engineering [1], [2], [7], [11]–[13], [15], the dynamics and control of multiagent systems have been studied in order to achieve a cooperative goal with decentralized control laws or neighbor-based rules. Particularly, synchronized motion of agents in the leader-following coordination has been studied widely. For example, leader-following phenomena of self-propelled particles was observed [10], while an active leader could be followed by the agents with dynamic neighbor-based rules [4].

In practical situations, interaction between individual agents may change over time. Different conditions on variable interconnection graphs to achieve the coordination have been explored in recent years. In [6], the states of all the jointly connected agents converged to the same value or the value of a given leader, where the motions of nearest neighbors were averaged. [9] reported a simple network model of agents interacting via time-dependent communication links based on graph theory and set-valued Lyapunov theory. Additionally, [8] showed the agreement of multiagent systems with switching interconnection structure and nonlinear agent dynamics. In the study of variable topologies, joint connection is an important condition because it does not require the connection of the time-varying interconnection topology at any moment. For multiagent systems with simple neighbor-based average rules under joint-connection-like conditions, many research works focused on each agent expressed as

$$\dot{\phi}_i = u_i \quad \phi_i, u_i \in R^M, M \geq 1, \qquad i = 1, \ldots, n \tag{1}$$

where ϕ_i is the state (e.g., position or angle) of agent i and u_i its local control input for $i = 1, \ldots, n$.

The objective of this note is to analyze the coordination of multiagent systems with jointlyconnected interconnection and individual agents in the form of vector double integrator

$$\begin{cases} \dot{\phi}_i = q_i \\ \dot{q}_i = u_i \end{cases} \quad \phi_i, q_i, u_i \in R^M, \qquad i = 1, \ldots, n \tag{2}$$

where ϕ_i and u_i are the position (or angle) and control input, respectively, and q_i its velocity (or angular velocity) of agent i for $i = 1, \ldots, n$ (referring to [11] and [14]). However, the analysis idea (mainly based on graph theory and related stochastic matrix) that served system (1) effectively cannot be extended straightforward to study system (2) with jointlyconnected interconnection structures since the system matrix related to dynamics (2) along with the nearest-neighbor average rule does not satisfy some properties directly associated with stochastic matrices. Therefore, a Lyapunov-based approach is proposed here. Although the existence of common Lyapunov function and the construction of common Lyapunov functions are usually difficult to solve for switching multiagent networks (as [6] pointed out), a common Lyapunov function is explicitly constructed in the case of switching jointlyconnected topologies and then a coordination problem that each agent expressed as (2) flocks to a desired state (given by a leader) with a neighbor-based rule is solved.

II. FORMULATION AND MAIN RESULT

Consider a dynamical system consisting of n agents and a leader (or food source) with help of graph theory (referring to [3] for the details). The interconnection topology of n agents can be conveniently described by graph $\mathcal{G} = \{V, \varepsilon\}$, where $V = \{v_i, i = 1, \ldots, n\}$ is the set of vertices (representing n agents) and $\varepsilon \subset V \times V$ is the set of edges of the graph. If $(v_i, v_j) \in \varepsilon$, then v_i is said to be a neighbor of v_j and the set of all neighbor vertices of vertex v_j is denoted by $\mathcal{N}_j = \{i | (v_i, v_j) \in \varepsilon\}$. The leader (labelled 0) is represented by vertex v_0, and the connection between the agents and the leader (or food source) is directed; namely, there are only edges from some agents to the leader, but there are no edges from the leader to any agents. Then, we have a simple graph $\bar{\mathcal{G}}$ with vertex set $\bar{\mathcal{V}} = \mathcal{V} \cup \{v_0\}$, which contains graph $\mathcal{G}$ of n agents and vertex v_0 (representing the leader) with directed edges, if any, from some vertices of $\mathcal{G}$ to the leader vertex. Note that there may not be any connection between the agents and the leader at some moment. By "the graph $\bar{\mathcal{G}}$ is connected," we mean that there is at least one directed edge from some vertices of each (connected) component of $\mathcal{G}$ to the leader vertex v_0.

A union graph of a collection of simple graphs $\bar{\mathcal{G}}_1, \ldots, \bar{\mathcal{G}}_m$, with the same vertex set $\bar{\mathcal{V}}$ for some $m \geq 1$ is defined as a simple graph, denoted by $\bar{\mathcal{G}}_{1,\ldots,m}$ with vertex set $\bar{\mathcal{V}}$ and edge set equaling the union of the edge sets of all the graphs in the collections, and connection weight between edge i and edge j is the sum of nonzero a_{ij}'s of $\bar{\mathcal{G}}_1, \ldots, \bar{\mathcal{G}}_m$

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and connection strength between edge i and the leader is the sum of nonzero b_i's of $\bar{\mathcal{G}}_1, \ldots, \bar{\mathcal{G}}_m$. Moreover, this collection, $\bar{\mathcal{G}}_1, \ldots, \bar{\mathcal{G}}_m$, is jointly connected if its union graph $\bar{\mathcal{G}}_{1,\ldots,m}$ is connected (see [6] for the details).

An angle between two nonzero vectors $y, z \in R^n$ is defined as $\theta_{y,z} = \arccos(\langle y, z\rangle / \|y\| \, \|z\|)$, where $\langle y, z\rangle = y^T z$ and $\|y\| = \sqrt{\langle y, y\rangle}$ denotes a norm of y. If $\langle y, z\rangle = 0$, then the two vectors y and z are orthogonal, denoted by $y \perp z$. A subspace $S \subset R^n$ is trivial if it is either $\{0\}$ or the whole space R^n. Let S and S' be two nontrivial subspaces. If $\langle s, s'\rangle = 0$ for any $s \in S$ and $s' \in S'$, then the two subspaces are orthogonal, denoted by $S \perp S'$. Then a new subspace $S^* = S + S'$ with $S \perp S'$ will be denoted by $S^* = S \oplus_\perp S'$. If $R^n = S \oplus_\perp S'$, then S' is called the orthogonal complement of S, denoted by $S_\perp$. Moreover, the angle between two subspaces S and S' is defined as

$$\theta_{S,S'} = \min_{s\in S, s'\in S'} \theta_{s,s'} = \min_{s\in S, s'\in S'} \arccos \frac{|\langle s, s'\rangle|}{\|s\|\|s'\|}$$

with $0 \leq \theta_{S,S'} \leq \pi/2$.

Consider two symmetric matrices G_1 and G_2, which are positive semi-definite. Clearly

$$rank(G_1 + G_2) \geq \max\{rank(G_1), rank(G_2)\}. \quad (3)$$

Denote $rank(G_1) = \tilde{r}_1$ and $rank(G_1 + G_2) = \tilde{r}_{1,2}$. It is not hard to get the following lemma.

Lemma 1: For any positive–semidefinite matrices G_1 and G_2, there is an orthogonal matrix U, such that

$$U^T(G_1 + G_2)U = \begin{pmatrix} \tilde{G}_{1,2} & 0 \\ 0 & 0 \end{pmatrix} \quad (4)$$

with

$$U^T G_1 U = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \quad U^T G_2 U = \begin{pmatrix} \tilde{G}_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\Lambda \in R^{\tilde{r}_1 \times \tilde{r}_1}$ is a diagonal matrix with positive diagonal elements, $\tilde{G}_{1,2} \in R^{\tilde{r}_{1,2} \times \tilde{r}_{1,2}}$ is a positive–definite matrix, and $\tilde{G}_2 \in R^{\tilde{r}_{1,2} \times \tilde{r}_{1,2}}$ is a positive–semidefinite matrix.

In this note, the neighboring graph of the multiagent system is time-varying (i.e., the neighbor of agent $i, \mathcal{N}_i(t)$, changes over time), which can be described by the change of interconnection weights

$$a_{ij}(t) = \begin{cases} \alpha_{ij}, & \text{if agent } i \text{ is connected to} \\ & \quad \text{agent } j \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$$
$$b_i(t) = \begin{cases} \beta_i, & \text{if agent } i \text{ is connected to} \\ & \quad \text{the leader at time } t \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha_{ij} > 0$ and $\beta_i > 0$ are fixed constants $(i, j = 1, \ldots, n)$. Denote $\bar{\mathcal{S}} = \{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \ldots, \bar{\mathcal{G}}_N\}$ as a set of the graphs of all possible topologies, and $\mathcal{P} = \{1, 2, \ldots, N\}$ as the index set.

Consider an infinite sequence of nonempty, bounded, and contiguous time-intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, with $t_0 = 0$ and $t_{k+1} - t_k \leq T$ $(k \geq 0)$ for some constant $T > 0$. Suppose that in each interval $[t_k, t_{k+1})$ there is a sequence of nonoverlapping subintervals

$$[t_{k_0}, t_{k_1}), \ldots, [t_{k_j}, t_{k_{j+1}}), \ldots, \left[t_{k_{m_k-1}}, t_{k_{m_k}}\right)$$
$$t_k = t_{k_0}, \; t_{k+1} = t_{k_{m_k}} \quad (5)$$

satisfying $t_{k_{j+1}} - t_{k_j} \geq \tau$, $0 \leq j < m_k$ for some integer $m_k \geq 0$ and given constant $\tau > 0$ such that, during each of such subintervals, the interconnection topology described by of $\bar{\mathcal{G}}$ does not change. It is easy to see that there are at most

$$m_* = \left[\frac{T}{\tau}\right] + 1 \quad (6)$$

($[T/\tau]$ denotes the maximum integer no larger than T/τ) subintervals in each interval. For convenience, let $\sigma : [0, \infty) \to \mathcal{P}$ be a piecewise constant switching signal with successive times to describe the topology switches between subintervals.

For simplicity, each agent of the considered multiagent system is in the form of a double integrator (2) with $M = 1$. In this scenario, we consider how the agents converge (or flock) to a static leader (or food source), whose state is denoted by a constant ϕ_0, under jointly-connected interconnection.

Here is our main result.

Theorem 1: Consider any given initial condition $(\phi_i(0), q_i(0))$ of system (2) for $i = 1, \ldots, n$, a switching signal $\sigma : [0, \infty) \to \mathcal{P}$ corresponding to an infinite sequence of time-intervals $[t_k, t_{k+1})$, $k = 0, 1, \cdots$, with $t_0 = 0$ and $t_{k+1} - t_k \leq T$ $(k \geq 0)$ for $T > 0$, and a sequence of nonoverlapping subintervals in the form of (5) with $t_{k_{j+1}} - t_{k_j} \geq \tau$ in each interval for $\tau > 0$. If the collection of simple graphs (each graph in the form of $\bar{\mathcal{G}}$ associated with the time-invariant interconnection topology in each subinterval) in $[t_k, t_{k+1})$ (for $k = 0, 1, \ldots$) is jointly connected, then

$$\lim_{t\to\infty} \phi_i(t) = \phi_0 \quad \lim_{t\to\infty} q_i(t) = 0, \qquad i = 1, \ldots, n$$

by taking a neighbor-based feedback rule

$$u_i(t) = -\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\left(\phi_i(t) - \phi_j(t)\right) - b_i(t)\left(\phi_i(t) - \phi_0\right) - \kappa q_i(t), \qquad i = 1, \ldots, n \quad (7)$$

where

$$\kappa \geq \alpha + 1 \quad (8)$$

with $\alpha := \max_{i=1,\ldots,n}\{\beta_i\} + 2\max_{i=1,\ldots,n} \sum_{j=1}^{n} \alpha_{ij} + 1$.

Let $s_i = \phi_i - \phi_0$ $(i = 1, \ldots, n)$ and

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^{2n} \quad x_i = \begin{pmatrix} s_i \\ q_i \end{pmatrix} \in R^2 \quad (9)$$

where x_i is the state of agent i $(i = 1, \ldots, n)$. Then, the closed-loop multiagent system can be expressed as

$$\dot{x} = F_\sigma x \quad (10)$$

with

$$F_\sigma = I_n \otimes \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix} + H_\sigma \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad H_\sigma = L_\sigma + B_\sigma$$

where $\otimes$ denotes the Kronecker product [5], $I_n \in R^{n\times n}$ is the identity matrix, and $\sigma(t)$ is the switching signal defined in Theorem 1. L_σ is the Laplacian of the switching graph $\mathcal{G}$ consisting of n follower-agents, and B_σ is a diagonal matrix whose ith diagonal entry is b_i. Notice that σ takes some constant value $p \in \mathcal{P}$ (i.e., the corresponding interconnection does not change) during a given subinterval. Therefore, L_p and B_p are time-invariant matrices, and then H_p associated with graph $\bar{\mathcal{G}}_p$ $(p \in \mathcal{P})$ is a time-invariant matrix.

The conditions in Theorem 1 are almost the same as those given in [6]. However, the agent dynamics under consideration here is in the form of (2), different from the one in the form of (1) discussed in [6]. In fact, this result may be viewed as an extension of the joint-connection discussion presented in [6]. However, the analysis idea of [6] cannot be extended straightforwardly to study system (10) since F_σ fails to satisfy some properties associated with Laplacian or stochastic matrices.

Obviously, H_p of system (10) is positive semidefinite because both L_p and B_p are positive semidefinite. Moreover, based on [4, Lemma 3], we can easily obtain the following.

Lemma 2: If $\bar{\mathcal{G}}_p$ is connected for some p, then this symmetric matrix H_p is positive definite. Moreover, let matrices $H_{i_1}, \ldots, H_{i_m}$ be associated with the graphs $\bar{\mathcal{G}}_{i_1}, \ldots, \bar{\mathcal{G}}_{i_m}$, respectively. If these graphs are jointly connected, then $\sum_{j=1}^{m_i} H_j$ is positive definite.

III. Lyapunov Function

To prove Theorem 1, we only need to prove $\lim_{t\to\infty} x(t) = 0$ of system (10). Since the method based on stochastic analysis (reported in [6]) cannot be applied to F_σ, we propose a Lyapunov-based approach for system (10) with each agent in the form of (2). Here, we first construct a common Lyapunov function $V = x^T P x$ for system (10), where

$$P = I_n \otimes \begin{pmatrix} \kappa & 1 \\ 1 & 1 \end{pmatrix} \in R^{2n\times 2n}, \qquad \kappa > 1. \tag{11}$$

Obviously, for any given orthogonal matrix $U_0 \in R^{n\times n}$, P is unchanged under $U_0 \otimes I_2$

$$(U_0 \otimes I_2)^T P (U_0 \otimes I_2) = \left(U_0^T U_0\right) \otimes P_0 = P \quad P_0 = \begin{pmatrix} \kappa & 1 \\ 1 & 1 \end{pmatrix} \tag{12}$$

and the vector x can be transformed to

$$x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix} = (U_0 \otimes I_2)^T x \in R^{2n}$$
$$x_l^0 = \begin{pmatrix} s_l^0 \\ q_l^0 \end{pmatrix} \in R^2, \qquad l = 1, \ldots, n. \tag{13}$$

Thus, setting $\tilde{V}(\xi) = \xi^T P_0 \xi$, $\xi \in R^2$, we have

$$\begin{aligned} V(x) &= x^T P x = \sum_{i=1}^{n} \tilde{V}(x_i) \\ &= \sum_{i=1}^{n} \tilde{V}\left(x_i^0\right) = (x^0)^T P x^0 = V(x^0) \end{aligned} \tag{14}$$

by (12). In fact, $\tilde{V}(x_i)$ can be regarded as the energy function of x_i or agent i and $V(x)$ can be viewed as the total energy function of system (10). Moreover, define an energy vector $\nu = (\nu_1, \ldots \nu_n)^T \in R^n$ with $\nu_i = \sqrt{\tilde{V}(x_i)}$ to show how the "energy" (Lyapunov function $V(x)$) of this multiagent system is distributed. Note that $V(x)$ and $\|\nu(x)\| = \sqrt{V(x)}$ keep unchanged through any orthogonal transformations in the form of $U_0 \otimes I_2$ according to (14).

For a given matrix H_p ($p \in \mathcal{P}$) of rank $r_p > 0$, there is an orthogonal matrix $U_0 = U_p$ such that

$$U_p^T H_p U_p = \begin{pmatrix} \Lambda_p & 0 \\ 0 & 0 \end{pmatrix} \quad \Lambda_p \in R^{r_p \times r_p} \tag{15}$$

where diagonal matrix $\Lambda_p = diag\{\lambda_1(H_p) \ldots \lambda_{r_p}(H_p)\}$ is positive definite with $\lambda_i(H_p)$ ($i = 1, \ldots, r_p$) denoting the nonzero eigenvalue of H_p. Correspondingly, we have $x^p = (U_p \otimes I_2)^T x$. Based on (15) and (14), $V(x) = V(x^p) = \sum_{i=1}^n \tilde{V}(x_i^p)$. Then, we have the following.

Lemma 3: Let H_p be the matrix associated with the subinterval $[t_{k_l}, t_{k_{l+1}})$ (for any $k \geq 0$ and appropriate l and with $t_{k_l} - t_{k_{l+1}} \geq \tau$). Then, with κ given in (8), we have

$$\dot{V}(x(t)) \leq 0 \qquad \forall t \in \left[t_{k_l}, t_{k_{l+1}}\right) \tag{16}$$

and, moreover

$$\tilde{V}\left(x_i^p\left(t_{k_{l+1}}\right)\right) \leq \tilde{V}\left(x_i^p\left(t_{k_l}\right)\right), \qquad i = 1, \ldots n \tag{17}$$
$$\tilde{V}\left(x_i^p\left(t_{k_{l+1}}\right)\right) \leq \gamma_0 \tilde{V}\left(x_i^p\left(t_{k_l}\right)\right)\Big), \qquad i = 1, \ldots, r_p \tag{18}$$

for some positive real number $\gamma_0 < 1$.

Proof: From (8), $\kappa > 1$, which secures the positive definiteness of P defined in (11).

Note that F_p is time-invariant during any given subinterval

$$\dot{V}(x)|_{(10)} = x^T\left(F_p^T P + P F_p\right)x = -x^T Q_p x, \quad t \in \left[t_{k_l}, t_{k_{l+1}}\right)$$

where

$$\begin{aligned} Q_p &= -\left(F_p^T P + P F_p\right) \\ &= I_n \otimes \begin{pmatrix} 0 & 0 \\ 0 & 2(\kappa-1) \end{pmatrix} + H_p \otimes \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Then, based on (15)

$$\begin{aligned} (U_p \otimes I_2)^T Q_p (U_p \otimes I_2) = I_n \otimes &\begin{pmatrix} 0 & 0 \\ 0 & -2(\kappa-1) \end{pmatrix} \\ &+ \begin{pmatrix} \Lambda_p & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{19}$$

where $\Lambda_p = diag\{\lambda_1(H_p) \ldots \lambda_{r_p}(H_p)\}$ is positive definite. It is not hard to see that all the eigenvalues of H_p ($\forall p \in \mathcal{P}$) are less than α, and then, by (8)

$$Q_p^i = \begin{pmatrix} 2\lambda_i(H_p) & \lambda_i(H_p) \\ \lambda_i(H_p) & 2(\kappa-1) \end{pmatrix}, \qquad i = 1, \ldots, r_p$$

is positive definite. Therefore, Q_p is positive semidefinite, which implies (16).

Moreover, from (19), we can easily obtain $\dot{\tilde{V}}(x_i^p) \leq 0$, ($i = 1, \ldots, n$), which leads to (17).

Note that $\dot{\tilde{V}}(x_i^p) = -(x_i^p)^T Q_p^i x_i^p$ ($i = 1, \ldots, r_p$). Denote $\mu_{i,\pm}$ as the eigenvalues of Q_p^i, $i = 1, \ldots, r_p$, and then

$$\mu_{i,\pm} = \kappa + \lambda_i(H_p) - 1 \pm \sqrt{(\lambda_i(H_p) - \kappa + 1)^2 + \lambda_i(H_p)^2}.$$

Set $\bar{\lambda} := \min\{\text{positive eigenvalues of } H_p \ \forall p \in \mathcal{P}\} > 0$. Clearly, the minimum eigenvalue μ_{min} of Q_p^i takes the form of $\mu_{i,-}$. Again, based on (8), it can be found that the eigenvalues of Q_p^i are no less than $\bar{\mu} := k + \bar{\lambda} - 1 - \sqrt{(\kappa - \bar{\lambda} - 1)^2 + \bar{\lambda}^2}$. In addition, the maximum eigenvalue of P_0 defined in (12) is $\lambda_+(P_0) = (\kappa + 2 + \sqrt{(\kappa-1)^2 + 4})/2$. Therefore

$$\dot{\tilde{V}}\left(x_i^p\right) \leq -\frac{\bar{\mu}}{\lambda_+(P_0)} \tilde{V}\left(x_i^p\right) := -\beta \tilde{V}\left(x_i^p\right) \quad i = 1, \ldots, r_p.$$

In this way, (18) follows with $\gamma_0 = e^{-\beta\tau} < 1$ because $t_{k_{l+1}} - t_{k_l} \geq \tau$. ∎

Lemma 3 shows that any interconnection topology cannot make the Lyapunov function V increase due to (16). Moreover, the selection of κ given in (8) does not depend on γ_0.

Remark 1: Take $V_{p,0}(x) = \sum_{i=r_p+1}^{n} \tilde{V}(x_i^p)$ and $V_{p,\perp}(x) = \sum_{i=1}^{r_p} \tilde{V}(x_i^p)$. According to Lemma 3, there is a positive constant $\gamma_0 < 1$ such that

$$\begin{aligned} V_{p,0}\left(x\left(t_{k_{l+1}}\right)\right) &\le V_{p,0}\left(x\left(t_{k_l}\right)\right) \\ V_{p,\perp}\left(x\left(t_{k_{l+1}}\right)\right) &\le \gamma_0 V_{p,\perp}\left(x\left(t_{k_l}\right)\right)). \end{aligned} \tag{20}$$

The next lemma is proposed to estimate the convergent rate in the following section.

Lemma 4: Set $0 < \delta < 1$. For a constant $0 < \gamma_0 < 1$, define a sequence of constants

$$\gamma_{l+1} = (1-\delta^2)(1-\gamma_l) + \gamma_l, \qquad l = 0, 1, \ldots. \tag{21}$$

Then, we have $\gamma_{l+1} > \gamma_l$ and $0 < \gamma_l < 1, l = 0, 1, \ldots$.

IV. Convergence Analysis

Take an interval $[t_k, t_{k+1})$ with m_k subintervals, and, without loss of generality, denote the matrix associated with the (time-invariant) interconnection graph on subinterval $[t_{k_{i-1}}, t_{k_i})$ by H_i $(i = 1, \cdots, m_k \le m_*)$. Note that interval $[t_k, t_{k_{j-1}})$ consists of $j-1$ subintervals: $[t_k, t_{k_1})$, $[t_{k_1}, t_{k_2})$, and $[t_{k_{j-2}}, t_{k_{j-1}})$. It is easy to see that $H_{\widehat{j-1}} = \sum_{i=1}^{j-1} H_i$ is a matrix associated with the union graph of interval $[t_k, t_{k_{j-1}})$ (for $2 \le j \le m_k$). For convenience, we denote the rank of H_i as r_i and the rank of $H_{\widehat{i}}$ as $r_{\widehat{i}} (i = 1, \cdots, m_k)$.

A recursive graph-based space decomposition technique is employed to check the convergence situation in different regions over different subintervals. We will derive the convergence of the considered system during $[t_k, t_{k_j})$ based on the results in $[t_k, t_{k_{j-1}})$ and $[t_{k_{j-1}}, t_{k_j})$ (for $j \le m_k$). As we discussed, we have two matrices $H_{\widehat{j-1}}$ and H_j for $[t_k, t_{k_{j-1}})$ and $[t_{k_{j-1}}, t_{k_j})$, respectively. Then, $H_{\widehat{j}} = H_{\widehat{j-1}} + H_j$ is associated with the union graph on $[t_k, t_{k_j})$.

According to Lemma 1 (by taking $G_1 = H_{\widehat{j-1}}$ and $G_2 = H_j$, along with an orthogonal matrix $U = U^1$), there is a normalized orthogonal basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ of R^n (that is, $\vec{e}_i \perp \vec{e}_l$ when $i \ne l$ and $\|\vec{e}_i\| = 1$, $i = 1, \ldots, n$) such that we can obtain three subspaces.

- $S_{\widehat{j},0}$ denotes the kernel of matrix $H_{\widehat{j}}$ with $\{\vec{e}^1_{r_{\widehat{j}}+1}, \cdots, \vec{e}^1_n\}$ as its orthogonal basis.
- $S_{\widehat{j-1},\perp}$ with $\{\vec{e}^1_1, \ldots, \vec{e}^1_{r_{\widehat{j-1}}}\}$ as its basis of the subspace spanned by eigenvectors corresponding to nonzero eigenvalues of $H_{\widehat{j-1}}$.
- $S_{\widehat{j-1},-}$ with $\{\vec{e}^i_{r_{\widehat{j-1}}+1}, \ldots, \vec{e}^i_{r_{\widehat{j}}}\}$ as its orthogonal basis: $H_{\widehat{j-1}} s = 0$ for any $s \in S_{\widehat{j-1},-}$.

Denote $S_{\widehat{j},\perp} = S_{\widehat{j-1},\perp} \oplus_\perp S_{\widehat{j-1},-}$, which is the orthogonal complement of $S_{\widehat{j},0}$. Moreover, we have a new expression of a vector, denoted by x^1

$$x^1 = (U^1 \otimes I_2)^T x = \begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix} \in R^{2n}. \tag{22}$$

Then, we define Lyapunov functions and related energy vectors in the subspaces

$$\begin{aligned} V_{\widehat{j},0}(x) &= \sum_{l=r_{\widehat{j}}+1}^{n} \tilde{V}\left(x_l^1\right),\ \nu_{\widehat{j},0} = \sum_{l=r_{\widehat{j}}+1}^{n} \sqrt{\tilde{V}(x_l^1)}\vec{e}^1_l \in R^n \\ V_{\widehat{j-1},\perp}(x) &= \sum_{l=1}^{r_{\widehat{j-1}}} \tilde{V}\left(x_l^1\right),\ \nu_{\widehat{j-1},\perp} = \sum_{l=1}^{r_{\widehat{j-1}}} \sqrt{\tilde{V}(x_l^1)}\vec{e}^1_l \in R^n \\ V_{\widehat{j-1},-}(x) &= \sum_{l=r_{\widehat{j-1}}+1}^{r_{\widehat{j}}} \tilde{V}\left(x_l^1\right),\ \nu_{\widehat{j-1},-} = \sum_{l=r_{\widehat{j-1}}+1}^{r_{\widehat{j}}} \sqrt{\tilde{V}(x_l^1)}\vec{e}^1_l. \end{aligned}$$

Furthermore, we have the kernel of $H_{\widehat{j-1}}$, denoted by $S_{\widehat{j-1},0}$ $(= S_{\widehat{j},0} \oplus_\perp S_{\widehat{j-1},-})$, which is the orthogonal complement of $S_{\widehat{j-1},\perp}$. Correspondingly, we have

$$V_{\widehat{j-1},0}(x) = \sum_{l=r_{\widehat{j-1}}+1}^{n} \tilde{V}\left(x_l^1\right).$$

On the other hand, recalling Lemma 1 (with taking $G_1 = H_j$ and $G_2 = H_{\widehat{j}}$ with an orthogonal matrix $U = U^2$), there are a normalized orthogonal basis denoted by $\{\vec{e}^2_1, \cdots, \vec{e}^2_n\}$, and a new expression of a vector in the new coordinates (similar to (22)): $x^2 = (U^2 \otimes I_2)^T x \in R^{2n}$. Also, we have three subspaces, still with the first subspace $S_{\widehat{j},0}$, the kernel of $H_{\widehat{j}}$. Thus, we can take $\vec{e}^1_{r_l} = \vec{e}^2_{r_l}$ for $l = \widehat{j}+1, \ldots, n$. The other two subspaces are

- $S_{j,\perp}$, with its orthogonal basis $\{\vec{e}^2_1, \ldots, \vec{e}^2_{r_j}\}$, is spanned by the eigenvectors corresponding to nonzero eigenvalues of H_j;
- $S_{j,-}$ is a subspace with $\{\vec{e}^2_{r_j+1}, \ldots, \vec{e}^2_{r_{\widehat{j}}}\}$ as its orthogonal basis: $H_j s = 0$ for any $s \in S_{j,-}$.

Similarly, define $V_{j,\perp}(x) = \sum_{l=1}^{r_j} \tilde{V}(x_l^2)$, $\nu_{j,\perp} = \sum_{l=1}^{r_j} \sqrt{\tilde{V}(x_l^2)}\vec{e}^2_l$ and $V_{j,-}(x) = \sum_{l=r_j+1}^{r_{\widehat{j}}} \tilde{V}(x_l^2)$, $\nu_{j,-} = \sum_{l=r_j+1}^{r_{\widehat{j}}} \sqrt{\tilde{V}(x_l^2)}\vec{e}^2_i$. Obviously

$$S_{\widehat{j},\perp} = S_{\widehat{j-1},\perp} \oplus_\perp S_{\widehat{j-1},-} = S_{j,\perp} \oplus_\perp S_{j,-}. \tag{23}$$

Therefore, $V = V_{\widehat{j},0} + V_{\widehat{j-1},\perp} + V_{\widehat{j-1},-} = V_{\widehat{j},0} + V_{j,\perp} + V_{j,-}$. For convenience, we denote

$$V_{\widehat{j},\perp}(x) = V_{\widehat{j-1},\perp} + V_{\widehat{j-1},-} = V_{j,\perp} + V_{j,-} \tag{24}$$

and, correspondingly, a vector $\nu_{\widehat{j},\perp} = \nu_{\widehat{j-1},\perp} + \nu_{\widehat{j-1},-} = \nu_{j,\perp} + \nu_{j,-} \in R^n$ with

$$\|\nu_{\widehat{j},\perp}\| = \sqrt{\|\nu_{\widehat{j-1},\perp}\|^2 + \|\nu_{\widehat{j-1},-}\|^2} = \sqrt{\|\nu_{j,\perp}\|^2 + \|\nu_{j,-}\|^2}. \tag{25}$$

Remark 2: It is not hard to show that $S_{\widehat{j-1},-} \cap S_{j,-} = \{0\}$. If not, there will be a nonzero vector $s \in S_{\widehat{j-1},-} \cap S_{j,-} \subset S_{\widehat{j},\perp}$ with $H_{\widehat{j-1}} s = 0$ and $H_j s = 0$. Then $(H_{\widehat{j-1}} + H_j)s = 0$, which leads to a contradiction because $s \in S_{\widehat{j},\perp}$.

By (3), $rank(H_{\widehat{j-1}} + H_j) \ge \max\{rank(H_{\widehat{j-1}}), rank(H_j)\}$, which leads to the following two cases: That is, Case (I): $r_{\widehat{j}} = \max\{r_{\widehat{j-1}}, r_j\}$; and Case (II):

$$r_{\widehat{j}} > \max\{r_{\widehat{j-1}}, r_j\}. \tag{26}$$

Remark 3: Let us consider Case (II). According to (26) and Remark 2, $S_{\widehat{j-1},-} \ne \{0\}$ and $S_{j,-} \ne \{0\}$, and the angle θ_j between these two subspaces is positive. Define a set $\hat{\mathcal{H}} = \{\sum_{i=1}^{\hat{m}} H_{l_i} \in R^{n\times n} | H_{l_i} \ne H_{l_{i+1}}, l_i \in \mathcal{P}, i \le \hat{m} \le m_*\}$, which is a finite set because $\mathcal{P}$ is finite.

Because $H_j, H_{\hat{j}} \in \hat{\mathcal{H}}$, $(j = 1, \ldots, m_k)$ and $\hat{\mathcal{H}}$ is finite, there is a lower bound of θ_* $(0 < \theta_* \leq \pi/2)$ such that

$$\min_{H_{\widehat{j-1}}, H_j \in \hat{\mathcal{H}} \text{ with satisfying (26)}} \theta_j > \theta_*.$$

According to (23), for any nonzero vector $z \in S_{\hat{j},\perp}$, we have $z = z_{l,\perp} + z_{l,-}$ for l representing either $\widehat{j-1}$ or j, with $z_{l,\perp} \in S_{l,\perp}$ and $z_{l,-} \in S_{l,-}$. Define two sets

$$S_l^\delta = \left\{ z \in S_{\hat{j},\perp} : z \neq 0 \; \& \; \frac{\|z_{l,\perp}\|}{\|z\|} \leq \delta \right\} \subset S_{\hat{j},\perp} \tag{27}$$

for $l = \widehat{j-1}$ or j, with positive constant $\delta < \sin(\theta_*/2)$. Clearly, $S_{\widehat{j-1},-} \subset S_{\widehat{j-1}}^\delta \cup \{0\}$ and $S_{j,-} \subset S_j^\delta \cup \{0\}$. Therefore, $S_{\widehat{j-1}}^\delta \cap S_j^\delta = \emptyset$.

We propose an assumption for the next lemma.

Assumption 1: There is a positive constant $\gamma_{j-2} < 1$ defined in (21) (with a fixed $j \geq 2$ and $\gamma_0 = e^{-\beta\tau}$ defined in Lemma 3) such that

$$V_{\widehat{j-1},0}\left(x\left(t_{k_{j-1}}\right)\right) \leq V_{\widehat{j-1},0}\left(x(t_k)\right) \tag{28}$$

and

$$V_{\widehat{j-1},\perp}\left(x\left(t_{k_{j-1}}\right)\right) \leq \gamma_{j-2} V_{\widehat{j-1},\perp}\left(x(t_k)\right). \tag{29}$$

Lemma 5: If Assumption 1 holds with γ_{j-2}, then Assumption 1 holds with γ_{j-1} from (21); that is

$$V_{\hat{j},0}\left(x\left(t_{k_j}\right)\right) \leq V_{\hat{j},0}\left(x(t_k)\right) \tag{30}$$

and

$$V_{\hat{j},\perp}\left(x\left(t_{k_j}\right)\right) \leq \gamma_{j-1} V_{\hat{j},\perp}\left(x(t_k)\right). \tag{31}$$

Proof: (30) is obvious by Lemma 3 and construction of $V_{\hat{j},0}$. Meanwhile, due to Lemma 3 and (24) along with the definitions of $V_{\widehat{j-1},\perp}$, $V_{\widehat{j-1},-}$, $V_{j,\perp}$, and $V_{j,-}$, we have

$$\dot{V}_{\hat{j},\perp} \leq 0 \quad t \in \left[t_k, t_{k_j}\right). \tag{32}$$

In what follows, we only need to prove (31).

Consider Case (I) first. If $r_{\hat{j}} = r_{\widehat{j-1}}$, then $S_{\hat{j},\perp} = S_{\widehat{j-1},\perp}$ and $S_{\hat{j},0} = S_{\widehat{j-1},0}$. Based on **Assumption 1** and $\gamma_{j-1} > \gamma_{j-2}$ (from Lemma 4), we have the conclusion. If $r_{\hat{j}} = r_j$, then we have (20) with $p = j$. Clearly, $S_{\hat{j},\perp} = S_{j,\perp}$, $S_{\hat{j},0} = S_{j,0}$, and $\gamma_{j-1} \geq \gamma_0$, which implies the conclusion.

Next, we study Case (II). With (25), for any $\nu_{\hat{j},\perp} \in S_{\hat{j},\perp}$, we have

$$\nu_{\hat{j},\perp} \notin S_l^\delta \quad \text{if and only if} \quad \frac{\|\nu_{l,-}\|}{\|\nu_{\hat{j},\perp}\|} \leq \sqrt{1-\delta^2}$$
$$\text{if and only if} \quad \frac{V_{l,-}}{V_{\hat{j},\perp}} \leq 1 - \delta^2 \tag{33}$$

with l representing either $\widehat{j-1}$ or j. Then, we discuss the following cases.

Case (II)a: If $V_{\hat{j},\perp}(t_{k_{j-1}}) \leq \gamma_{j-2} V_{\hat{j},\perp}(t_k)$, then we have (31) because $V_{\hat{j},\perp}(t_{k_j}) \leq V_{\hat{j},\perp}(t_{k_{j-1}})$ by (32) and $\gamma_{j-2} \leq \gamma_{j-1}$.

Case (II)b: If $V_{\hat{j},\perp}(t_{k_{j-1}}) > \gamma_{j-2} V_{\hat{j},\perp}(t_k)$, then, with (29) in **Assumption 1**, we have

$$\frac{V_{\widehat{j-1},\perp}\left(t_{k_{j-1}}\right)}{V_{\hat{j},\perp}\left(t_{k_{j-1}}\right)} \leq \frac{V_{\widehat{j-1},\perp}(t_k)}{V_{\hat{j},\perp}(t_k)}. \tag{34}$$

Thus, we have two subcases.

Subcase b.1 (when $\nu_{\hat{j},\perp}(t_k) \notin S_{\widehat{j-1}}^\delta$): From (33), we have $V_{\widehat{j-1},-}(t_k)/V_{\hat{j},\perp}(t_k) \leq 1 - \delta^2$. Then, from **Assumption 1**, (21), and $V_{\hat{j},\perp} = V_{\widehat{j-1},\perp} + V_{\widehat{j-1},-}$

$$\begin{aligned}\frac{V_{\hat{j},\perp}\left(t_{k_{j-1}}\right)}{V_{\hat{j},\perp}(t_k)} &\leq \max_{\nu_{\hat{j},\perp}(t_k) \notin S_{\widehat{j-1}}^\delta} \frac{V_{\widehat{j-1},-}(t_k) + \gamma_{j-2} V_{\widehat{j-1},\perp}(t_k)}{V_{\hat{j},\perp}(t_k)} \\ &\leq (1-\delta^2)(1-\gamma_{j-2}) + \gamma_{j-2} = \gamma_{j-1}.\end{aligned} \tag{35}$$

According to (32), $V_{\hat{j},\perp}(t_{k_j}) \leq V_{\hat{j},\perp}(t_{k_{j-1}})$ in $[t_{k_{j-1}}, t_{k_j})$, which implies (31) in light of (35).

Subcase b.2 (when $\nu_{\hat{j},\perp}(t_k) \in S_{\widehat{j-1}}^\delta$): According to (34), $\nu_{\hat{j},\perp}(t_{k_{j-1}}) \in S_{\widehat{j-1}}^\delta$ if $\nu_{\hat{j},\perp}(t_k) \in S_{\widehat{j-1}}^\delta$. Similar analysis leads to $V_{\hat{j},\perp}(t_{k_j}) \leq \gamma_1 V_{\hat{j},\perp}(t_k)$, which implies (31) since $\gamma_{j-1} \geq \gamma_1$.

In a sum of the previous cases, (31) holds for Case (II). Thus, (31) is proved and the proof is completed. ■

Then, we give the proof of Theorem 1.

Proof of Theorem 1: Consider interval $[t_k, t_{k+1})$ consisting of a series of subintervals $[t_{k_l}, t_{k_{l+1}})$ $(k_0 = k, k_{m_k} = k+1)$ without topological switching in-between (corresponding to a switching signal σ). Without loss of generality, a sequence of matrices, corresponding to m_k graphs $\bar{\mathcal{G}}_{k_1}, \ldots, \bar{\mathcal{G}}_{k_{m_k}}$ in these m_k subintervals, are denoted by $H_1, H_2, \ldots, H_{m_k}$, respectively. Set

$$\gamma = \gamma_{m_*} \tag{36}$$

according to (6) and (21) with $\gamma_0 = e^{-\beta\tau}$ given in Lemma 3 and δ given in Remark 3.

We claim that, for any given switching signal σ and any initial value $x_0 = x(0)$,

$$V\left(x(t_{k+1})\right) \leq \gamma V\left(x(t_k)\right). \tag{37}$$

To prove (37), a recursive procedure based on space decomposition is given as follows.

Step 1: Take the first subinterval $[t_k, t_{k_1})$. According to Remark 1 with taking $p = 1$, **Assumption 1** holds for $j = 1$.

Step $p(\geq 2)$: With **Assumption 1** for $j = p - 1$ in Step $p - 1$, we can prove **Assumption 1** for $j = p$ by Lemma 5.

Step m_k: Since a collection of m_k graphs, $H_1, H_2, \ldots, H_{m_k}$, are jointly connected in $[t_k, t_{k+1})$, $\sum_{p=1}^{m_k} H_p$ is positive definite by Lemma 2. Therefore, $V = V_{\widehat{m_k},\perp}$. Note that $\gamma = \gamma_{m_*} \geq \gamma_{m_k-1}$ according to Lemma 4 and (36). From Lemma 5, $V(x(t_{k+1})) \leq V(x(t_{k_{m_k}})) = V_{\widehat{m_k},\perp}(x(t_{k+1})) \leq \gamma_{m_k-1} V_{\widehat{m_k},\perp}(x(t_k)) = \gamma V(x(t_k))$, which leads to (37).

Therefore, for any $t > 0$, let j_t be the largest nonnegative integer such that $t_{j_t} \leq t$. By (37) and (16), $V(x(t)) \leq V(x(t_{j_t})) \leq \gamma^{j_t} V(x(0))$. Since $j_t \to \infty$ as $t \to \infty$, we have $\lim_{t\to\infty} V(x(t)) = 0$, and the conclusion follows. ■

The proposed method can be also used for the multiagent system with the agent dynamics expressed in the form of (1), and the results are consistent with those in [6]. The detailed analysis is omitted due to the space limitations. In addition, it is quite straightforward to extend our results to the case for $\phi_i \in R^M$ with $M > 1$.

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Global Stabilization With Low Computational Cost of the Discrete-Time Chain of Integrators by Means of Bounded Controls

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***Abstract*—This note proposes a bounded nonlinear control law composed of saturation functions for the discrete-time chain of integrators. A dynamical adaptation rule of the saturation levels involved in the control law is proposed to improve the closed-loop performances. The note unifies the original work of Yang *et al.* (1997) with static saturation level and convergence improvements that recently appeared in the continuous time case. The possible ranges for the controller's parameters are extended with respect to existing results.**

***Index Terms*—Bounded control, chain of integrators, saturation functions.**

I. INTRODUCTION

Practical control applications obviously require bounded control in order to fit into the physical limits of the actuators (see, for instance, recent books [1], [2], or the special issue [3] and the chronological bibliography therein). Among the numerous existing methods, one can find the model predictive control (MPC). It is based on an online computation of an open-loop optimal input over a prediction horizon. The first step of the resulting optimal sequence is applied, and then the prediction horizon is shifted forward. The optimization problem is resolved again, and so on. MPC is a well-known method used for stabilizing linear systems with constrained control inputs [4]–[7]. However, due to the intensive computations, this method is not always applicable to fast systems. Moreover, the optimal solution may be discontinuous as it is for the optimal time problem. The linear anti-windup compensation is widely used. The saturation effect is compensated by means of a linear feedback (see [8] and [9] in continuous time and [10] and [11] in discrete time). Unfortunately, as mentioned by Megretski [12], a rigorous stability and robustness analysis is hard to carry out. Low-gain control also gave rise to much literature [12]–[15]. In this scheme, the saturation of a linear controller is usually obtained by solving a Riccati equation which depends on a specific parameter adapted online (without adaptation, only semiglobal stabilization is achieved) [12], [13], [16]. Unfortunately, in order to insure global asymptotic stability, a convex optimization problem must be solved at each time instant, a drawback that reduces the number of embedded and fast applications based on this type of control. Teel in [17] has proposed a nonlinear globally stabilizing control law composed of nested saturation functions for the continuous linear chain of integrators. Various works extended Teel's initial result to general controllable linear systems (in continuous time [18] or in discrete time [19]) and linear systems subject to measurement bounds [20]. The complexity of these methods is close to those for the unconstrained one. As mentioned in [21], a comparison

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